Informed Search

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Heuristic

From Wikipedia:

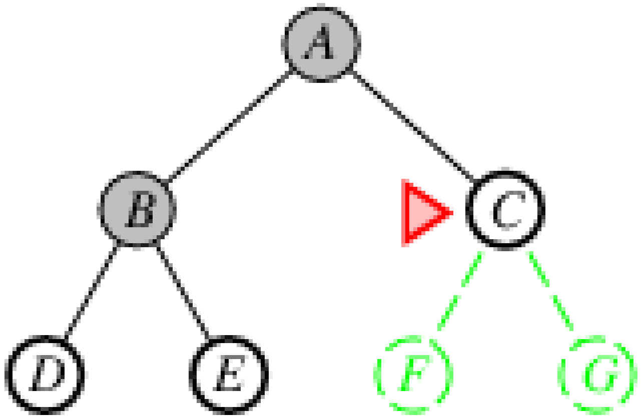
any approach to problem solving, learning, or discovery
that employs a practical method
not guaranteed to be optimal or perfect
but sufficient for the immediate goals

Outline

- Best-first search
- A* search
- Iterative improvement algorithms
  - hill-climbing
  - simulated annealing
  - genetic algorithms

best-first search

Review: Tree Search



- Search space is in form of a tree
- Strategy is defined by picking the **order of node expansion**

Best-First Search

- **Idea**: use an **evaluation function** for each node
  - estimate of "desirability"

⇒ Expand most desirable unexpanded node

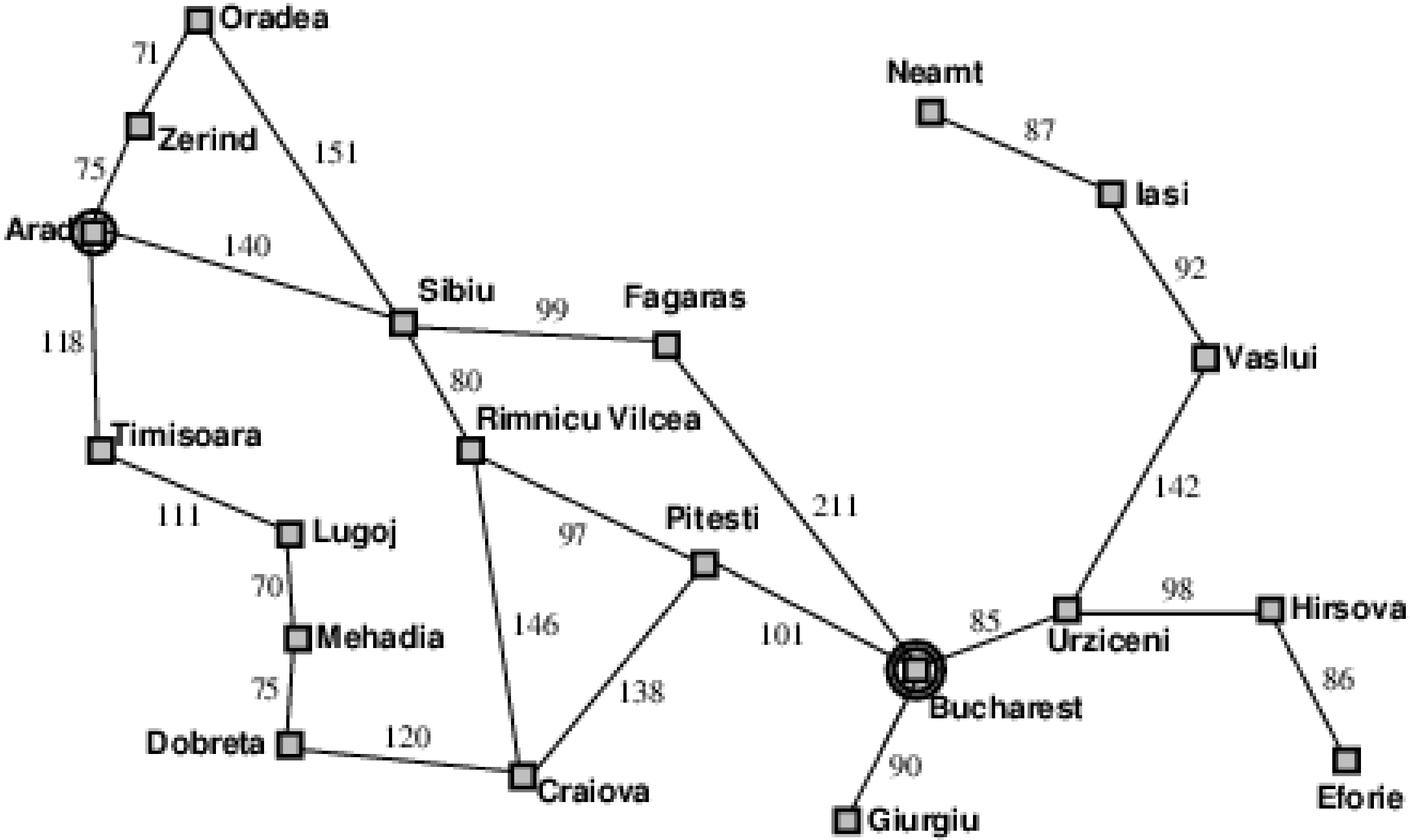
- **Implementation**:
  fringe is a queue sorted in decreasing order of desirability

- Special cases
  - greedy search
  - A* search

Romania

Legend:
International border
Motorway
Motorway under construction
Main road
Secondary road
Railroad
Airport
Port
Cities and towns (nr. of inhabitants):
> 1.000.000
300.000 - 1.000.000
100.000 - 300.000
50.000 - 100.000
20.000 - 50.000
< 20.000
BUCHAREST National capital
Major urban concentration (over 500.000 inhabitants)
Scale of altitudes and depths:
3000 m
2500 m
2000 m
1500 m
1000 m
500 m
250 m
100 m
50 m
0 m
-500 m
-1000 m
-1500 m
-2000 m
Scale: 1:1.800.000
0 (km) 100
0 (mi) 60
STEREO 70 projection, S-42 (Romania) datum
UKRAINE
HUNGARY
MOLDOVA
SERBIA
BULGARIA
BLACK SEA
TRANSYLVANIAN PLATEAU
MOLDAVIAN PLATEAU
WALLACHIAN PLAIN
DANUBE DELTA
DOBRUJAN PLATEAU
BUCHAREST
CHIȘINĂU
BELGRADE
CLUJ-NAPOCA
TIMIȘOARA
IAȘI
CONSTANȚA
CRAIOVA
BRAȘOV
GALAȚI
PLOIEȘTI
ORADEA
BRĂILA
ARAD
PITEȘTI
SIBIU
BACĂU
TÂRGU MUREȘ
BAIA MARE
BUZĂU
SATU MARE
BOTOȘANI
DEBRECEN
NYÍREGYHÁZA
SZEGED
CHERNIVTSI
TIRASPOL
RUSE
PLEVEN
KRAGUJEVAC

Romania


Oradea
71
Zerind
151
75
Arad
140
Sibiu
99
Fagaras
118
80
Rimnicu Vilcea
Timisoara
111
Lugoj
97
Pitesti
211
70
Mehadia
146
101
75
138
Dobreta
120
Craiova
Neamt
87
Iasi
92
Vaslui
142
85
Urziceni
98
Hirsova
86
Bucharest
90
Giurgiu
Eforie

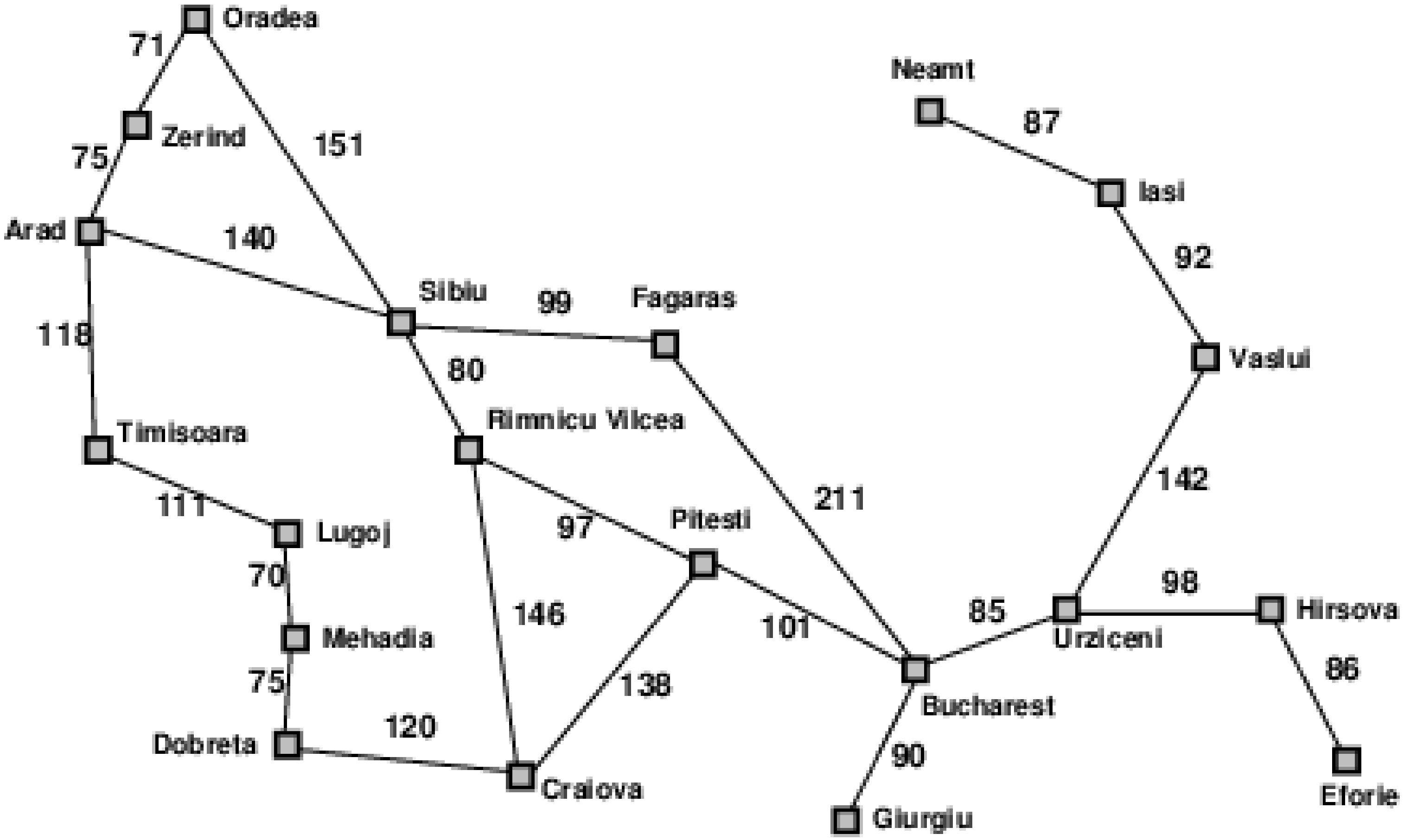
Greedy Search

- State evaluation function $h(n)$ (**heuristic**)
  = estimate of cost from n to the closest goal

- E.g., $h_{\text{SLD}}(n)$ = straight-line distance from n to Bucharest

- Greedy search expands the node that **appears** to be closest to goal

Romania with Step Costs in km



Straight–line distance to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
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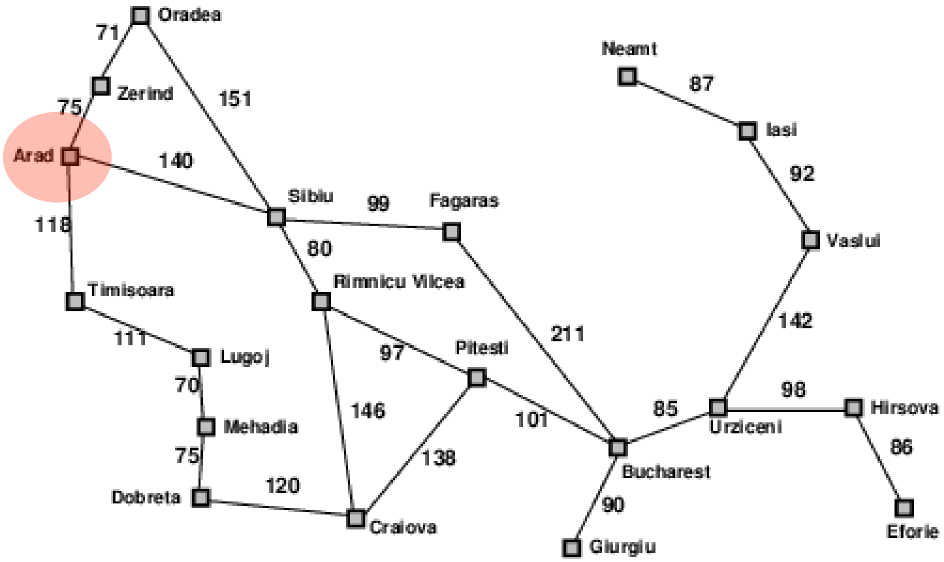
Greedy Search Example



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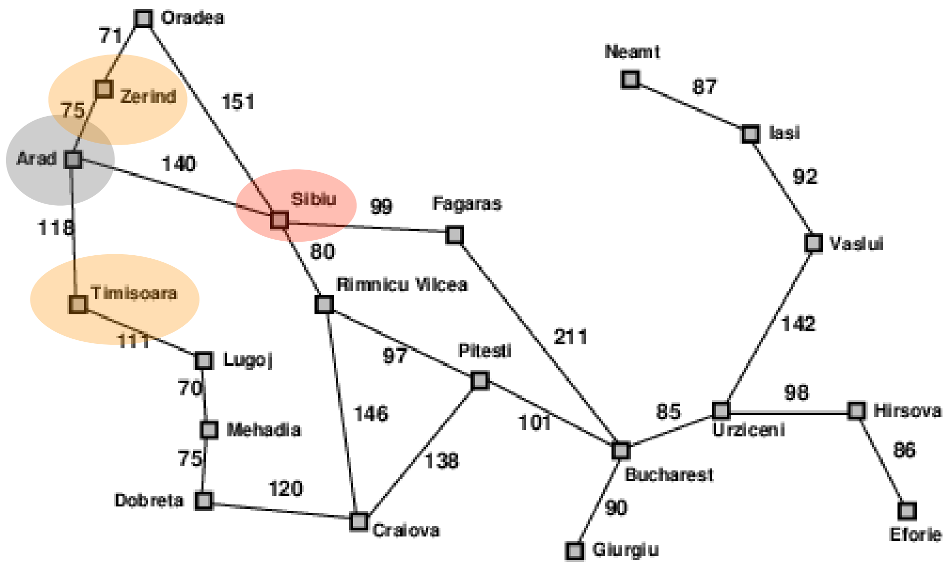

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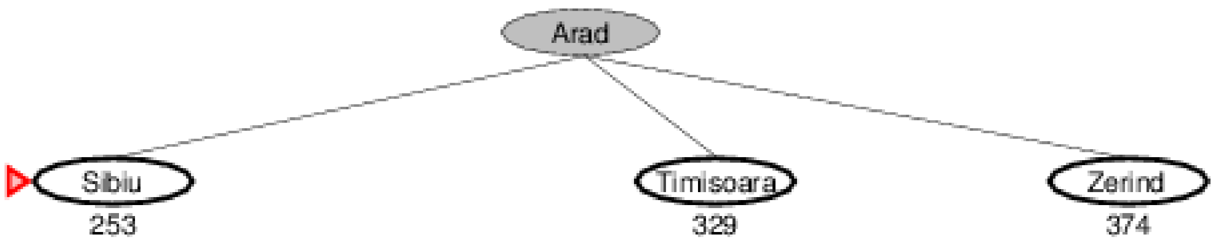
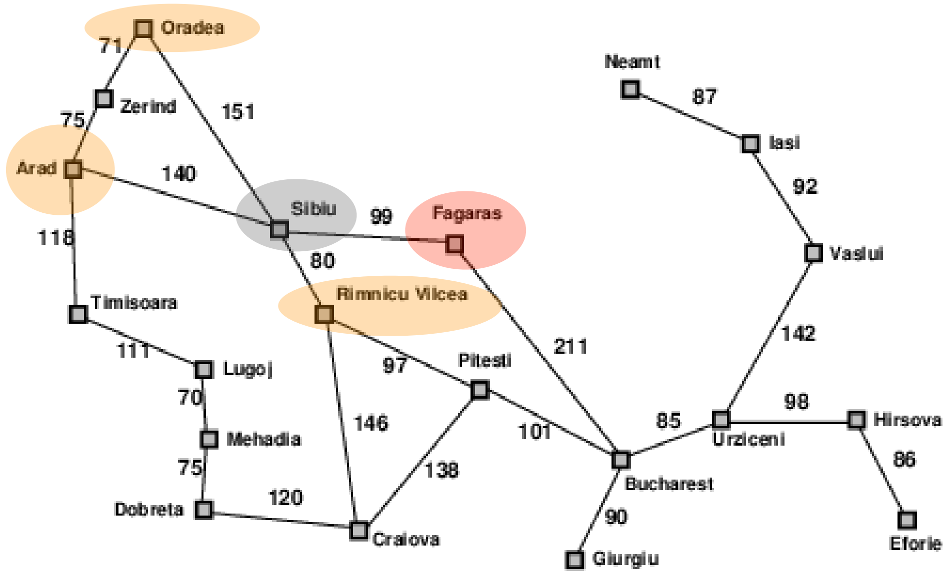


Greedy Search Example



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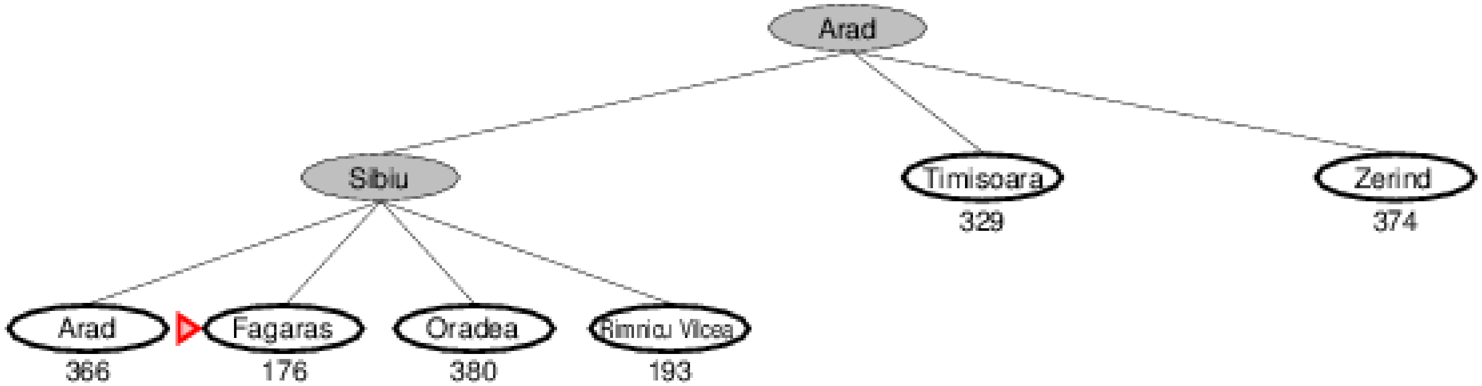

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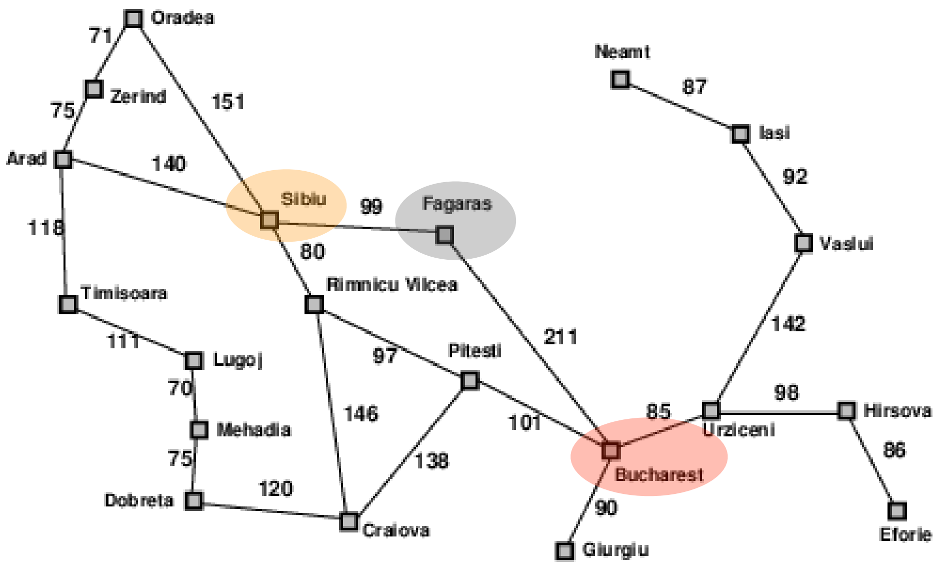

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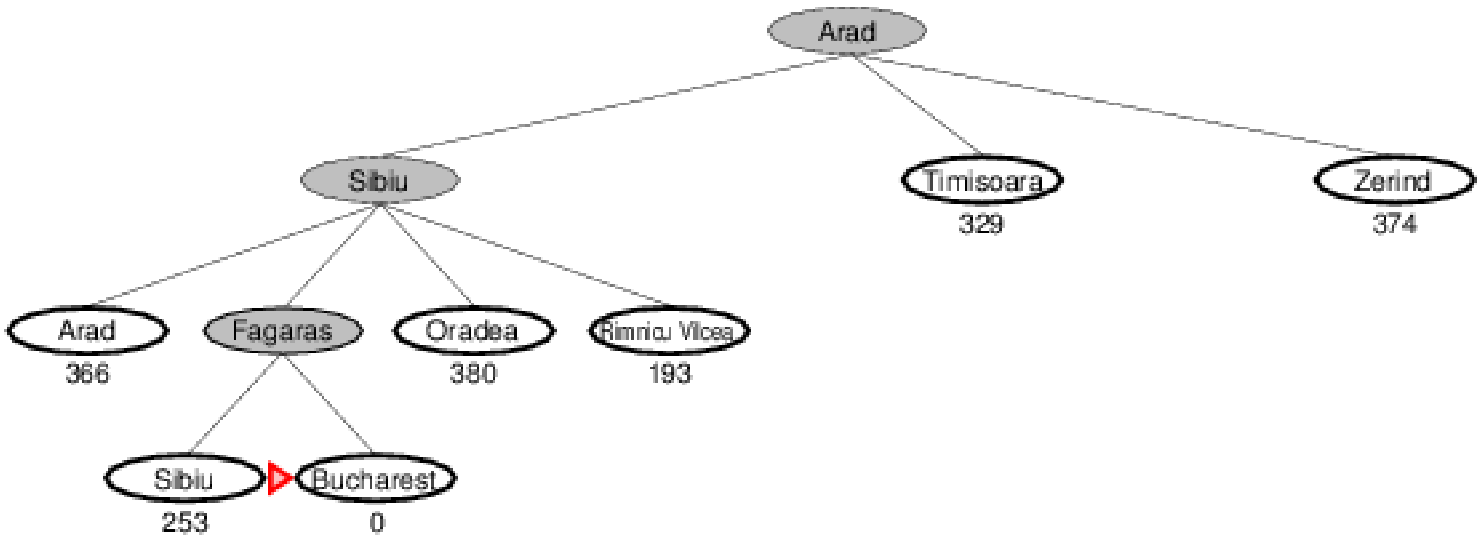
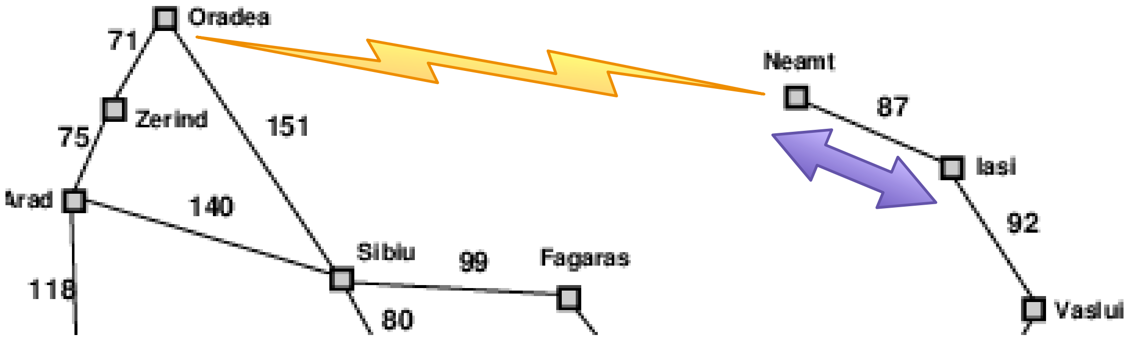


Properties of Greedy Search

- Complete? No, can get stuck in loops, e.g., with Oradea as goal,
  Iasi → Neamt → Iasi → Neamt →



  Complete in finite space with repeated-state checking

- Time? $O(b^m)$, but a good heuristic can give dramatic improvement

- Space? $O(b^m)$—keeps all nodes in memory

- Optimal? No

a* search

A* Search

- Idea: avoid expanding paths that are already expensive

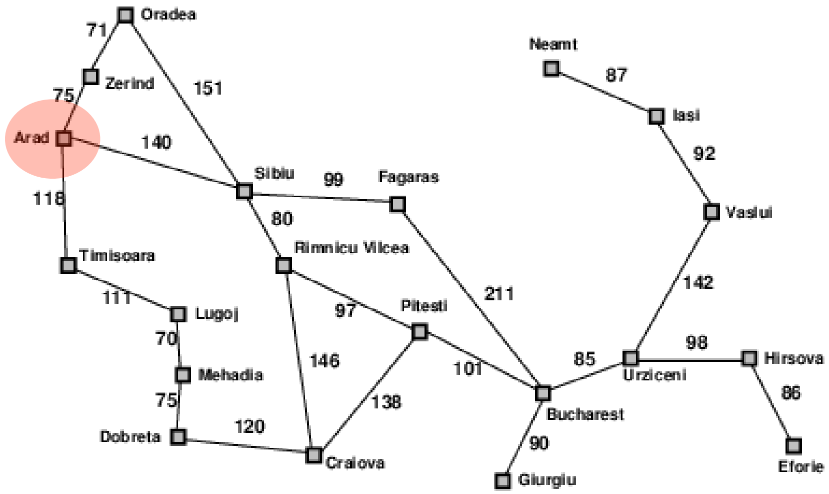
- State evaluation function $f(n) = g(n) + h(n)$
  - $g(n)$ = cost so far to reach n
  - $h(n)$ = estimated cost to goal from n
  - $f(n)$ = estimated total cost of path through n to goal

- A* search uses an **admissible** heuristic
  - i.e., $h(n) \leq h^*(n)$ where $h^*(n)$ is the **true** cost from n
  - also require $h(n) \geq 0$, so $h(G) = 0$ for any goal G

- E.g., $h_{\text{SLD}}(n)$ **never overestimates** the actual road distance

- Theorem: A* search is optimal

A* Search Example



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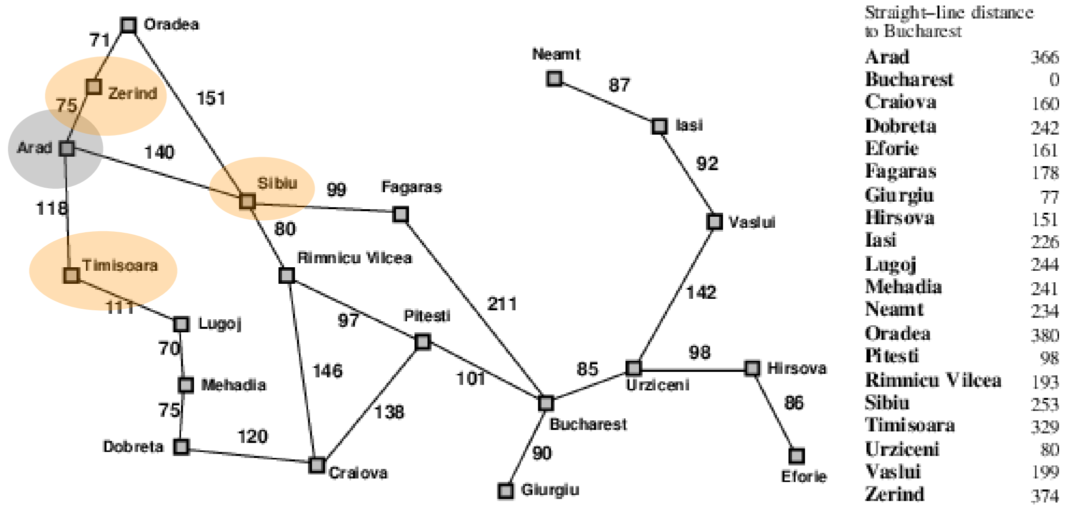

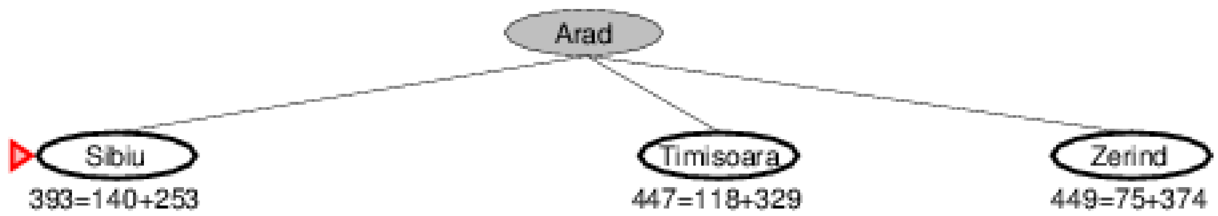

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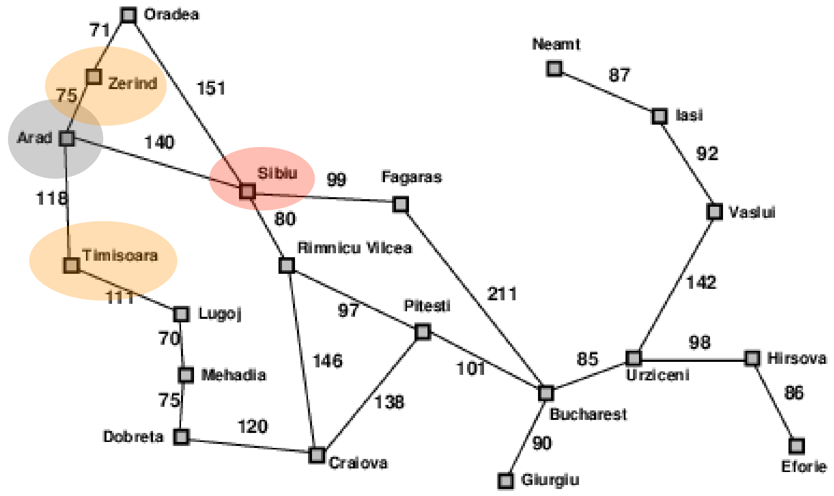

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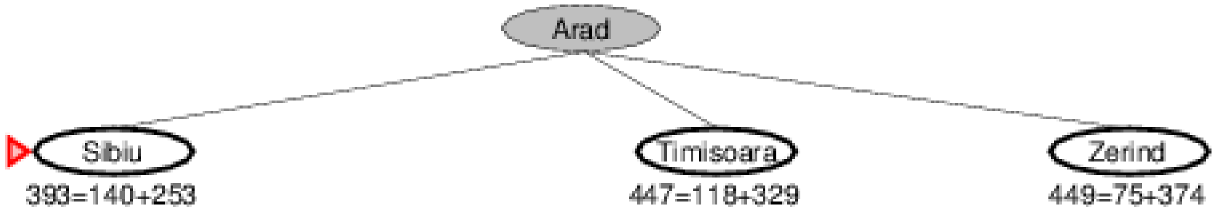

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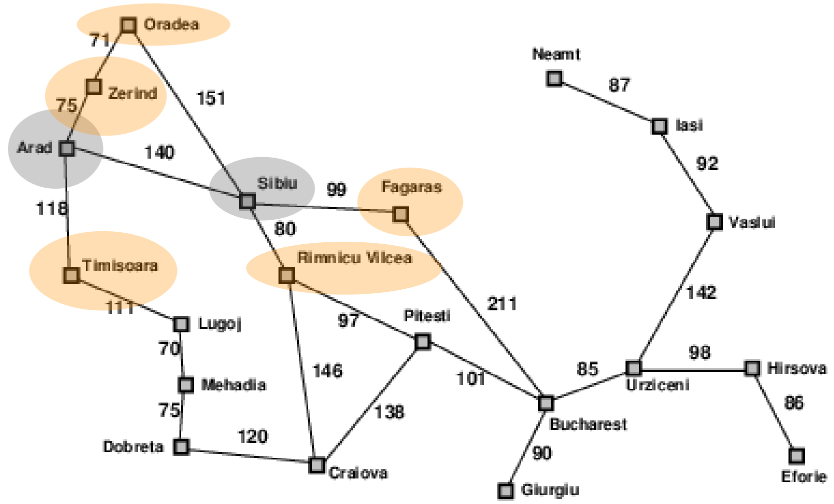

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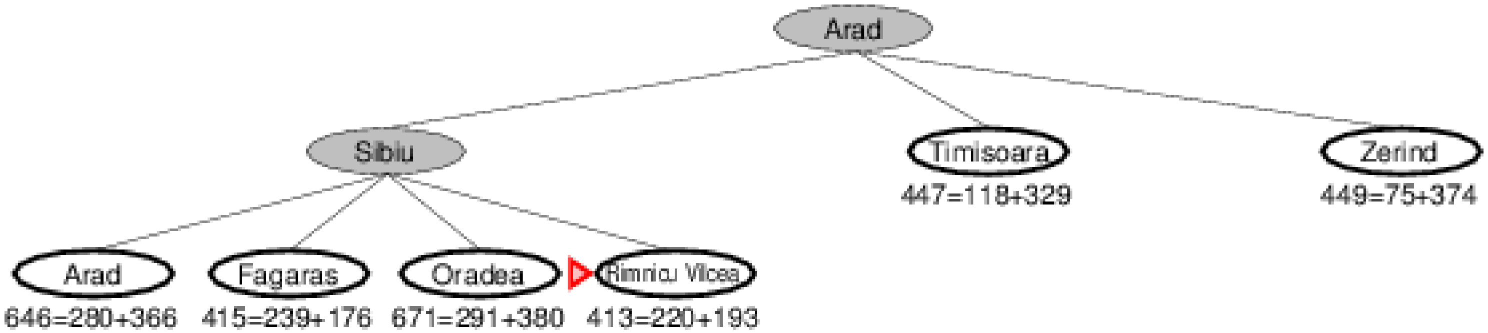

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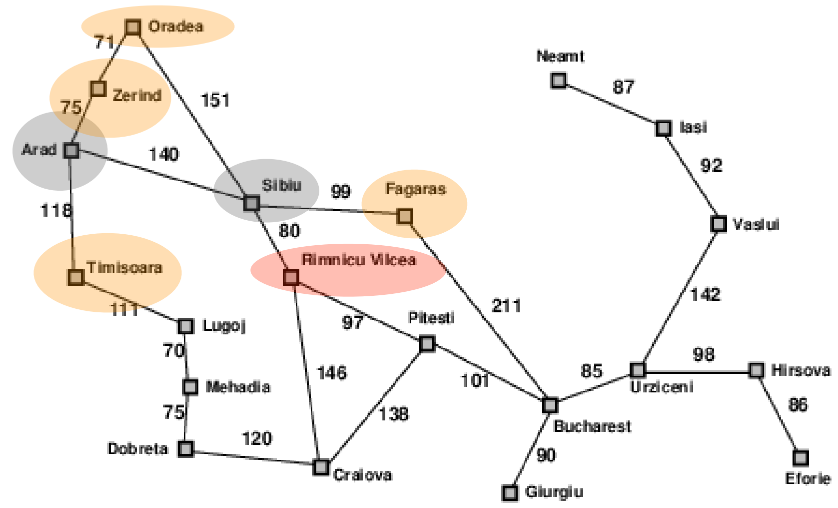

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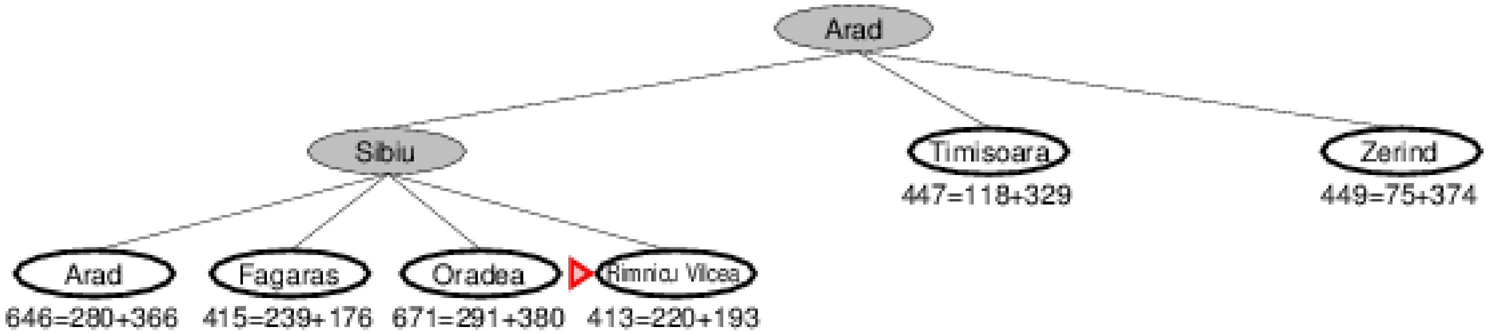

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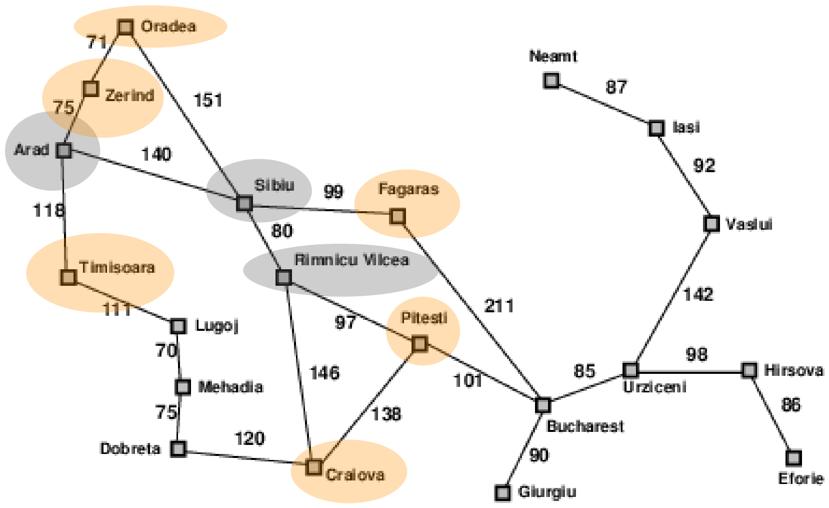

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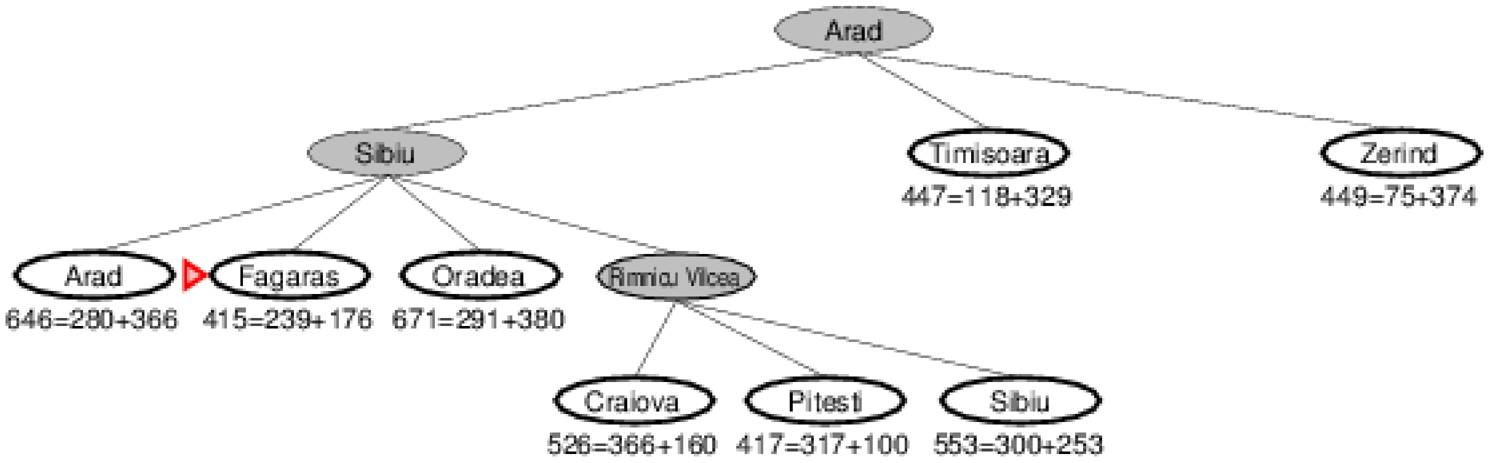

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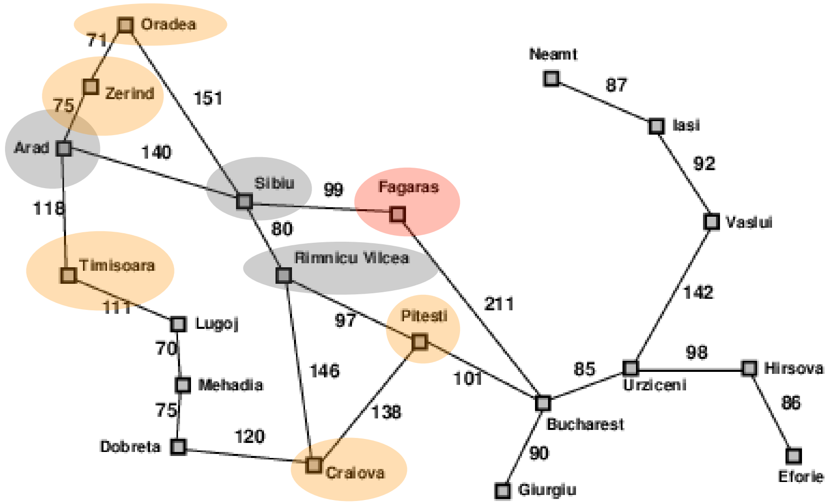

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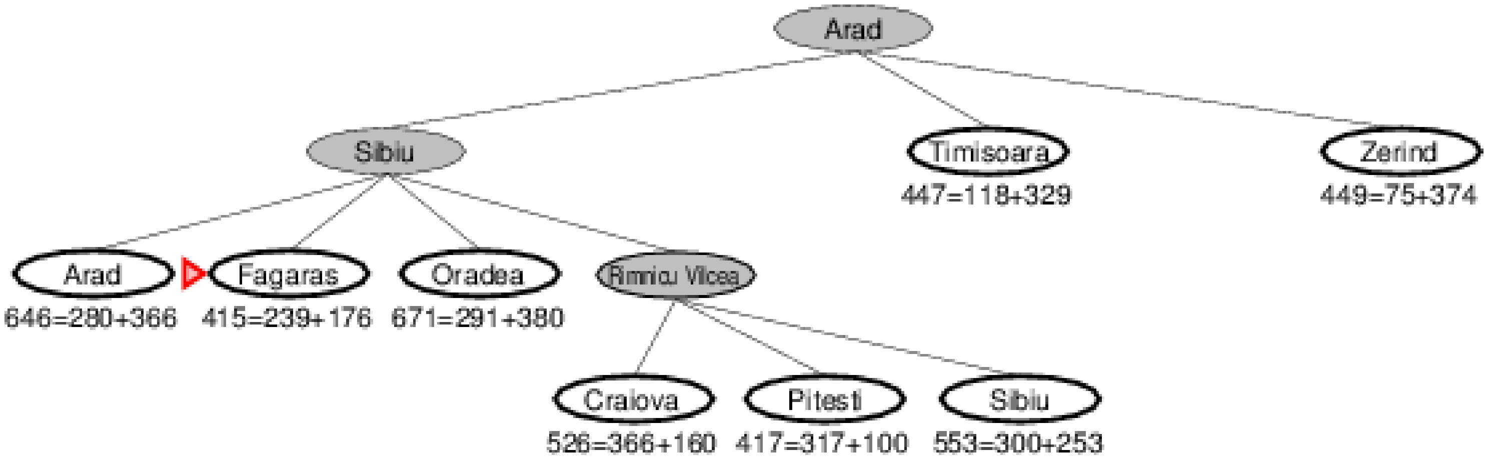
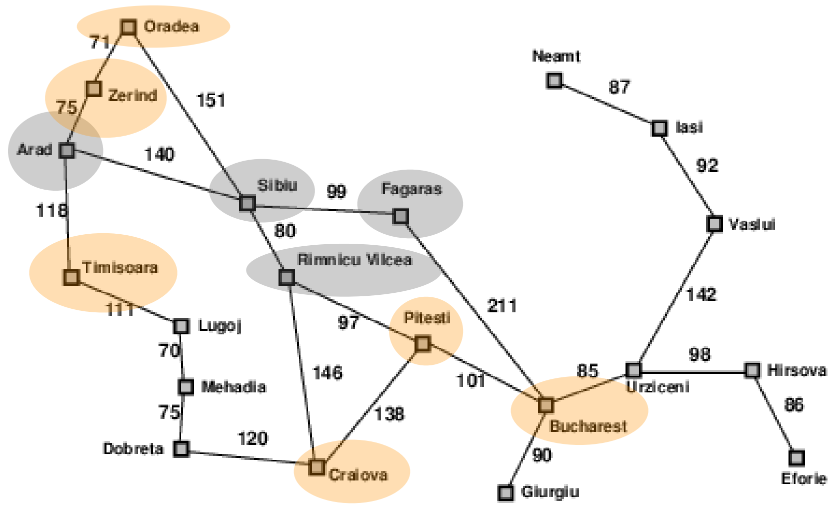


A* Search Example



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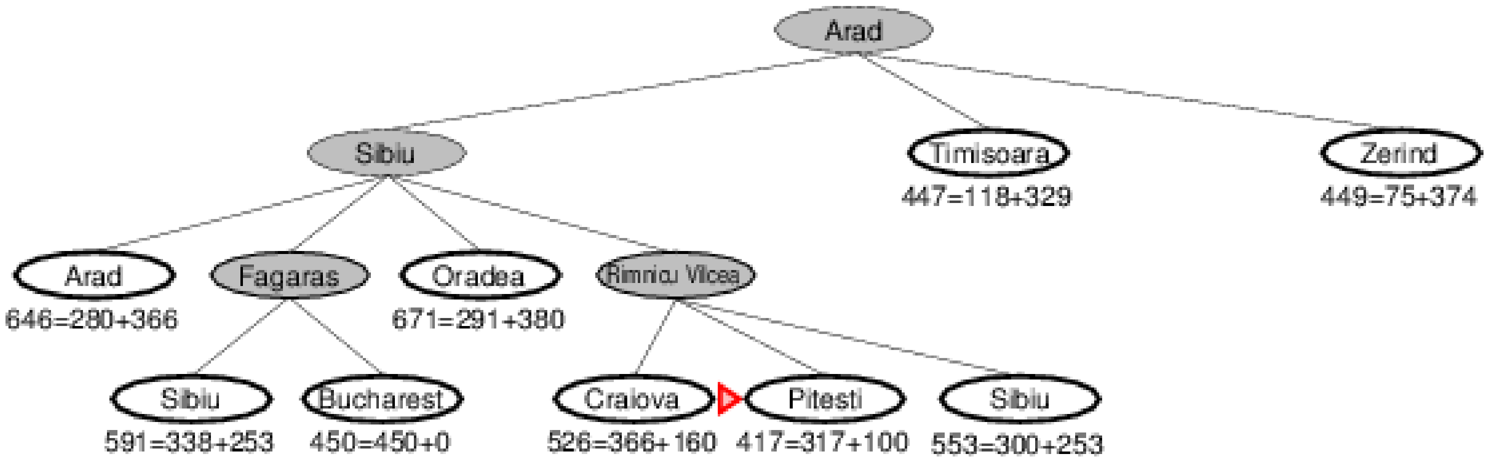

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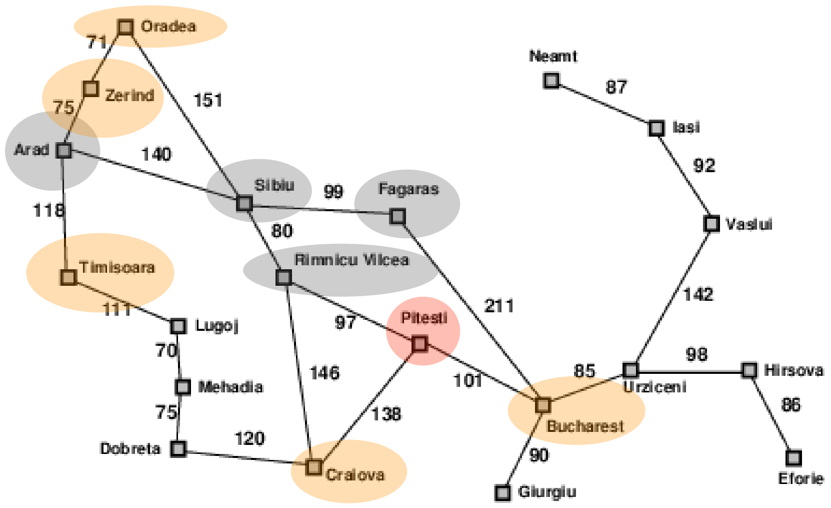

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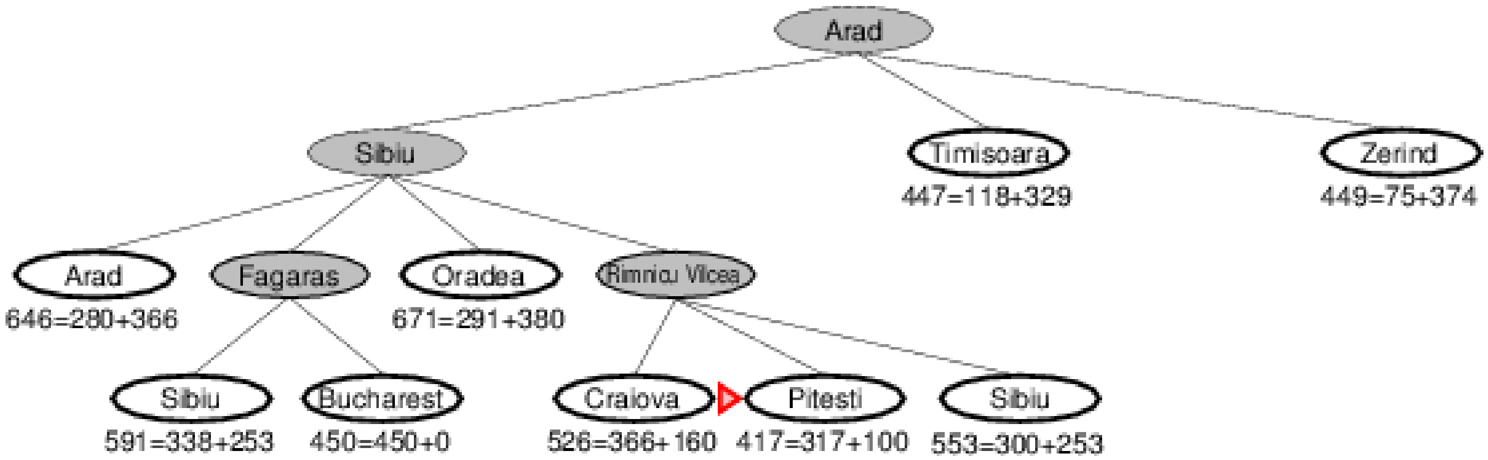

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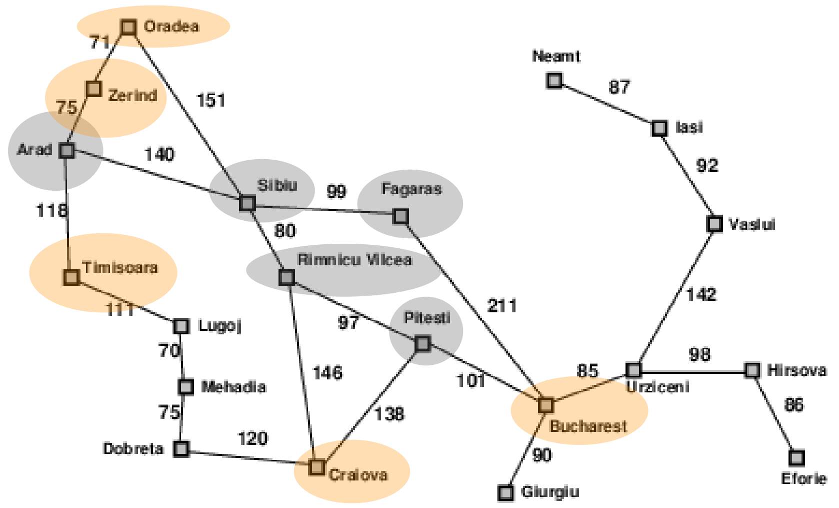

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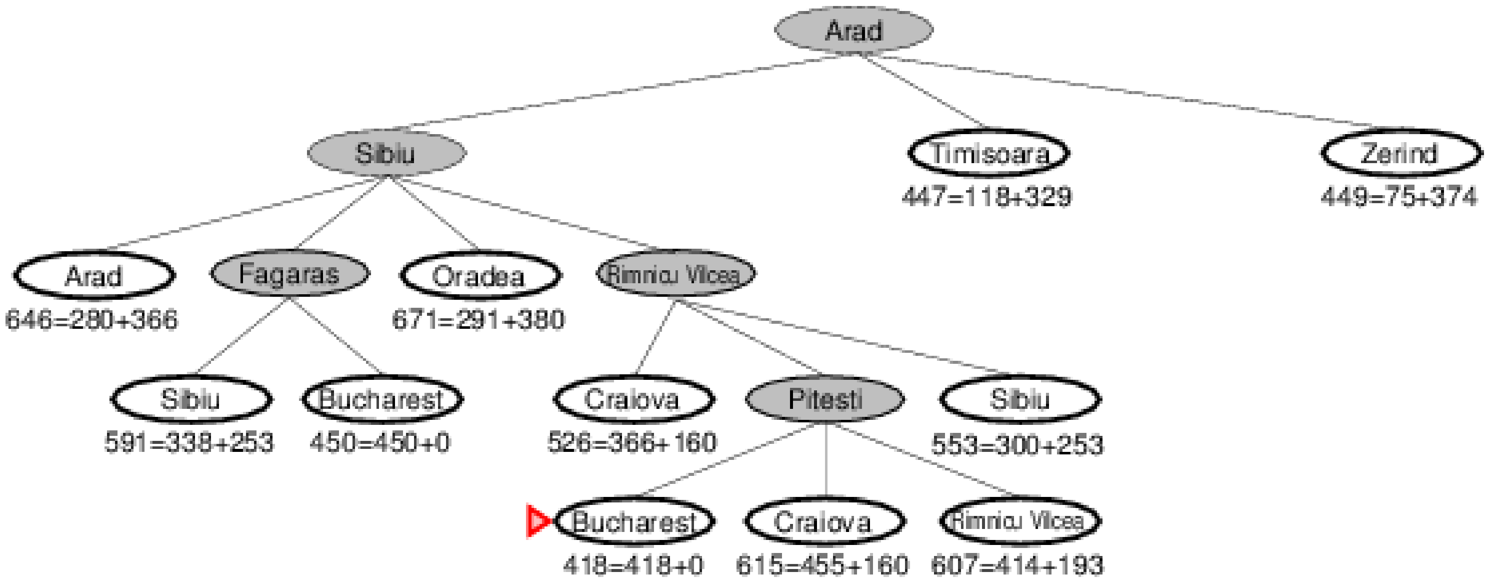

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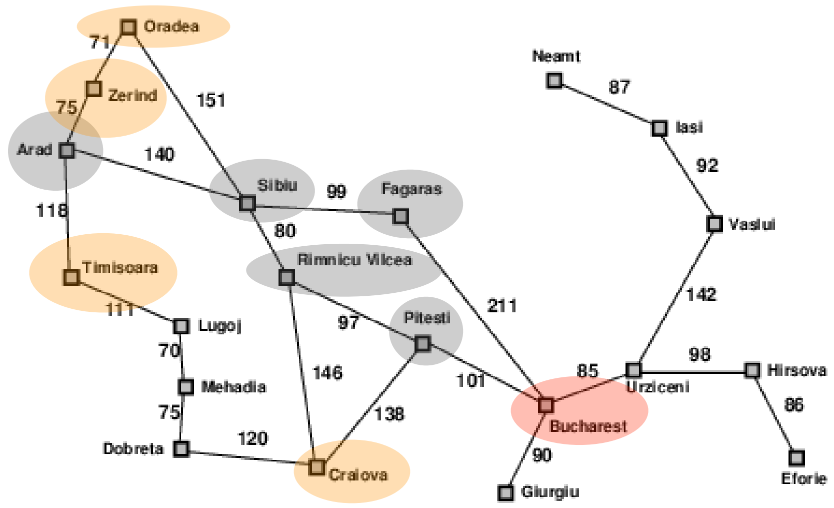

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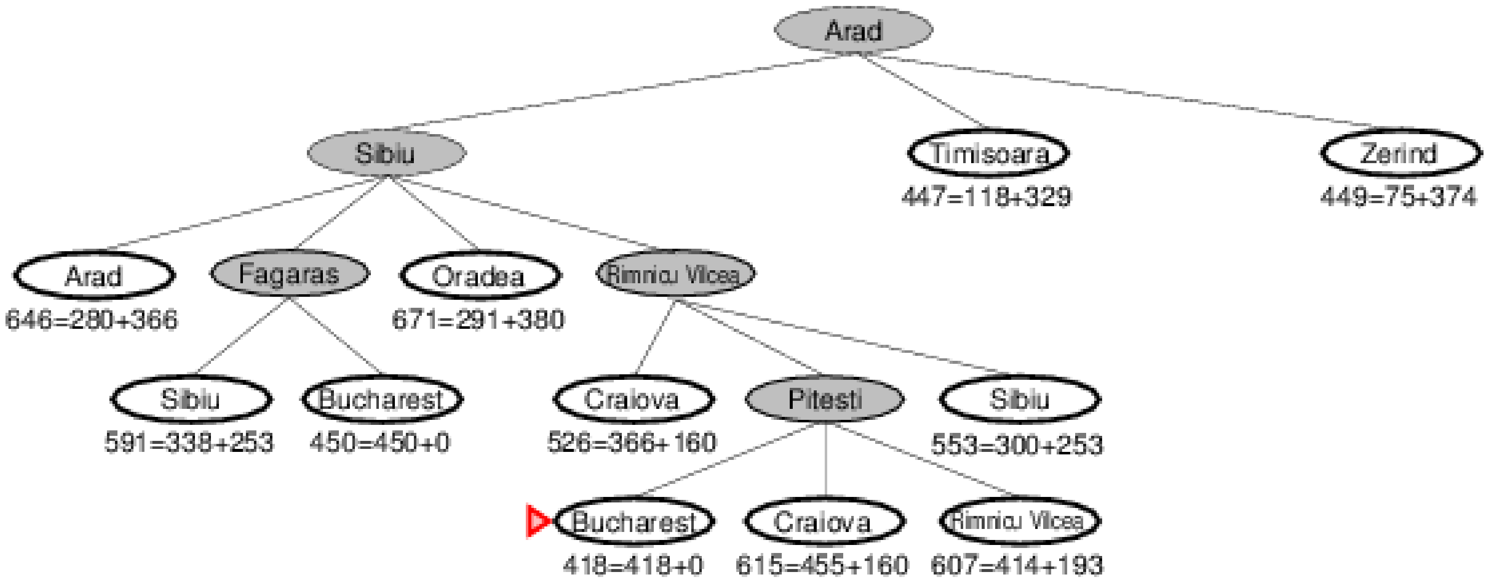
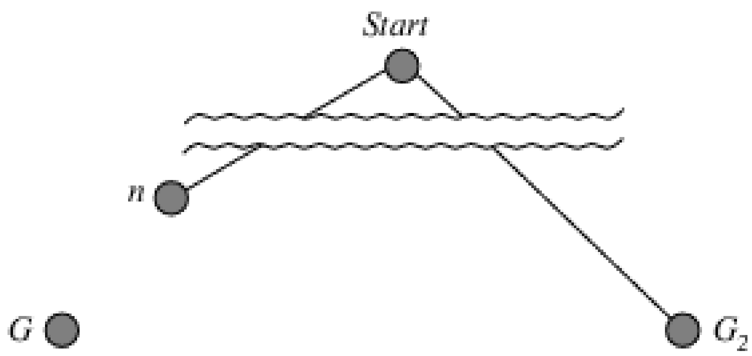


Optimality of A*

- Suppose some suboptimal goal G_2 has been generated and is in the queue
- Let n be an unexpanded node on a shortest path to an optimal goal G



$$\begin{array}{rcll} f(G_2) & = & g(G_2) & \text{since } h(G_2) = 0 \\ & > & g(G) & \text{since } G_2 \text{ is suboptimal} \\ & \geq & f(n) & \text{since } h \text{ is admissible} \end{array}$$

- Since $f(G_2) > f(n)$, A* will never terminate at G_2

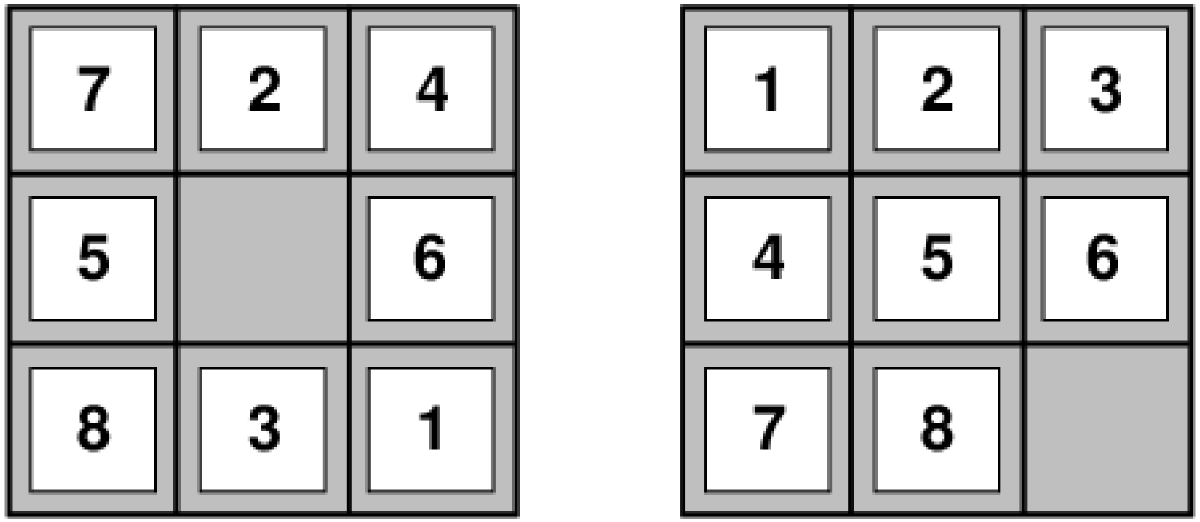
Properties of A*

- Complete? Yes, unless there are infinitely many nodes with $f \leq f(G)$
- Time? Exponential in [relative error in h × length of solution]
- Space? Keeps all nodes in memory
- Optimal? Yes—cannot expand f_{i+1} until f_i is finished

  A* expands all nodes with $f(n) < C^*$
  A* expands some nodes with $f(n) = C^*$
  A* expands no nodes with $f(n) > C^*$

Admissible Heuristics

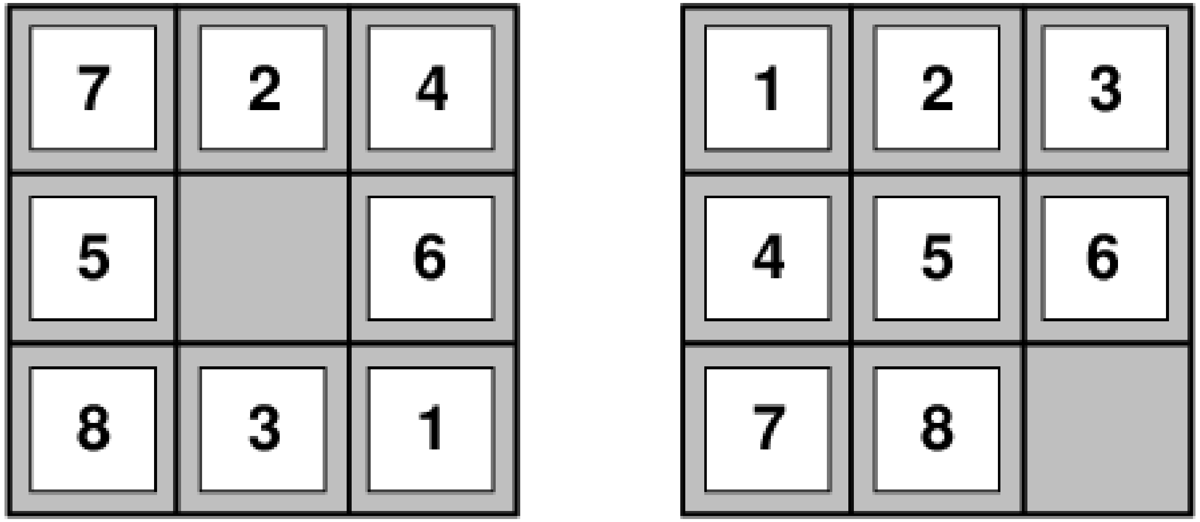
- E.g., for the 8-puzzle



Start State Goal State

Admissible Heuristics

- E.g., for the 8-puzzle
  - $h_1(n)$ = number of misplaced tiles
  - $h_2(n)$ = total Manhattan distance
    (i.e., no. of squares from desired location of each tile)



Start State Goal State

- $h_1(S) =?$
- $h_2(S) =?$

Admissible Heuristics

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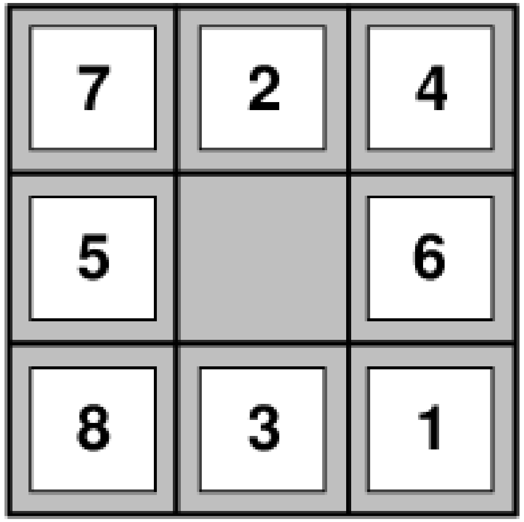
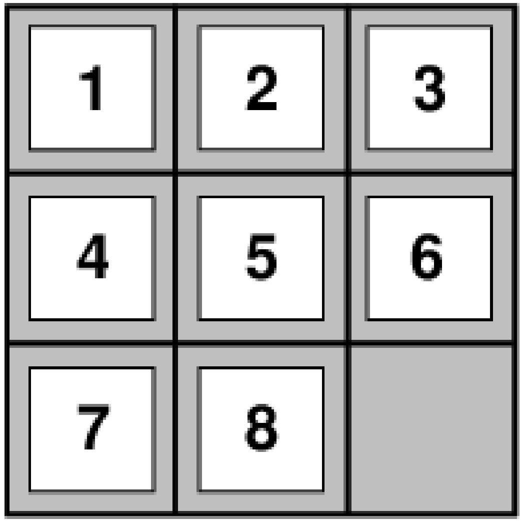


Start State



Goal State

- $h_1(S) =?$ 6
- $h_2(S) =?$ 4+0+3+3+1+0+2+1 = 14

Dominance

- If $h_2(n) \geq h_1(n)$ for all n (both admissible)
  $\rightarrow$ h_2 dominates h_1 and is better for search

- Typical search costs (d = depth of solution for 8-puzzle)

$d = 14$	IDS = 3,473,941 nodes
	$A^*(h_1)$ = 539 nodes
	$A^*(h_2)$ = 113 nodes
$d = 24$	IDS ≈ 54,000,000,000 nodes
	$A^*(h_1)$ = 39,135 nodes
	$A^*(h_2)$ = 1,641 nodes

- Given any admissible heuristics h_a, h_b,

$$h(n) = \max(h_a(n), h_b(n))$$

  is also admissible and dominates h_a, h_b

Relaxed Problems

- Admissible heuristics can be derived from the **exact** solution cost of a **relaxed** version of the problem

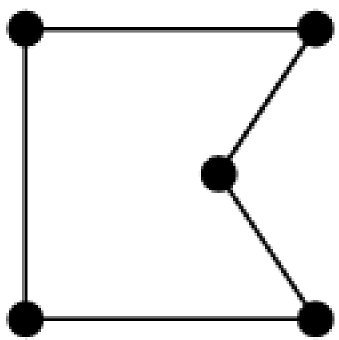
- If the rules of the 8-puzzle are relaxed so that a tile can move **anywhere**
  $\Rightarrow h_1(n)$ gives the shortest solution

- If the rules are relaxed so that a tile can move to **any adjacent square**
  $\Rightarrow h_2(n)$ gives the shortest solution

- Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem

Relaxed Problems

- Well-known example: travelling salesperson problem (TSP)
- Find the shortest tour visiting all cities exactly once



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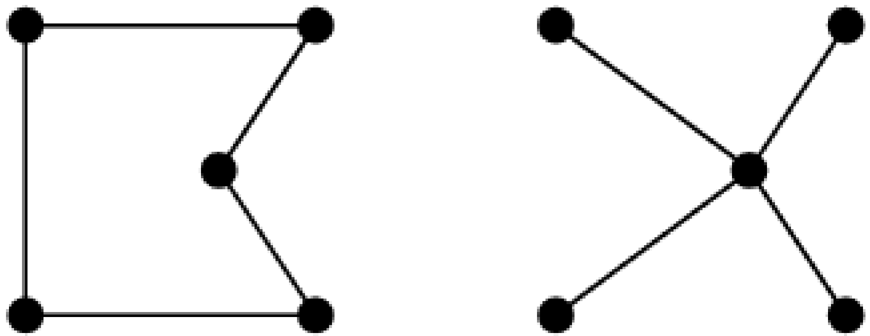


- Minimum spanning tree
  - connects all vertices without cycles, with the minimum total edge weight
  - can be computed in $O(n^2)$
  - is a lower bound on the shortest (open) tour

Summary: A*

- Heuristic functions estimate costs of shortest paths
- Good heuristics can dramatically reduce search cost
- Greedy best-first search expands lowest h
  - incomplete and not always optimal
- A* search expands lowest $g + h$
  - h is never an over-estimate
  - complete and optimal
  - also optimally efficient (up to tie-breaks, for forward search)
- Admissible heuristics can be derived from exact solution of relaxed problems

iterative improvement algorithms

Iterative Improvement Algorithms

- In many optimization problems, **path** is irrelevant;
  the goal state itself is the solution

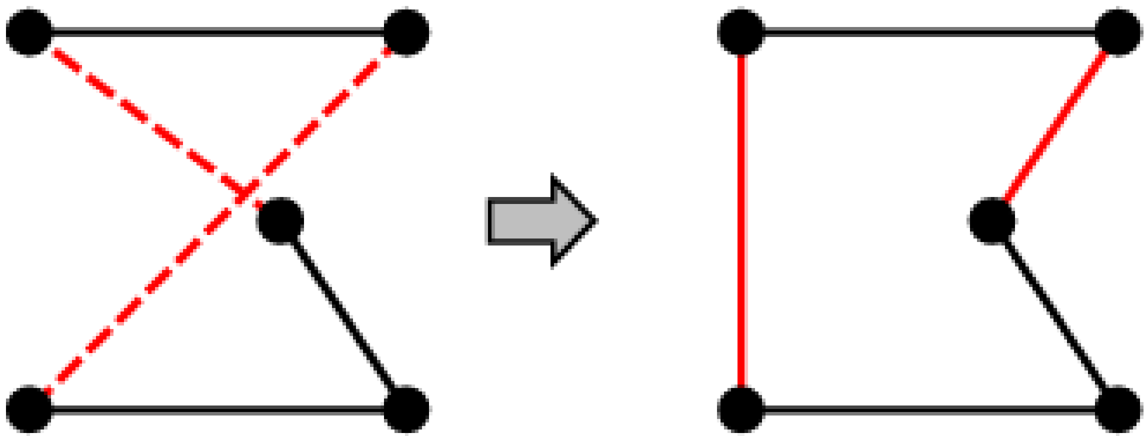
- Then state space = set of "complete" configurations
  - find **optimal** configuration, e.g., TSP
  - find configuration satisfying constraints, e.g., timetable

- In such cases, can use **iterative improvement** algorithms
  → keep a single "current" state, try to improve it

- Constant space, suitable for online as well as offline search

Example: Travelling Salesperson Problem

- Start with any complete tour, perform pairwise exchanges

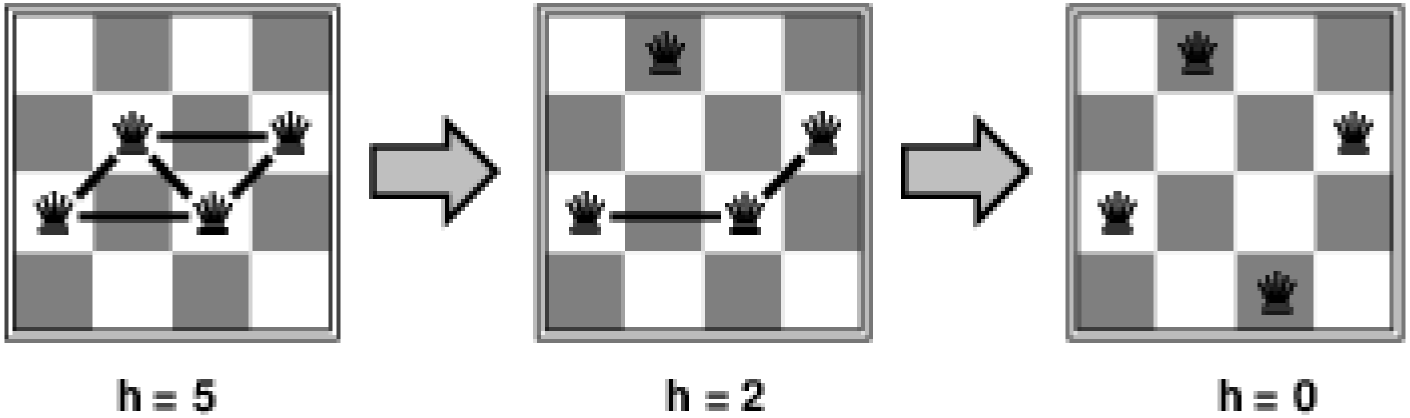


- Variants of this approach get within 1% of optimal quickly with 1000s of cities

Example: n-Queens

- Put n queens on an $n \times n$ board with no two queens on the same row, column, or diagonal

- Move a queen to reduce number of conflicts (h)



- Almost always solves n-queens problems almost instantaneously for very large n, e.g., $n = 1$ million

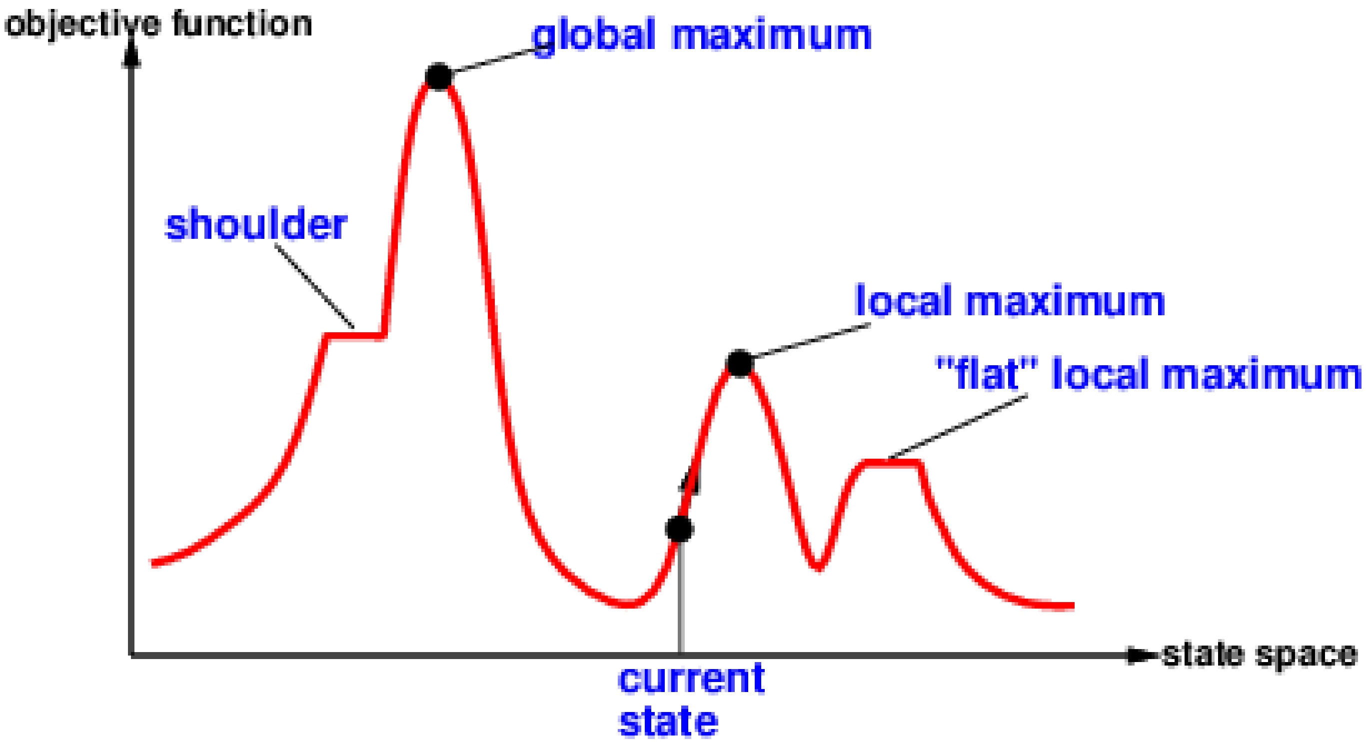
Hill-Climbing

- For instance: Gradient Ascent (or Descent)
- "Like climbing Everest in thick fog with amnesia"

1. Start state = a solution (maybe randomly generated)
2. Consider neighboring states, e.g.,
   - move a queen
   - pairwise exchange in traveling salesman problem
3. No better neighbors? Done.
4. Adopt best neighbor state
5. Go to step 2

Hill-Climbing

- Useful to consider state space landscape



- **Random-restart hill climbing** overcomes local maxima—trivially complete
- **Random sideways moves** ☺ escape from shoulders ☹ loop on flat maxima

Local Beam Search

- **Idea**: keep k states instead of 1; choose top k of all their successors
- Not the same as k searches run in parallel!

- **Problem**: quite often, all k states end up on same local hill
- **Idea**: choose k successors randomly, biased towards good ones

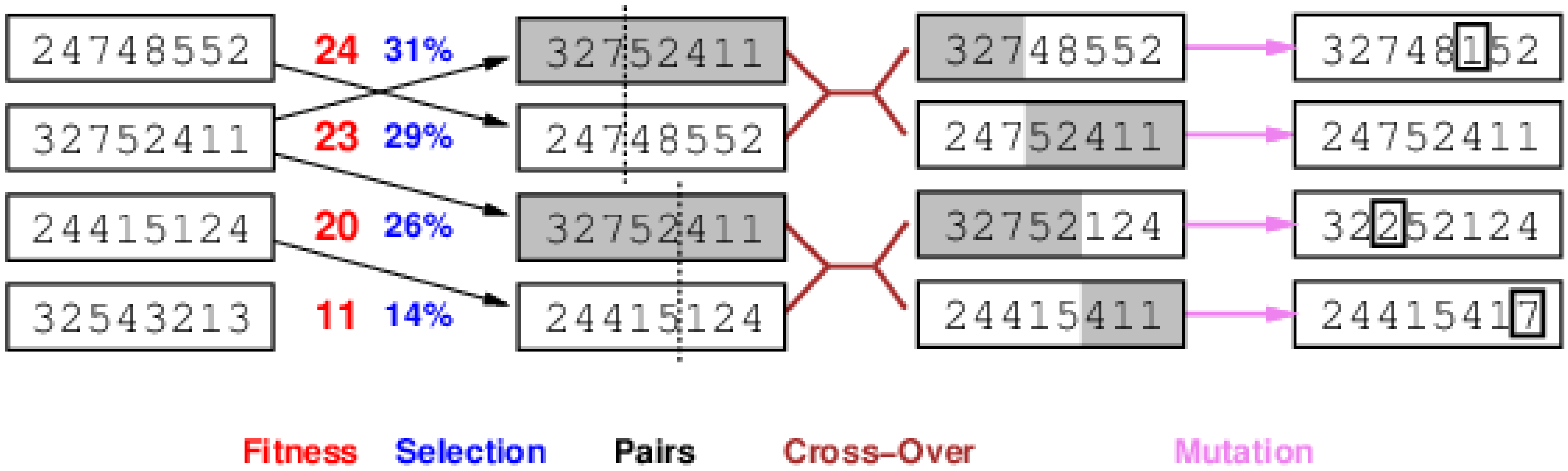
Simulated Annealing

- Idea: escape local maxima by allowing some "bad" moves
- **But gradually decrease their size and frequency**

- Iterate, reduce temperature T over time
  - compute **best** greedy move
  - draw **random** move
  - compute difference in value ΔE = value(**random**) – value(**best**)
  - with probability $e^{\frac{\Delta E}{T}}$: return **random**
  - else: return **best**

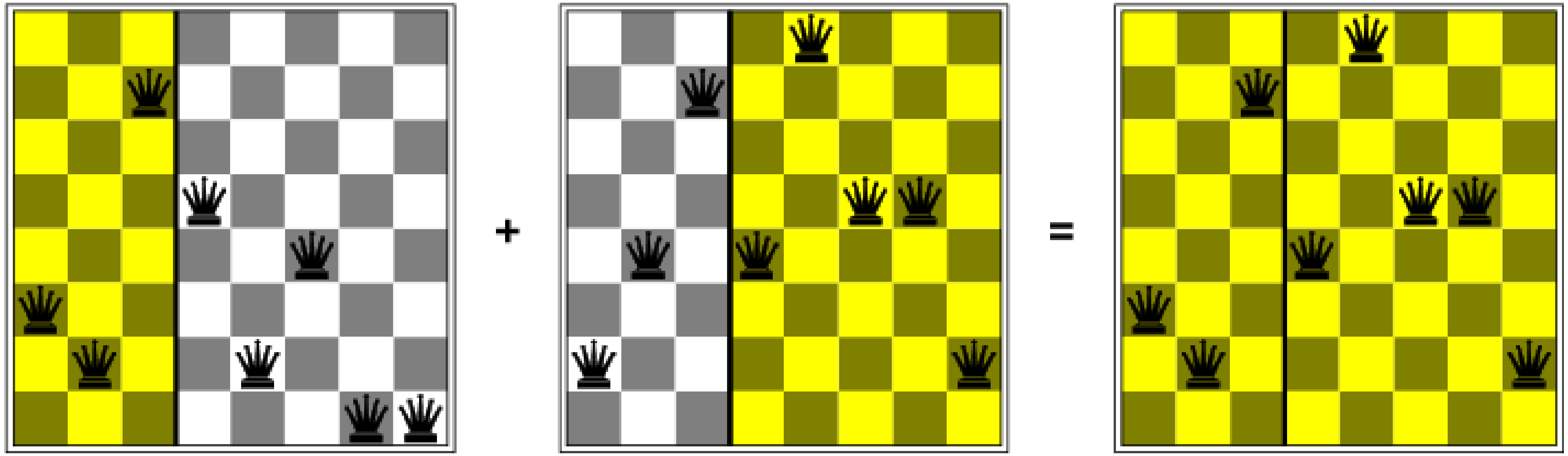
Genetic Algorithms

- Stochastic local beam search + generate successors from **pairs** of states



Genetic Algorithms

- GAs require states encoded as strings (GPs use programs)
- Crossover helps **iff substrings are meaningful components**



Summary

- Exact search
  - exhaustive exploration of the search space
  - search with heuristics: A*

- Approximate search
  - hill-climbing
  - simulated annealing
  - genetic algorithms