Geometry Processing (601.458/658)

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Outline

Recall

Math Review

Inhomogeneous/Anisotropic Geometry Processing

Recall

Notation:

Given a vector space V and a symmetric bilinear form $B \in \mathrm{Hom}(V, V^*)$ we overload notation, representing B as a real-valued function on pairs of vectors, $B: V \times V \rightarrow \mathbb{R}$, with:

$$B(u, v) \equiv [B(u)](v)$$

For an inner-product space $\{V, B: V \rightarrow V^*\}$ we denote the inner-product $\langle \cdot, \cdot \rangle_B: V \times V \rightarrow \mathbb{R}$ and norm $\|\cdot\|_B: V \rightarrow \mathbb{R}^{\geq 0}$ as:

$$\langle u, v \rangle_B \equiv B(u, v)$$
$$\|v\|_B \equiv \sqrt{B(v, v)}$$

Recall

Given an inner-product space $\{V, B: V \to V^*\}$ and given a vector $u \in V$, we can define a new bilinear form:

$$[\tilde{B}(v)](w) \equiv \langle v, w \rangle_B + \langle u, v \rangle_B \cdot \langle u, w \rangle_B$$

Claim:

This is also symmetric and positive definite.

Symmetry follows from the symmetry of B and commutativity of multiplication.

Positive definiteness follows from:

$$\begin{aligned}[\tilde{B}(v)](v) &= \|v\|_B^2 + \langle u, v \rangle_B \cdot \langle u, v \rangle_B \\ &= \|v\|_B^2 + (\langle u, v \rangle_B)^2 \\ &> 0\end{aligned}$$

Recall

Given an inner-product space $\{V, B: V \to V^*\}$ and given a vector $u \in V$, we can define a new inner-product:

$$\langle v, w \rangle_{\tilde{B}} \equiv \langle v, w \rangle_B + \langle u, v \rangle_B \cdot \langle u, w \rangle_B$$

Note:

If v is perpendicular to u (i.e. $\langle u, v \rangle_B = 0$) then:

$$\begin{aligned} \|v\|_{\tilde{B}}^2 &= \|v\|_B^2 + \langle u, v \rangle_B \cdot \langle u, v \rangle_B \\ &= \|v\|_B^2 \end{aligned}$$

$\Rightarrow$ Distances are not changed along the perpendicular.

Recall

Given an inner-product space $\{V, B: V \to V^*\}$ and given a vector $u \in V$, we can define a new inner-product:

$$\langle v, w \rangle_{\tilde{B}} \equiv \langle v, w \rangle_B + \langle u, v \rangle_B \cdot \langle u, w \rangle_B$$

Note:

If v is parallel to u (i.e. $v = \lambda \cdot u$) then:

$$\begin{aligned} \|v\|_{\tilde{B}}^2 &= \|v\|_B^2 + \langle u, v \rangle_B \cdot \langle u, v \rangle_B \\ &= \|v\|_B^2 + \langle u, \lambda \cdot u \rangle_B \cdot \langle u, \lambda \cdot u \rangle_B \\ &= \|v\|_B^2 + \lambda^2 \cdot \|u\|_B^2 \cdot \|u\|_B^2 \end{aligned}$$

$\Rightarrow$ Distances are increased along u, with the scale factor growing with the magnitude of u.

Recall

We denote by $\mathbb{T} \subset \mathbb{E}^2$ the unit right triangle:

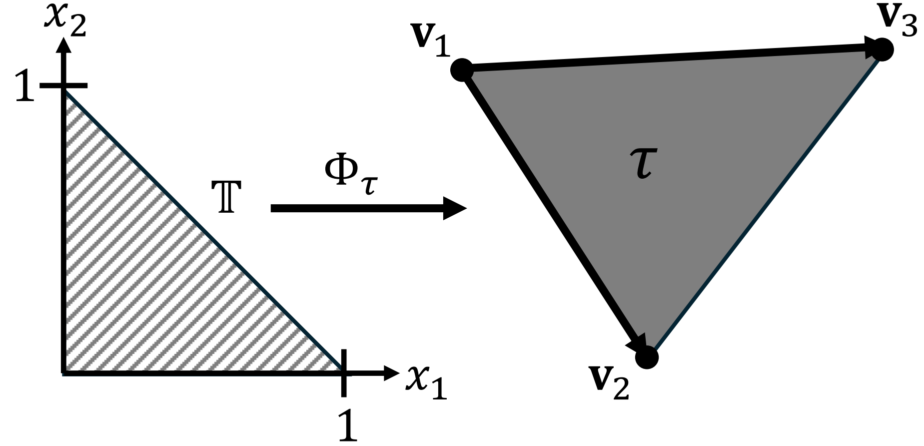
$$\mathbb{T} = \{(x_1, x_2) \in [0,1]^2 | x_1 + x_2 \leq 1\}$$

Given a triangle $\tau \subset \mathbb{E}^3$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{E}^3$, we parameterize:

$$\Phi_\tau(x_1, x_2) = \mathbf{v}_1 + x_1 \cdot (\mathbf{v}_2 - \mathbf{v}_1) + x_2 \cdot (\mathbf{v}_3 - \mathbf{v}_1)$$

W.r.t. the cartesian basis $\{\partial_1|_\mathbf{p}, \partial_2|_\mathbf{p}\}$, this gives the inner-product matrix:

$$\mathbf{g}_\tau = \begin{pmatrix} \|\mathbf{v}_2 - \mathbf{v}_1\|^2 & \langle \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1 \rangle \\ \langle \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1 \rangle & \|\mathbf{v}_3 - \mathbf{v}_1\|^2 \end{pmatrix}$$



$$\mathbf{g}_\tau = \begin{pmatrix} \|\mathbf{v_2}-\mathbf{v_1}\|^2 & \langle \mathbf{v_2}-\mathbf{v_1}, \mathbf{v_3}-\mathbf{v_1}\rangle \\ \langle \mathbf{v_3}-\mathbf{v_1}, \mathbf{v_2}-\mathbf{v_1}\rangle & \|\mathbf{v_3}-\mathbf{v_1}\|^2 \end{pmatrix}$$

Recall

Using the inner-product, we integrate the hat basis functions $\{\psi_i\}$ (and their differentials) to construct the **per-triangle** mass and stiffness matrices:

$$\mathbf{m}_{ij}^\tau = \sqrt{\det(\mathbf{g}_\tau)} \cdot \int_{\mathbb{T}} \psi_i \cdot \psi_j \cdot \omega_E$$

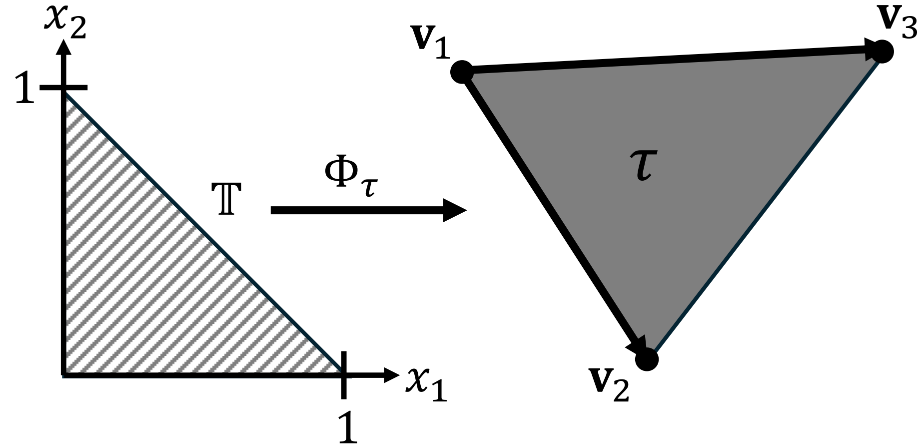
$$\mathbf{s}_{ij}^\tau = \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j{}^\top \cdot \omega_E \right\rangle_F$$

The mass matrix is authalic:

It only depends on the area of the triangle and not on the angles.

The stiffness matrix is conformal:

It only depends on the angles and not on the area



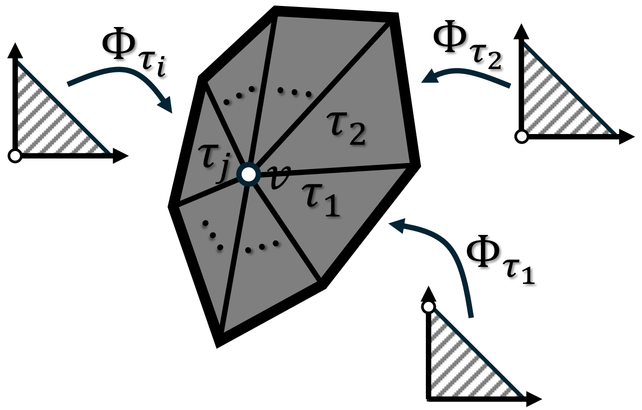
In particular:

Scaling the mesh by λ scales the mass by λ^2 and does not change the stiffness.

Recall

Using finite-element assembly, we construct the **per-mesh** mass and stiffness matrices $\mathbf{M}, \boldsymbol{S} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ by iterating over the triangles and contributing to the coefficients associated with pairs of vertices:

$$\mathbf{M}_{uv} = \sum_{\tau\in\mathcal{T}(u)\cap\mathcal{T}(v)} \mathbf{m}^{\tau}_{\tau(u),\tau(v)} \quad \text{and} \quad \mathbf{S}_{uv} = \sum_{\tau\in\mathcal{T}(u)\cap\mathcal{T}(v)} \mathbf{s}^{\tau}_{\tau(u),\tau(v)}$$



Outline

Recall

Math Review

Inhomogeneous/Anisotropic Geometry Processing

Math Review

Solving the Quadratic:

Given a quadratic polynomial:

$$Q(x) = ax^2 + bx + c$$

the roots/zeros of the polynomial are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The *discriminant*, $(b^2 - 4ac)$, determines the number of roots:

$(b^2 - 4ac) > 0$: Two roots

$(b^2 - 4ac) = 0$: One root

$(b^2 - 4ac) < 0$: No roots

$\Rightarrow$ A quadratic polynomial will have both positive and negative values if and only if its discriminant is positive.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Math Review

Cauchy-Schwarz Inequality:

Given an inner-product space $\{V, B\colon V \to V^*\}$, for all $u, v \in V$ we have:

$$|\langle u, v\rangle_B| \le \|u\|_B \cdot \|v\|_B$$

Proof:

If $\langle u, v\rangle_B = 0$, the statement is trivially true.

Define the sign of the inner-product as:

$$\text{sign}(\langle u, v\rangle_B) = \begin{cases} 1 & \text{if } \langle u, v\rangle_B > 0 \\ -1 & \text{if } \langle u, v\rangle_B < 0 \end{cases}$$

Consider the quadratic polynomial:

$$\begin{aligned} Q(x) &= \|u + x \cdot \text{sign}(\langle u, v\rangle_B) \cdot v\|_B^2 \\ &= \|v\|_B^2 \cdot x^2 + 2 \cdot \text{sign}(\langle u, v\rangle_B) \cdot \langle u, v\rangle_B \cdot x + \|u\|_B^2 \\ &= \|v\|_B^2 \cdot x^2 + 2 \cdot |\langle u, v\rangle_B| \cdot x + \|u\|_B^2 \end{aligned}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Math Review

Cauchy-Schwarz Inequality:

Given an inner-product space $\{V, B: V \to V^*\}$, for all $u, v \in V$ we have:

$$|\langle u, v\rangle_B| \leq \|u\|_B \cdot \|v\|_B$$

Proof:

$$\begin{aligned} Q(x) &= \|u + x \cdot \text{sign}(\langle u, v\rangle_B) \cdot v\|_B^2 \\ &= \|v\|_B^2 \cdot x^2 + 2 \cdot |\langle u, v\rangle_B| \cdot x + \|u\|_B^2 \end{aligned}$$

Since $Q(x)$ is the square norm of a vector, it cannot be negative

$\Rightarrow$ The discriminant cannot be positive:

$$0 \geq (2|\langle u, v\rangle_B|)^2 - 4 \cdot \|u\|_B^2 \cdot \|v\|_B^2$$

$$\Updownarrow$$

$$\|u\|_B^2 \cdot \|v\|_B^2 \geq (\langle u, v\rangle_B)^2$$

$$\Updownarrow$$

$$\|u\|_B \cdot \|v\|_B \geq |\langle u, v\rangle_B|$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Math

Note: We have equality if and only if the discriminant is zero.

$$\Updownarrow$$

There exists a value of x for which the quadratic evaluates to zero.

Cauchy-Schw

$$\Updownarrow$$

Given an in There is a value $\alpha \in \mathbb{R}$ such that $\|u + \alpha \cdot v\| = 0$ ve have:

$|\langle u, v$ $\Updownarrow$ $\|_B$

$$u + \alpha \cdot v = 0$$

Proof:

$$\Updownarrow$$

The vectors u and v are linearly dependent.

Since $Q(x)$ is the square norm of a vector, it cannot be negative

$\Rightarrow$ The discriminant cannot be positive:

$$0 \geq (2|\langle u, v\rangle_B|)^2 - 4 \cdot \|u\|_B^2 \cdot \|v\|_B^2$$
$$\Updownarrow$$
$$\|u\|_B^2 \cdot \|v\|_B^2 \geq (\langle u, v\rangle_B)^2$$
$$\Updownarrow$$
$$\|u\|_B \cdot \|v\|_B \geq |\langle u, v\rangle_B|$$

Math Review

Definition:

A *metric space* is a set S combined with a real-valued *distance function*, $d: S \times S \to \mathbb{R}$, satisfying:

Identity: $d(x, x) = 0$ for all $x \in S$

Positivity: $d(x, y) > 0$ for all $x, y \in S$ with $x \neq y$

Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in S$

Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$

Math Review

Given an inner-product space $\{V, B: V \to V^*\}$, we can define a notion of distance between two points:

$$d(u, v) \equiv \|u - v\|_B$$

Claim:

The function $d: V \times V \to \mathbb{R}$ makes V a metric space.

Math Review

Given an inner-product space $\{V, B: V \rightarrow V^*\}$, we can define a notion of distance between two points:

$$d(u, v) \equiv \|u - v\|_B$$

Proof (identity):

For all $v \in V$ we have:

$$\begin{aligned} d(v, v) &= \|v - v\|_B \\ &= \|0\|_B \\ &= \sqrt{[B(0)](0)} \\ &= 0 \end{aligned}$$

Math Review

Given an inner-product space $\{V, B: V \to V^*\}$, we can define a notion of distance between two points:

$$d(u, v) \equiv \|u - v\|_B$$

Proof (positivity):

For all $u, v \in V$ we have:

$$\begin{aligned} d(u, v) &= \|u - v\|_B \\ &= \sqrt{[B(u - v)](u - v)} \\ &> 0 \end{aligned}$$

whenever $u - v \neq 0$ (by positive definiteness of B).

Or, equivalently, when $u \neq v$.

Math Review

Given an inner-product space $\{V, B\colon V \to V^*\}$, we can define a notion of distance between two points:

$$d(u,v) \equiv \|u - v\|_B$$

Proof (symmetry):

For all $u, v \in V$ we have:

$$\begin{aligned} d(u,v) &= \|u - v\|_B \\ &= \sqrt{[B(u-v)](u-v)} \\ &= \sqrt{(-1)^2 \cdot [B(v-u)](v-u)} \\ &= \sqrt{[B(v-u)](v-u)} \\ &= \|v - u\|_B \\ &= d(v,u) \end{aligned}$$

Math Review

Given an inner-product space $\{V, B: V \to V^*\}$, we can define a notion of distance between two points:

$$d(u,v) \equiv \|u-v\|_B$$

Proof (triangle inequality):

Need to show that for all $u, v, w \in V$ we have:

$$d(u,w) \leq d(u,v) + d(v,w)$$

$$\Updownarrow$$

$$\|u-w\|_B \leq \|u-v\|_B + \|v-w\|_B$$

$$\Updownarrow$$

$$\|u-w\|_B^2 \leq (\|u-v\|_B + \|v-w\|_B)^2$$

$$\Updownarrow$$

$$\begin{aligned}\|u-w\|_B^2 \leq\ & \|u-v\|_B^2 + \|v-w\|_B^2 + 2\cdot\|u-v\|_B\cdot\|v-w\|_B \\ =\ & \|u-v\|_B^2 + \|v-w\|_B^2 + 2\cdot\langle u-v, v-w\rangle_B - 2\langle u-v, v-w\rangle_B + 2\cdot\|u-v\|_B\cdot\|v-w\|_B \\ =\ & \|(u-v)+(v-w)\|_B^2 - 2\cdot\langle u-v, v-w\rangle_B + 2\cdot\|u-v\|_B\cdot\|v-w\|_B \\ =\ & \|u-w\|_B^2 - 2\cdot\langle u-v, v-w\rangle_B + 2\cdot\|u-v\|_B\cdot\|v-w\|_B\end{aligned}$$

$$\Updownarrow$$

$$\langle u-v, v-w\rangle_B \leq \|u-v\|_B\cdot\|v-w\|_B$$

Math Review

Given an inner-product space $\{V, B: V \to V^*\}$, we can define a notion of distance between two points:

$$d(u, v) \equiv \|u - v\|_B$$

Proof (triangle inequality):

Need to show that for all $u, v, w \in V$ we have:

$$d(u, w) \leq d(u, v) + d(v, w)$$

$$\Updownarrow$$

$$\langle u - v, v - w \rangle_B \leq \|u - v\|_B \cdot \|v - w\|_B$$

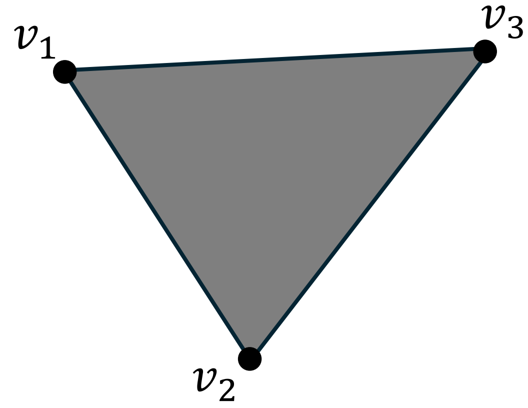
This follows from the Cauchy-Schwarz Inequality.

Math Review

Note:

Given a triangle with vertices v_1, v_2, and v_3, the triangle inequality states:

$$d(v_1, v_2) \leq d(v_1, v_3) + d(v_3, v_2)$$
$$d(v_3, v_1) \leq d(v_3, v_2) + d(v_2, v_1)$$
$$d(v_2, v_3) \leq d(v_2, v_1) + d(v_1, v_3)$$
$$\Updownarrow$$
$$d(v_1, v_2) - d(v_1, v_3) \leq d(v_3, v_2)$$
$$d(v_3, v_1) - d(v_2, v_1) \leq d(v_3, v_2)$$
$$d(v_2, v_3) \leq d(v_2, v_1) + d(v_1, v_3)$$
$$\Updownarrow$$
$$d(v_1, v_2) - d(v_1, v_3) \leq d(v_2, v_3)$$
$$d(v_3, v_1) - d(v_1, v_2) \leq d(v_2, v_3)$$
$$d(v_2, v_3) \leq d(v_1, v_2) + d(v_1, v_3)$$
$$\Updownarrow$$
$$|d(v_1, v_2) - d(v_1, v_3)| \leq d(v_2, v_3) \leq |d(v_1, v_2) + d(v_1, v_3)|$$



Math Review

Note:

Given a triangle with vertices v_1, v_2, and v_3, the triangle inequality states:

$$|d(v_1, v_2) - d(v_1, v_3)| \leq \boxed{d(v_2, v_3) \leq |d(v_1, v_2) + d(v_1, v_3)|}$$

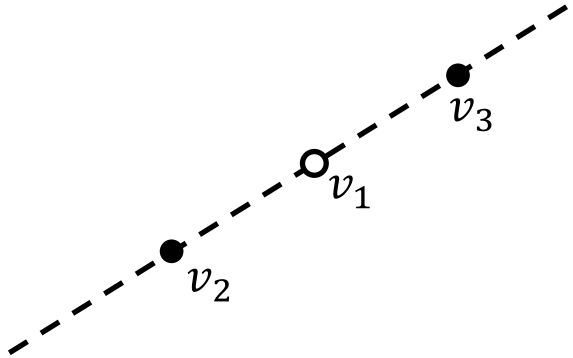
Informally:

Consider the extreme configurations with v_1 on the line between v_2 and v_3.

Case 1:

v_1 is **inside** the segment between v_2 and v_3

$\Rightarrow$ The distance between v_2 and v_3 is the sum of the distances to v_1.



Math Review

Note:

Given a triangle with vertices v_1, v_2, and v_3, the triangle inequality states:

$$\boxed{|d(v_1,v_2) - d(v_1,v_3)| \leq d(v_2,v_3)} \leq |d(v_1,v_2) + d(v_1,v_3)|$$

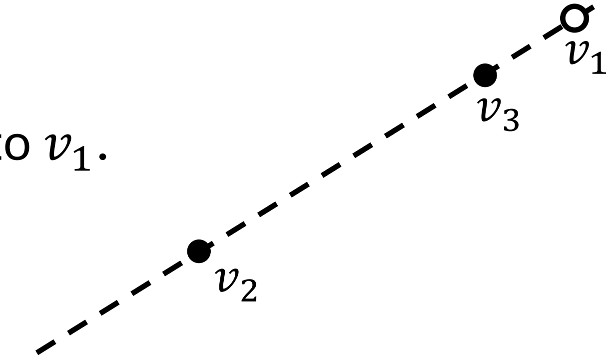
Informally:

Consider the extreme configurations with v_1 on the line between v_2 and v_3.

Case 2:

v_1 is **outside** the segment between v_2 and v_3 (w.l.o.g. closer to v_3)

$\Rightarrow$ The distance between v_2 and v_3 is the difference of the distances to v_1.



Math Review

Claim:

Given a **two-dimensional** vector space V and a symmetric bilinear form $B: V \rightarrow V^*$, the bilinear form is positive semi-definite if and only if:

There exists some basis $\{v_1, v_2\} \subset V$ such that:

1. The bilinear form is positive semi-definite on the basis vectors:
$$B(v_1, v_1), B(v_2, v_2) \geq 0$$
2. The bilinear form satisfies the Cauchy-Schwarz inequality on the basis:
$$B(v_1, v_2) \leq \sqrt{B(v_1, v_1)} \cdot \sqrt{B(v_2, v_2)}$$

Proof (⇒):

Suppose $B: V \rightarrow V^*$ is positive semi-definite, then for **any** basis $\{v_1, v_2\} \subset V$, it will satisfy positive semi-definite and Cauchy-Schwarz.

$$B(v_1, v_1), B(v_2, v_2) \geq 0$$
$$B(v_1, v_2) \leq \sqrt{B(v_1, v_1)} \cdot \sqrt{B(v_2, v_2)}$$

Math Review

Proof (⇐):

Suppose there exists a basis $\{v_1, v_2\} \subset V$ satisfying positive semi-definite and Cauchy-Schwarz.

Since $\{v_1, v_2\}$ is a basis, any $v \in V$ can be expressed as:

$$v = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$.

⇒ It suffices to show that for any $\alpha_1, \alpha_2 \in \mathbb{R}$ we have:

$$B(\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2, \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2) \geq 0$$

$$B(v_1, v_1), B(v_2, v_2) \geq 0$$
$$B(v_1, v_2) \leq \sqrt{B(v_1, v_1)} \cdot \sqrt{B(v_2, v_2)}$$

Math Review

Proof (⇐):

It suffices to show that for any $\alpha_1, \alpha_2 \in \mathbb{R}$ we have:

$$B(\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2, \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2) \geq 0$$

Expanding gives:

$$\begin{aligned} B(\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2, \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2) = & \\ &= \alpha_1^2 \cdot B(v_1, v_1) + \alpha_2^2 \cdot B(v_2, v_2) + 2 \cdot \alpha_1 \cdot \alpha_2 \cdot B(v_1, v_2) \\ &\geq \alpha_1^2 \cdot B(v_1, v_1) + \alpha_2^2 \cdot B(v_2, v_2) - 2 \cdot |\alpha_1| \cdot |\alpha_2| \cdot |B(v_1, v_2)| \\ &\geq \alpha_1^2 \cdot B(v_1, v_1) + \alpha_2^2 \cdot B(v_2, v_2) - 2 \cdot |\alpha_1| \cdot |\alpha_2| \cdot \sqrt{B(v_1, v_1)} \cdot \sqrt{B(v_2, v_2)} \\ &= \left(|\alpha_1| \cdot \sqrt{B(v_1, v_1)} - |\alpha_2| \cdot \sqrt{B(v_2, v_2)}\right)^2 \\ &\geq 0 \end{aligned}$$

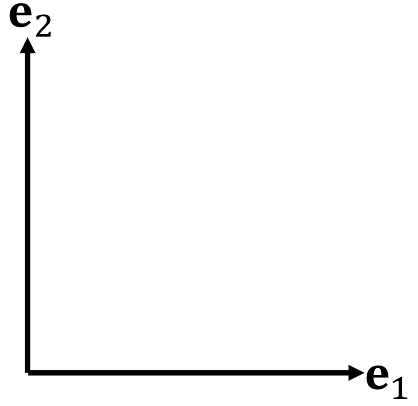
Math Review

Recall:

On $\mathbb{R}^2$, we have the cartesian basis $\{\mathbf{e}_1 = (1,0)^\top, \mathbf{e}_2 = (0,1)^\top\}$.

$\Rightarrow$ Any symmetric bilinear form $B: \mathbb{R}^2 \to (\mathbb{R}^2)^*$ is determined by its expression as a matrix $\mathbf{B} \in \mathbb{R}^{2\times 2}$ w.r.t. the basis:

$$\mathbf{B}_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle_B$$



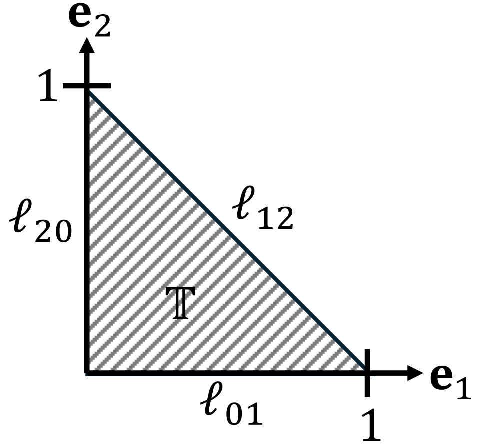
$\mathbf{B}_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle_B$

Math Review

Note:

Given a symmetric bilinear form $B: \mathbb{R}^2 \to (\mathbb{R}^2)^*$, we can assign non-Euclidean (square) lengths to the edges of the triangle $\mathbb{T} \subset \mathbb{R}^2$:

$$\ell_{01}^2 = \|\mathbf{e}_1\|_B^2 = \mathbf{B}_{11}$$
$$\ell_{20}^2 = \|\mathbf{e}_2\|_B^2 = \mathbf{B}_{22}$$
$$\begin{aligned}\ell_{12}^2 &= \|\mathbf{e}_2 - \mathbf{e}_1\|_B^2 \\ &= \|\mathbf{e}_1\|_B^2 + \|\mathbf{e}_2\|_B^2 - 2 \cdot \langle \mathbf{e}_1, \mathbf{e}_2 \rangle_B \\ &= \mathbf{B}_{11} + \mathbf{B}_{22} - 2 \cdot \mathbf{B}_{12}\end{aligned}$$



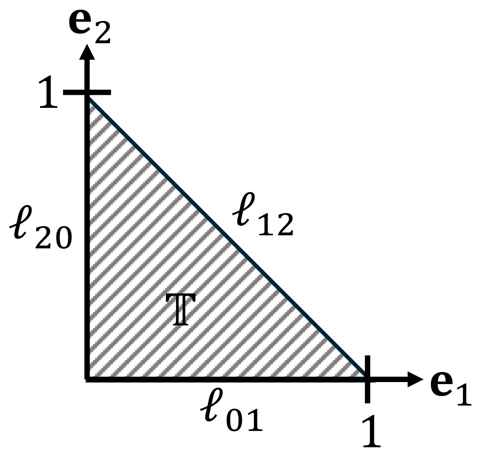
$\mathbf{B}_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle_B$

Math Review

Conversely:

Given non-Euclidean (square) edge-lengths, we can define a symmetric bilinear form:

$$\mathbf{B}_{11} = \|\mathbf{e}_1\|_B^2 = \ell_{01}^2$$
$$\mathbf{B}_{22} = \|\mathbf{e}_2\|_B^2 = \ell_{20}^2$$
$$\begin{aligned}\mathbf{B}_{12} = \mathbf{B}_{21} &= \langle \mathbf{e}_1, \mathbf{e}_2 \rangle_B \\ &= \frac{\|\mathbf{e}_1\|_B^2 + \|\mathbf{e}_2\|_B^2 - \|\mathbf{e}_1\|_B^2 - \|\mathbf{e}_2\|_B^2 + 2 \cdot \langle \mathbf{e}_1, \mathbf{e}_2 \rangle_B}{2} \\ &= \frac{\|\mathbf{e}_1\|_B^2 + \|\mathbf{e}_2\|_B^2 - \|\mathbf{e}_2 - \mathbf{e}_1\|_B^2}{2} \\ &= \frac{\ell_{01}^2 + \ell_{20}^2 - \ell_{12}^2}{2}\end{aligned}$$



Math Review

Claim:

A symmetric bilinear form is **positive semi-definite** if and only the associated edge lengths satisfy the **triangle inequality**:

$$|\ell_{01} - \ell_{20}| \leq \ell_{12} \leq |\ell_{01} + \ell_{20}|$$

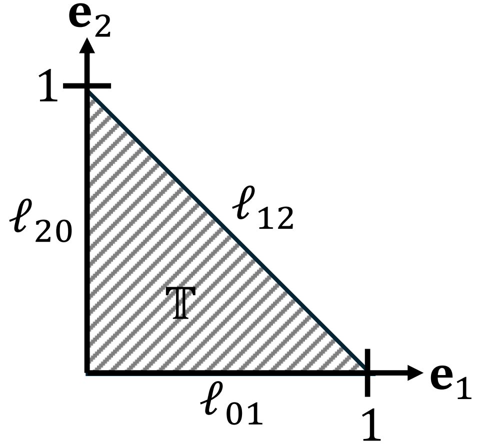
Proof (⇒):

If it's positive semi-definite, the distance function:

$$d(u,v) \equiv \|u - v\|_B$$

makes V a metric-space.

⇒ The distances satisfy the triangle inequality.



Math Review

$$\mathbf{B}_{11} = \ell_{01}^2$$
$$\mathbf{B}_{22} = \ell_{20}^2$$
$$\mathbf{B}_{12} = \frac{\ell_{01}^2 + \ell_{20}^2 - \ell_{12}^2}{2}$$

$$|\ell_{01} - \ell_{20}| \leq \ell_{12} \leq |\ell_{01} + \ell_{20}|$$

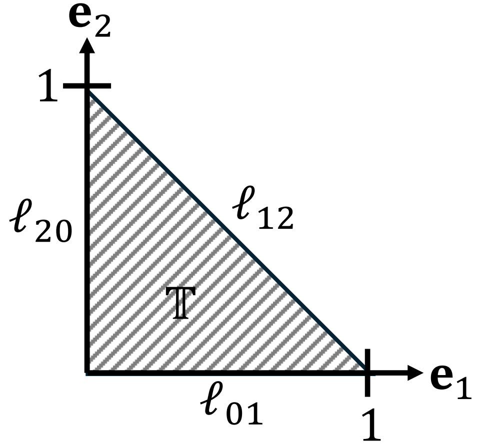
Proof (⇐):

To show that the bilinear form is positive semi-definite, it suffices to show:

Positive semi-definite: $\mathbf{B}_{11}, \mathbf{B}_{22} \geq 0$

Cauchy-Schwarz: $|\mathbf{B}_{12}| \leq \sqrt{\mathbf{B}_{11}} \cdot \sqrt{\mathbf{B}_{22}}$

Positive semi-definite follows from the definition of $\mathbf{B}_{11}$ and $\mathbf{B}_{22}$ as the squares of the edge lengths.



Math Review

$$\mathbf{B}_{11} = \ell_{01}^2$$
$$\mathbf{B}_{22} = \ell_{20}^2$$
$$\mathbf{B_{12}} = \frac{\ell_{01}^2 + \ell_{20}^2 - \ell_{12}^2}{2}$$

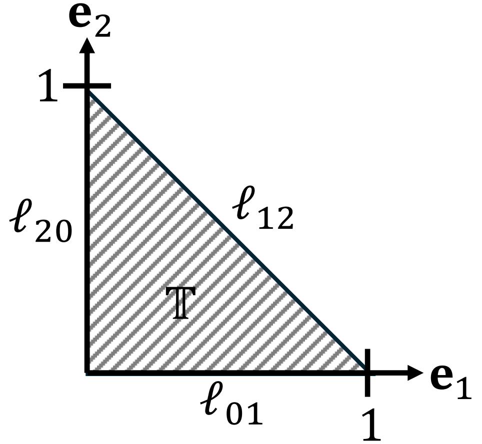
$$\boxed{|\ell_{01} - \ell_{20}| \leq \ell_{12}} \leq |\ell_{01} + \ell_{20}|$$

Proof (⇐):

To show Cauchy-Schwarz: $|\mathbf{B}_{12}| \leq \sqrt{\mathbf{B}_{11}} \cdot \sqrt{\mathbf{B}_{22}}$ consider the sign of $\mathbf{B}_{12}$:

Case 1 ($\mathbf{B}_{12} > 0$):

$$\begin{aligned}
|\mathbf{B}_{12}| &= \mathbf{B}_{12} \\
&= \frac{\ell_{01}^2 + \ell_{20}^2 - \ell_{12}^2}{2} \\
&= \frac{\ell_{01}^2 + \ell_{20}^2 - 2 \cdot \ell_{01} \cdot \ell_{20} + 2 \cdot \ell_{01} \cdot \ell_{20} - \ell_{12}^2}{2} \\
&= \frac{(\ell_{01} - \ell_{20})^2 + 2 \cdot \ell_{01} \cdot \ell_{20} - \ell_{12}^2}{2} \\
&\leq \frac{\ell_{12}^2 + 2 \cdot \ell_{01} \cdot \ell_{20} - \ell_{12}^2}{2} \\
&= \ell_{01} \cdot \ell_{20} \\
&= \sqrt{\mathbf{B}_{11}} \cdot \sqrt{\mathbf{B}_{22}}
\end{aligned}$$



$$\mathbf{B}_{11} = \ell_{01}^2$$
$$\mathbf{B}_{22} = \ell_{20}^2$$
$$\mathbf{B}_{12} = \frac{\ell_{01}^2 + \ell_{20}^2 - \ell_{12}^2}{2}$$

Math Review

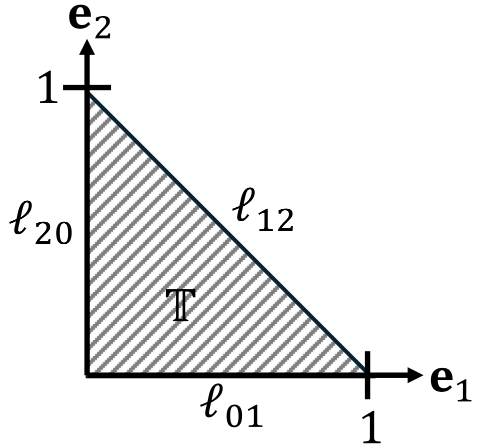
$$|\ell_{01} - \ell_{20}| \leq \boxed{\ell_{12} \leq |\ell_{01} + \ell_{20}|}$$

Proof (⇐):

To show Cauchy-Schwarz: $|\mathbf{B}_{12}| \leq \sqrt{\mathbf{B}_{11}} \cdot \sqrt{\mathbf{B}_{22}}$ consider the sign of $\mathbf{B}_{12}$:

Case 2 ($\mathbf{B}_{12} < 0$):

$$\begin{aligned}
|\mathbf{B}_{12}| &= -\mathbf{B}_{12} \\
&= \frac{\ell_{12}^2 - \ell_{01}^2 - \ell_{20}^2}{2} \\
&\leq \frac{(\ell_{01} + \ell_{20})^2 - \ell_{01}^2 - \ell_{20}^2}{2} \\
&= \frac{\ell_{01}^2 + \ell_{20}^2 + 2 \cdot \ell_{01} \cdot \ell_{20} - \ell_{01}^2 - \ell_{20}^2}{2} \\
&= \ell_{01} \cdot \ell_{20} \\
&= \sqrt{\mathbf{B}_{11}} \cdot \sqrt{\mathbf{B}_{22}}
\end{aligned}$$



Outline

Recall

Math Review

Inhomogeneous/Anisotropic Geometry Processing

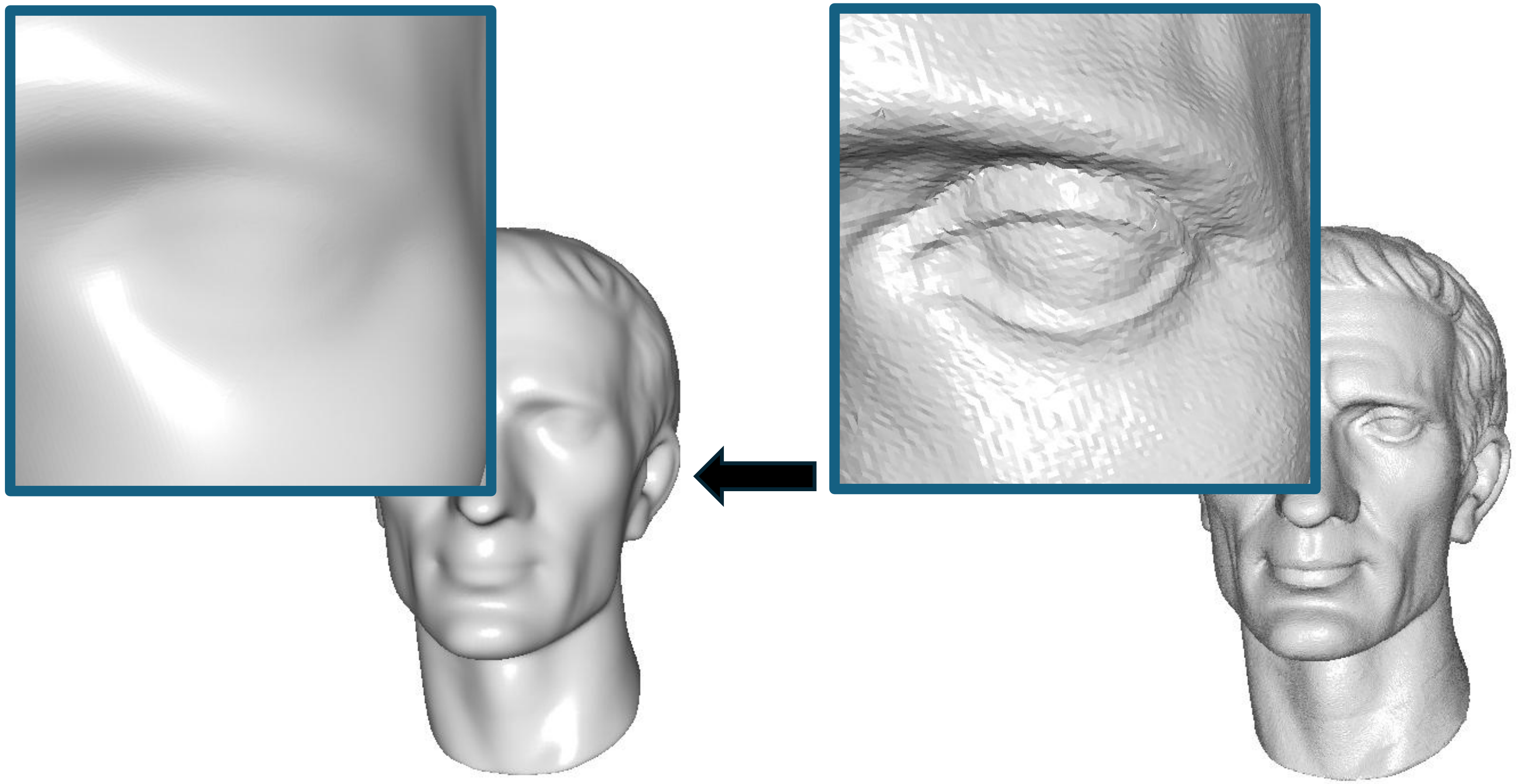
Inhomogeneous Geometry Processing

Challenge:

Given a noisy mesh, diffuse the vertex positions.

✓This removes the noise

✗This removes detail



Inhomogeneous Geometry Processing

Challenge:

Given a noisy mesh, diffuse the vertex positions **while preserving detail**.

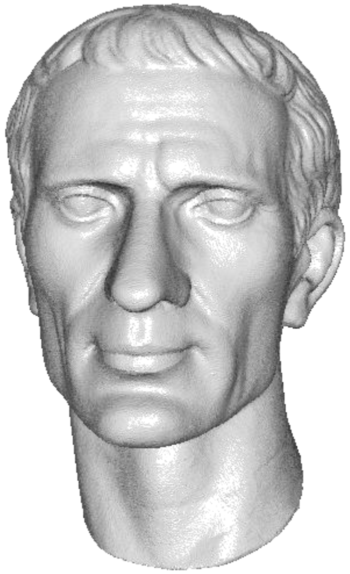
Recall:

We can estimate the per-triangle curvatures using the spectral decomposition of the shape operator:

$$I\Big|_{\mathbf{p}}^{-1} \circ II\Big|_{\mathbf{p}} : T_{\mathbf{p}}\mathbb{T} \rightarrow T_{\mathbf{p}}\mathbb{T}$$

At every triangle we get principal curvatures $\kappa_1, \kappa_2 \in \mathbb{R}$ describing how the normal/surface curves over the triangle.

$\kappa_1^2 + \kappa_2^2$



$\Rightarrow$ Can use per-triangle *total curvature,* $\kappa_1^2 + \kappa_2^2$, as a measure of detail.

Inhomogeneous Geometry Processing

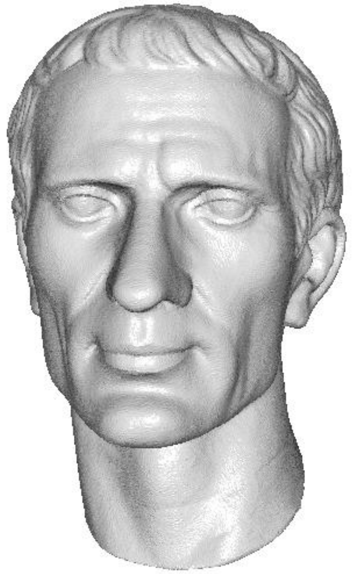
Challenge:

Given a noisy mesh and per-triangle detail-weighting function:

$$w\colon \mathcal{T} \to \mathbb{R}^{>0}$$

diffuse the vertices of the mesh *inhomogeneously*, diffusing more in regions where the weight is low and less where it is high.

$\kappa_1^2 + \kappa_2^2$



Inhomogeneous Geometry Processing

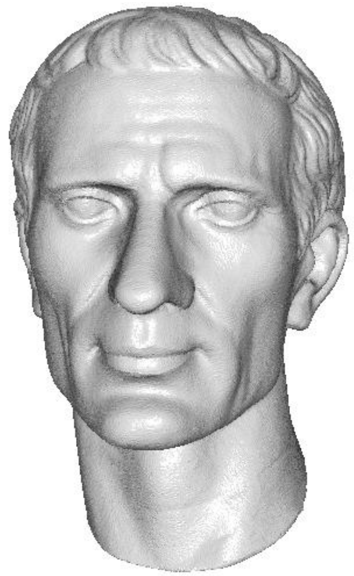
Recall:

Given a signal $f^0 \in V$, we diffuse by minimizing the gradient-domain energy:

$$E(f) = \int_{\mathcal{M}} (f - f^0)^2 + \varepsilon \cdot \langle df, df \rangle_{\mathcal{M}}$$
$$= \sum_{\tau} \int_{\tau} (f - f^0)^2 + \varepsilon \cdot \langle df, df \rangle_{\tau}$$

Setting $\mathbf{M}, \mathbf{S} \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$ to be the mass and stiffness matrices and $\mathbf{f}, \mathbf{f}^0 \in \mathbb{R}^{|\mathcal{V}|}$ to be the expressions of the functions w.r.t. the hat-basis, this gives the linear system:

$$(\mathbf{M} + \varepsilon \cdot \mathbf{S}) \cdot \mathbf{f} = \mathbf{M} \cdot \mathbf{f}^0$$



Inhomogeneous Geometry Pro

$$\mathbf{m}_{ij}^{\tau} = \sqrt{\det(\mathbf{g}_\tau)} \cdot \int_{\mathbb{T}} \psi_i \cdot \psi_j \cdot \omega_E$$

$$\mathbf{s}_{ij}^{\tau} = \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^\top \cdot \omega_E \right\rangle_F$$

$$E(f) = \sum_\tau \int_\tau (f - f^0)^2 + \alpha \cdot \langle df, df \rangle_\tau$$

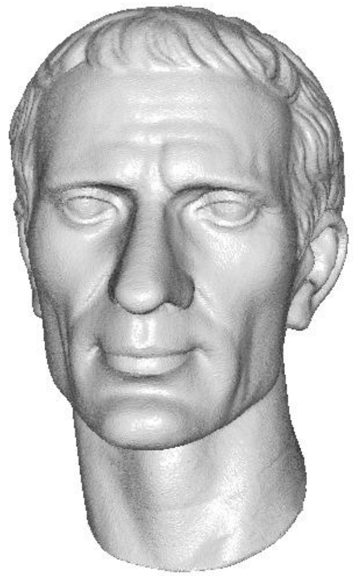
$$(\mathbf{M} + \varepsilon \cdot \mathbf{S}) \cdot \mathbf{f} = \mathbf{M} \cdot \mathbf{f}^0$$

Given a detail-weighting function $w\colon \mathcal{T} \to \mathbb{R}^{>0}$, we can minimize the weighted gradient-domain energy:

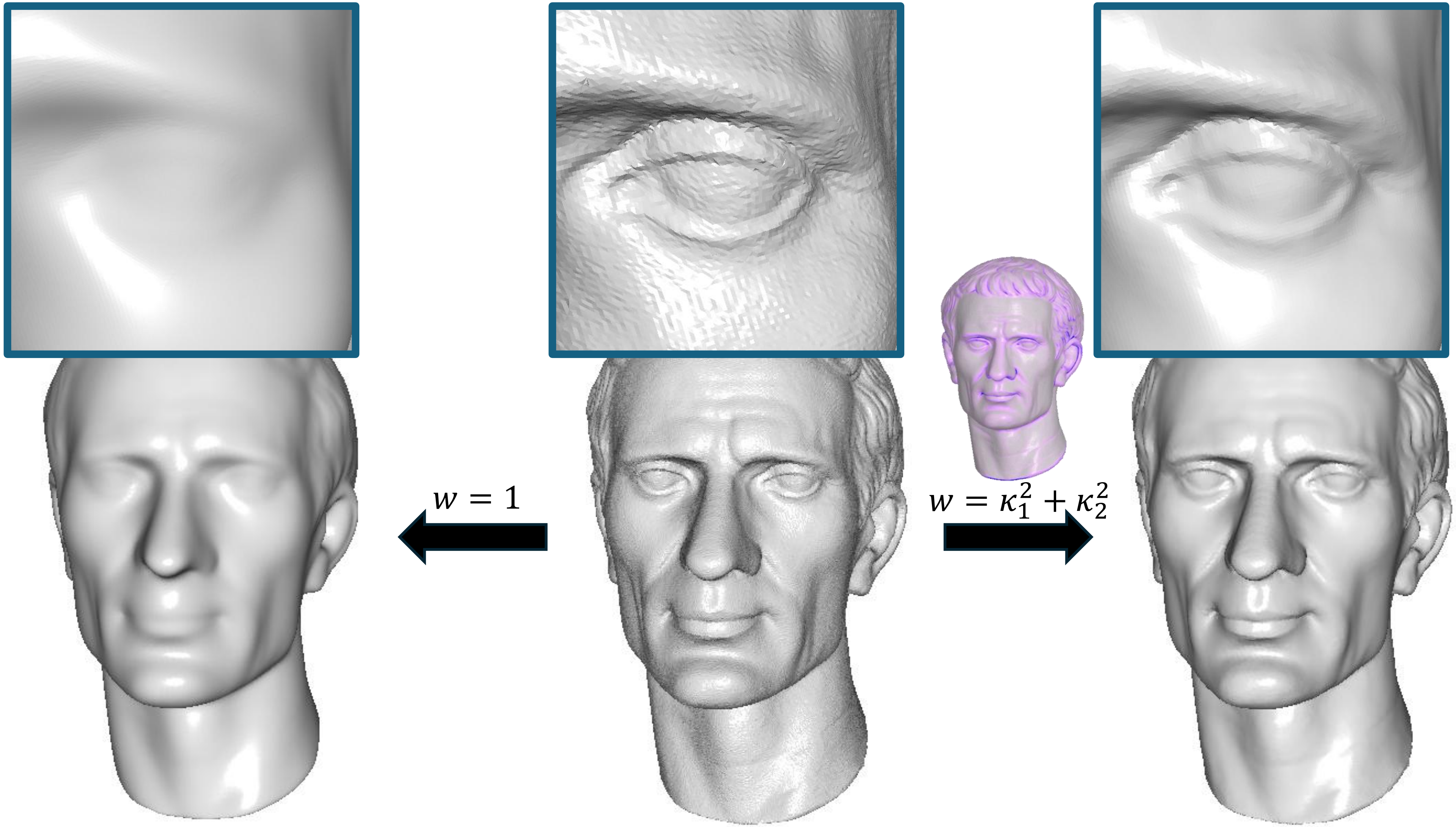
$$E(f) = \sum_\tau \int_\tau w(\tau) \cdot (f - f^0)^2 + \alpha \cdot \langle df, df \rangle_\tau$$

In the context of finite-element assembly, this amounts to weighting the triangles' contributions to the mesh's mass matrix:

$$\widetilde{\mathbf{M}}_{uv} = \sum_{\tau \in \mathcal{T}(u) \cap \mathcal{T}(v)} w(\tau) \cdot \mathbf{m}_{\tau(u),\tau(v)}^{\tau}$$



Inhomogeneous Geometry Processing



Inhomogeneous Geometry Pro

$$\mathbf{m}_{ij}^{\tau} = \sqrt{\det(\mathbf{g}_{\tau})} \cdot \int_{\mathbb{T}} \psi_i \cdot \psi_j \cdot \omega_E$$

$$\mathbf{s}_{ij}^{\tau} = \left\langle \sqrt{\det(\mathbf{g}_{\tau})} \cdot \mathbf{g}_{\tau}^{-1}, \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j{}^{\top} \cdot \omega_E \right\rangle_F$$

$$E(f) = \sum_{\tau} \int_{\tau} w(\tau) \cdot (f - f^0)^2 + \alpha \cdot \langle df, df \rangle_{\tau}$$

$$\left(\widetilde{\mathbf{M}} + \varepsilon \cdot \mathbf{S}\right) \cdot \mathbf{f} = \widetilde{\mathbf{M}} \cdot \mathbf{f}^0$$

We can solve the weighted gradient-domain problem, by incorporating the weights within the finite-element assembly:

$$\widetilde{\mathbf{M}}_{uv} = \sum_{\tau \in \mathcal{T}(u) \cap \mathcal{T}(v)} w(\tau) \cdot \mathbf{m}_{\tau(u),\tau(v)}^{\tau}$$

$\Leftrightarrow$ We can solve by pre-weighting the triangles' mass matrices:

$$\widetilde{\mathbf{m}}^{\tau} = w(\tau) \cdot \mathbf{m}^{\tau}$$

Inhomogeneous Geometry Pro

$$\mathbf{m}_{ij}^{\tau} = \sqrt{\det(\mathbf{g}_\tau)} \cdot \int_{\mathbb{T}} \psi_i \cdot \psi_j \cdot \omega_E$$

$$\mathbf{s}_{ij}^{\tau} = \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^{\top} \cdot \omega_E \right\rangle_F$$

$$E(f) = \sum_\tau \int_\tau w(\tau) \cdot (f - f^0)^2 + \alpha \cdot \langle df, df \rangle_\tau$$

$$\left(\widetilde{\mathbf{M}} + \varepsilon \cdot \mathbf{S}\right) \cdot \mathbf{f} = \widetilde{\mathbf{M}} \cdot \mathbf{f}^0$$

We can solve the weighted gradient-domain problem, by pre-weighting the triangles' mass matrices:

$$\widetilde{\mathbf{m}}^\tau = w(\tau) \cdot \mathbf{m}^\tau$$

Recall:

The mass matrix scales with the inner-product, the stiffness matrix is authalic (i.e. scale independent).

⇔ This is equivalent so scaling the inner-product:

$$\tilde{\mathbf{g}}^\tau = w(\tau) \cdot \mathbf{g}^\tau$$

$$\mathbf{m}_{ij}^{\tau} = \sqrt{\det(\mathbf{g}_\tau)} \cdot \int_{\mathbb{T}} \psi_i \cdot \psi_j \cdot \omega_E$$

$$\mathbf{s}_{ij}^{\tau} = \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^\top \cdot \omega_E \right\rangle_F$$

Inhomogeneous Geometry Pro

$$E(f) = \sum_\tau \int_\tau w(\tau) \cdot (f - f^0)^2 + \alpha \cdot \langle df, df \rangle_\tau$$

$$\left(\widetilde{\mathbf{M}} + \varepsilon \cdot \mathbf{S}\right) \cdot \mathbf{f} = \widetilde{\mathbf{M}} \cdot \mathbf{f}^0$$

We can perform inhomogeneous geometry processing by modifying the inner-products on the triangles.

We can ... ghting the

$\Downarrow$

We are no longer working with the pull-back of the Euclidean inner-product on $\mathbb{R}^3$

Recall:

The mass matrix scales with the inner-product, the stiffness matrix is authalic (i.e. scale independent).

$\Leftrightarrow$ This is equivalent so scaling the inner-product:

$$\tilde{\mathbf{g}}^\tau = w(\tau) \cdot \mathbf{g}^\tau$$

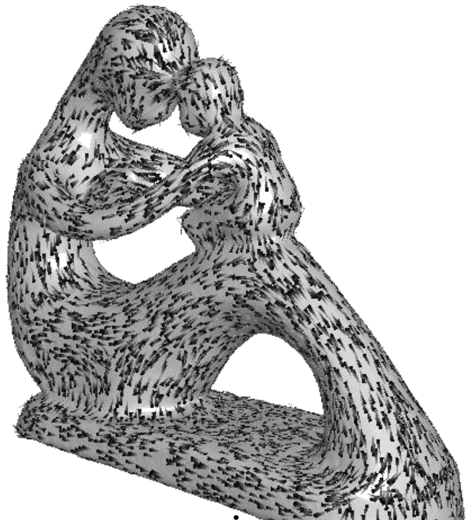
Anisotropic Geometry Processing

Recall:

We can estimate the per-triangle curvatures directions using the spectral decomposition of the shape operator:

$$I\Big|_{\mathbf{p}}^{-1} \circ II\Big|_{\mathbf{p}} : T_{\mathbf{p}}\mathbb{T} \to T_{\mathbf{p}}\mathbb{T}$$

⇒ Up to sign, we can assign a min/max curvature direction to each triangle.*



min

max

*Ignoring umbilic points.

Anisotropic Geometry Processing

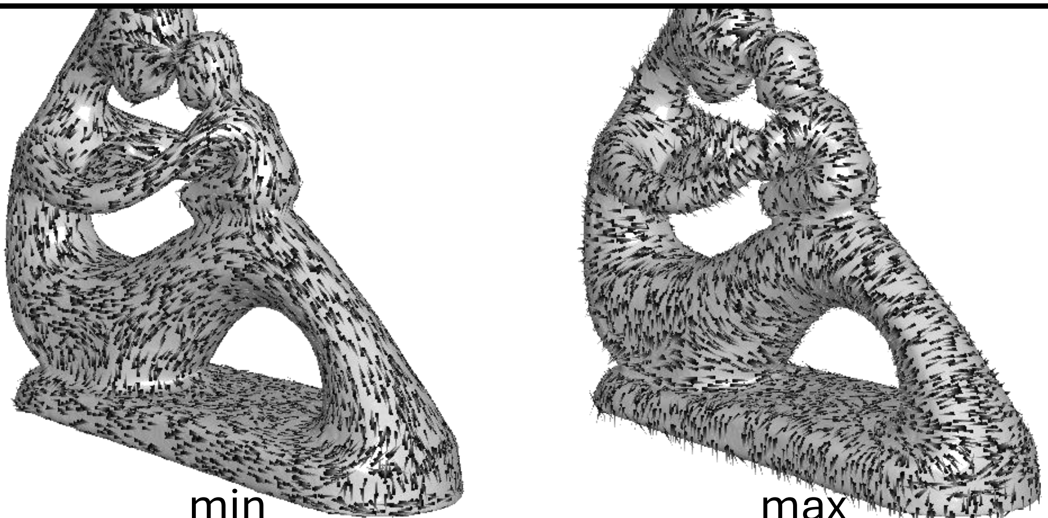
Recall:

We can estimate the per-triangle curvatures directions using the spectral decomposition of the shape operator:

$$I\Big|_{\mathbf{p}}^{-1} \circ II\Big|_{\mathbf{p}} : T_{\mathbf{p}}\mathbb{T} \to T_{\mathbf{p}}\mathbb{T}$$

⇒ Up to sign, ch triangle.*

How to visualize the curvature directions?

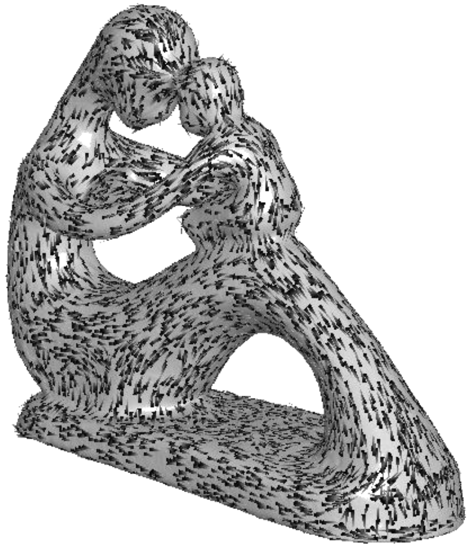


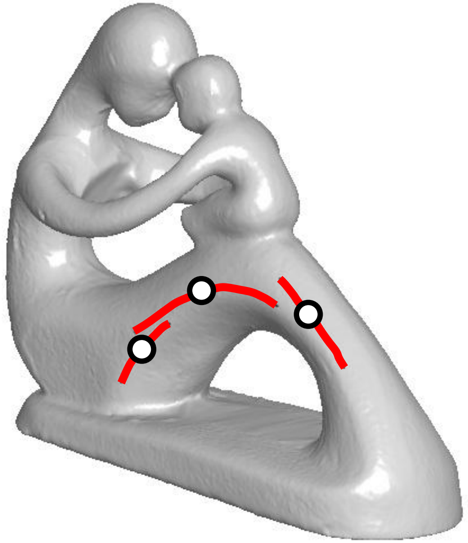
*Ignoring umbilic points.

[Cabral *et al.*, 1993]

Key Idea:

Given a vector field on a surface, can trace flow-lines that follow the field.





[Cabral *et al.*, 1993]

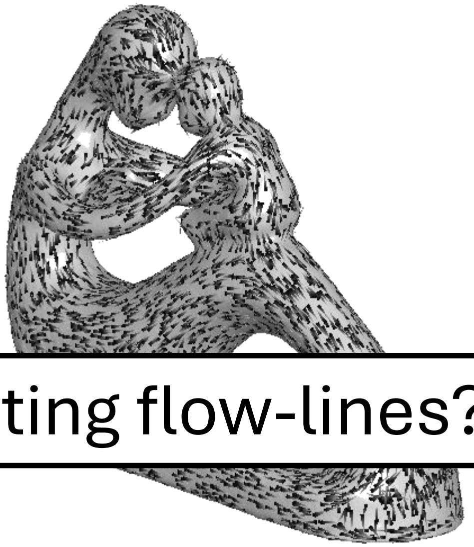
Key Idea:

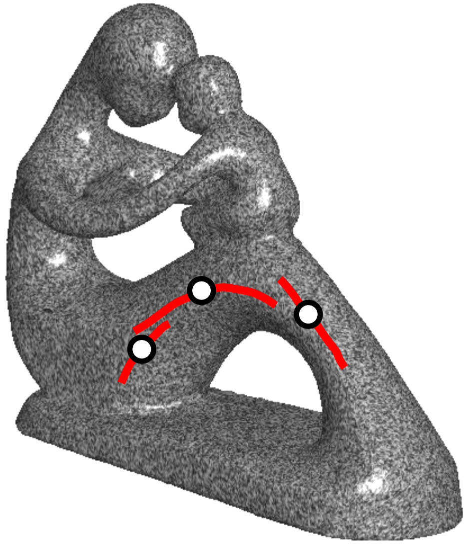
Given a vector field on a surface, can trace flow-lines that follow the field.

Given an initially random signal, we can generate a new signal whose value at a point is obtained by following the flow-line through the point and averaging the random signal.

⇒ Nearby points on the same flow-line will be assigned similar values.

⇒ Get a streaked visualization with the streaks following the flow-lines.





Can we do this without explicitly computing flow-lines?

[Deiwald *et al.*, 2000]

Recall:

Given the unit-right triangle $\mathbb{T} \subset \mathbb{R}^2$ and an inner-product pulled-back from embedding into a triangle $\tau \subset \mathbb{R}^3$:

$$g_\tau\Big|_{\mathbf{p}} : T_{\mathbf{p}}\mathbb{T} \to T_{\mathbf{p}}^*\mathbb{T}$$

we use the inner-product g_τ to measure lengths/angles of tangent vectors.

⇒ Given a tangent vector $u \in T_{\mathbf{p}}\mathbb{T}$, we can define a new inner-product:

$$\tilde{g}_\tau\Big|_{\mathbf{p}}(v,w) \equiv g_\tau\Big|_{\mathbf{p}}(v,w) + g_\tau\Big|_{\mathbf{p}}(u,v) \cdot g_\tau\Big|_{\mathbf{p}}(u,w)$$

For vectors perpendicular to u, distances are unchanged.

For vectors parallel to u, distances are increased.

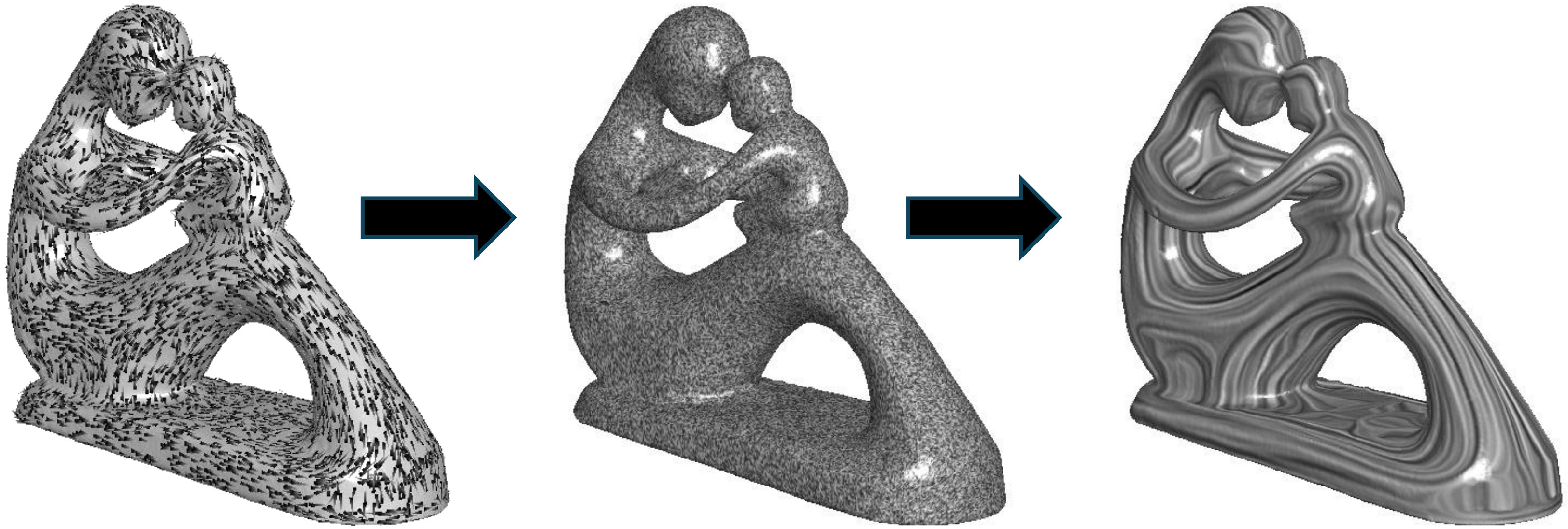
⇒ Diffusing w.r.t. the new inner-product smooths less along direction v.

[Deiwald *et al.*, 2000]

Approach:

- Modify the inner-product to increase distances along the **perpendicular** direction
- Start with an initially random signal
- Diffuse using the modified inner-product (and gradient-domain sharpen)

Since distances aligned with the vector field will be (relatively) short this has the effect of blurring more along the vector field



[Deiwald *et al.*, 2000]

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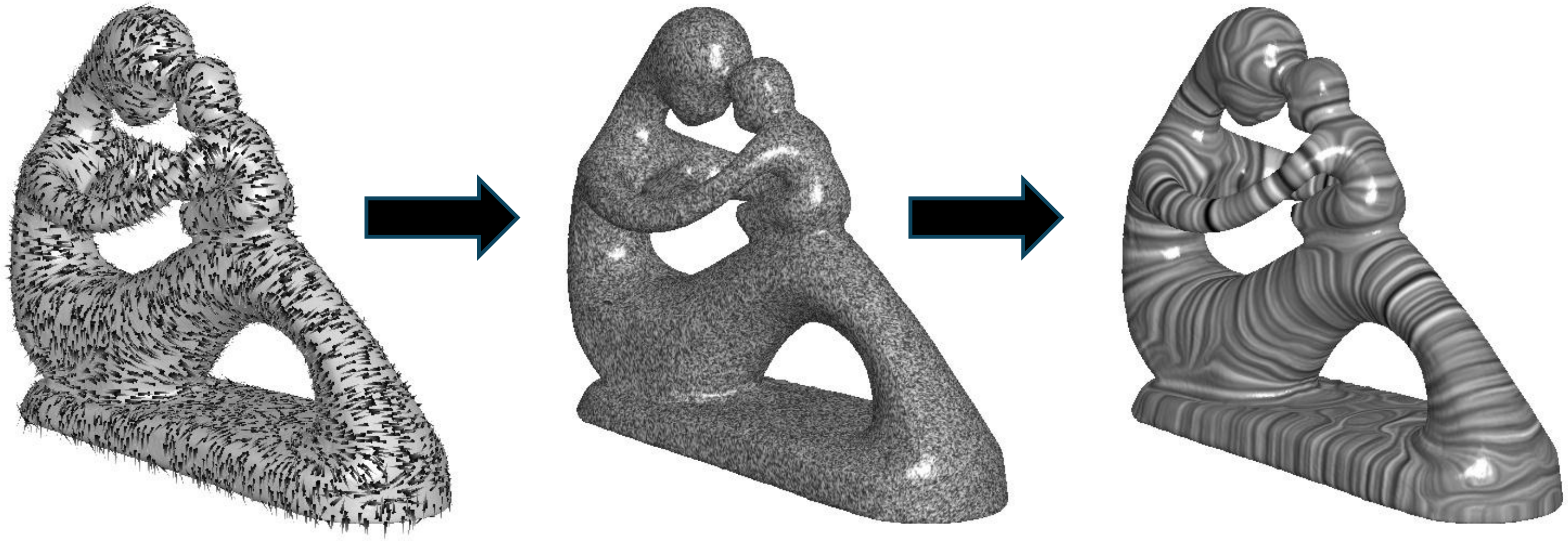
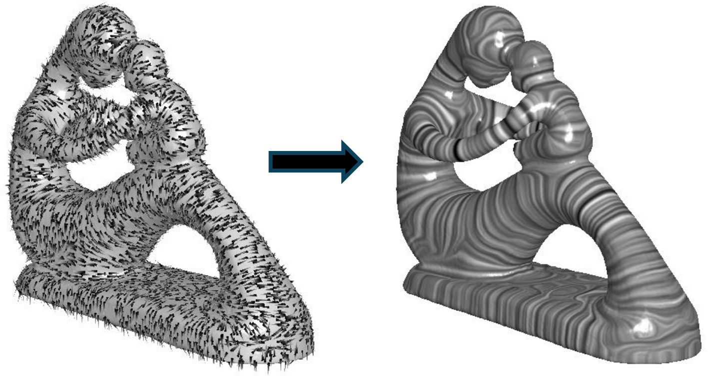


[Deiwald *et al.*, 2000]

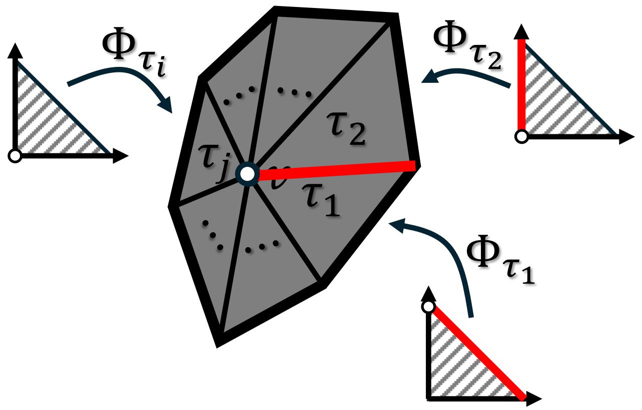


What did we break?

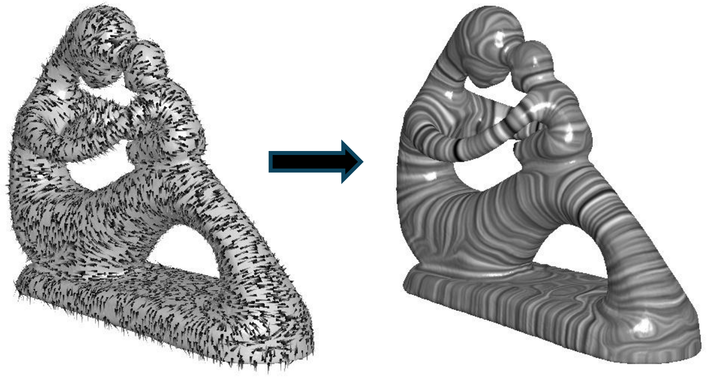
To perform anisotropic diffusion, we independently modify the inner-product of every mesh triangle.

$\Rightarrow$ An edge may be assigned different lengths by its two incident triangles.

For gradient-domain processing, this is not an issue.
But it may be for other types of geometry-processing.



[Deiwald *et al.*, 2000]



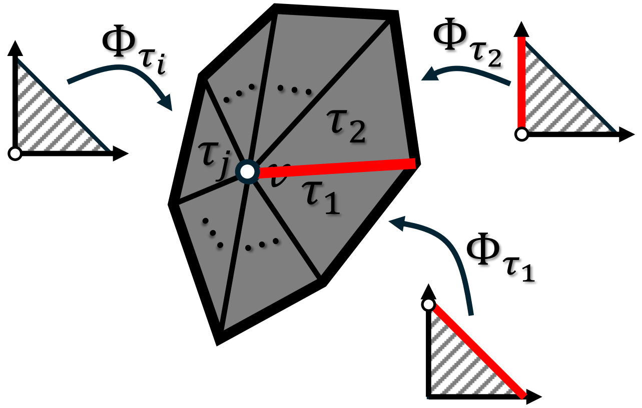
What did we break?

An edge may be assigned different lengths by its two incident triangles.

Recall:

Assigning an inner-product is equivalent to prescribing positive edge-lengths to triangles that satisfy the triangle inequality.

- ✓ Using edge-lengths as degrees of freedom ensures that the associated inner-product is consistent on edges.*
- ✗ Enforcing the triangle inequality can be challenging as satisfying the triangle inequality on one edge can have cascading effects on nearby edges.



*This halves the number of degrees of freedom: 3 dofs/triangle → 1 dof per edge.