January 23, 2018

1 Problem 1 (25 points)

1.1 (15 points)

For each statement below explain if it is true or false and prove your answer. Be as precise as you can. The base of log is 2 unless stated otherwise.

1. $2^{n^2} = \Theta(3^{n+\sqrt{n}})$
2. $n^{n^2} = \Omega(e^{n^3})$
3. $2^{2n} = \Theta(2^{n+2})$
4. $\log \log n = O(\log \frac{\sqrt{n}}{3\log n} + \log(n - \log n))$
5. $n^{n \log n} = \Omega(e^{n^{2-n \log n}})$
6. $n! = \omega(2^n)$
7. Let $f, g$ be positive functions with $g(n) = o(f(n))$. Then $f(n)g(n) = o(f(n))$. 

8. Let $f, g$ be positive functions. If $f(n) + g(n) = \Omega(f(n))$ then $g(n) = O((f(n))^2)$.

9. Let $f, g, h$ be positive functions. If $g(n) = o(f(n))$ and $f(n) = O(h(n))$, then $g(n) = o(h(n))$.

10. Let $f$ be positive function. Then $f(n) = O((f(n))^2)$.

1.2 (10 points)

1. Prove that 
$$\sum_{i=1}^{n} \frac{1}{i} = O(\log n).$$

2 Problem 2(25 Points)

2.1 (9 points)
Prove by induction that 
$$\sum_{i=0}^{n-1} \binom{i}{k} = \binom{n}{k+1}$$
for $n \geq 1, 0 \leq k < n.$
(Hint: Pascal’s rule states that for $1 \leq k \leq n, \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$)

2.2 (16 points)

1. Let $A, B, C$ be sets. Prove that
   (a) $A \setminus (A \setminus B) = A \cap B$
   (b) $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$
   (c) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

2. Suppose that a random machine outputs each number from 1 to $x - 1$ with equal probability. What is probability that the output is coprime with $x$, where $x = 3^n 5^m 7^k$ and $n, m, k$ are nonnegative integers.

3. We have $n$ balls. Each ball, independently and randomly, is placed into one of $n$ bins. What is the probability that there are no empty bins at the end of our experiment?

4. Prove the pigeon-hole principle, which states that the maximum of a set of numbers is greater than or equal to the arithmetic mean of the set. (Hint: assume the contrary and derive a contradiction.)