Geometry Processing (601.458/658)

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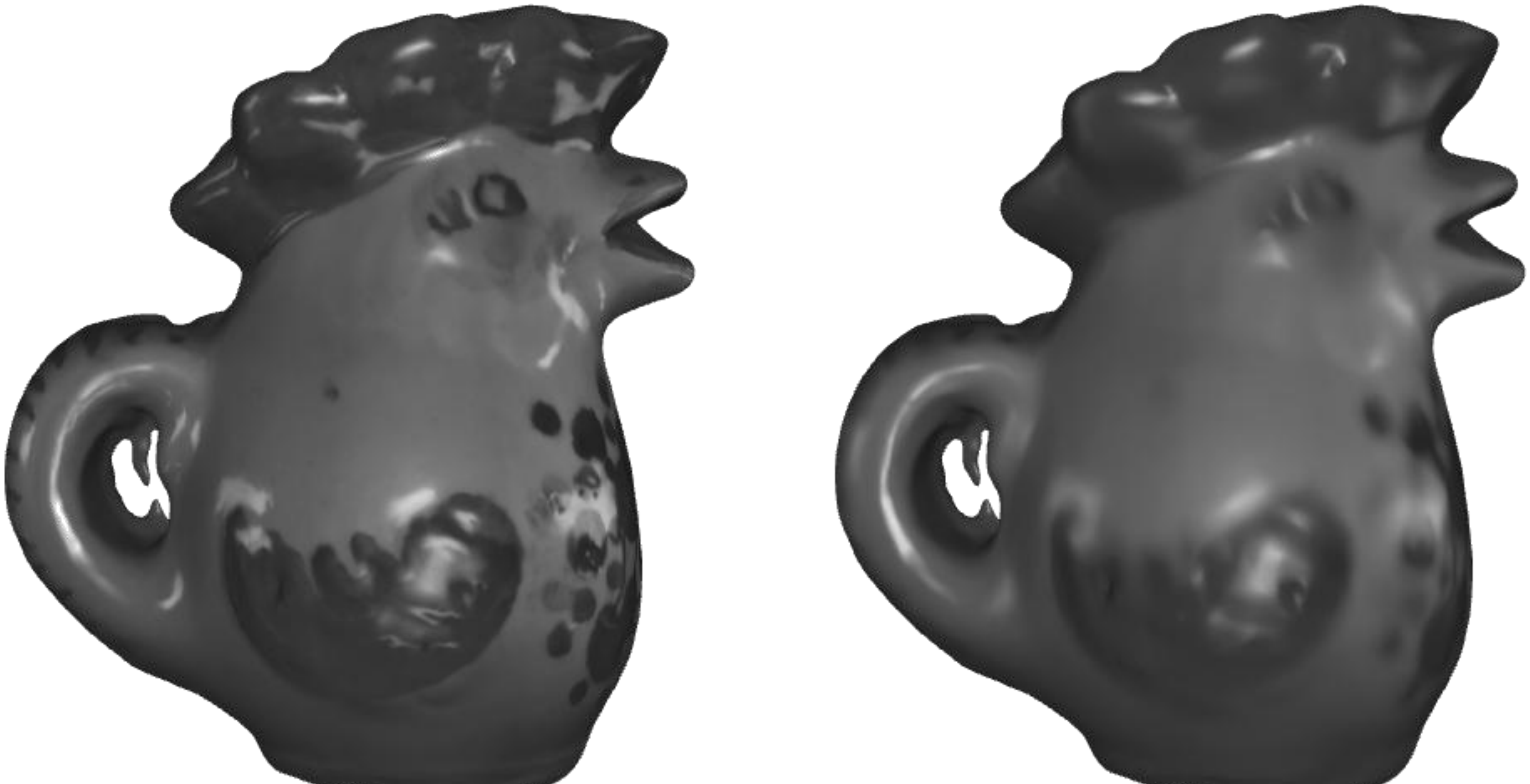
Outline

Applications

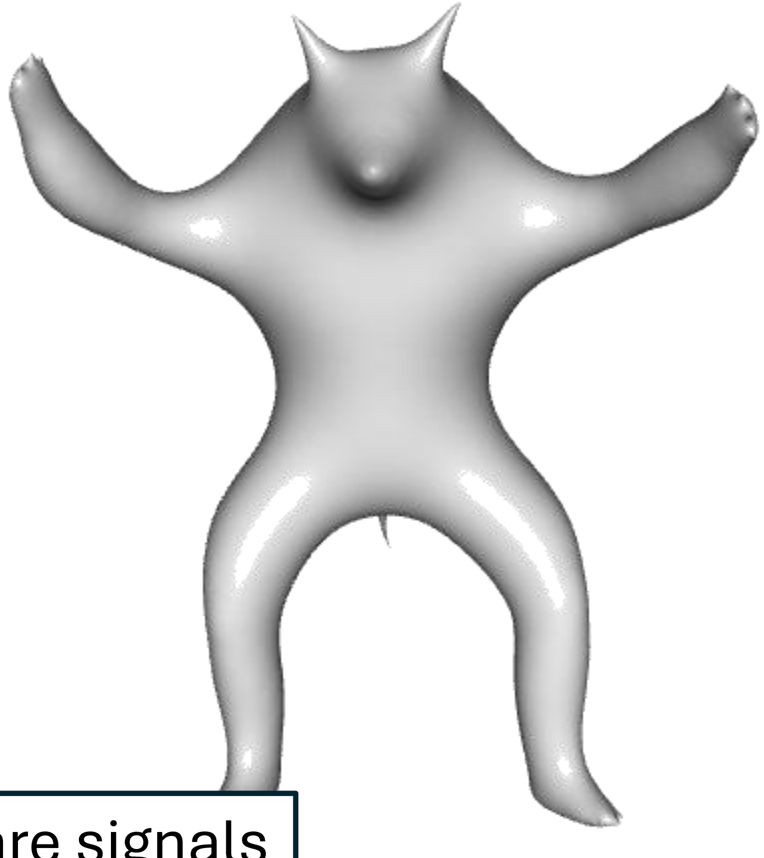
Take-Aways

What Next?

Applications: Signal Smoothing

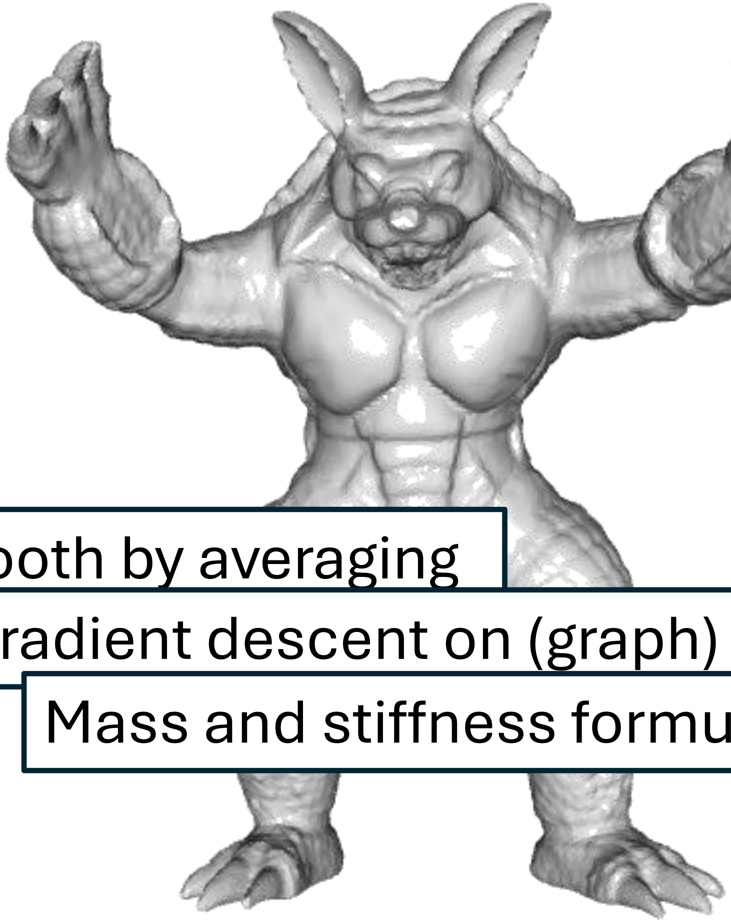


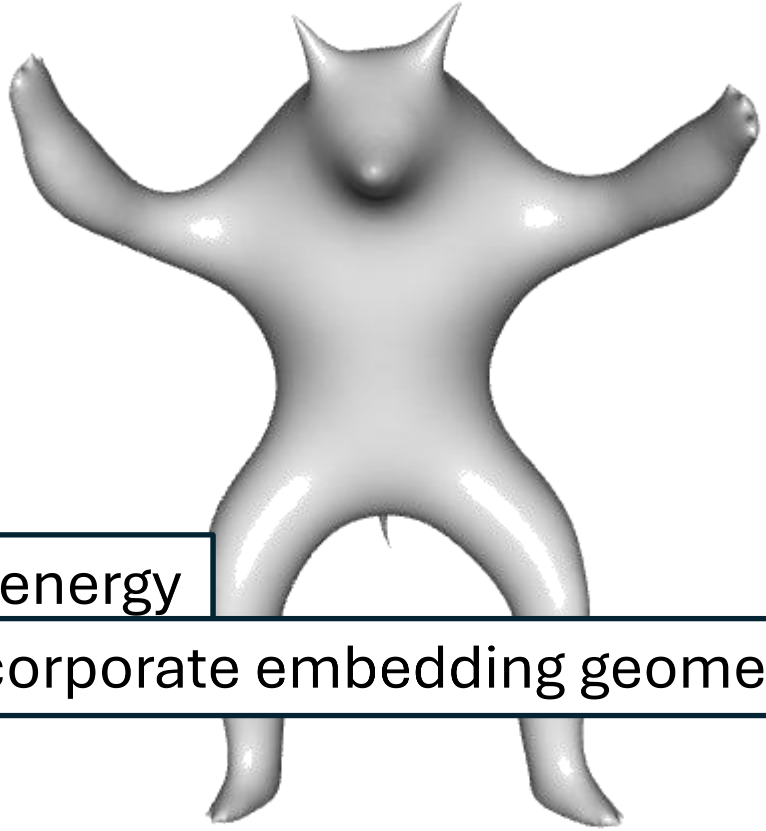
Applications: Geometry Smoothing



Vertex positions are signals

Applications: Geometry Smoothing



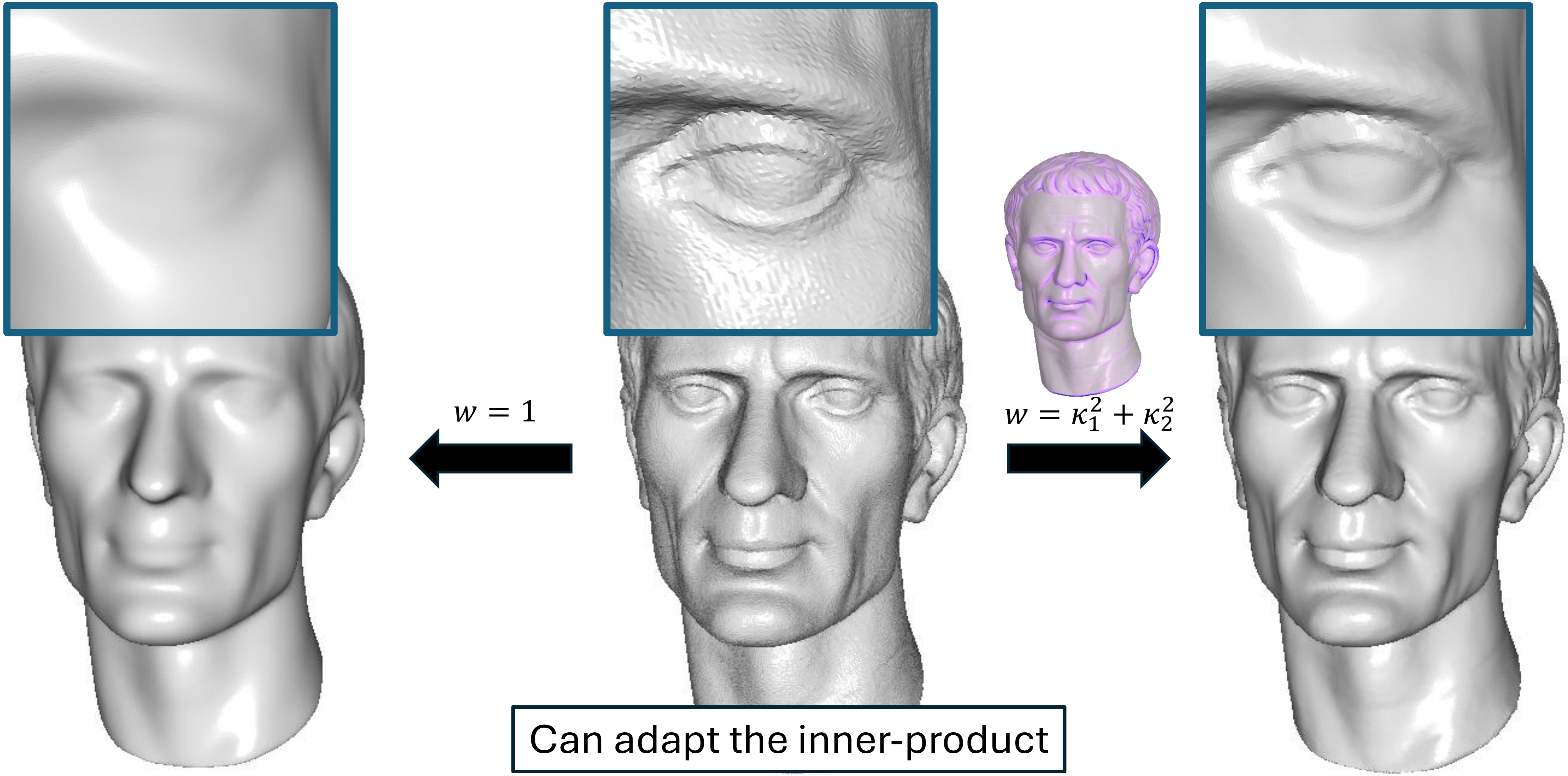


Smooth by averaging

Gradient descent on (graph) Dirichlet energy

Mass and stiffness formulation incorporate embedding geometry

Applications: Inhomogeneous Smoothing

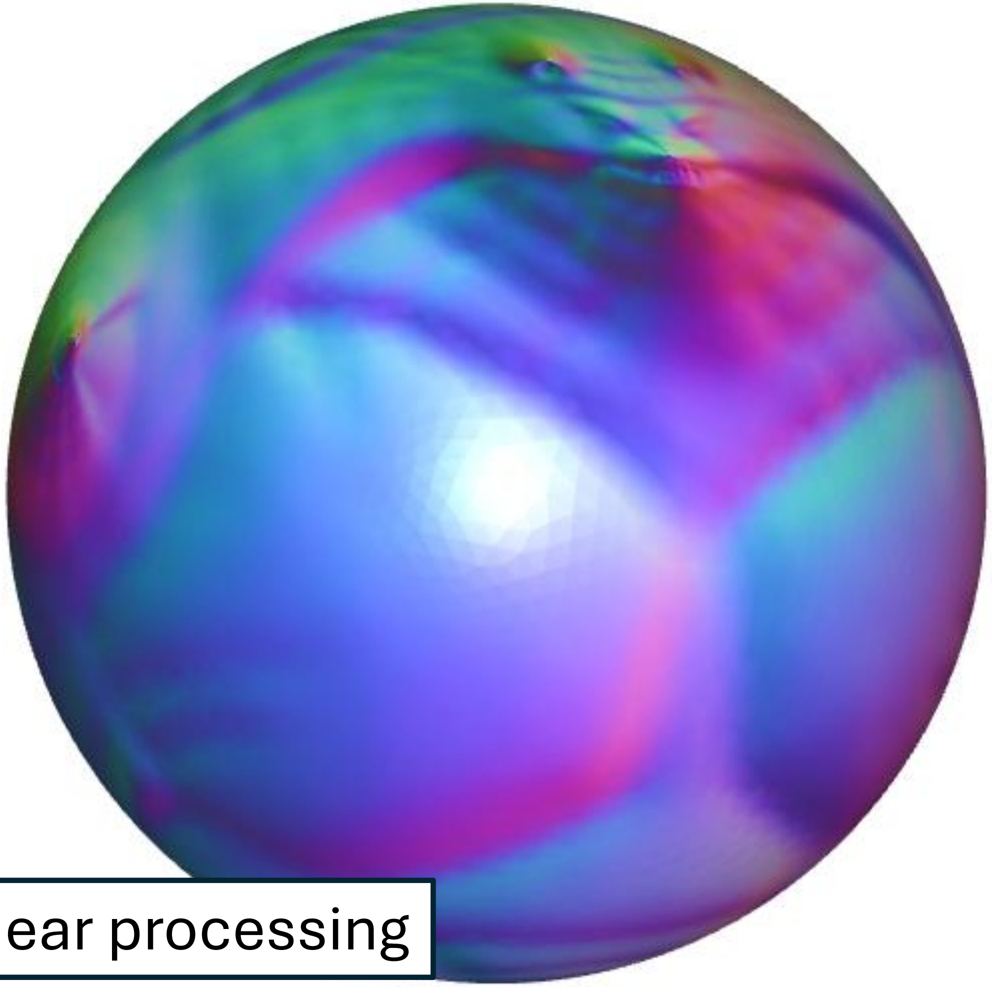


Can adapt the inner-product

Applications: Anisotropic Smoothing

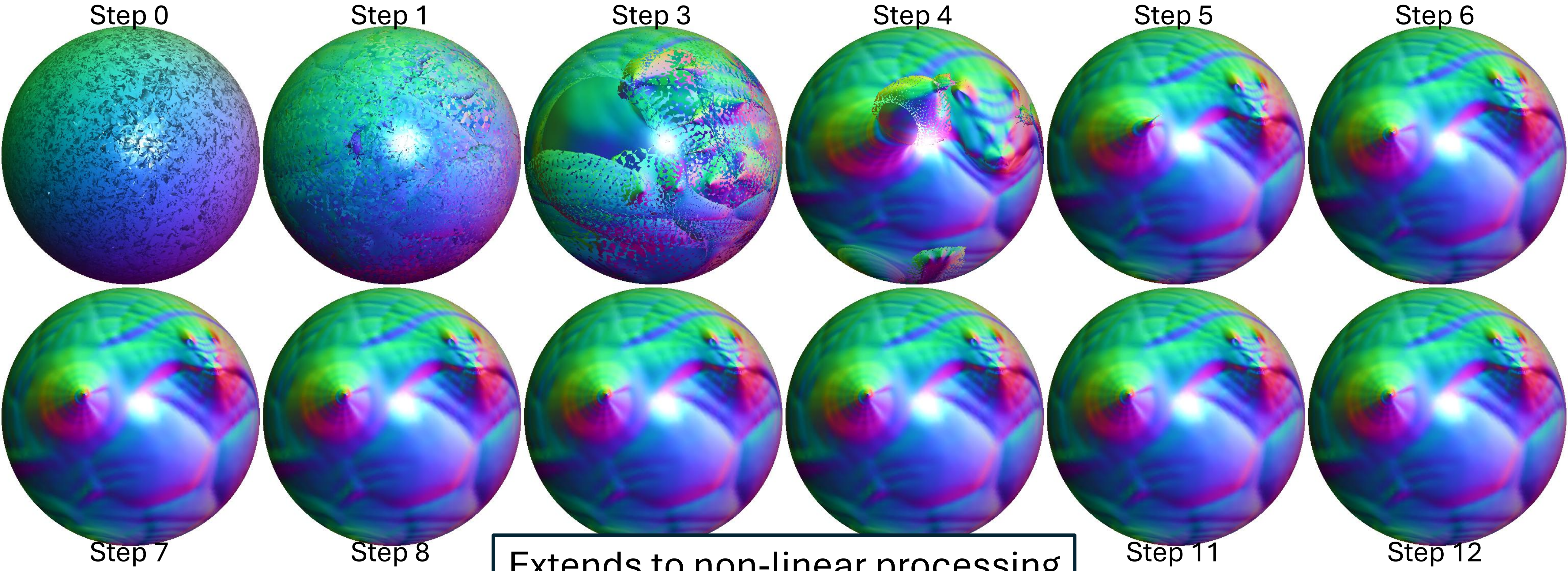
Can adapt the inner-product

Applications: Geometry Smoothing (non-linear)



Extends to non-linear processing

Applications: Normal Smoothing (non-linear)



Extends to non-linear processing

Applications: Smoothing

Though these are all variants of smoothing, getting them to work requires understanding:

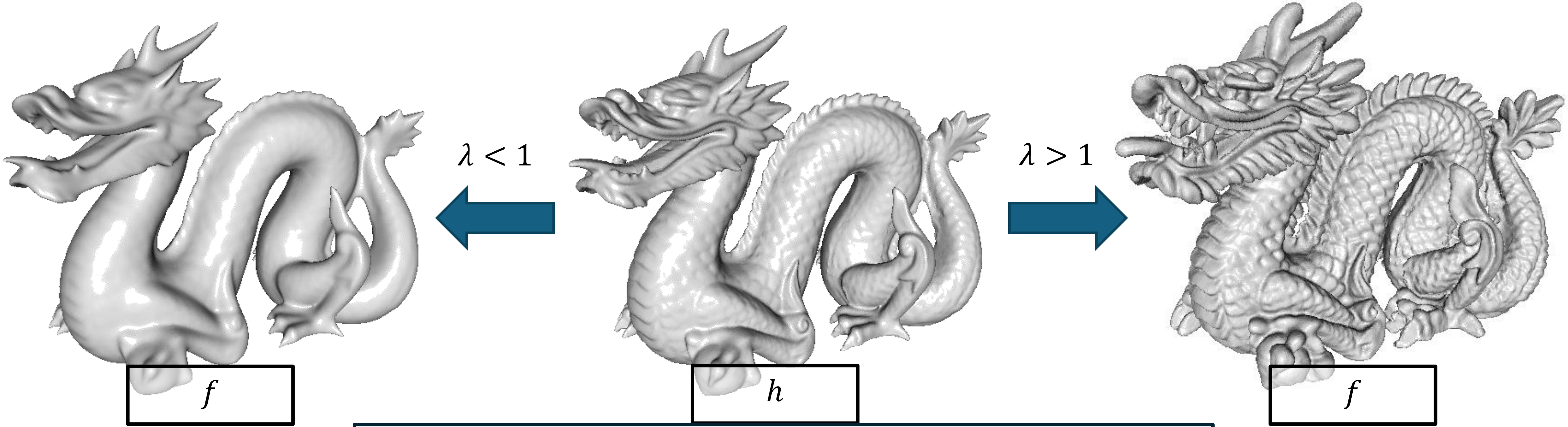
Discrete spaces of functions on meshes

Discrete spaces of co-tangent vector-fields on meshes

The differential taking us from one space to the other

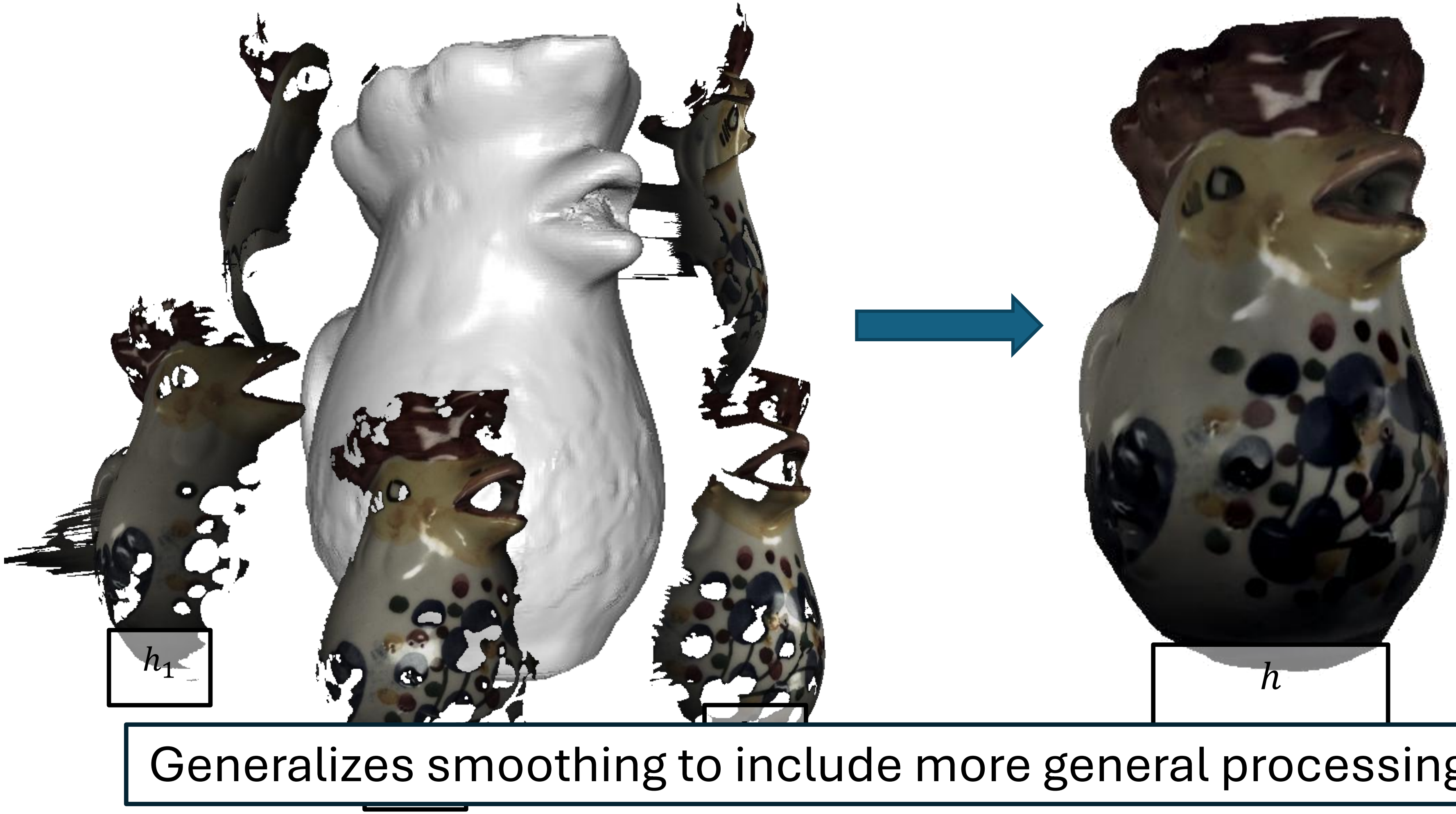
The inner-products on these spaces of functions,
and how they are defined by the inner-product on the tangent spaces.

Applications: Gradient-Domain Processing



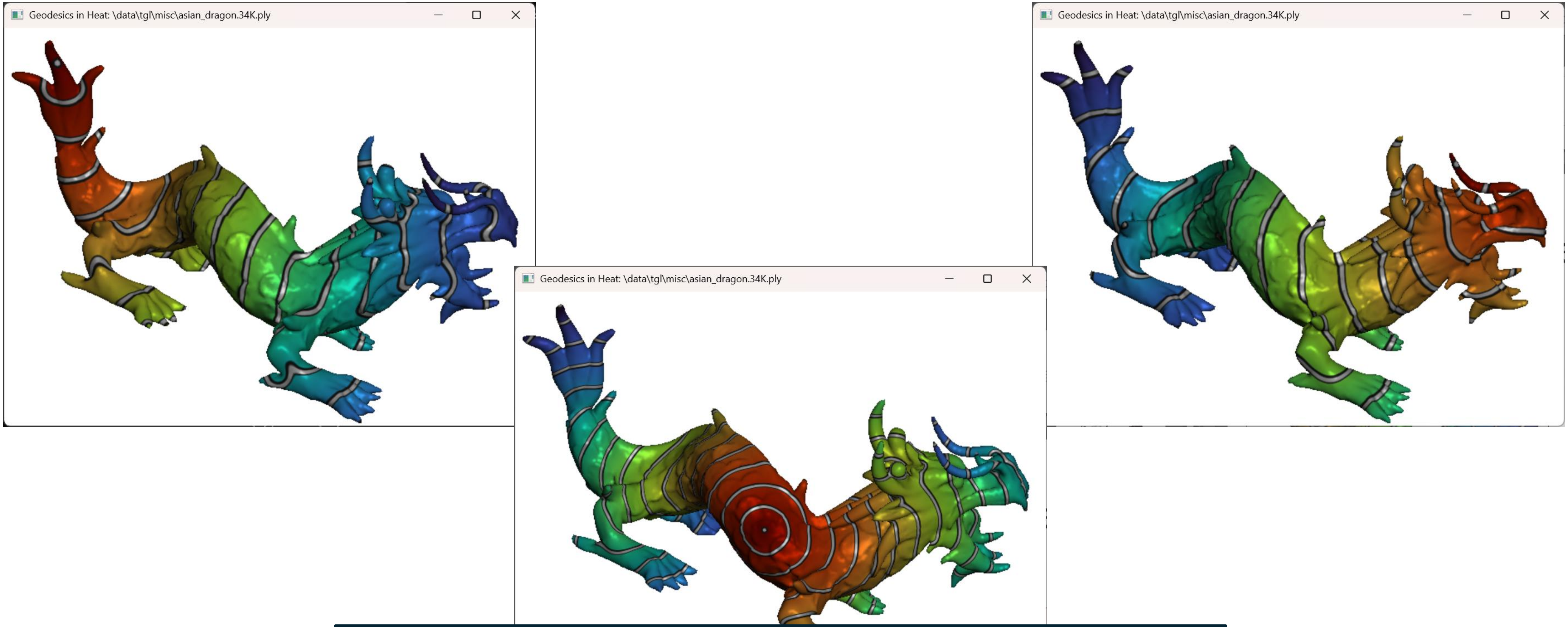
Generalizes smoothing to include sharpening

Applications: Gradient-Domain Processing



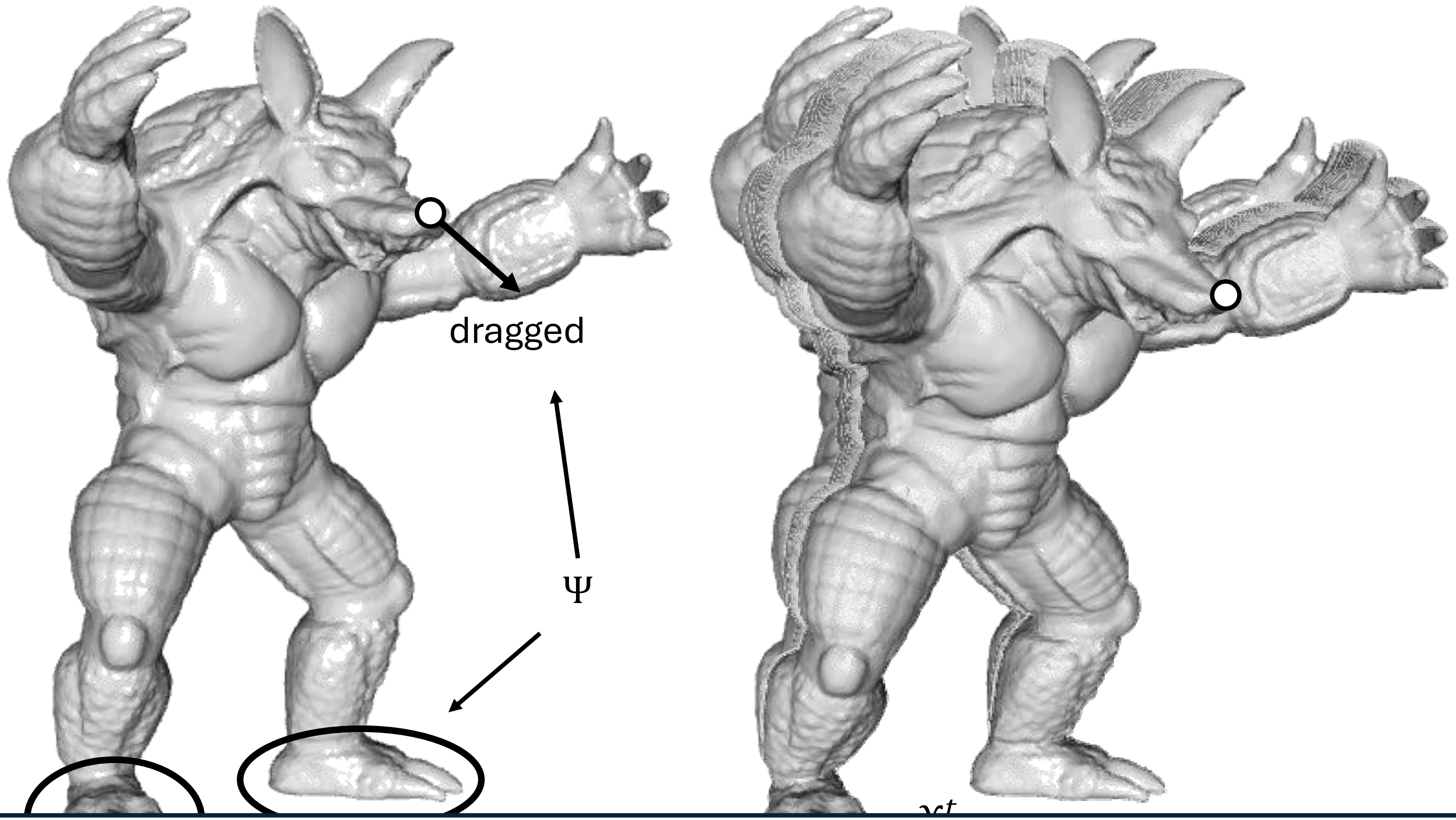
Generalizes smoothing to include more general processing

Applications: Geodesics in Heat



Gradient-domain smoothing and derivative-fitting

Applications: Surface Deformation



Local non-linear factorizing + global gradient-domain derivative-fitting

Applications: Wave Propagation

Mass/stiffness processing generalizes to other systems

Take-Aways

A lot of geometry processing can be formulated as solution of linear systems expressed in terms of the mass and stiffness matrices.

Though spectral decomposition may be impractical for processing geometry, it is essential to understanding how the systems behave.

- What is gradient-domain processing actually doing?
- Why is implicit time-stepping unconditionally stable?

Take-Aways

We worked with three types of inner-product spaces.

$\{T_{\mathbf{p}}\mathbb{T}, g_{\mathbf{p}}: T_{\mathbf{p}}\mathbb{T} \to T_{\mathbf{p}}^{*}\mathbb{T}\}$:

Tangent vectors at the point $\mathbf{p} \in \mathbb{T}$, with inner-product $g_{\mathbf{p}}$ defined by pulling back the Euclidean inner-product using the differential of the parametrization $\Phi_{\tau}: \mathbb{T} \to \tau$.

$\{V, M: V \to V^{*}\}$:

Piecewise-linear functions on the mesh, with inner-product M defined by integrating over the mesh.

$\downarrow \; d \in \mathrm{Hom}(V, \overline{V})$

$\{\overline{V}, \overline{M}: \overline{V} \to \overline{V}^{*}\}$:

Piecewise-constant cotangent vector fields on the mesh, with inner-product $\overline{M}$ defined by integrating over the mesh.

Take-Aways

Essential to getting things to work is getting a clear picture on what types of algebraic object we are working with.

We can/should think of bilinear maps $B: V \times V \to \mathbb{R}$ as linear maps $B: V \to V^*$.

It **does** makes sense to talk about a bilinear map $B: V \times V \to \mathbb{R}$ being symmetric.
It **does** makes sense to talk about a linear map $B: V \to V^*$ being symmetric.
It **does not** make sense to talk about an endomorphism being symmetric.

It **does** makes sense to talk about the eigenvectors/values of an endomorphism.
It **does not** make sense to talk about the eigenvectors/values of a homomorphism.

The differential **is not** a tangent vector describing the "direction of steepest change".
The gradient **is** the tangent vector describing the "direction of steepest change".

The determinant **does not** define the volume of a parallelepiped.
It **does** define the change relative to an existing measure of volume.

Take-Aways

Essential to getting things to work is getting a clear picture of the types of algebraic objects we are working with.

When working with matrices and column vectors, it is easy to lose sight of the picture.

Taking a transpose of matrix could mean:

- Take the dual (w.r.t. to a general basis)
- Invert an orthogonal map (w.r.t. to an orthonormal basis)
- Compute the adjoint (w.r.t. to two orthonormal bases)

The identity matrix can represent:

- The identity map (an endomorphism)
- The expression of an inner-product w.r.t. an orthonormal basis (a bilinear form)

Take-Aways

Matrices allow us to type-check dimensions:

$$\mathbf{v} = \underbrace{\underbrace{\underbrace{\underbrace{\boldsymbol{\iota}}_{\in\mathbb{R}^{n\times k}} \cdot (\underbrace{\boldsymbol{\iota}^\top \cdot \mathbf{B}\cdot\boldsymbol{\iota}}_{\in\mathbb{R}^{k\times k}})^{-1}}_{\in\mathbb{R}^{n\times k}} \cdot \underbrace{\underbrace{\boldsymbol{\iota}^\top}_{\in\mathbb{R}^{k\times n}}}_{\in\mathbb{R}^{k\times n}} \cdot \underbrace{(\underbrace{\mathbf{l}}_{\in\mathbb{R}^n} - \underbrace{\mathbf{B}\cdot\hat{\mathbf{v}}}_{\in\mathbb{R}^n})}_{\in\mathbb{R}^n}}_{\in\mathbb{R}^n} + \underbrace{\underbrace{\underbrace{\hat{\mathbf{v}}}_{\in\mathbb{R}^n}}_{\in\mathbb{R}^n}}_{\in\mathbb{R}^n}}_{\in\mathbb{R}^n}$$

- $\boxed{\mathbf{B} \in \mathbb{R}^{n\times n}}$
- $\boldsymbol{\iota} \in \mathbb{R}^{n\times k}$
- $\boldsymbol{\iota}^\top \in \mathbb{R}^{k\times n}$
- $\boxed{\mathbf{l} \in \mathbb{R}^{n}}$
- $\hat{\mathbf{v}} \in \mathbb{R}^{n}$

Take-Aways

The algebraic picture allows us to check the spaces themselves:

$$\mathbf{v} = \boldsymbol{\iota} \cdot (\boldsymbol{\iota}^{\top} \cdot \mathbf{B} \cdot \boldsymbol{\iota})^{-1} \cdot \Big(\boldsymbol{\iota}^{\top} \cdot (\mathbf{l} - \mathbf{B} \cdot \hat{\mathbf{v}})\Big) + \hat{\mathbf{v}}$$

$$v = \underbrace{\underbrace{\underbrace{\underbrace{\iota}_{\in \mathrm{Hom}(K,V)}}_{\in \mathrm{Hom}(K,V)} \Big(\underbrace{(\underbrace{\iota^* \circ B \circ \iota}_{\in \mathrm{Hom}(K,K^*)})^{-1}}_{\in \mathrm{Hom}(K^*,K)}}_{\in \mathrm{Hom}(K^*,V)} \circ \underbrace{\underbrace{\underbrace{\iota^*}_{\in \mathrm{Hom}(V^*,K^*)}(\underbrace{l - B \circ \hat{v}}_{\in V^*})}_{\in K^*}}_{\in K^*} \Big) + \underbrace{\underbrace{\underbrace{\hat{v}}_{\in V}}_{\in V}}_{\in V}}_{\in V}$$

- $\boxed{B \in \mathrm{Hom}(V, V^*)}$
- $\iota \in \mathrm{Hom}(K, V)$
- $\iota^* \in \mathrm{Hom}(V^*, K^*)$
- $\boxed{l \in V^*}$
- $\hat{v} \in V$

$\boxed{\text{Denoting } K \equiv \mathrm{Ker}(C)}$

What Next?

Other discretizations of function spaces

Anisotropic geometry processing

Higher dimensions

Other PDEs/energies

Higher-order systems

Vector-field processing

Relating to differential geometry

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