Problem 1 (10 points)

You are given an array $A$ of $n$ integer. All numbers except for $O(n^{1-\varepsilon})$ are in the range $[1, 1000n^2 - n]$. Find an algorithm that sorts an array $A$ in $O(n)$ time in the worst case given that $\varepsilon \in (0, 1)$. Provide running time analysis and correctness proof for your algorithm.

Problem 2 (10 points)

In class we considered the algorithm SELECT, which determines the $i$th smallest element in the array of size $n$ in $O(n)$ time for the worst case input. The first step of this algorithm is division into $n/5$ groups of 5 elements each. Consider other two versions of this algorithm: the first one uses division into $n/3$ groups of 3 elements each and the second one uses division into $n/7$ groups of 7 elements each. Otherwise both algorithms implement the same routine as SELECT. Which algorithm is asymptotically faster, the one with division into groups of 3, groups of 5 (SELECT) or groups of 7? Prove your statement.
Problem 3 (10 points)

Professor asked Bob to find the median of an integer array $A$, of size $n$, which is stored on the lab server. Bob has access to the server, however, his rights are very limited: he can only read data from the server, but cannot write to the server. The array is so large that Bob can not just copy it to his machine. Bob’s computer has only $O(\log n)$ memory. Help Bob to develop an efficient algorithm which finds the median of $A$. Provide a correctness proof and running time analysis. Full score will be given for $O(n \log n)$ expected time algorithm.

Problem 4 (20 points)

Resolve the following recurrences. Use Master theorem, if applicable. In all examples, assume that $T(1) = 1$. To simplify your analysis, you can assume that $n = a^k$ for some $a, k$.

1. $T(n) = 2T(n/8) + n^{\frac{5}{2}} \log n \log \log n$
2. $T(n) = nT(n/6)$
3. $T(n) = 16T(n/2) + \sum_{i=1}^{n} i^3 \ln(e + \frac{1}{i})$
4. $T(n) = 8T(n/4) + n$
5. $T(n) = 4T(n - 2) + 2^{2n}$
6. $T(n) = 8T(n/2) + n^3$
7. $T(n) = T(n/2) + \log n$
8. $T(n) = T(n - 1) + T(n - 2)$