Probabilistic Reasoning

Philipp Koehn

25 March 2025

Outline

- Uncertainty
- Probability
- Inference
- Independence and Bayes' Rule

uncertainty

Uncertainty

- Let action A_t = leave for airport t minutes before flight
  Will A_t get me there on time?

- Problems
  - partial observability (road state, other drivers' plans, etc.)
  - noisy sensors (WBAL traffic reports)
  - uncertainty in action outcomes (flat tire, etc.)
  - immense complexity of modelling and predicting traffic

- Hence a purely logical approach either
  1. risks falsehood: "A_{25} will get me there on time"
  2. leads to conclusions that are too weak for decision making:

     "A_{25} will get me there on time if there's no accident on the bridge
     and it doesn't rain and my tires remain intact etc etc."

Methods for Handling Uncertainty

- **Default** or **nonmonotonic** logic:
  Assume my car does not have a flat tire
  Assume A_{25} works unless contradicted by evidence
  Issues: What assumptions are reasonable? How to handle contradiction?

- **Rules with fudge factors**:
  $A_{25} \mapsto_{0.3} AtAirportOnTime$
  $Sprinkler \mapsto_{0.99} WetGrass$
  $WetGrass \mapsto_{0.7} Rain$
  Issues: Problems with combination, e.g., $Sprinkler$ causes $Rain$?

- **Probability**
  Given the available evidence,
  A_{25} will get me there on time with probability 0.04
  Mahaviracarya (9th C.), Cardamo (1565) theory of gambling

probability

Probability

- Probabilistic assertions **summarize** effects of
  laziness: failure to enumerate exceptions, qualifications, etc.
  ignorance: lack of relevant facts, initial conditions, etc.

- **Subjective** or **Bayesian** probability:
  Probabilities relate propositions to one's own state of knowledge
  e.g., $P(A_{25}|\text{no reported accidents}) = 0.06$

- Might be learned from past experience of similar situations

- Probabilities of propositions change with new evidence:
  e.g., $P(A_{25}|\text{no reported accidents, 5 a.m.}) = 0.15$

Making Decisions under Uncertainty

- Suppose I believe the following:

$$
\begin{aligned}
P(A_{25} \text{ gets me there on time}|\ldots) &= 0.04 \\
P(A_{90} \text{ gets me there on time}|\ldots) &= 0.70 \\
P(A_{120} \text{ gets me there on time}|\ldots) &= 0.95 \\
P(A_{1440} \text{ gets me there on time}|\ldots) &= 0.9999
\end{aligned}
$$

- Which action to choose?

- Depends on my **preferences** for missing flight vs. airport cuisine, etc.

- **Utility theory** is used to represent and infer preferences

- **Decision theory** = utility theory + probability theory

Probability Basics

- Begin with a set Ω—the **sample space**
  e.g., 6 possible rolls of a die.
  $\omega \in \Omega$ is a **sample point**/**possible world**/**atomic event**

- A **probability space** or **probability model** is a sample space with an assignment $P(\omega)$ for every $\omega \in \Omega$ s.t.
  $0 \leq P(\omega) \leq 1$
  $\sum_\omega P(\omega) = 1$
  e.g., $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$.

- An **event** A is any subset of Ω

$$P(A) = \sum_{\{\omega \in A\}} P(\omega)$$

- E.g., $P(\text{die roll} \leq 3) = P(1) + P(2) + P(3) = 1/6 + 1/6 + 1/6 = 1/2$

Random Variables

- A **random variable** is a function from sample points to some range, e.g., the reals or Booleans

  e.g., $Odd(1) = true$.

- P induces a **probability distribution** for any random variable X:

$$P(X = x_i) = \sum_{\{\omega : X(\omega) = x_i\}} P(\omega)$$

- E.g., $P(Odd = true) = P(1) + P(3) + P(5) = 1/6 + 1/6 + 1/6 = 1/2$

Propositions

- Think of a proposition as the event (set of sample points) where the proposition is true

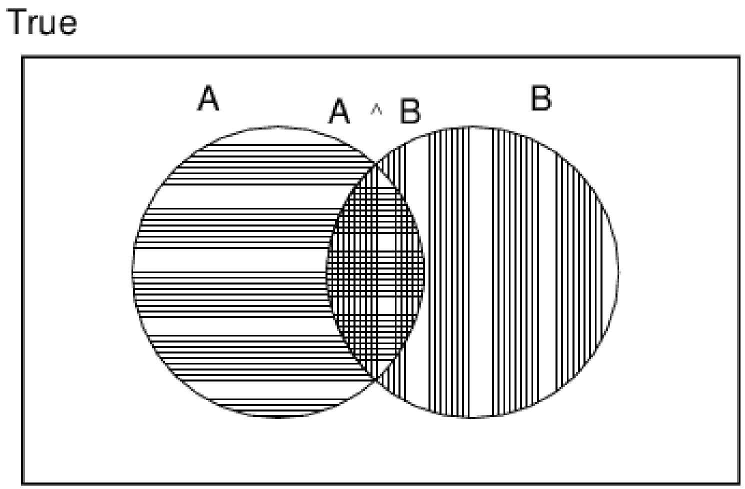
- Given Boolean random variables A and B:
  event a = set of sample points where $A(\omega) = true$
  event $\neg a$ = set of sample points where $A(\omega) = false$
  event $a \wedge b$ = points where $A(\omega) = true$ and $B(\omega) = true$

- Often in AI applications, the sample points are **defined** by the values of a set of random variables, i.e., the sample space is the Cartesian product of the ranges of the variables

- With Boolean variables, sample point = propositional logic model
  e.g., $A = true$, $B = false$, or $a \wedge \neg b$.
  Proposition = disjunction of atomic events in which it is true
  e.g., $(a \vee b) \equiv (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b)$
  $\implies P(a \vee b) = P(\neg a \wedge b) + P(a \wedge \neg b) + P(a \wedge b)$

Why use Probability?

- Logically related events have related probabilities
- E.g., $P(a \vee b) = P(a) + P(b) - P(a \wedge b)$



Syntax for Propositions

- **Propositional** or **Boolean** random variables
  e.g., $Cavity$ (do I have a cavity?)
  $Cavity = true$ is a proposition, also written $cavity$

- **Discrete** random variables (finite or infinite)
  e.g., $Weather$ is one of $\langle sunny, rain, cloudy, snow \rangle$
  $Weather = rain$ is a proposition
  Values must be exhaustive and mutually exclusive

- **Continuous** random variables (bounded or unbounded)
  e.g., $Temp = 21.6$; also allow, e.g., $Temp < 22.0$.

- Arbitrary Boolean combinations of basic propositions

Joint Probability

- **Prior** or **unconditional probabilities** of propositions
  e.g., $P(Cavity = true) = 0.1$ and $P(Weather = sunny) = 0.72$
  correspond to belief prior to arrival of any (new) evidence

- **Probability distribution** gives values for all possible assignments:
  $\mathbf{P}(Weather) = \langle 0.72, 0.1, 0.08, 0.1 \rangle$ (normalized, i.e., sums to 1)

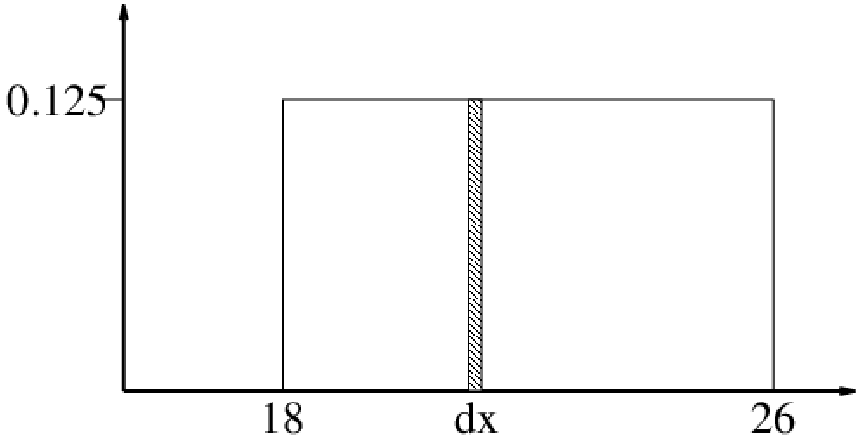
- **Joint probability distribution** for a set of random variables gives the probability of every every sample point
  $\mathbf{P}(Weather, Cavity)$ = a 4×2 matrix of values:

$Weather =$	$sunny$	$rain$	$cloudy$	$snow$
$Cavity = true$	0.144	0.02	0.016	0.02
$Cavity = false$	0.576	0.08	0.064	0.08

- **Every question about an event can be answered by the joint distribution because every event is a sum of sample points**

Probability for Continuous Variables

- Example:
  $P(X = x) = U[18, 26](x)$ = uniform density between 18 and 26

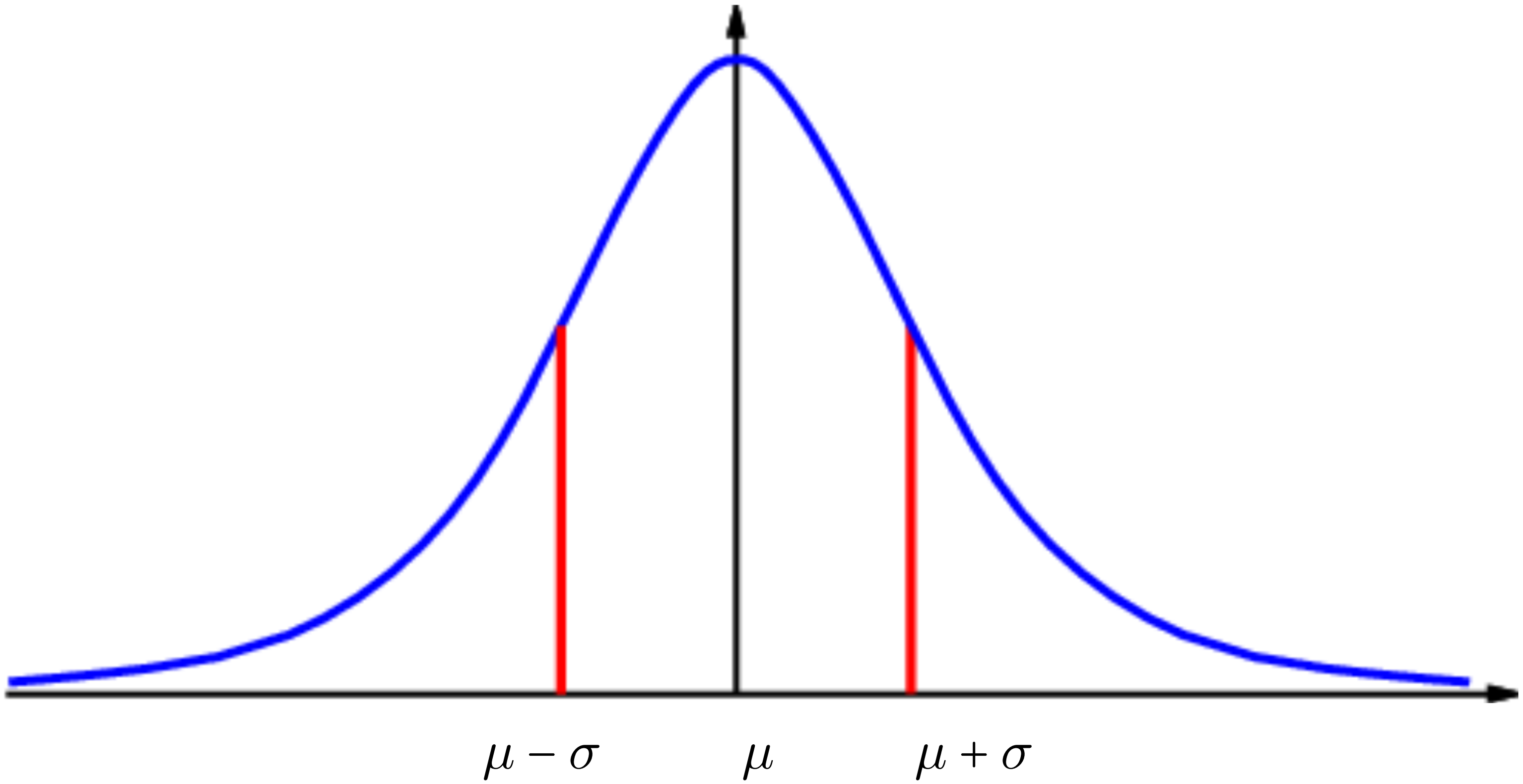


- Here P is a **density**; integrates to 1.
  $P(X = 20.5) = 0.125$ really means

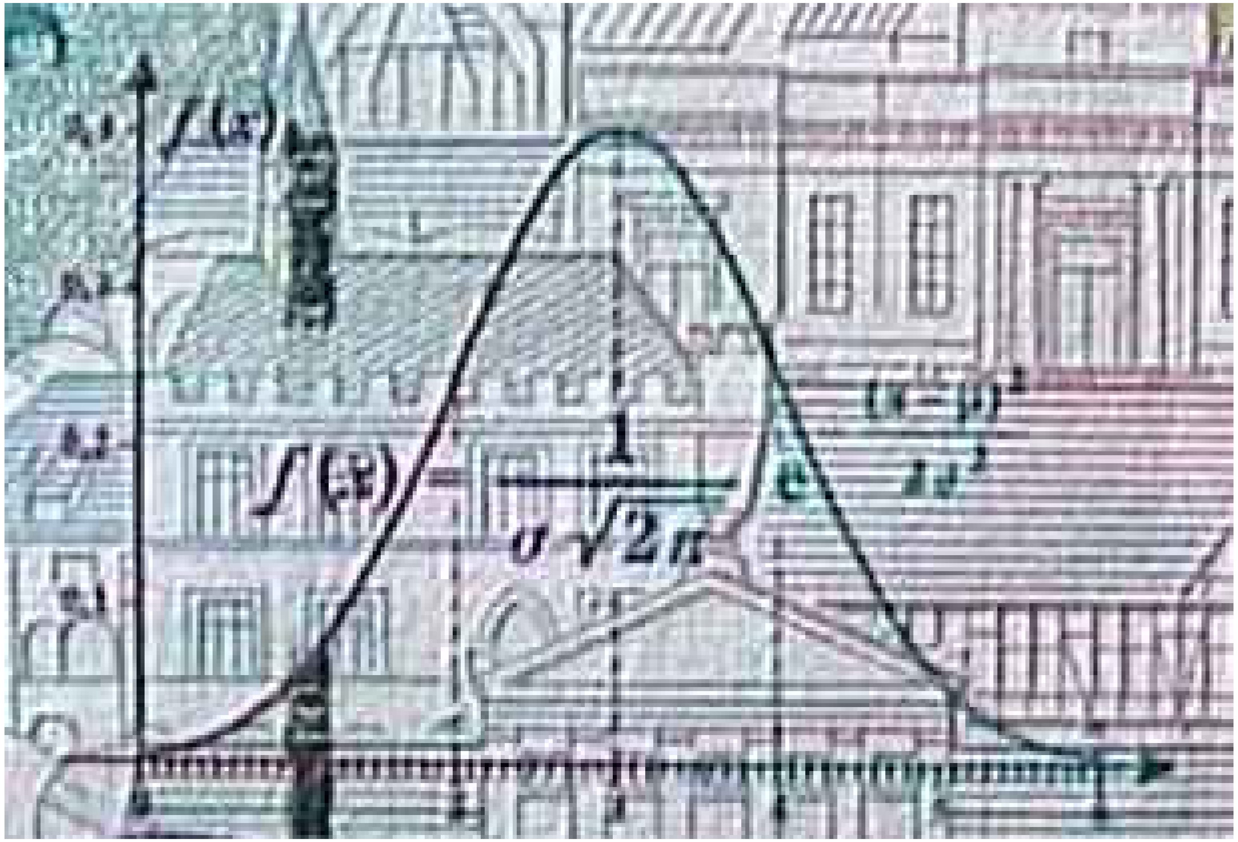
$$\lim_{\epsilon \to 0} \frac{P(20.5 - \epsilon \leq X \leq 20.5 + \epsilon)}{2\epsilon} = 0.125$$

Gaussian Density

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$




GN4480100S8
DEUTSCHE BUNDESBANK
Banknote
10
ZEHN DEUTSCHE MARK
1777–1855 Carl Friedr. Gauß
$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
Deutsche Bundesbank
Frankfurt am Main
1. September 1999
GN4480100S8


f(x)

inference

Conditional Probability

- **Conditional** or **posterior probabilities**
  e.g., $P(cavity|toothache) = 0.8$

- If we know more, e.g., $cavity$ is also given, then we have
  $P(cavity|toothache, cavity) = 1$

  Note: the less specific belief **remains valid** after more evidence arrives,
  but is may be less **useful**

- New evidence may be irrelevant, allowing simplification, e.g.,
  $P(cavity|toothache, RavensWin) = P(cavity|toothache) = 0.8$

  This kind of inference, sanctioned by domain knowledge, is crucial

Conditional Probability

- Definition of conditional probability:

$$P(a|b) = \frac{P(a \wedge b)}{P(b)} \text{ if } P(b) \neq 0$$

- **Product rule** follows from this:
  $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

- A general version holds for whole distributions, e.g.,
  $\mathbf{P}(Weather, Cavity) = \mathbf{P}(Weather|Cavity)\mathbf{P}(Cavity)$

- **Chain rule** is derived by successive application of product rule:

$$\begin{aligned} \mathbf{P}(X_1, \ldots, X_n) &= \mathbf{P}(X_1, \ldots, X_{n-1})\, \mathbf{P}(X_n|X_1, \ldots, X_{n-1}) \\ &= \mathbf{P}(X_1, \ldots, X_{n-2})\, \mathbf{P}(X_{n-1}|X_1, \ldots, X_{n-2})\, \mathbf{P}(X_n|X_1, \ldots, X_{n-1}) \\ &= \ldots \\ &= \prod_{i=1}^{n} \mathbf{P}(X_i|X_1, \ldots, X_{i-1}) \end{aligned}$$

Example: Joint Distribution

- Start with the joint distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- For any proposition ϕ, sum the atomic events where it is true:

  $P(\phi) = \sum_{\omega:\omega\models\phi} P(\omega)$

(catch = dentist's steel probe gets caught in cavity)

Inference by Enumeration

- Start with the joint distribution:

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	.108	.012	.072	.008
$\neg$ *cavity*	.016	.064	.144	.576

- For any proposition ϕ, sum the atomic events where it is true

$P(\phi) = \sum_{\omega:\omega\models\phi} P(\omega)$

$P(toothache) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$

Inference by Enumeration

- Start with the joint distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

- For any proposition ϕ, sum the atomic events where it is true:

$P(\phi) = \sum_{\omega:\omega\models\phi} P(\omega)$

$P(cavity \lor toothache) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$

Inference by Enumeration

- Start with the joint distribution:

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	.108	.012	.072	.008
$\neg$ *cavity*	.016	.064	.144	.576

- Can also compute conditional probabilities:

$$
\begin{aligned}
P(\neg cavity | toothache) &= \frac{P(\neg cavity \wedge toothache)}{P(toothache)} \\
&= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4
\end{aligned}
$$

Normalization

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- Denominator can be viewed as a normalization constant α

$$\begin{aligned}
\mathbf{P}(Cavity|toothache) &= \alpha\, \mathbf{P}(Cavity, toothache) \\
&= \alpha\, [\mathbf{P}(Cavity, toothache, catch) + \mathbf{P}(Cavity, toothache, \neg catch)] \\
&= \alpha\, [\langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle] \\
&= \alpha\, \langle 0.12, 0.08 \rangle = \langle 0.6, 0.4 \rangle
\end{aligned}$$

- General idea: compute distribution on query variable by fixing **evidence variables** and summing over **hidden variables**

Inference by Enumeration

- Let $\mathbf{X}$ be all the variables.
  Typically, we want the posterior joint distribution of the **query variables** $\mathbf{Y}$ given specific values $\mathbf{e}$ for the **evidence variables** $\mathbf{E}$

- Let the **hidden variables** be $\mathbf{H} = \mathbf{X} - \mathbf{Y} - \mathbf{E}$

- Sum out the hidden variables:

$$\mathbf{P}(\mathbf{Y}|\mathbf{E}=\mathbf{e}) = \alpha \mathbf{P}(\mathbf{Y},\mathbf{E}=\mathbf{e}) = \alpha \sum_{\mathbf{h}} \mathbf{P}(\mathbf{Y},\mathbf{E}=\mathbf{e},\mathbf{H}=\mathbf{h})$$

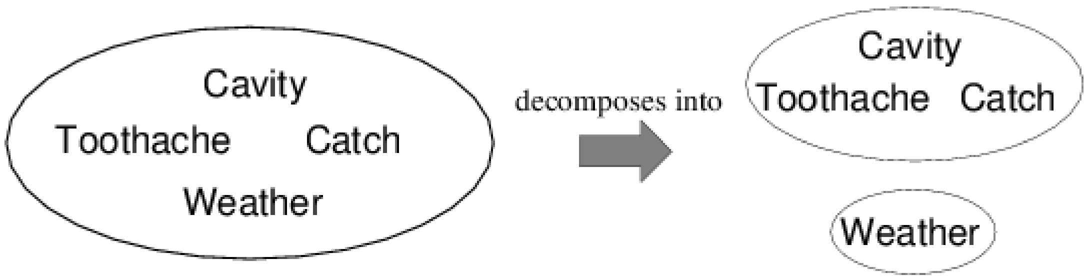
- The terms in the summation are joint entries because $\mathbf{Y}$, $\mathbf{E}$, and $\mathbf{H}$ together exhaust the set of random variables

- Obvious problems
  - Worst-case time complexity $O(d^n)$ where d is the largest arity
  - Space complexity $O(d^n)$ to store the joint distribution

independence

Independence

- A and B are **independent** iff
  $\mathbf{P}(A|B) = \mathbf{P}(A)$ or $\mathbf{P}(B|A) = \mathbf{P}(B)$ or $\mathbf{P}(A,B) = \mathbf{P}(A)\mathbf{P}(B)$



- $\mathbf{P}(Toothache, Catch, Cavity, Weather)$
  $= \mathbf{P}(Toothache, Catch, Cavity)\mathbf{P}(Weather)$

- 32 entries reduced to 12

- Absolute independence powerful but rare

- Dentistry is a large field with hundreds of variables,
  none of which are independent. What to do?

Conditional Independence

- $\mathbf{P}(Toothache, Cavity, Catch)$ has $2^3 - 1$ = 7 independent entries

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  (1) $P(catch|toothache, cavity) = P(catch|cavity)$

- The same independence holds if I haven't got a cavity:
  (2) $P(catch|toothache, \neg cavity) = P(catch|\neg cavity)$

- $Catch$ is **conditionally independent** of $Toothache$ given $Cavity$:
  $\mathbf{P}(Catch|Toothache, Cavity) = \mathbf{P}(Catch|Cavity)$

- Equivalent statements:
  $\mathbf{P}(Toothache|Catch, Cavity) = \mathbf{P}(Toothache|Cavity)$
  $\mathbf{P}(Toothache, Catch|Cavity) = \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)$

Conditional Independence

- Write out full joint distribution using chain rule:

$$\mathbf{P}(Toothache, Catch, Cavity)$$
$$= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch, Cavity)$$
$$= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)$$
$$= \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)$$

- I.e., 2 + 2 + 1 = 5 independent numbers (equations 1 and 2 remove 2)

- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in n to linear in n.

- **Conditional independence is our most basic and robust form of knowledge about uncertain environments.**

bayes rule

Bayes' Rule

- Product rule $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

$$\Longrightarrow \textbf{Bayes' rule } P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

- Or in distribution form

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha \mathbf{P}(X|Y)\mathbf{P}(Y)$$

Bayes' Rule

- Useful for assessing **diagnostic** probability from **causal** probability

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

- E.g., let M be meningitis, S be stiff neck:

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

- Note: posterior probability of meningitis still very small!

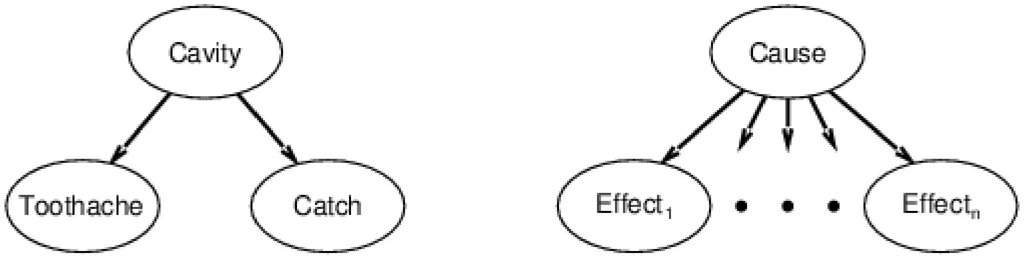
Bayes' Rule and Conditional Independence

- Example of a **naive Bayes** model

$$\begin{aligned}\mathbf{P}(Cavity|toothache \wedge catch) \\ &= \alpha\,\mathbf{P}(toothache \wedge catch|Cavity)\mathbf{P}(Cavity) \\ &= \alpha\,\mathbf{P}(toothache|Cavity)\mathbf{P}(catch|Cavity)\mathbf{P}(Cavity)\end{aligned}$$

- Generally:

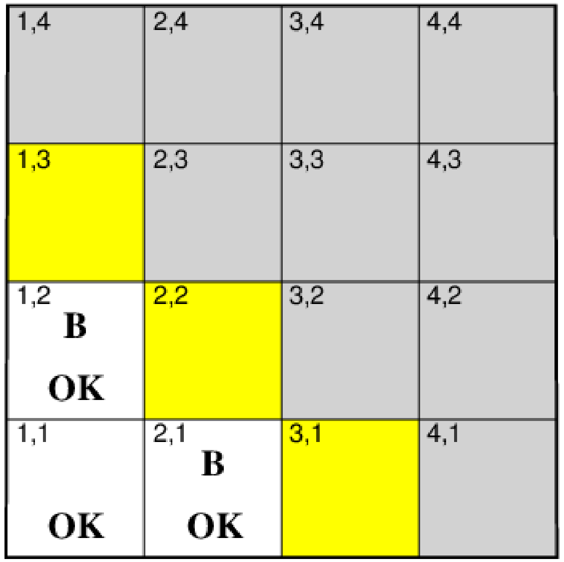
$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause)\prod_i \mathbf{P}(Effect_i|Cause)$$



- Total number of parameters is **linear** in n

wampus world

Wumpus World



- $P_{ij} = true$ iff $[i, j]$ contains a pit

- $B_{ij} = true$ iff $[i, j]$ is breezy

  Include only $B_{1,1}, B_{1,2}, B_{2,1}$ in the probability model

Specifying the Probability Model

- The full joint distribution is $\mathbf{P}(P_{1,1},\ldots,P_{4,4},B_{1,1},B_{1,2},B_{2,1})$
- Apply product rule: $\mathbf{P}(B_{1,1},B_{1,2},B_{2,1}\,|\,P_{1,1},\ldots,P_{4,4})\mathbf{P}(P_{1,1},\ldots,P_{4,4})$

  This gives us: $P(Effect|Cause)$
- First term: 1 iff pits are adjacent to breezes, 0 otherwise
- Second term: pits are placed randomly,
  independent of each other,
  probability 0.2 per square:

$$\mathbf{P}(P_{1,1},\ldots,P_{4,4}) = \prod_{i,j=1,1}^{4,4}\mathbf{P}(P_{i,j}) = 0.2^n \times 0.8^{16-n}$$

for n pits.

Observations and Query

- We know the following facts:

  $b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

  $known = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$

- Query is $\mathbf{P}(P_{1,3}|known, b)$

- Define $Unknown$ = P_{ij} other than $P_{1,3}$ and $Known$

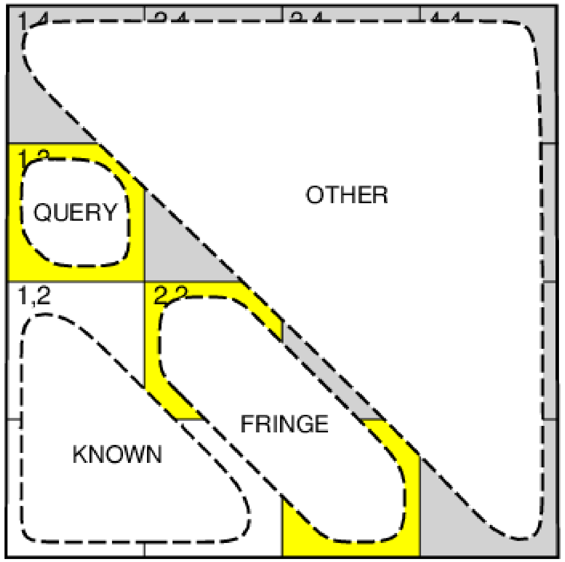
- For inference by enumeration, we have

$$\mathbf{P}(P_{1,3}|known, b) = \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, known, b)$$

- Grows exponentially with number of squares!

Using Conditional Independence

- Basic insight: observations are conditionally independent of other hidden squares given neighbouring hidden squares



- Define $Unknown = Fringe \cup Other$
  $\mathbf{P}(b|P_{1,3}, Known, Unknown) = \mathbf{P}(b|P_{1,3}, Known, Fringe)$

- Manipulate query into a form where we can use this!

Using Conditional Independence

$$\mathbf{P}(P_{1,3}|known,b) = \alpha \sum_{unknown} \mathbf{P}(P_{1,3},unknown,known,b)$$

$$= \alpha \sum_{unknown} \mathbf{P}(b|P_{1,3},known,unknown)\mathbf{P}(P_{1,3},known,unknown)$$

$$= \alpha \sum_{fringe}\sum_{other} \mathbf{P}(b|known,P_{1,3},fringe,other)\mathbf{P}(P_{1,3},known,fringe,other)$$

$$= \alpha \sum_{fringe}\sum_{other} \mathbf{P}(b|known,P_{1,3},fringe)\mathbf{P}(P_{1,3},known,fringe,other)$$

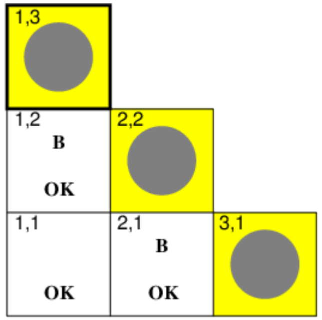
$$= \alpha \sum_{fringe} \mathbf{P}(b|known,P_{1,3},fringe) \sum_{other} \mathbf{P}(P_{1,3},known,fringe,other)$$

$$= \alpha \sum_{fringe} \mathbf{P}(b|known,P_{1,3},fringe) \sum_{other} \mathbf{P}(P_{1,3})P(known)P(fringe)P(other)$$

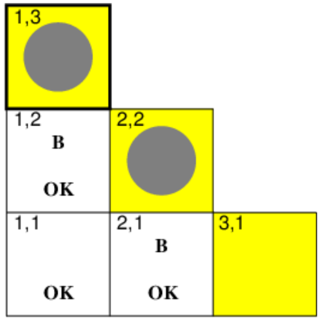
$$= \alpha\, P(known)\mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b|known,P_{1,3},fringe)P(fringe) \sum_{other} P(other)$$

$$= \alpha'\, \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b|known,P_{1,3},fringe)P(fringe)$$

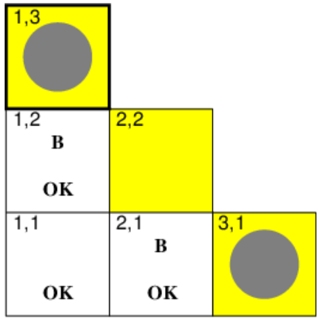
Using Conditional Independence



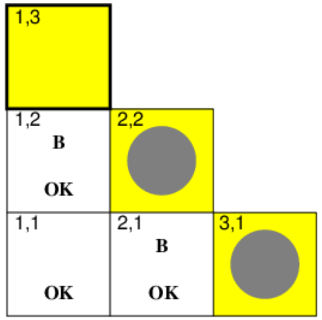
0.2 x 0.2 = 0.04



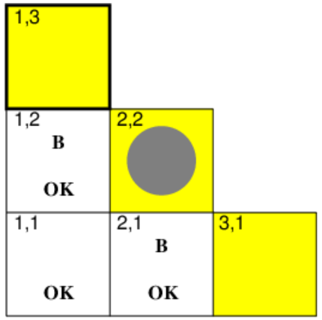
0.2 x 0.8 = 0.16



0.8 x 0.2 = 0.16



0.2 x 0.2 = 0.04



0.2 x 0.8 = 0.16

$$
\begin{aligned}
\mathbf{P}(P_{1,3}|known, b) &= \alpha' \langle 0.2(0.04 + 0.16 + 0.16),\ 0.8(0.04 + 0.16) \rangle \\
&\approx \langle 0.31, 0.69 \rangle
\end{aligned}
$$

$$
\mathbf{P}(P_{2,2}|known, b) \approx \langle 0.86, 0.14 \rangle
$$

Summary

- Probability is a rigorous formalism for uncertain knowledge
- **Joint probability distribution** specifies probability of every **atomic event**
- Queries can be answered by summing over atomic events
- For nontrivial domains, we must find a way to reduce the joint size
- **Independence** and **conditional independence** provide the tools