First Order Logic

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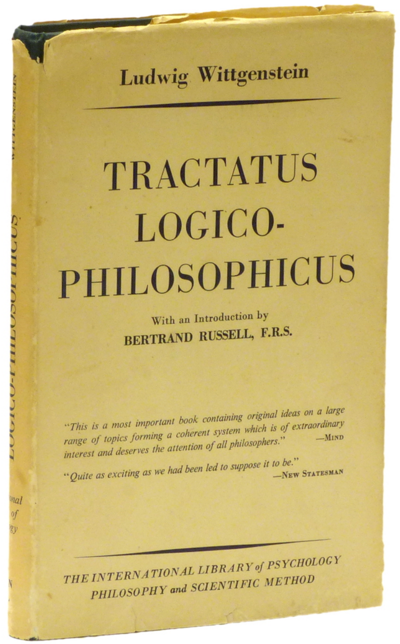
The World is Defined by Facts



1. The world is everything that is the case.

2. What is the case (a fact) is the existence of states of affairs.

3. A logical picture of facts is a thought.

4. A thought is a proposition with a sense.

5. A proposition is a truth-function of elementary propositions.
   (An elementary proposition is a truth-function of itself.)

6. The general form of a proposition is the general form of a truth function, which is: $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$. This is the general form of a proposition.

7. Whereof one cannot speak, thereof one must be silent.

Outline

- Why first order logic?
- Syntax and semantics of first order logic
- Fun with sentences
- Wumpus world in first order logic

why?

Pros and Cons of Propositional Logic

- **PRO:** Propositional logic is **declarative**: pieces of syntax correspond to facts
- **PRO:** Propositional logic allows partial/disjunctive/negated information
  (unlike most data structures and databases)
- **PRO:** Propositional logic is **compositional**:
  meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- **PRO:** Meaning in propositional logic is **context-independent**
  (unlike natural language, where meaning depends on context)
- **CON:** Propositional logic has very limited expressive power
  (unlike natural language)
  E.g., cannot say "pits cause breezes in adjacent squares"
  except by writing one sentence for each square

First-Order Logic

- Propositional logic: world contains **facts**

- First-order logic: the world contains **objects**, **relations**, and **functions**

- **Objects**: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .

- **Relations**: red, round, bogus, prime, multistoried . . .,
  brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .

- **Functions**: father of, best friend, third inning of, one more than, end of . . .

More Logics

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Higher-order logic:

relations and functions operate not only on objects,
but also on relations and functions

syntax and semantics

Syntax of FOL: Basic Elements

- Constants: $KingJohn,\ 2,\ UCB, \ldots$
- Predicates: $Brother,\ >, \ldots$
- Functions: $Sqrt,\ LeftLegOf, \ldots$
- Variables: $x,\ y,\ a,\ b, \ldots$
- Connectives: $\wedge \ \vee \ \neg \ \implies \ \Leftrightarrow$
- Equality: $=$
- Quantifiers: $\forall \ \exists$

Atomic Sentences

- Atomic sentence = $predicate(term_1, \ldots, term_n)$
  or $term_1 = term_2$
- Term = $function(term_1, \ldots, term_n)$
  or $constant$ or $variable$
- E.g., $Brother(KingJohn, RichardTheLionheart)$
  $>(Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex Sentences

- Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \implies S_2, \quad S_1 \Leftrightarrow S_2$$

- For instance

$$Sibling(KingJohn, Richard) \implies Sibling(Richard, KingJohn)$$

Truth in First-Order Logic

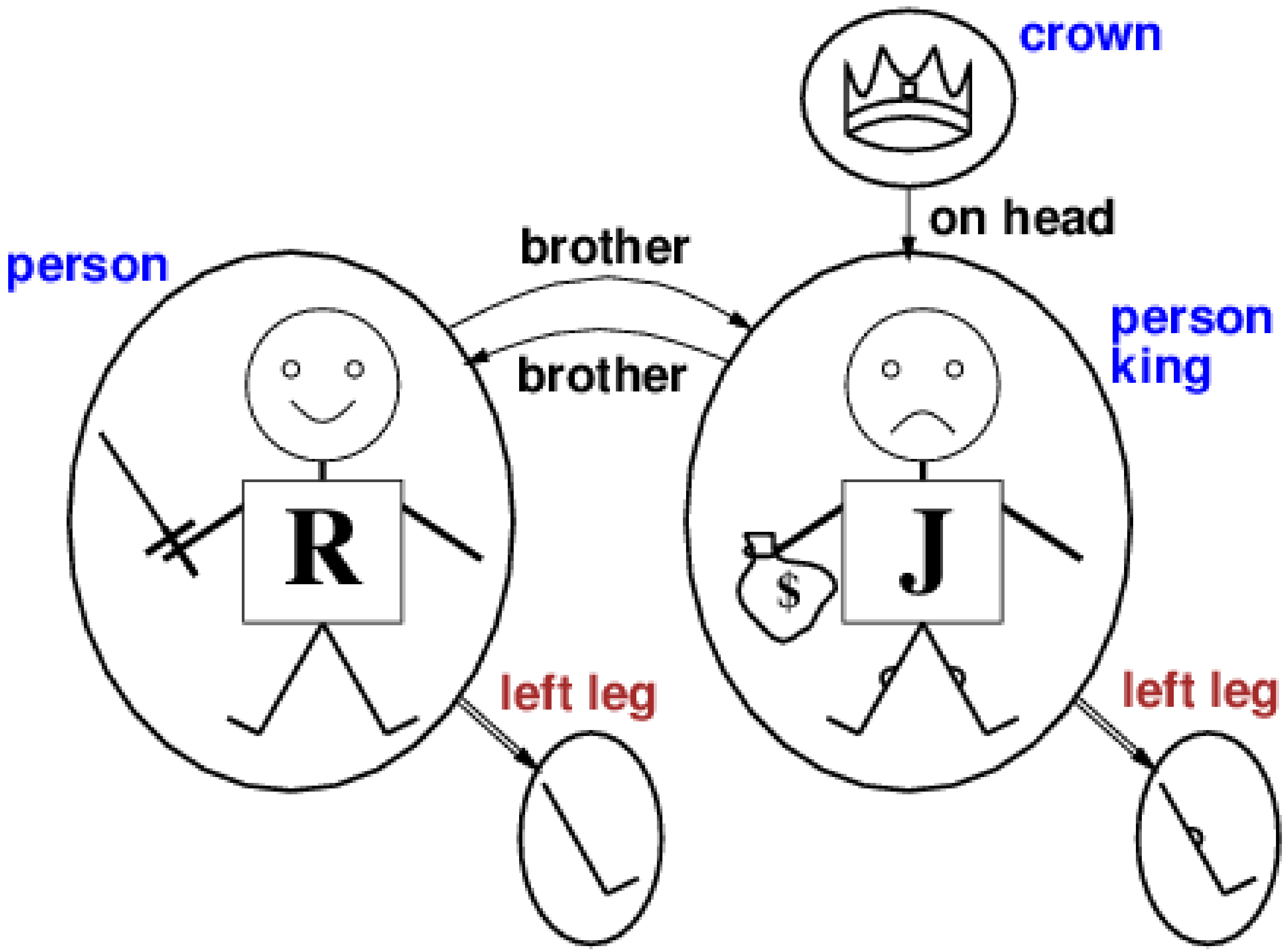
- Sentences are true with respect to a **model** and an **interpretation**

- **Model** contains objects (**domain elements**) and relations among them

- **Interpretation** specifies referents for
  - **constant symbols** → **objects**
  - **predicate symbols** → **relations**
  - **function symbols** → **functional relations**

- An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
  iff the **objects** referred to by $term_1, \ldots, term_n$
  are in the **relation** referred to by $predicate$

Models for FOL: Example



Truth Example

- Object symbols
  - $Richard$ → **Richard the Lionheart**
  - $John$ → **the evil King John**

- Predicat symbol
  - $Brother$ → **the brotherhood relation**

- Atomic sentence
  - $Brother(Richard, John)$

  true iff **Richard the Lionheart** and **the evil King John**
  are in **the brotherhood relation** in the model

Models for FOL: Lots!

- Entailment in propositional logic can be computed by enumerating models

- We **can** enumerate the FOL models for a given KB vocabulary:

```
For each number of domain elements n from 1 to ∞
   For each k-ary predicate P_k in the vocabulary
      For each possible k-ary relation on n objects
         For each constant symbol C in the vocabulary
            For each choice of referent for C from n objects ...
```

- Computing entailment by enumerating FOL models is not easy!

Universal Quantification

- Syntax: $\forall \langle variables \rangle \ \langle sentence \rangle$
- Everyone at JHU is smart:
  $\forall \, x \ \ At(x, JHU) \implies Smart(x)$
- $\forall \, x \ \ P$ is true in a model m iff P is true with x being **each** possible object
- **Roughly** speaking, equivalent to the **conjunction** of **instantiations** of P

$$
\begin{aligned}
& (At(KingJohn, JHU) \implies Smart(KingJohn)) \\
\wedge \ & (At(Richard, JHU) \implies Smart(Richard)) \\
\wedge \ & (At(Jane, JHU) \implies Smart(Jane)) \\
\wedge \ & \ldots
\end{aligned}
$$

A Common Mistake to Avoid

- Typically " $\implies$ " is the main connective with $\forall$

- Common mistake: using "$\wedge$" as the main connective with $\forall$:

$$\forall\, x \;\; At(x, JHU) \wedge Smart(x)$$

means "Everyone is at JHU and everyone is smart"

- Correct

$$\forall\, x \;\; At(x, JHU) \implies Smart(x)$$

means "For everyone, if she is at JHU, then she is smart"

Existential Quantification

- Syntax: $\exists \langle variables \rangle \; \langle sentence \rangle$

- Someone at JHU is smart:

  $\exists x \;\; At(x, JHU) \wedge Smart(x)$

- $\exists x \;\; P$ is true in a model m iff P is true with x being **some** possible object

- **Roughly** speaking, equivalent to the **disjunction** of **instantiations** of P

$$
\begin{array}{ll}
 & (At(KingJohn, JHU) \wedge Smart(KingJohn)) \\
\vee & (At(Richard, JHU) \wedge Smart(Richard)) \\
\vee & (At(JHU, JHU) \wedge Smart(JHU)) \\
\vee & \ldots
\end{array}
$$

Another Common Mistake to Avoid

- Typically "$\wedge$" is the main connective with $\exists$
- Common mistake: using "$\implies$" as the main connective with $\exists$:

$$\exists\, x \;\; At(x, JHU) \implies Smart(x)$$

is true if there is anyone who is **not** at JHU

- Correct

$$\exists\, x \;\; At(x, JHU) \wedge Smart(x)$$

is true if there is someone who is at JHU and smart

Properties of Quantifiers

- $\forall x \ \forall y$ is the same as $\forall y \ \forall x$
- $\exists x \ \exists y$ is the same as $\exists y \ \exists x$
- $\exists x \ \forall y$ is **not** the same as $\forall y \ \exists x$
- $\exists x \ \forall y \ Loves(x, y)$
  "There is a person who loves everyone in the world"
- $\forall y \ \exists x \ Loves(x, y)$
  "Everyone in the world is loved by at least one person"
- **Quantifier duality**: each can be expressed using the other
- $\forall x \ Likes(x, IceCream) \qquad \neg\exists x \ \neg Likes(x, IceCream)$
- $\exists x \ Likes(x, Broccoli) \qquad \neg\forall x \ \neg Likes(x, Broccoli)$

Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object

- For instance
  - $1 = 2$ and $\forall\, x \;\; \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
  - $2 = 2$ is true

  (note: syntax does not imply anything about the semantics of 1, 2, $Sqrt(x)$, etc.)

- Definition of (full) $Sibling$ in terms of $Parent$

$$\forall\, x, y \;\; Sibling(x,y) \;\Leftrightarrow\; [\neg(x = y) \wedge \exists\, m, f \;\; \neg(m = f) \wedge Parent(m,x) \wedge Parent(f,x) \wedge Parent(m,y) \wedge Parent(f,y)]$$

fun with sentences

Fun with Sentences

- Brothers are siblings

  $\forall\, x, y\;\; Brother(x, y) \implies Sibling(x, y)$

- "Sibling" is symmetric

  $\forall\, x, y\;\; Sibling(x, y) \Leftrightarrow Sibling(y, x)$

- One's mother is one's female parent

  $\forall\, x, y\;\; Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y))$

- A first cousin is a child of a parent's sibling

  $\forall\, x, y\;\; FirstCousin(x, y) \Leftrightarrow \exists\, p, ps\;\; Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$

Lincoln Quote

You can fool all the people some of the time,
and some of the people all the time,
but you cannot fool all the people all the time.

$$\forall\, p\ \exists\, t\ \ Fool(p,t)$$
$$\wedge$$
$$\exists\, p\ \forall\, t\ \ Fool(p,t)$$
$$\wedge$$
$$\neg\ \forall\, p\ \forall\, t\ \ Fool(p,t)$$

Donkey Sentences

- *Every farmer owns a donkey.*
  - $\forall\, f\ \ (Farmer(f) \implies \exists\, d\ \ (Donkey(d) \wedge Own(f,d)))$
  - $\exists\, d\ \ (Donkey(d) \wedge \forall\, f\ \ (Farmer(f) \wedge Own(f,d)))$

- *Every human lives on a planet.*
  - $\exists\, p\ \ (Planet(p) \wedge \forall\, h\ \ (Human(h) \wedge LivesOn(h,p)))$

- *Every farmer who owns a donkey feeds it.*
  - $\forall\, f\ \ Farmer(f) \wedge \exists\, d\ \ (Donkey(d) \wedge Own(f,d) \implies Feeds(f,d))$
    but what if a farmer has a donkey d_1 and a pig d_2 and he feeds neither $Donkey(d_2) \wedge Own(f,d_2) \implies Feeds(f,d_2)$ is true ($false \wedge true \implies false$)
  - $\forall\, f\ \ \forall\, d\ \ (Farmer(f) \wedge Donkey(d) \wedge Own(f,d) \implies Feeds(f,d))$
    but this means "Every farmer feeds every donkey he owns."

Natural Language

- First order logic is close to the semantics of natural language
- But there are limitations
  - *"There is at least one thing John has in common with Peter."*
    Requires a quantifier over predicates.
  - *"The cake is very good."*
    $\exists\, c\;\; Cake(c) \land Good(c)$ but not $Very(c)$
    Functions and relations cannot be qualified.
- Natural language sentences are often intentionally vague and ambiguous

wampus world

Knowledge Base for the Wumpus World

- **"Perception"**: at current square, three perceptions expressed as variables
  - s either $Smell$ or $\neg Smell$
  - b either $Breeze$ or $\neg Breeze$
  - g either $Glitter$ or $\neg Glitter$

  $Percept([s, b, g], t)$ at time t

- Shorthands
  - $\forall\, b, g, t \;\; Percept([Smell, b, g], t) \implies Smelt(t)$
  - $\forall\, s, b, t \;\; Percept([s, b, Glitter], t) \implies AtGold(t)$

- **Reflex**: $\forall\, t \;\; AtGold(t) \implies Action(Grab, t)$

- **Reflex with internal state**: do we have the gold already?
  $\forall\, t \;\; AtGold(t) \wedge \neg Holding(Gold, t) \implies Action(Grab, t)$

- $Holding(Gold, t)$ cannot be observed
  $\Rightarrow$ keeping track of change is essential

Deducing Hidden Properties

- Properties of locations:
  $\forall\, x,t \;\; At(Agent,x,t) \land Smelt(t) \implies Smelly(x)$
  $\forall\, x,t \;\; At(Agent,x,t) \land Breeze(t) \implies Breezy(x)$

- Squares are breezy near a pit

- **Diagnostic** rule—infer cause from effect
  $$\forall\, y \;\; Breezy(y) \implies \exists\, x \;\; Pit(x) \land Adjacent(x,y)$$

- **Causal** rule—infer effect from cause
  $$\forall\, x,y \;\; Pit(x) \land Adjacent(x,y) \implies Breezy(y)$$

- Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

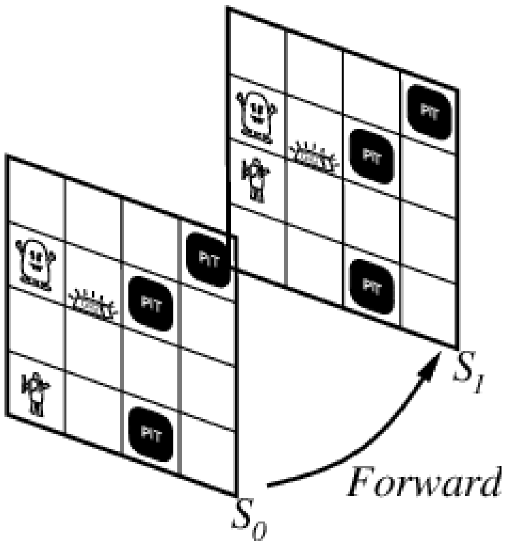
- **Definition** for the $Breezy$ predicate:
  $$\forall\, y \;\; Breezy(y) \Leftrightarrow [\exists\, x \;\; Pit(x) \land Adjacent(x,y)]$$

States and Fluents

- By acting, the agent moves through a sequence of situations s
- Fluents: aspects of the world that may change
  - current position
  - having an arrow
  - holding the gold
- Taking actions requires updates to the fluents

Keeping Track of Change

- Facts hold in **situations**, rather than eternally
  E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$
- **Situation calculus** is one way to represent change in FOL:
  Adds a situation argument to each non-eternal predicate
  E.g., Now in $Holding(Gold, Now)$ denotes a situation
- Situations are connected by the $Result$ function
  $s' = Result(a, s)$ is the situation that results from doing a in s



Describing Actions

- "Effect" axiom—describe changes due to action
  $\forall\, s \;\; AtGold(s) \implies Holding(Gold, Result(Grab, s))$

- "Frame" axiom—describe **non-changes** due to action
  $\forall\, s \;\; HaveArrow(s) \implies HaveArrow(Result(Grab, s))$

- **Frame problem**: find an elegant way to handle non-change
  (a) **representation**: too many frame axioms
  (b) **inference**: too many repeated "copy-overs" to keep track of state

- **Qualification problem**: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or ...

- **Ramification problem**: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, ...

Describing Actions

- **Successor-state axioms** solve the representational frame problem

- Each axiom is "about" a **predicate** (not an action per se):

$$\text{P true afterwards} \quad \Leftrightarrow \quad [\text{an action made P true}$$
$$\vee \quad \text{P true already and no action made P false}]$$

- For holding the gold:

$$\forall\, a, s \;\; Holding(Gold, Result(a, s)) \;\Leftrightarrow$$
$$[(a = Grab \wedge AtGold(s))$$
$$\vee\, (Holding(Gold, s) \wedge a \neq Release)]$$

Making Plans

- Initial condition in KB:
  $At(Agent, [1,1], S_0)$
  $At(Gold, [1,2], S_0)$

- Query: $Ask(KB, \exists\, s\;\; Holding(Gold, s))$
  i.e., in what situation will I be holding the gold?

- Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$
  i.e., go forward and then grab the gold

- This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Making Plans: A Better Way

- Represent **plans** as action sequences $[a_1, a_2, \ldots, a_n]$
- $PlanResult(p, s)$ is the result of executing p in s
- Then the query $Ask(KB, \exists\, p \;\; Holding(Gold, PlanResult(p, S_0)))$
  has the solution $\{p/[Forward, Grab]\}$
- Definition of $PlanResult$ in terms of $Result$:
  $\forall\, s \;\; PlanResult([\,], s) = s$
  $\forall\, a, p, s \;\; PlanResult([a|p], s) = PlanResult(p, Result(a, s))$
- **Planning systems** are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

Summary

- First-order logic
  - objects and relations are semantic primitives
  - syntax: constants, functions, predicates, equality, quantifiers
- Increased expressive power: sufficient to define wumpus world
- Situation calculus
  - conventions for describing actions and change in FOL
  - can formulate planning as inference on a situation calculus KB