Bayesian Networks

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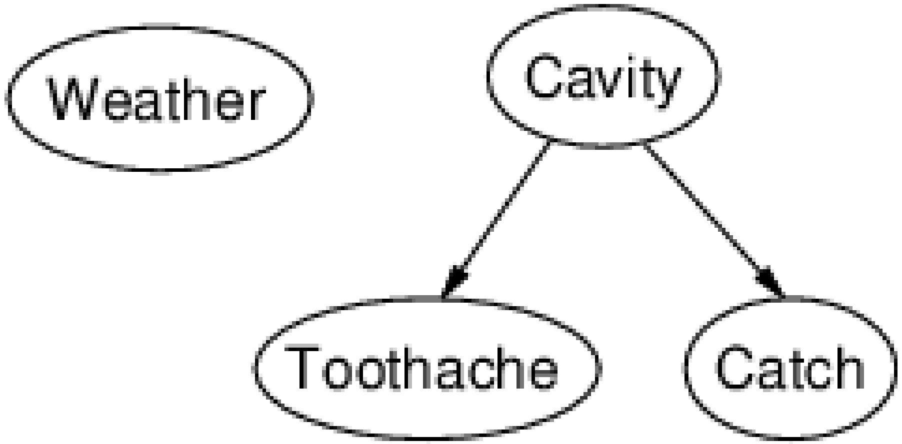
27 March 2025

Bayesian Network Example

- Topology of network encodes conditional independence assertions:



- $Weather$ is independent of the other variables

- $Toothache$ and $Catch$ are conditionally independent given $Cavity$

Bayesian Networks

- Graphical notation for conditional independence assertions
  $\rightarrow$ compact specification of full joint distributions

- Syntax
  - a set of nodes, one per variable
  - a directed, acyclic graph (link $\approx$ "directly influences")
  - a conditional distribution for each node given its parents:
    $\mathbf{P}(X_i|Parents(X_i))$

- In the simplest case, conditional distribution represented as a **conditional probability table** (CPT) giving the distribution over X_i for each combination of parent values

Example

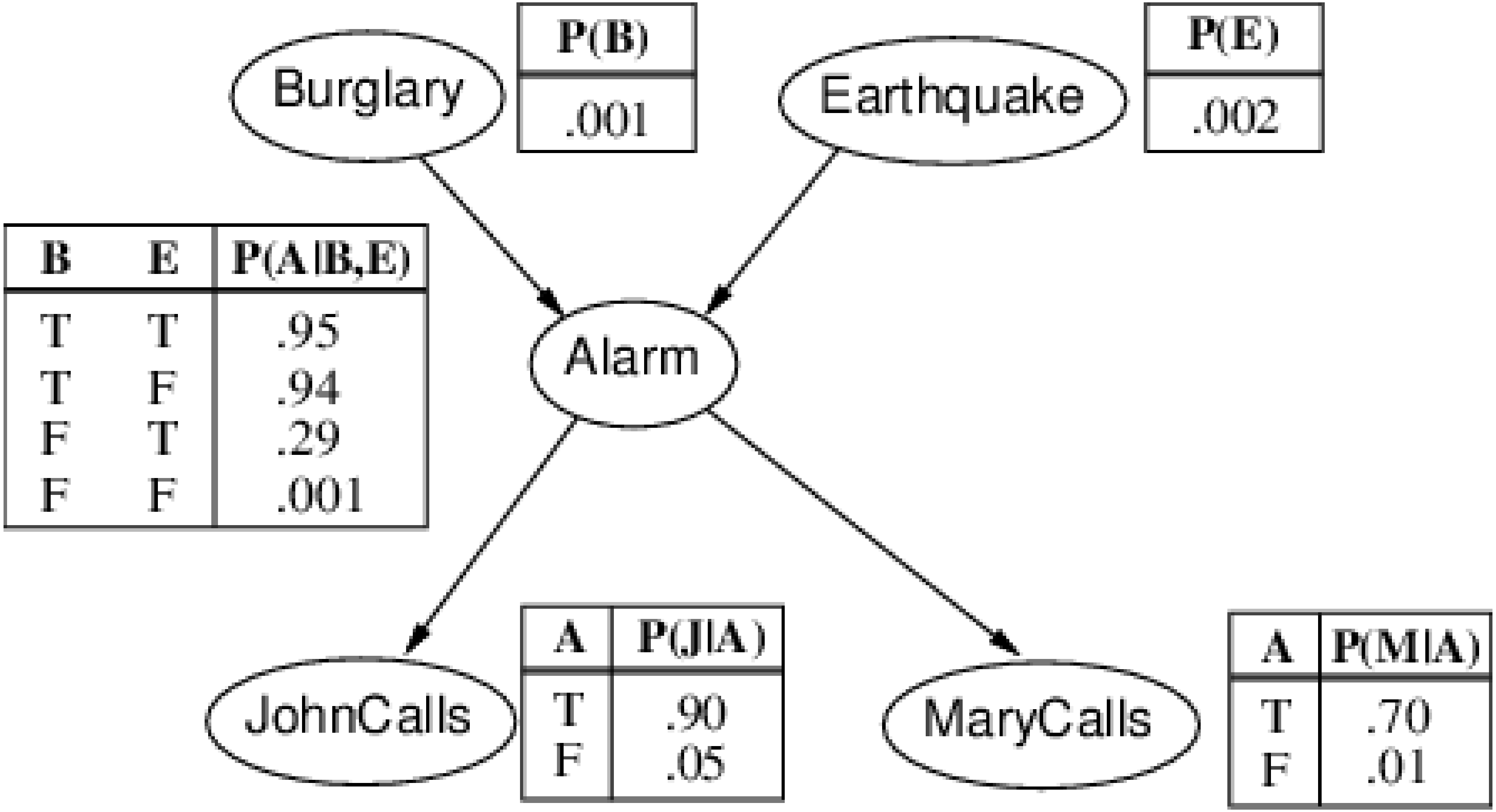
- *I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes.*

  Is there a burglar?

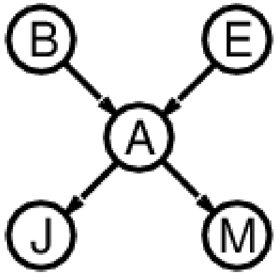
- Variables: $Burglar$, $Earthquake$, $Alarm$, $JohnCalls$, $MaryCalls$

- Network topology reflects "causal" knowledge
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call

Example


Burglary
P(B)
.001
Earthquake
P(E)
.002
B E P(A|B,E)
T T .95
T F .94
F T .29
F F .001
Alarm
JohnCalls
A P(J|A)
T .90
F .05
MaryCalls
A P(M|A)
T .70
F .01

Compactness



- A conditional probability table for Boolean X_i with k Boolean parents has 2^k rows for the combinations of parent values

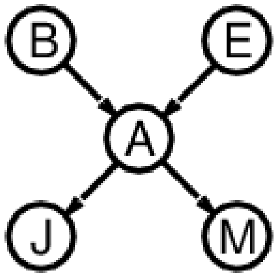
- Each row requires one number p for $X_i = true$
(the number for $X_i = false$ is just $1 - p$)

- If each variable has no more than k parents,
the complete network requires $O(n \cdot 2^k)$ numbers

- I.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution

- For burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$)

Global Semantics



- Full joint distribution = product of local conditional distributions

$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

- E.g., $P(j \land m \land a \land \neg b \land \neg e)$

$$\begin{aligned} &= P(j|a)P(m|a)P(a|\neg b, \neg e)P(\neg b)P(\neg e) \\ &= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 \\ &\approx 0.00063 \end{aligned}$$

Constructing Bayesian Networks

- Need a method such that a series of locally testable assertions of conditional independence guarantees the required global semantics

```
1.  Choose an ordering of variables X_1, ..., X_n
2.  For i = 1 to n
       add X_i to the network
       select parents from X_1, ..., X_{i-1} such that
          P(X_i|Parents(X_i)) = P(X_i|X_1, ..., X_{i-1})
```

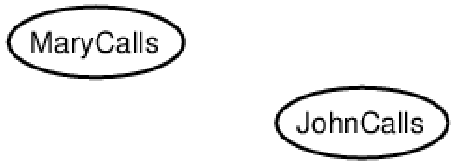
- This choice of parents guarantees the global semantics:

$$\mathbf{P}(X_1, \ldots, X_n) = \prod_{i=1}^{n} \mathbf{P}(X_i | X_1, \ldots, X_{i-1}) \quad \text{(chain rule)}$$

$$= \prod_{i=1}^{n} \mathbf{P}(X_i | Parents(X_i)) \quad \text{(by construction)}$$

Example

- Suppose we choose the ordering M, J, A, B, E



- $P(J|M) = P(J)$?

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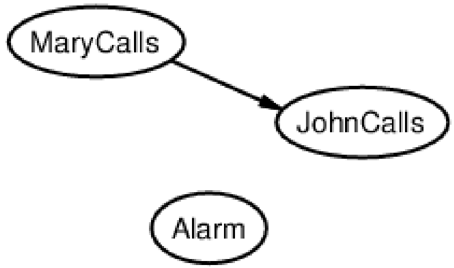

- $P(J|M) = P(J)$? No
- $P(A|J,M) = P(A|J)$? $P(A|J,M) = P(A)$?

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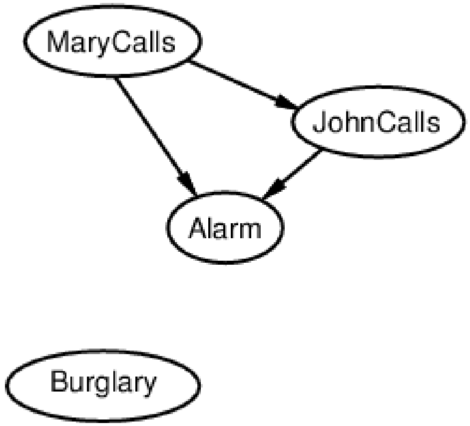

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- $P(B|A,J,M) = P(B|A)$?
- $P(B|A,J,M) = P(B)$?

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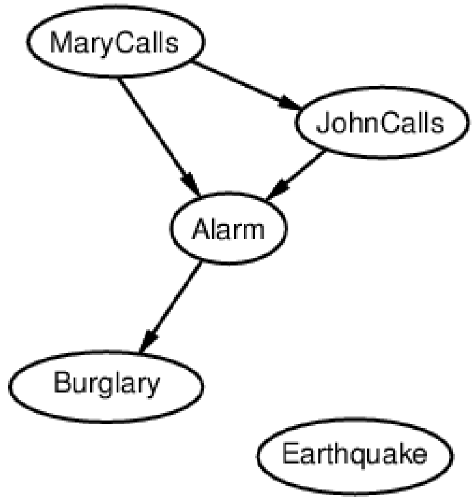

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- $P(B|A,J,M) = P(B)$? No
- $P(E|B,A,J,M) = P(E|A)$?
- $P(E|B,A,J,M) = P(E|A,B)$?

Example

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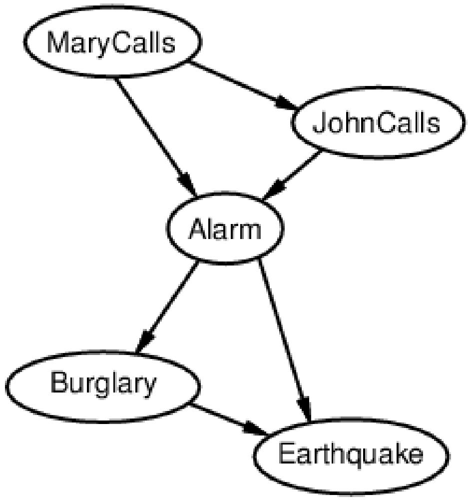

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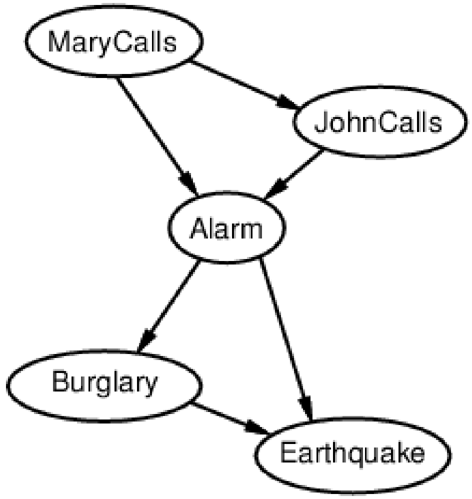
Example



- Deciding conditional independence is hard in noncausal directions

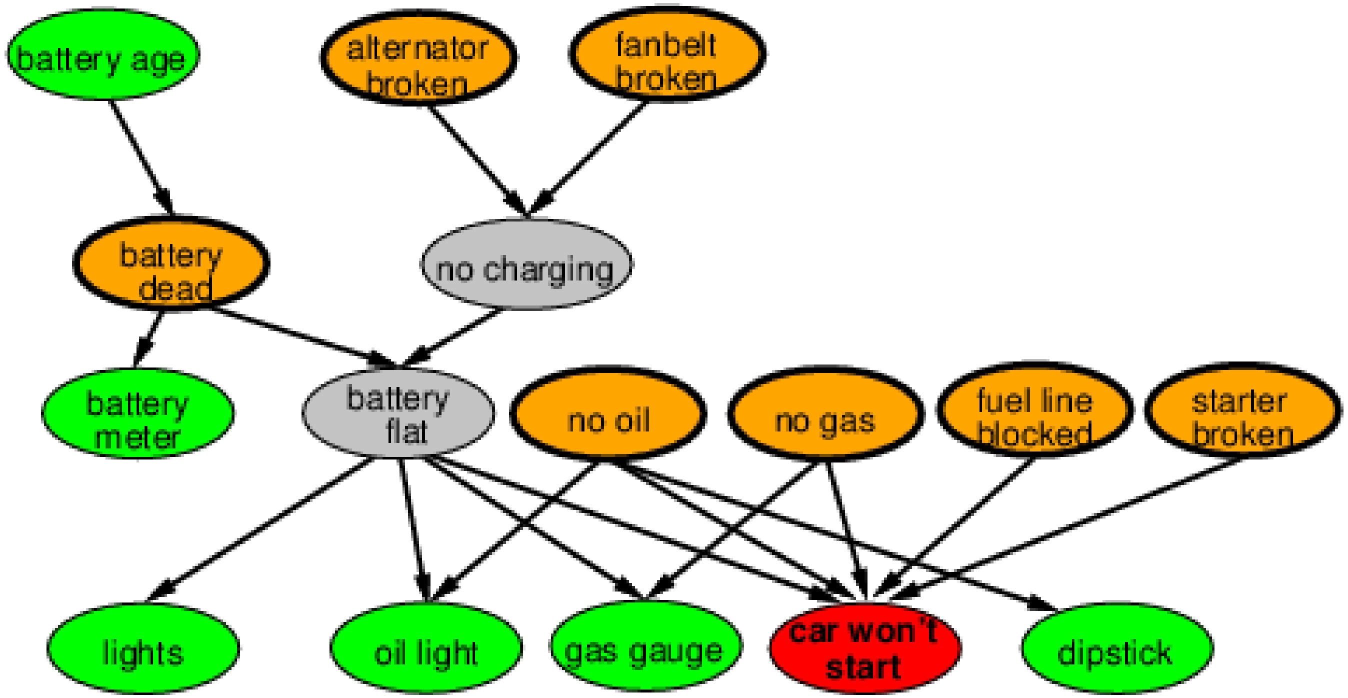
  (Causal models and conditional independence seem hardwired for humans!)

- Assessing conditional probabilities is hard in noncausal directions

- Network is less compact: $1 + 2 + 4 + 2 + 4 = 13$ numbers needed

Example: Car Diagnosis

- Initial evidence: car won't start
  - testable variables
  - "broken, so fix it" variables
  - hidden variables ensure sparse structure, reduce parameters



inference

Inference Tasks

- **Simple queries**: compute posterior marginal $\mathbf{P}(X_i|\mathbf{E}=\mathbf{e})$
  e.g., $P(NoGas|Gauge=empty, Lights=on, Starts=false)$

- **Conjunctive queries**: $\mathbf{P}(X_i, X_j|\mathbf{E}=\mathbf{e}) = \mathbf{P}(X_i|\mathbf{E}=\mathbf{e})\mathbf{P}(X_j|X_i, \mathbf{E}=\mathbf{e})$

- **Optimal decisions**: decision networks include utility information;
  probabilistic inference required for $P(outcome|action, evidence)$

- **Value of information**: which evidence to seek next?

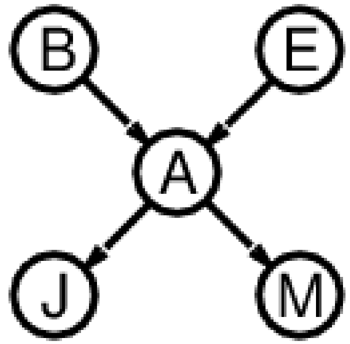
- **Sensitivity analysis**: which probability values are most critical?

- **Explanation**: why do I need a new starter motor?

Inference by Enumeration

- Sum out variables from the joint probability distribution

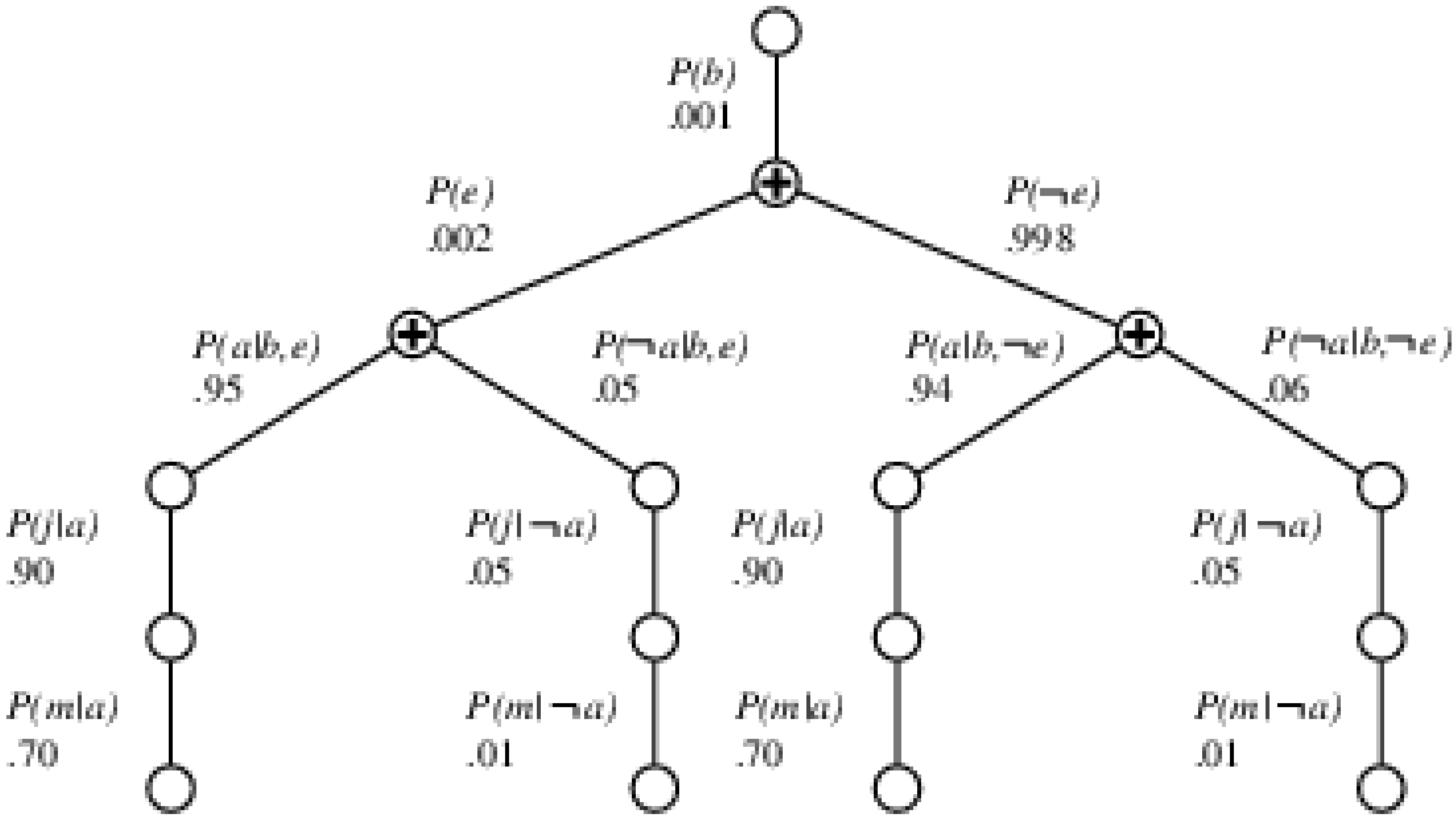
- Simple query on the burglary network
  $\mathbf{P}(B|j,m)$
  $= \mathbf{P}(B,j,m)/P(j,m)$
  $= \alpha\mathbf{P}(B,j,m)$
  $= \alpha \sum_e \sum_a \mathbf{P}(B,e,a,j,m)$



- Rewrite full joint entries using product of CPT entries:
  $\mathbf{P}(B|j,m)$
  $= \alpha \sum_e \sum_a \mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)$
  $= \alpha\mathbf{P}(B) \sum_e P(e) \sum_a \mathbf{P}(a|B,e)P(j|a)P(m|a)$

- Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

Evaluation Tree



- Enumeration is inefficient: repeated computation
  e.g., computes $P(j|a)P(m|a)$ for each value of e

Inference by Variable Elimination

- Variable elimination: carry out summations right-to-left, storing intermediate results (**factors**) to avoid recomputation

$$\mathbf{P}(B|j,m)$$

$$= \alpha \underbrace{\mathbf{P}(B)}_{B} \textstyle\sum_e \underbrace{P(e)}_{E} \textstyle\sum_a \underbrace{\mathbf{P}(a|B,e)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M}$$

$$= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a \mathbf{P}(a|B,e) P(j|a) f_M(a)$$

$$= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a \mathbf{P}(a|B,e) f_J(a) f_M(a)$$

$$= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a f_A(a,b,e) f_J(a) f_M(a)$$

$$= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) f_{\bar{A}JM}(b,e) \text{ (sum out } A)$$

$$= \alpha \mathbf{P}(B) f_{\bar{E}\bar{A}JM}(b) \text{ (sum out } E)$$

$$= \alpha f_B(b) \times f_{\bar{E}\bar{A}JM}(b)$$

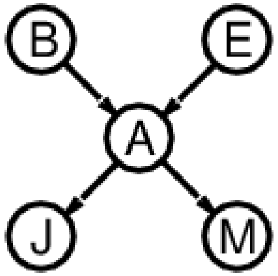
Variable Elimination: Core Operations

- **Summing out** a variable from a product of factors:
  move any constant factors outside the summation
  add up submatrices in pointwise product of remaining factors

$\sum_x f_1 \times \cdots \times f_k$
$= f_1 \times \cdots \times f_i \times \sum_x \; f_{i+1} \times \cdots \times f_k$
$= f_1 \times \cdots \times f_i \times f_{\bar{X}}$

assuming $f_1, \ldots, f_i$ do not depend on $\bar{X}$

Irrelevant Variables



- Consider the query $P(JohnCalls|Burglary = true)$

$$P(J|b) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e) P(J|a) \sum_m P(m|a)$$

  $\sum_m P(m|a) = 1$; M is **irrelevant** to the query

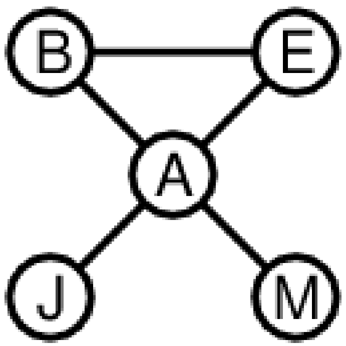
- Theorem 1: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup \mathbf{E})$

- Here
  - $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$
  - $Ancestors(\{X\} \cup \mathbf{E}) = \{Alarm, Earthquake\}$

  ⇒ $MaryCalls$ is irrelevant

Irrelevant Variables

- Definition: moral graph of Bayes net: marry all parents and drop arrows
- Definition: **A** is m-separated from **B** by **C** iff separated by **C** in the moral graph
- Theorem 2: Y is irrelevant if m-separated from X by **E**

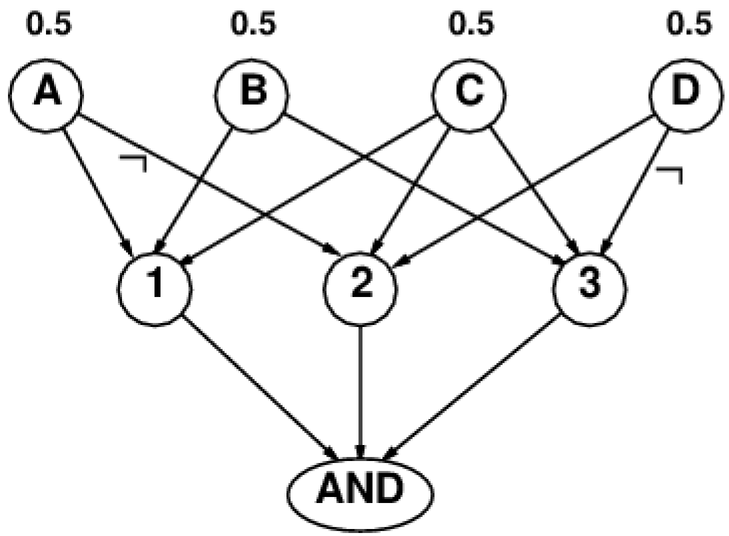


- For $P(JohnCalls|Alarm = true)$, both $Burglary$ and $Earthquake$ are irrelevant
  (**A** = $JohnCalls$, **B** = $Burglary$ and $Earthquake$, **C** = $Alarm$)

Complexity of Exact Inference

- **Singly connected** networks (or **polytrees**)
  - any two nodes are connected by at most one (undirected) path
  - time and space cost of variable elimination are $O(d^k n)$
- **Multiply connected** networks
  - can reduce 3SAT to exact inference $\implies$ NP-hard

1. A v B v C
2. C v D v ¬A
3. B v C v ¬D



approximate inference

Inference by Stochastic Simulation

- Basic idea
  - Draw N samples from a sampling distribution S
  - Compute an approximate posterior probability $\hat{P}$
  - Show this converges to the true probability P

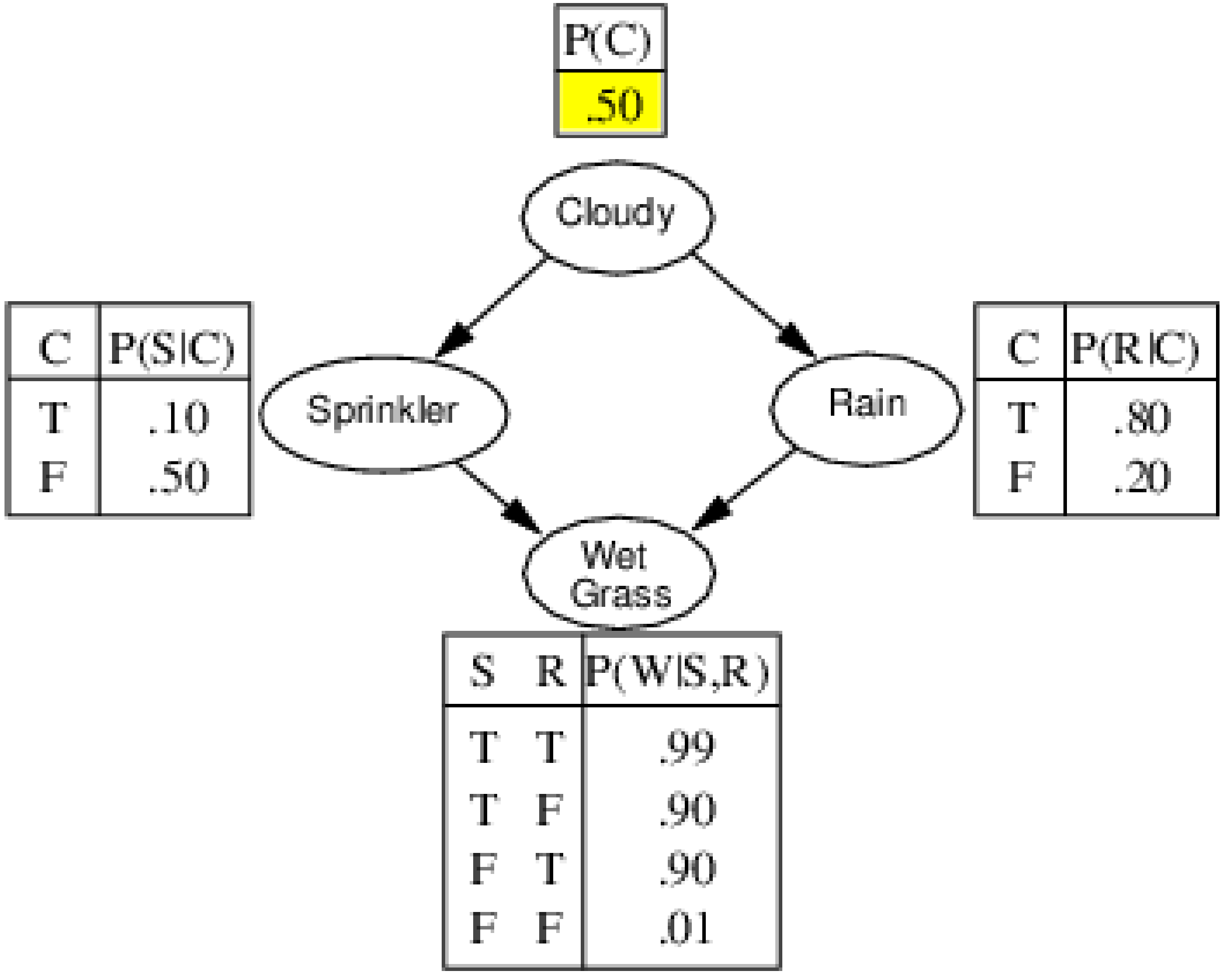
- Outline
  - Sampling from an empty network
  - Rejection sampling: reject samples disagreeing with evidence
  - Likelihood weighting: use evidence to weight samples
  - Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

Sampling from an Empty Network

function PRIOR-SAMPLE(*bn*) **returns** an event sampled from *bn*
  inputs: *bn*, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  x ← an event with n elements
  for $i = 1$ **to** n **do**
    $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
      given the values of $Parents(X_i)$ in **x**
  return x

Example


P(C)
.50
Cloudy
C P(S|C)
T .10
F .50
Sprinkler
Rain
C P(R|C)
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F .20
Wet Grass
S R P(W|S,R)
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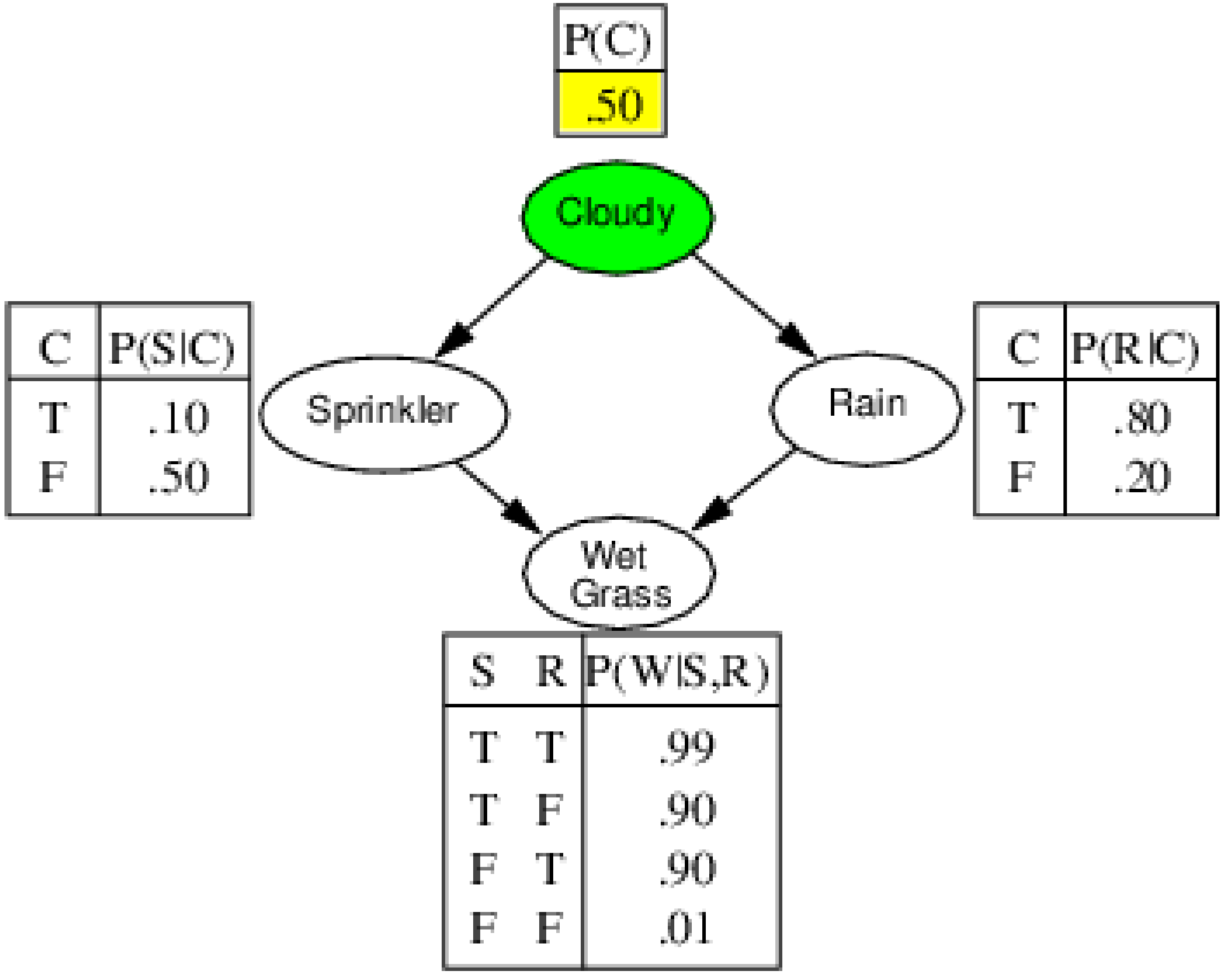
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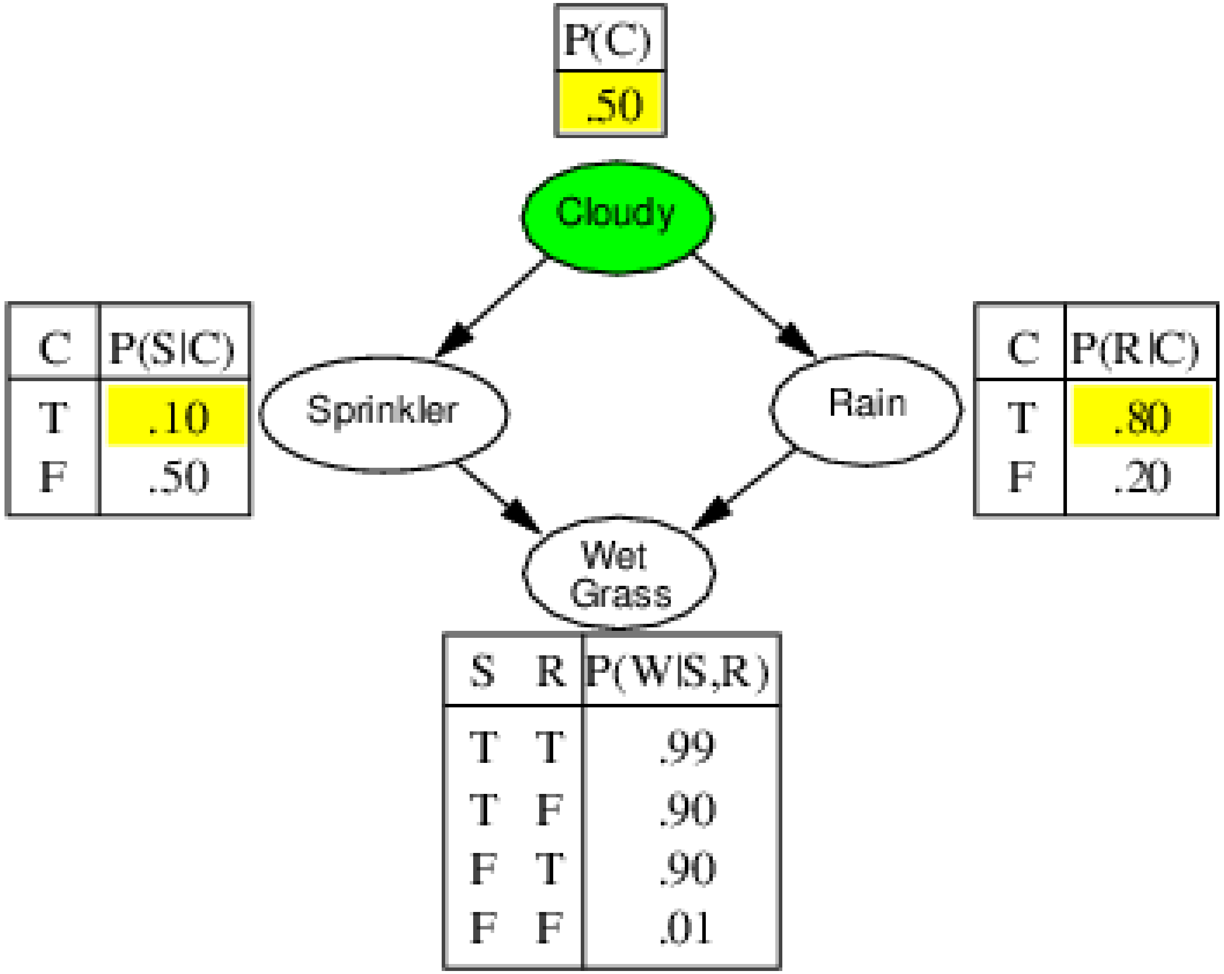

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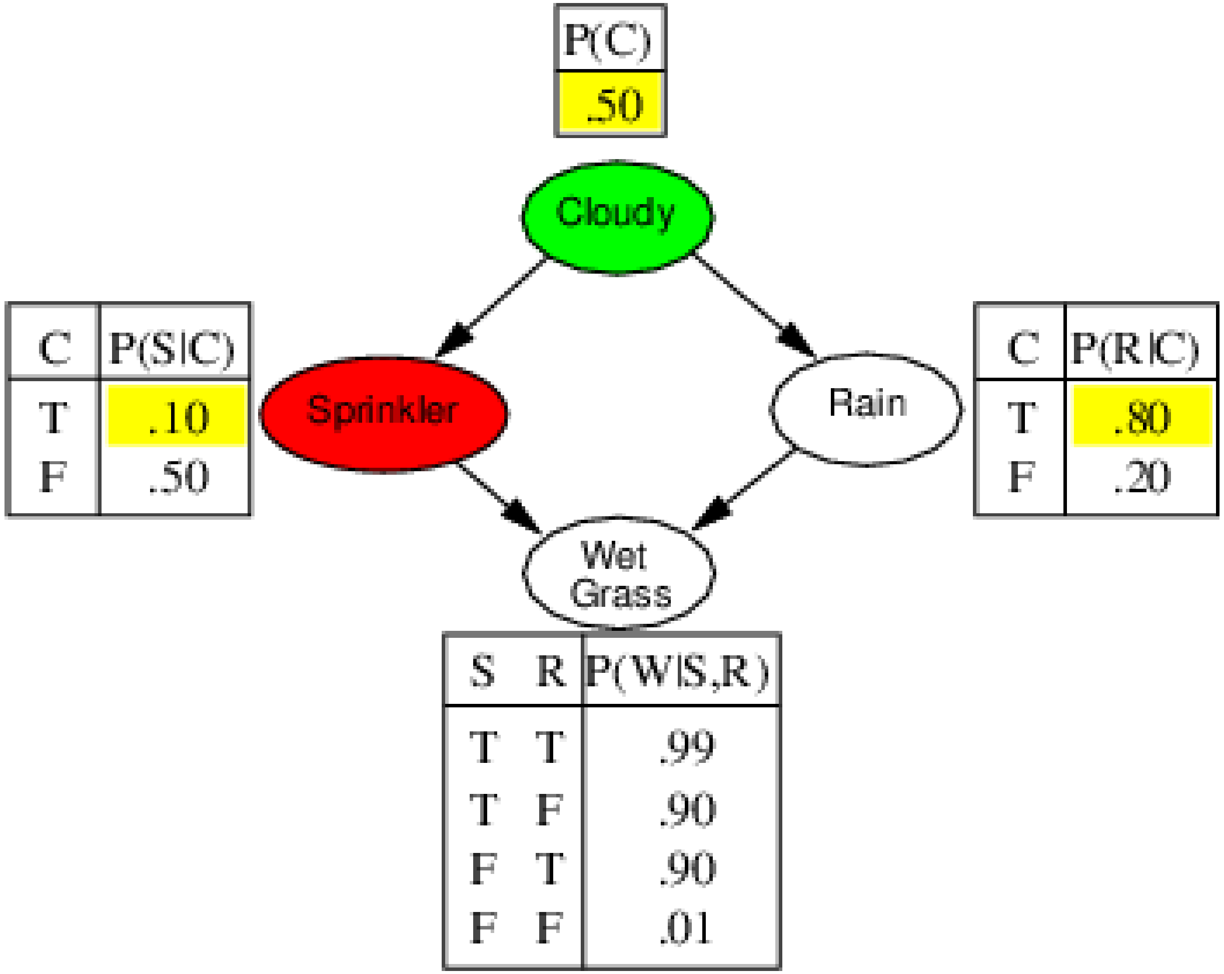

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S	R	P(W|S,R)
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T	F	.90
F	T	.90
F	F	.01

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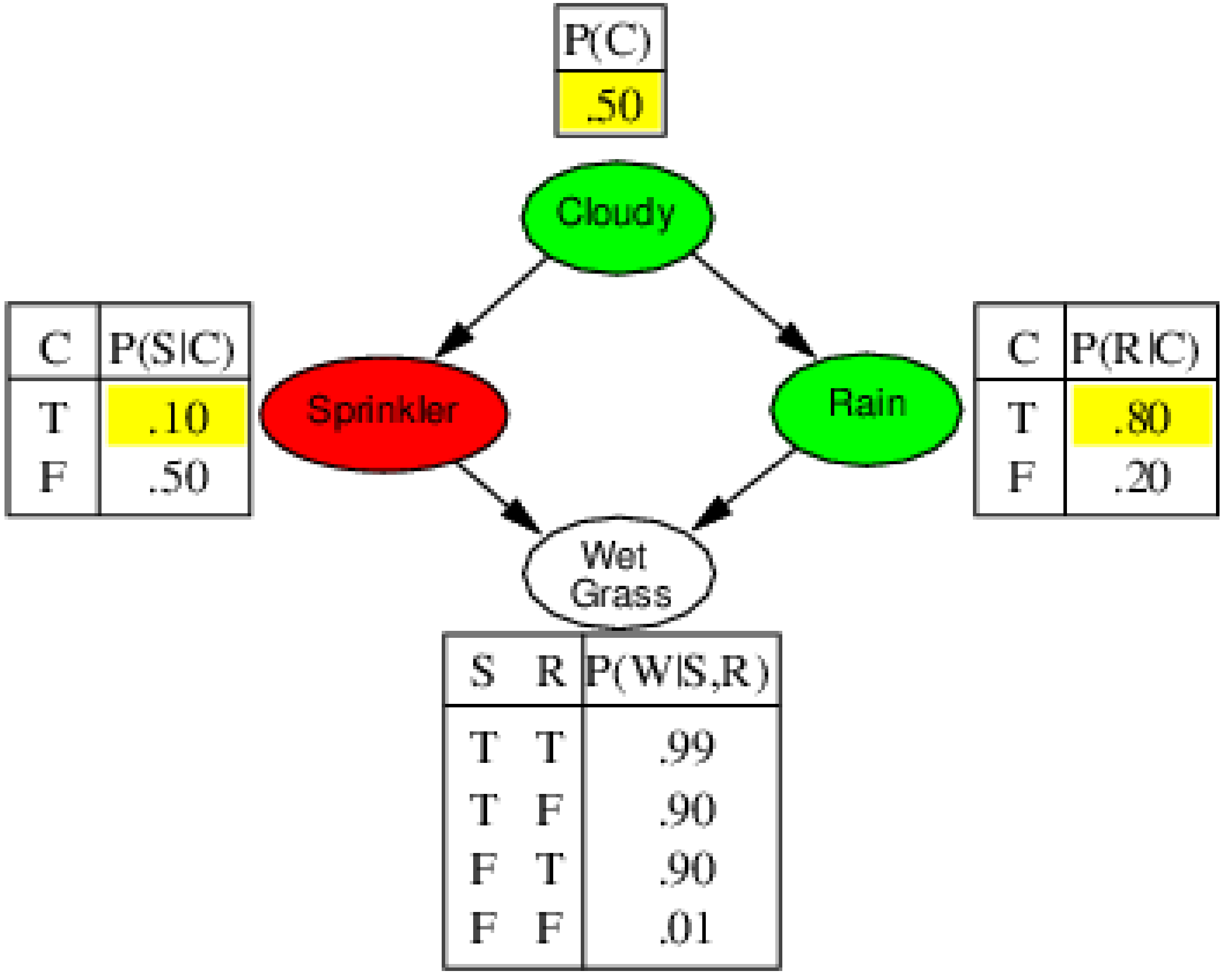
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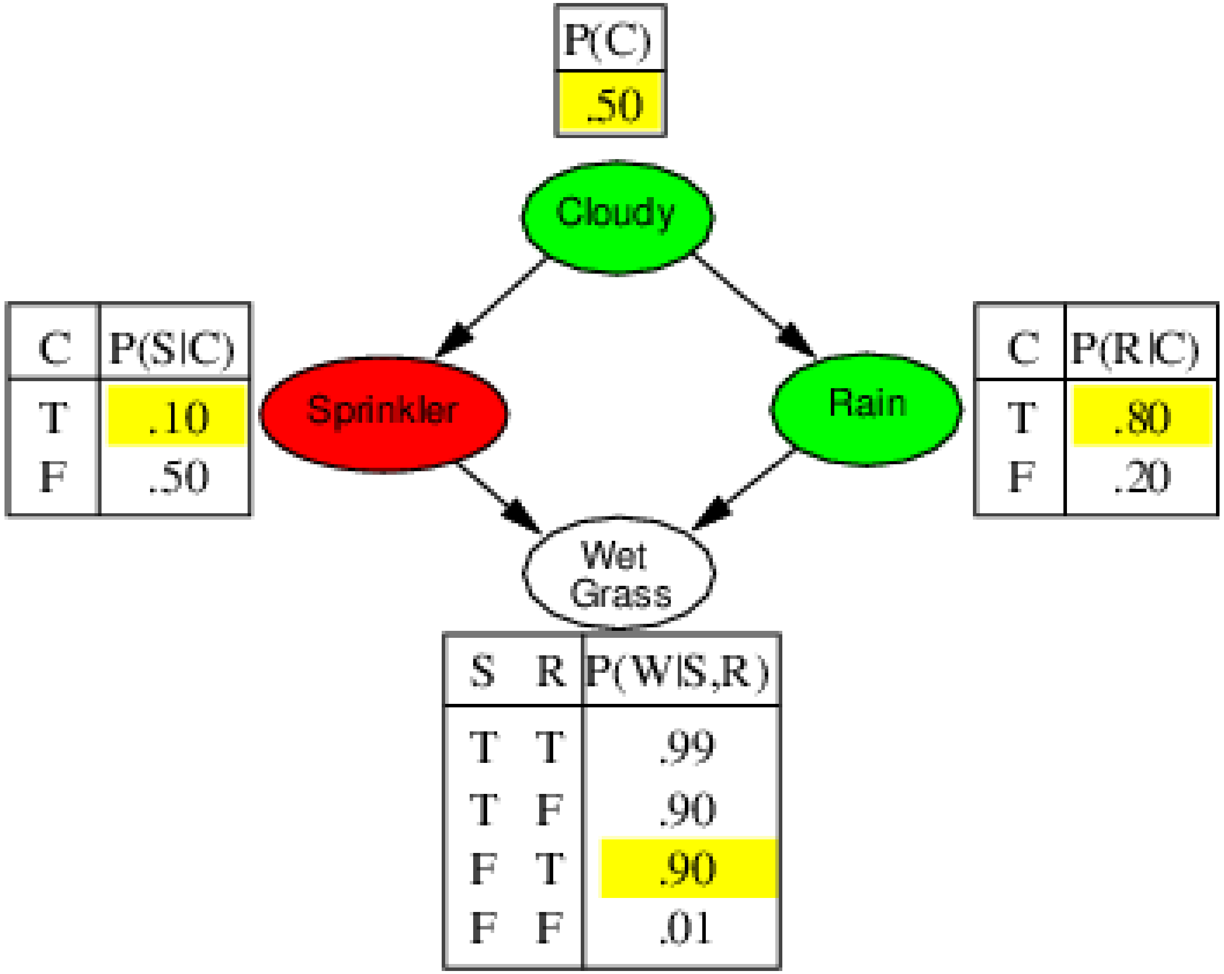
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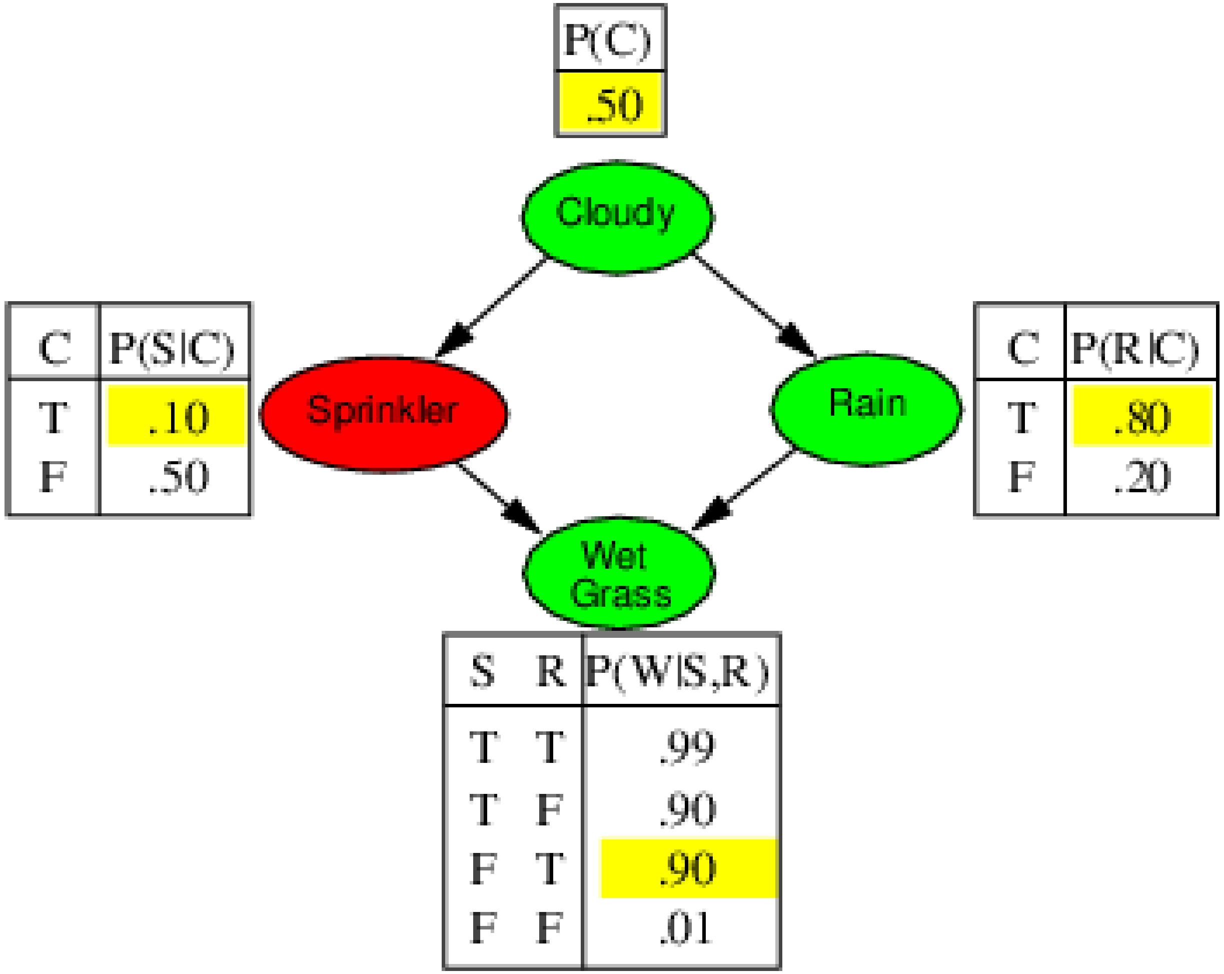
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Sampling from an Empty Network

- Probability that PRIORSAMPLE generates a particular event
  $S_{PS}(x_1 \dots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i)) = P(x_1 \dots x_n)$
  i.e., the true prior probability

- E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

- Let $N_{PS}(x_1 \dots x_n)$ be the number of samples generated for event $x_1, \dots, x_n$
  Let N be to total number of samples

- Then we have
$$\begin{aligned} \lim_{N \to \infty} \hat{P}(x_1, \dots, x_n) &= \lim_{N \to \infty} N_{PS}(x_1, \dots, x_n)/N \\ &= S_{PS}(x_1, \dots, x_n) \\ &= P(x_1 \dots x_n) \end{aligned}$$

- That is, estimates derived from PRIORSAMPLE are **consistent**

- Shorthand: $\hat{P}(x_1, \dots, x_n) \approx P(x_1 \dots x_n)$

Rejection Sampling

- Now, we want to sample with some evidence $\mathbf{e}$ given
- For example: $\mathbf{P}(Rain|Sprinkler = true)$

  $\mathbf{e} = \{Sprinkler = true\}$
- Idea
  - sample as before
  - reject any generated sample that is inconsistent with the evidence
  - normalize over surviving samples

Rejection Sampling

- $\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

```
function REJECTION-SAMPLING(X, e, bn, N) returns an estimate of P(X|e)
   local variables: N, a vector of counts over X, initially zero

   for j = 1 to N do
       x ← PRIOR-SAMPLE(bn)
       if x is consistent with e then
          N[x] ← N[x]+1 where x is the value of X in x
   return NORMALIZE(N[X])
```

- E.g., estimate $\mathbf{P}(Rain|Sprinkler = true)$ using 100 samples
  27 samples have $Sprinkler = true$
  Of these, 8 have $Rain = true$ and 19 have $Rain = false$

- $\hat{\mathbf{P}}(Rain|Sprinkler = true) = \text{NORMALIZE}(\langle 8, 19 \rangle) = \langle 0.296, 0.704 \rangle$

Analysis of Rejection Sampling

- $\hat{\mathbf{P}}(X|\mathbf{e}) = \alpha \mathbf{N}_{PS}(X,\mathbf{e})$ (algorithm defn.)
  $= \mathbf{N}_{PS}(X,\mathbf{e})/N_{PS}(\mathbf{e})$ (normalized by $N_{PS}(\mathbf{e})$)
  $\approx \mathbf{P}(X,\mathbf{e})/P(\mathbf{e})$ (property of PRIORSAMPLE)
  $= \mathbf{P}(X|\mathbf{e})$ (defn. of conditional probability)

- Hence rejection sampling returns consistent posterior estimates

- Problem: hopelessly expensive if $P(\mathbf{e})$ is small

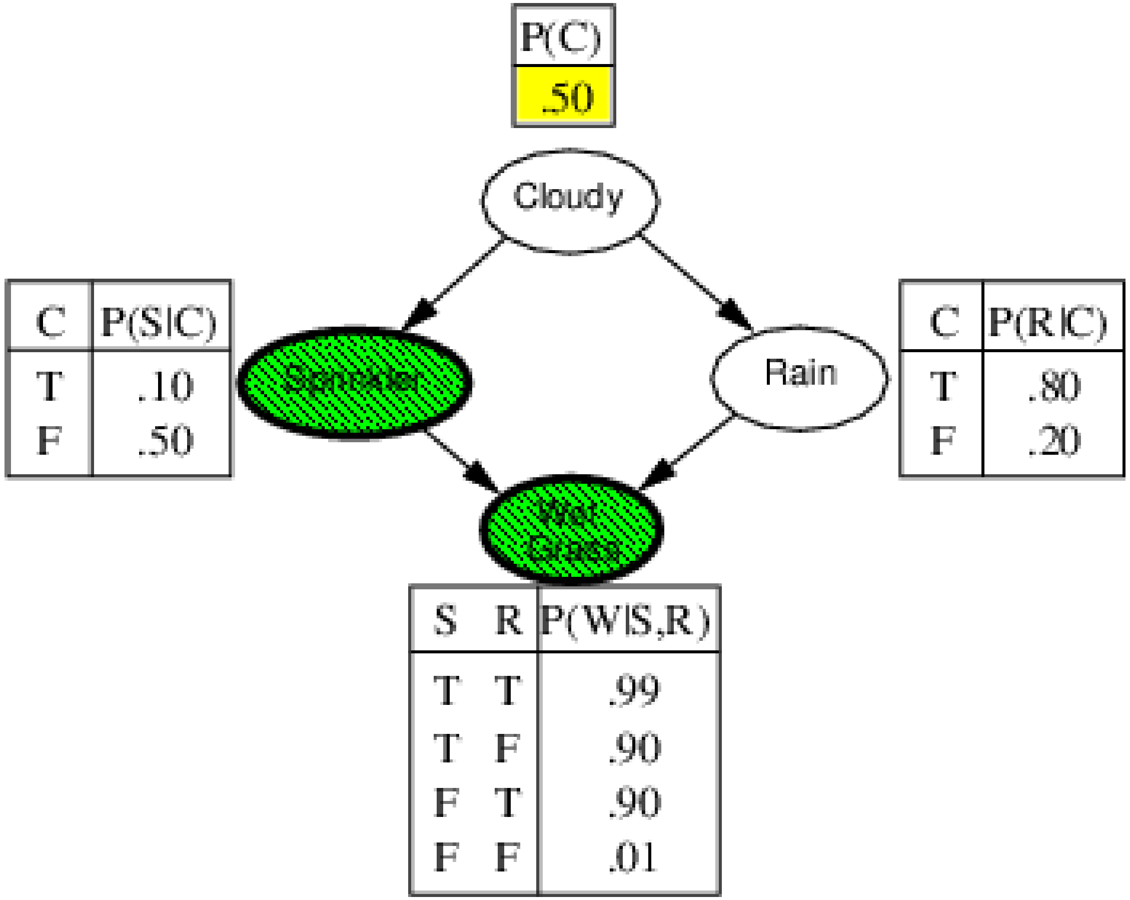
- $P(\mathbf{e})$ drops off exponentially with number of evidence variables!

Likelihood Weighting

- Idea
  - fix evidence variables
  - sample only non-evidence variables
  - weight each sample by the likelihood it accords the evidence

- Example: $\mathbf{P}(Rain|Sprinkler = true, WetGrass = true)$
  - $\mathbf{e} = \{Sprinkler = true, WetGrass = true\}$ → we weight these
  - we sample the other variables

Likelihood Weighting Example



$w = 1.0$

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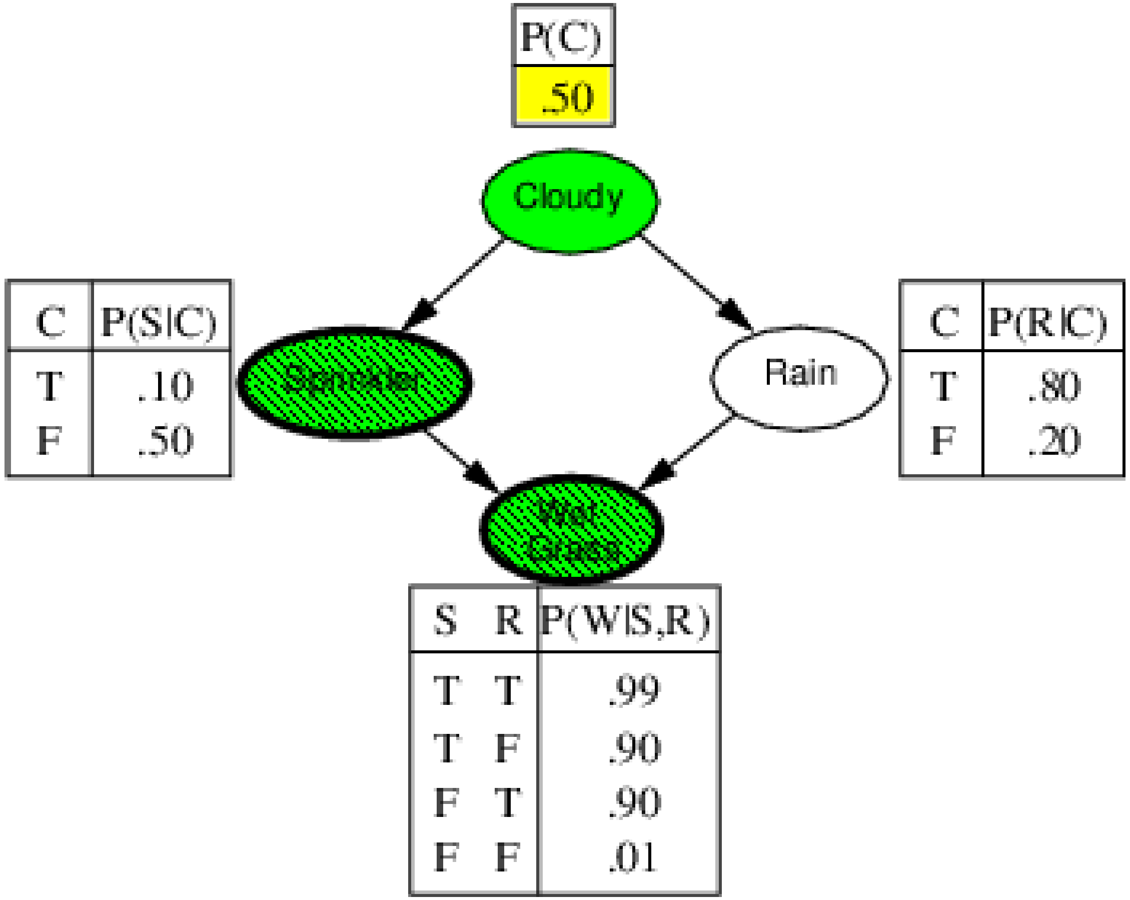

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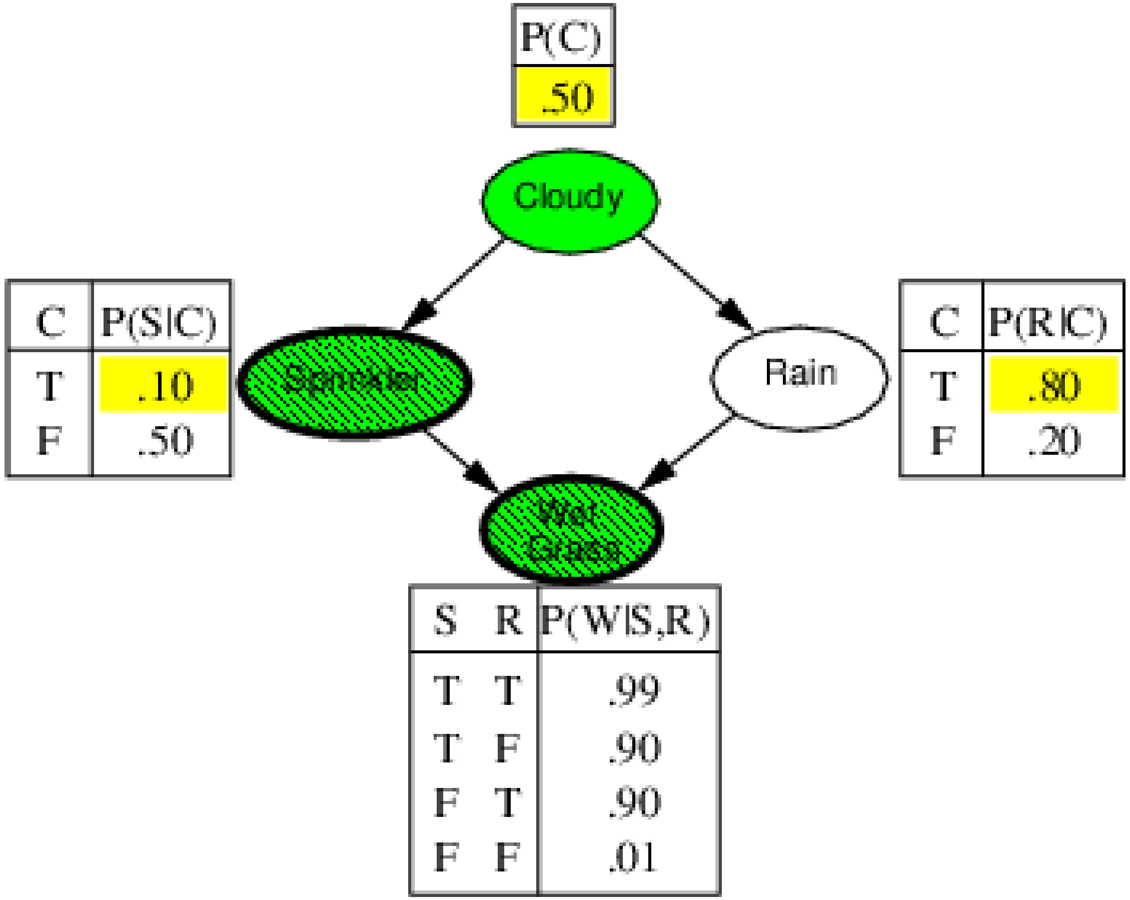

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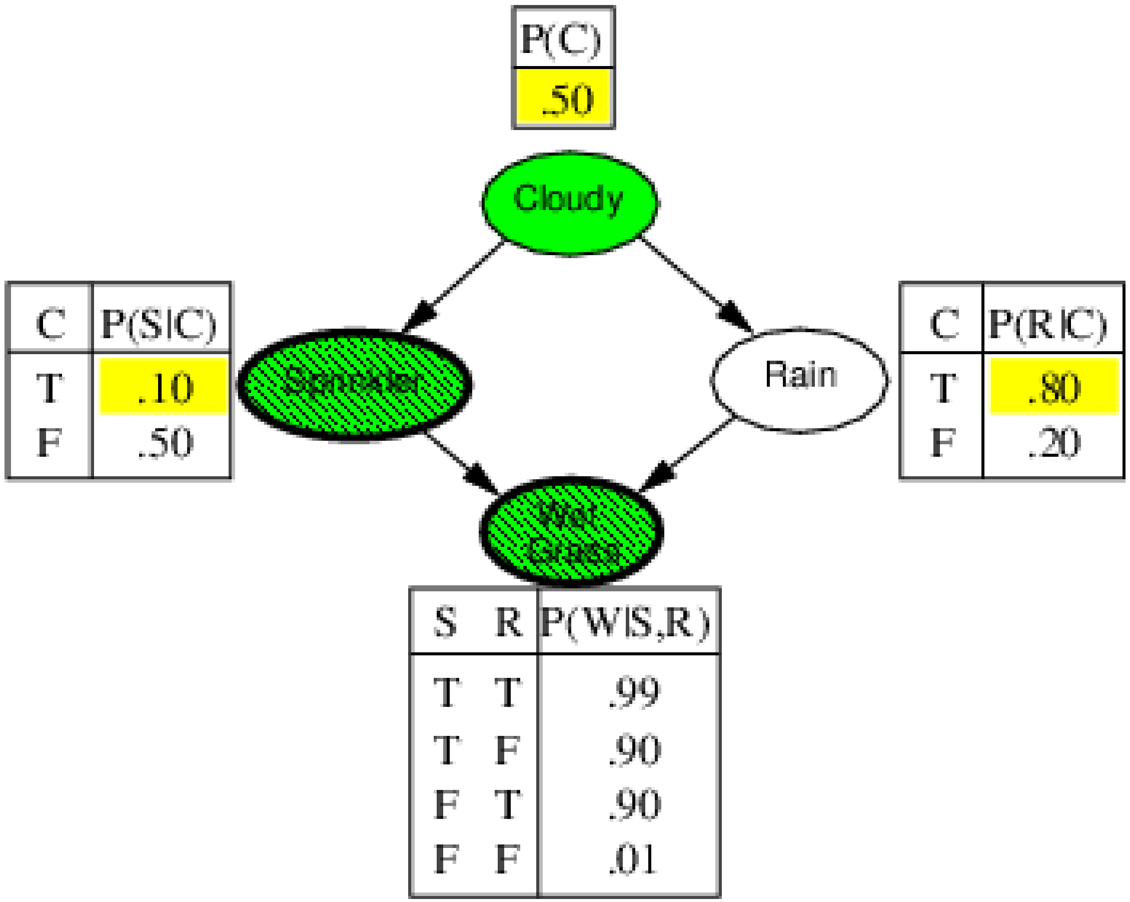
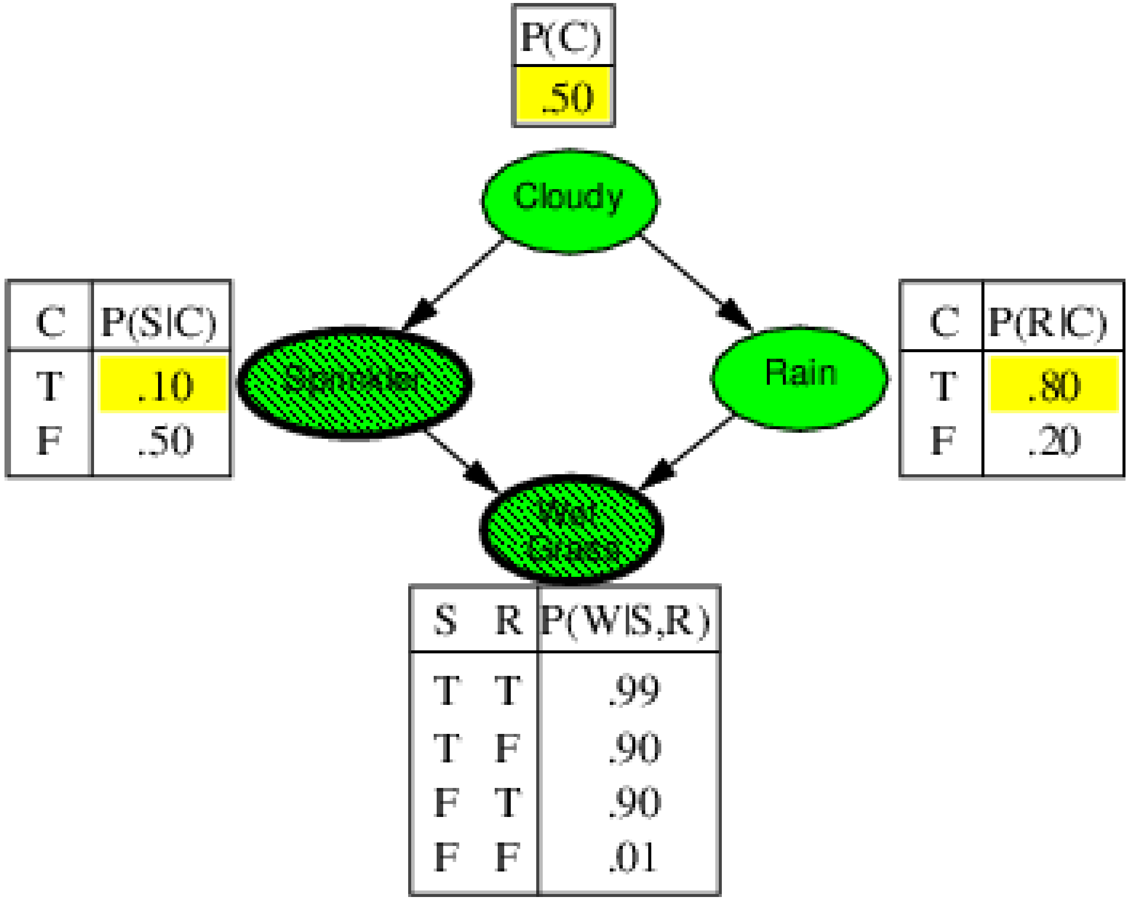


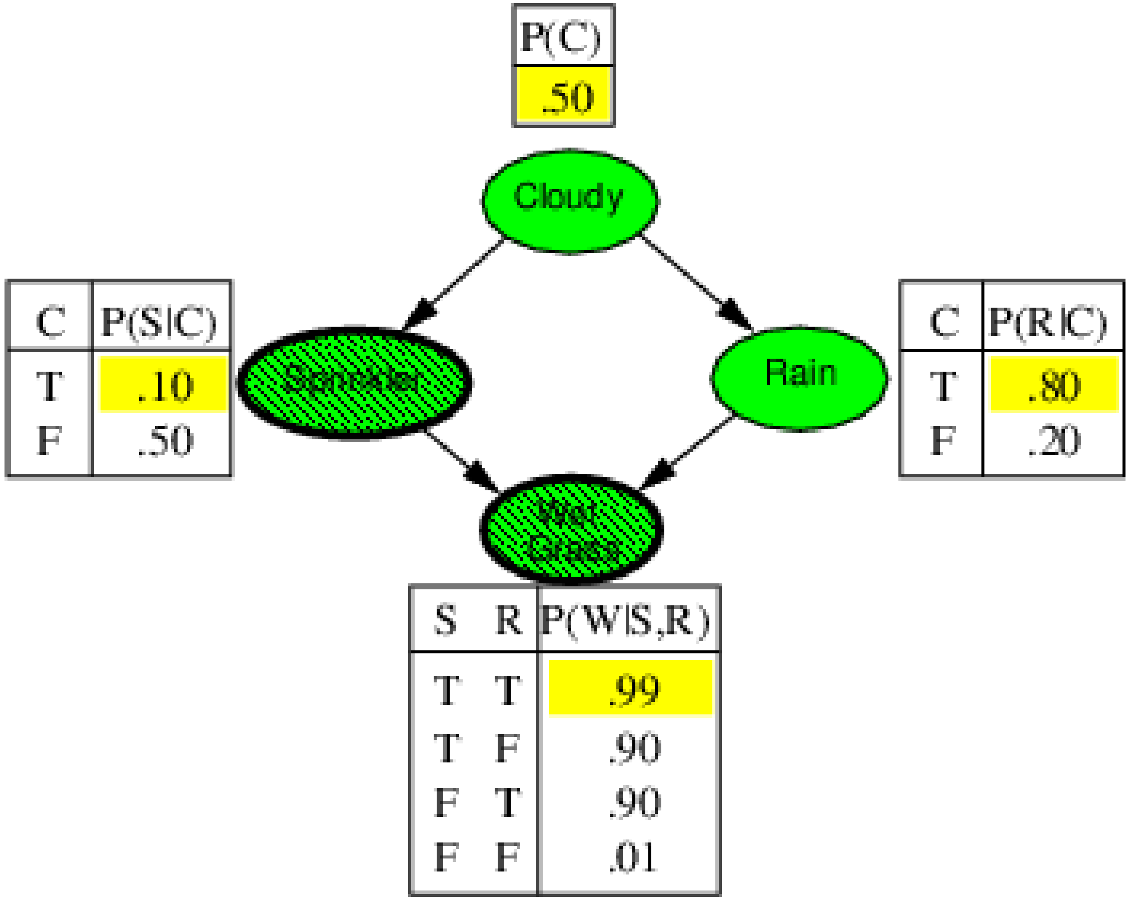
$$w = 1.0 \times 0.1$$

Likelihood Weighting Example



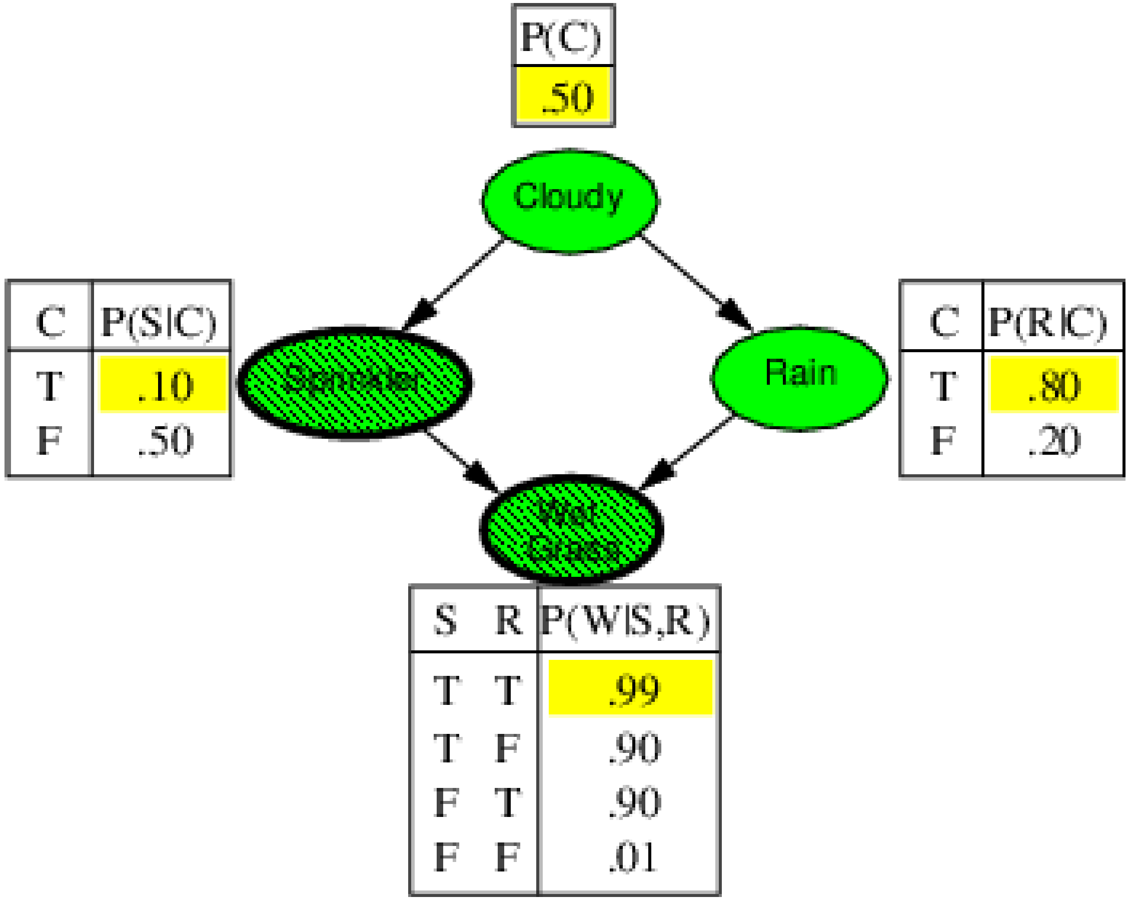
$$w = 1.0 \times 0.1$$

Likelihood Weighting Example



$$w = 1.0 \times 0.1$$

Likelihood Weighting Example



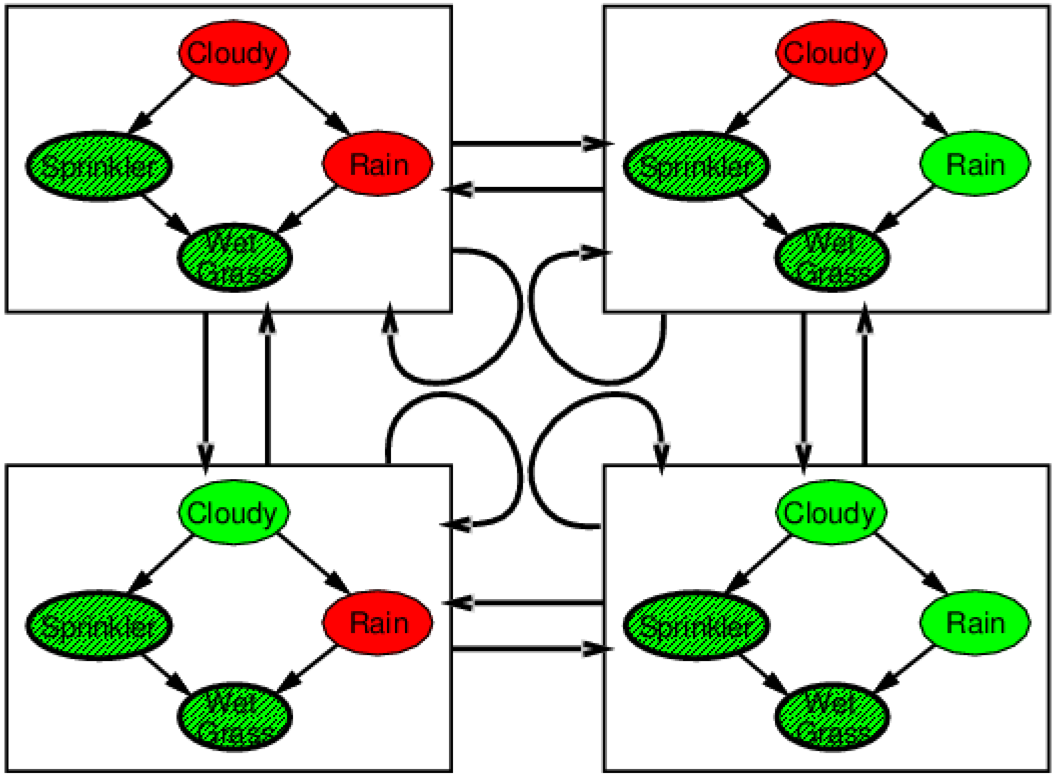
$$w = 1.0 \times 0.1 \times 0.99 = 0.099$$

Approximate Inference using MCMC

- "State" of network = current assignment to all variables
- Generate next state by sampling one variable
  Sample each variable in turn, keeping evidence fixed
- Can also choose a variable to sample at random each time

The Markov Chain

- With $Sprinkler = true, WetGrass = true$, there are four states:



- Wander about for a while, average what you see

MCMC Example

- Estimate $\mathbf{P}(Rain|Sprinkler = true, WetGrass = true)$

- Sample $Cloudy$ or $Rain$ given its Markov blanket, repeat.
  Count number of times $Rain$ is true and false in the samples.

- E.g., visit 100 states
  31 have $Rain = true$, 69 have $Rain = false$

- $\hat{\mathbf{P}}(Rain|Sprinkler = true, WetGrass = true)$
  $= \textsc{Normalize}(\langle 31, 69 \rangle) = \langle 0.31, 0.69 \rangle$

- Theorem: chain approaches **stationary distribution**:
  long-run fraction of time spent in each state is exactly
  proportional to its posterior probability

Summary

- Bayes nets provide a natural representation for (causally induced) conditional independence
- Generally easy for (non)experts to construct
- Exact inference by variable elimination
  - polytime on polytrees, NP-hard on general graphs
  - space = time, very sensitive to topology
- Approximate inference by LW, MCMC
  - LW does poorly when there is lots of (downstream) evidence
  - LW, MCMC generally insensitive to topology
  - Convergence can be very slow with probabilities close to 1 or 0
  - Can handle arbitrary combinations of discrete and continuous variables