Geometry Processing (601.458/658)

Misha Kazhdan

Outline

Recall

Differential Geometry Review

Approximating Curvature

Recall

Given an inner-product space $\{V, B: V \to V^*\}$ and given a bilinear form $L: V \to V^*$, the bilinear form is symmetric:

$$B^* = B$$

if and only if the endomorphism:

$$B^{-1} \circ L: V \to V$$

is self-adjoint.

If $L: V \to V^*$ is a symmetric bilinear form, there exists an orthonormal (generalized) eigen-basis $\{\{v_1, \lambda_1\}, \dots, \{v_n, \lambda_n\}\}$ for V:

$$(B^{-1} \circ L)(v_i) = \lambda_i \cdot v_i$$

$$\Updownarrow$$

$$L(v_i) = \lambda_i \cdot B(v_i)$$

Recall

Note:

Given a vector space V and an endomorphism $L: V \to V$, if $v_i, v_j \in V$ are eigenvectors with the same eigenvalue, $\lambda = \lambda_i = \lambda_j$, then:

$$\begin{aligned} L(\alpha_i \cdot v_i + \alpha_j \cdot v_j) &= \alpha_i \cdot L(v_i) + \alpha_j \cdot L(v_j) \\ &= \alpha_i \cdot \lambda \cdot v_i + \alpha_j \cdot \lambda \cdot v_j \\ &= \lambda \cdot (\alpha_i \cdot v_i + \alpha_j \cdot v_j) \end{aligned}$$

$\Rightarrow$ The linear combination of eigenvectors with the same eigenvalue is itself an eigenvector (with the same eigenvalue).

Similarly, given an inner-product space $\{V, B: V \to V^*\}$ and a symmetric bilinear form $L: V \to V^*$, if $v_i, v_j \in V$ are generalized eigenvectors with the same generalized eigenvalue, $\lambda = \lambda_i = \lambda_j$, any linear combination will be a generalized eigenvector (with the same generalized eigenvalue).

Recall

Given a (smooth) function $\Phi: \mathbb{R} \to \mathbb{R}^n$, we can express $\Phi = (\Phi_1, \dots, \Phi_n)^\top$, with $\Phi_i: \mathbb{R} \to \mathbb{R}$, as a collection of coordinate functions:

$$\Phi(\mathbf{p}) \equiv \begin{pmatrix} \Phi_1(\mathbf{p}) \\ \vdots \\ \Phi_n(\mathbf{p}) \end{pmatrix}$$

The differential of Φ at $\mathbf{p} \in \mathbb{R}$ is the column vector:

$$\mathbf{d}\Phi\Big|_{\mathbf{p}} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial x}\Big|_{\mathbf{p}} \\ \vdots \\ \frac{\partial \Phi_n}{\partial x}\Big|_{\mathbf{p}} \end{pmatrix} \in \mathbb{R}^n$$

Recall

Given a (smooth) function $\Phi: \mathbb{R}^m \rightarrow \mathbb{R}^n$, we can express $\Phi = (\Phi_1, \dots, \Phi_n)^\top$, with $\Phi_i: \mathbb{R}^m \rightarrow \mathbb{R}$, as a collection of coordinate functions:

$$\Phi(\mathbf{p}) \equiv \begin{pmatrix} \Phi_1(\mathbf{p}) \\ \vdots \\ \Phi_n(\mathbf{p}) \end{pmatrix}$$

The differential of Φ at $\mathbf{p} \in \mathbb{R}^m$ is the matrix:

$$\mathbf{d}\Phi\Big|_{\mathbf{p}} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial x_1}\Big|_{\mathbf{p}} & \cdots & \frac{\partial \Phi_1}{\partial x_m}\Big|_{\mathbf{p}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_n}{\partial x_1}\Big|_{\mathbf{p}} & \cdots & \frac{\partial \Phi_n}{\partial x_m}\Big|_{\mathbf{p}} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

Recall

Given a (smooth) function $\Phi: \mathbb{R}^m \to \mathbb{R}^n$, a point $\mathbf{p} \in \mathbb{R}^m$ and a column vector $\mathbf{v} \in \mathbb{R}^m$, the *change at* $\mathbf{p}$ *along direction* $\boldsymbol{v}$ is:

$$\left.\frac{\partial \Phi}{\partial \mathbf{v}}\right|_{\mathbf{p}} = \mathbf{d}\Phi\Big|_{\mathbf{p}} \cdot \mathbf{v}$$

More generally, given a (smooth) function $\Phi: V \to W$ between vector spaces, and vectors $v, w \in V$ the change at v along direction w is:

$$\left.\frac{\partial \Phi}{\partial w}\right|_{v} = d\Phi\Big|_{v}(w)$$

Recall

Given (smooth) functions $\Phi, \Psi: \mathbb{R}^m \to \mathbb{R}^n$, writing $\Phi = (\Phi_1, \ldots, \Phi_n)^\top$ and $\Phi = (\Psi_1, \ldots, \Psi_n)^\top$, we have the (Euclidean) inner-product $f: \mathbb{R}^m \to \mathbb{R}$:

$$f(\mathbf{p}) \equiv \langle \Phi(\mathbf{p}), \Psi(\mathbf{p}) \rangle = \sum_{i=1}^{n} \Phi_i(\mathbf{p}) \cdot \Psi_i(\mathbf{p})$$

Taking the derivative at $\mathbf{p} \in \mathbb{R}^m$ in direction $\mathbf{v} \in \mathbb{R}^m$:

$$\begin{aligned}
\frac{\partial f}{\partial \mathbf{v}}\Big|_{\mathbf{p}} &= \frac{\partial}{\partial \mathbf{v}}\Big|_{\mathbf{p}} \left(\sum_{i=1}^{n} \Phi_i(\mathbf{p}) \cdot \Psi_i(\mathbf{p}) \right) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{v}}\Big|_{\mathbf{p}} \left(\Phi_i(\mathbf{p}) \cdot \Psi_i(\mathbf{p}) \right) \\
&= \sum_{i=1}^{n} \frac{\partial \Phi_i}{\partial \mathbf{v}}\Big|_{\mathbf{p}} \cdot \Psi_i(\mathbf{p}) + \Phi_i(\mathbf{p}) \cdot \frac{\partial \Psi_i}{\partial \mathbf{v}}\Big|_{\mathbf{p}} \\
&= \left\langle \frac{\partial \Phi}{\partial \mathbf{v}}\Big|_{\mathbf{p}}, \Psi(\mathbf{p}) \right\rangle + \left\langle \Phi(\mathbf{p}), \frac{\partial \Psi}{\partial \mathbf{v}}\Big|_{\mathbf{p}} \right\rangle
\end{aligned}$$

Recall

Given a (smooth) function $\Phi: \mathbb{R}^m \to \mathbb{R}^n$, the partial derivatives commute:

$$\left.\frac{\partial^2 \Phi}{\partial \mathbf{u} \partial \mathbf{v}}\right|_{\mathbf{p}} = \left.\frac{\partial^2 \Phi}{\partial \mathbf{v} \partial u}\right|_{\mathbf{p}}, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^m$$

More generally, given a (smooth) function $\Phi: V \to W$ between vector spaces, the partial derivatives commute:

$$\left.\frac{\partial^2 \Phi}{\partial v \partial w}\right|_{u} = \left.\frac{\partial^2 \Phi}{\partial w \partial v}\right|_{u}, \qquad \forall u, v \in V$$

Recall

We denote by $\mathbb{T} \subset \mathbb{E}^2$ the unit right triangle:

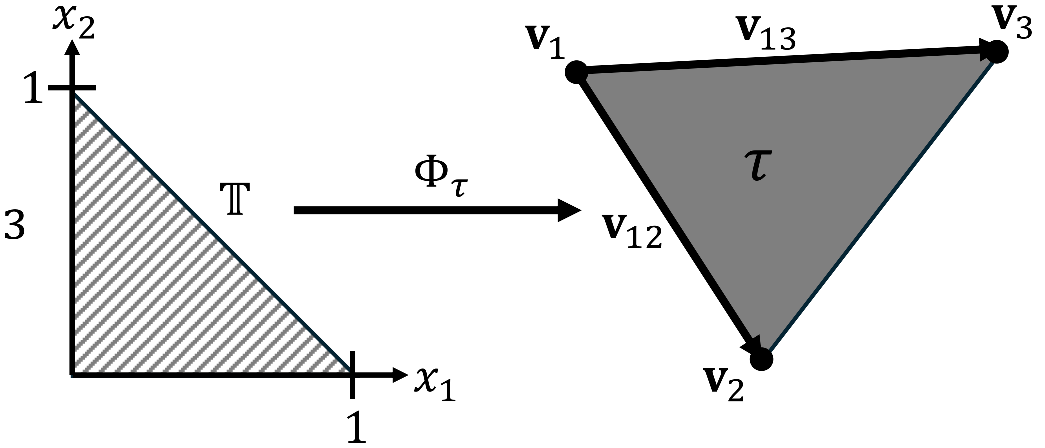
$$\mathbb{T} = \{(x_1, x_2) \in [0,1]^2 | x_1 + x_2 \leq 1\}$$

Given a triangle $\tau \subset \mathbb{E}^3$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{E}^3$, we parameterize the triangle over the unit-right triangle as:

$$\Phi_\tau(x_1, x_2) = \mathbf{v}_1 + x_1 \cdot (\mathbf{v}_2 - \mathbf{v}_1) + x_2 \cdot (\mathbf{v}_3 - \mathbf{v}_1)$$

Denoting by $\mathbf{v}_{ij} = \mathbf{v}_j - \mathbf{v}_i$ the direction from vertex $\mathbf{v}_i$ to vertex $\mathbf{v}_j$:

$$\Phi_\tau(x_1, x_2) = \mathbf{v}_1 + x_1 \cdot \mathbf{v}_{12} + x_2 \cdot \mathbf{v}_{13}$$

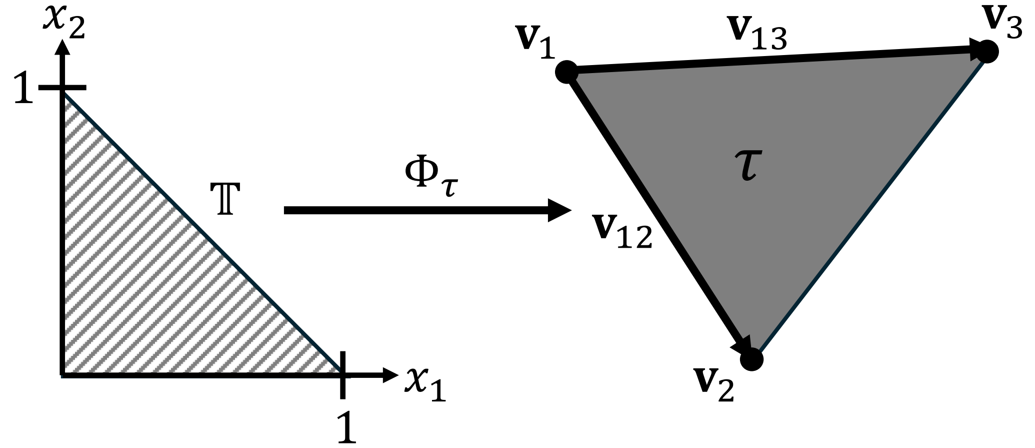


Recall

$$\Phi_\tau(x_1, x_2) = \mathbf{v}_1 + x_1 \cdot \mathbf{v}_{12} + x_2 \cdot \mathbf{v}_{13}$$

For $\mathbf{p} \in \mathbb{T}$, the differential $\mathbf{d}\Phi_\tau|_\mathbf{p} \in \mathbb{R}^{3\times 2}$, w.r.t. the cartesian basis $\{\partial_1|_\mathbf{p}, \partial_2|_\mathbf{p}\} \subset T_\mathbf{p}\mathbb{T}$, is the matrix:

$$\mathbf{d}\Phi_\tau\Big|_\mathbf{p} = (\mathbf{v}_{12} \quad \mathbf{v}_{13})$$



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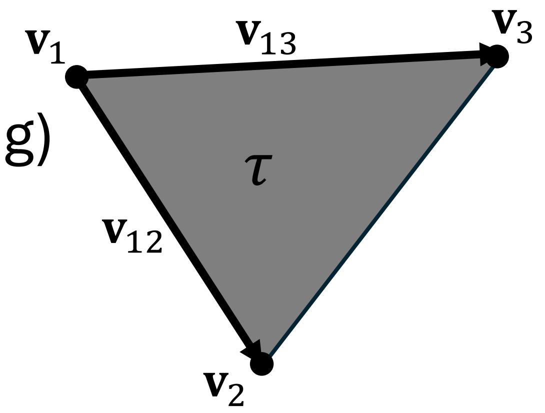
$$\mathbf{d}\Phi_\tau\Big|_\mathbf{p} = \begin{pmatrix} \mathbf{v}_{12} & \mathbf{v}_{13} \end{pmatrix}$$

and the pulled-back inner-product is:

$$\mathbf{g}_\tau = \begin{pmatrix} \|\mathbf{v}_{12}\|^2 & \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ \langle \mathbf{v}_{13}, \mathbf{v}_{12} \rangle & \|\mathbf{v}_{13}\|^2 \end{pmatrix}$$

Taking the cross-product of the edge directions (and normalizing) we get the embedded triangle's norma:

$$n_\tau\Big|_\mathbf{p} = \frac{\mathbf{v}_{12} \times \mathbf{v}_{13}}{\|\mathbf{v}_{12} \times \mathbf{v}_{13}\|}$$



Recall

$$\Phi_\tau(x_1, x_2) = \mathbf{v}_1 + x_1 \cdot \mathbf{v}_{12} + x_2 \cdot \mathbf{v}_{13}$$

For $\mathbf{p} \in$ … s

$\{\partial_1|_\mathbf{p}, \partial_2$ …

Because the differential $\mathbf{d}\Phi_\tau|_\mathbf{p}$ is constant for all $\mathbf{p} \in \tau$, so is the inner-product $\mathbf{g}_\tau|_\mathbf{p}$ and the normal $n_\tau|_\mathbf{p}$.

$$\mathbf{d}\Phi_\tau\big|_\mathbf{p} = (\mathbf{v}_{12} \quad \mathbf{v}_{13})$$

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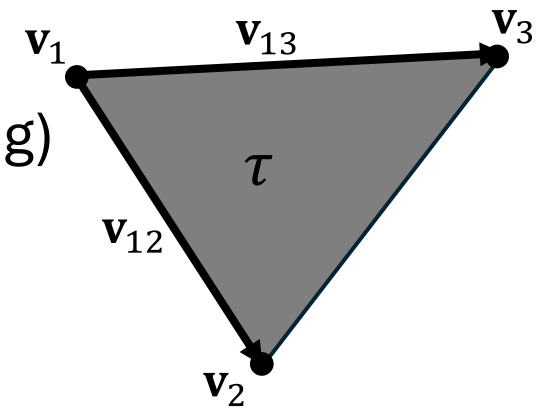

Outline

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Differential Geometry Review

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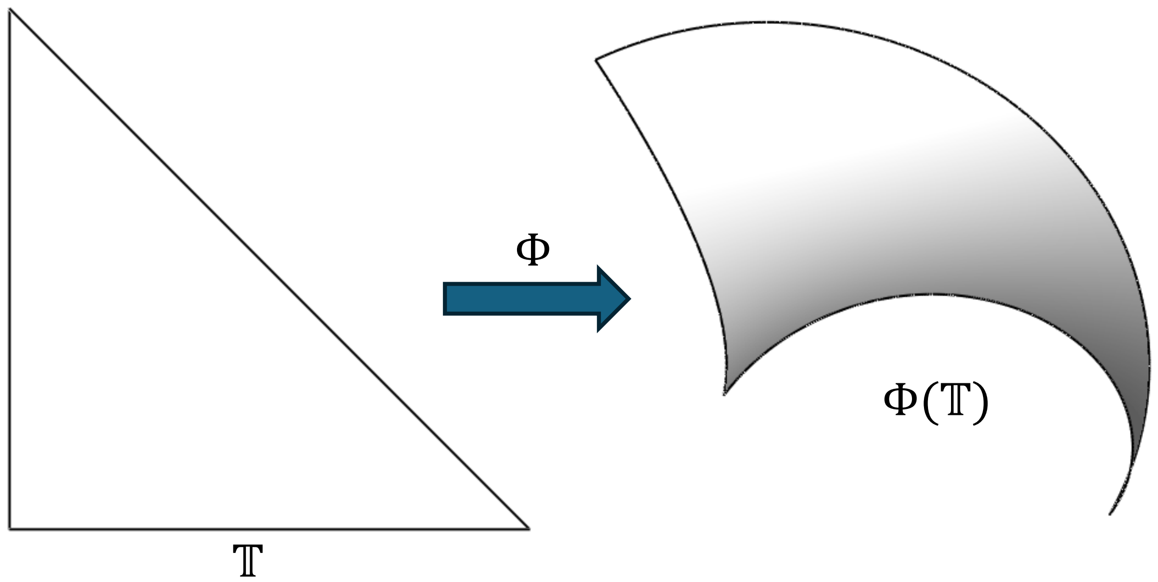
Differential Geometry

We will consider non-flat surface parameterizations:

$$\Phi: \mathbb{T} \to \mathbb{R}^3$$

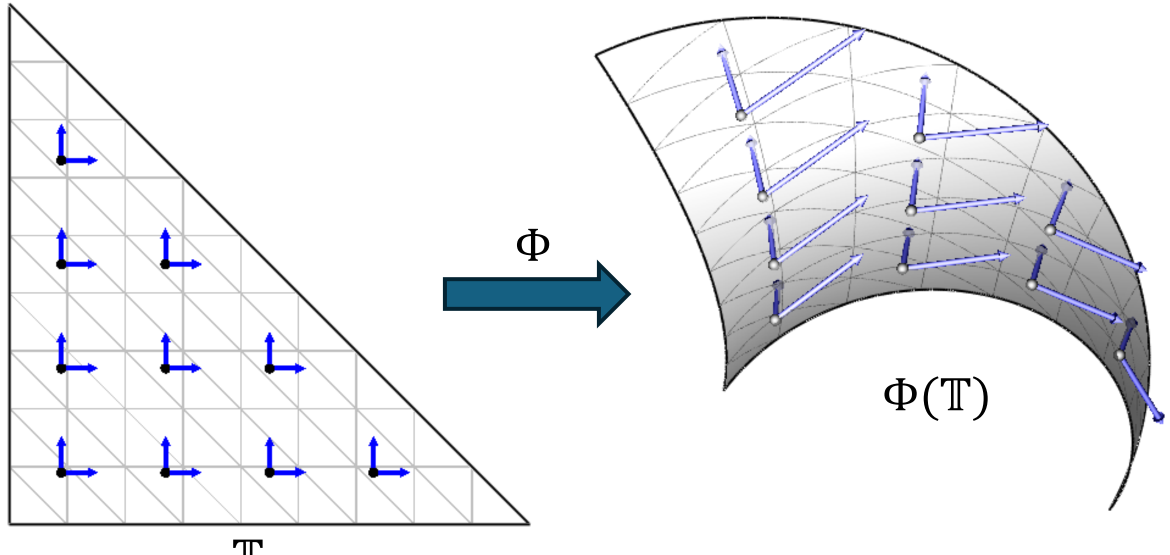


Differential Geometry

We will consider non-flat surface parameterizations:

$$\Phi\colon \mathbb{T} \rightarrow \mathbb{R}^3$$

Using the differential, we have a mapping from the cartesian directions $\{\partial_1|_{\mathbf{p}}, \partial_2|_{\mathbf{p}}\}$ in $T_{\mathbf{p}}\mathbb{T}$ to $\mathbb{R}^3$, for every $\mathbf{p} \in \mathbb{T}$.



Differential Geometry

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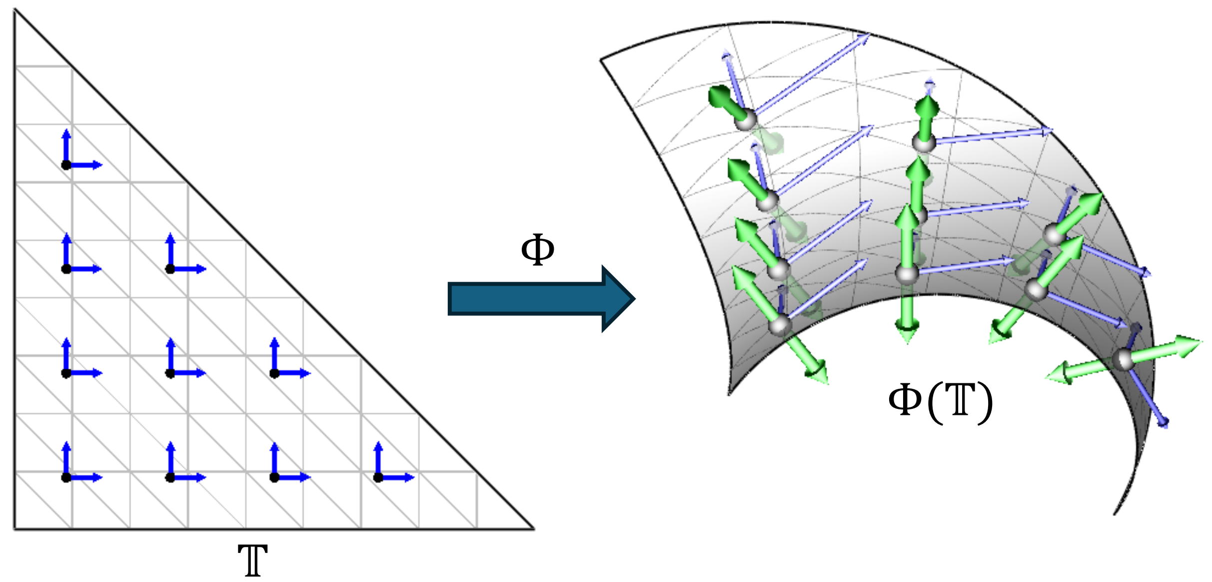
Normalizing the cross-product, gives the normal/*Gauss map* $N: \mathbb{T} \to \mathbb{R}^3$.



Differential Geometry

Since the differential $\mathbf{d}\Phi$ is not constant,
the inner-product $\mathbf{g}_\tau: \mathbb{T} \to \mathbb{R}^{2\times 2}$ and Gauss map $N: \mathbb{T} \to \mathbb{R}^3$ won't be either.

$$\Phi: \mathbb{T} \to \mathbb{R}^3$$

Using the differential, we have a mapping from the cartesian directions $\{\partial_1|_\mathbf{p}, \partial_2|_\mathbf{p}\}$ in $T_\mathbf{p}\mathbb{T}$ to $\mathbb{R}^3$, for every $\mathbf{p} \in \mathbb{T}$.

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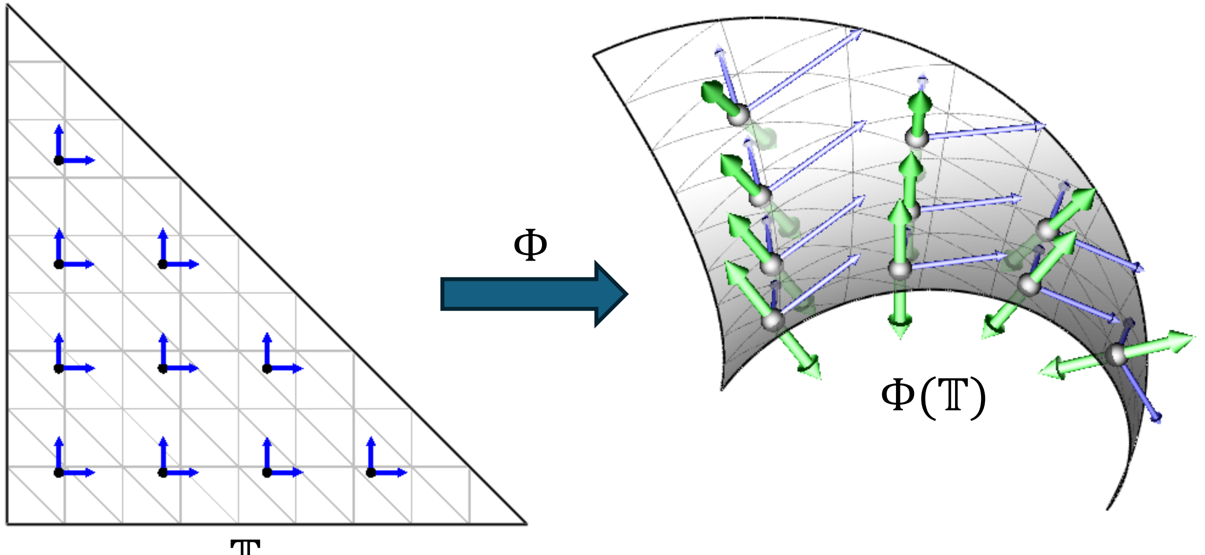

Differential Geometry

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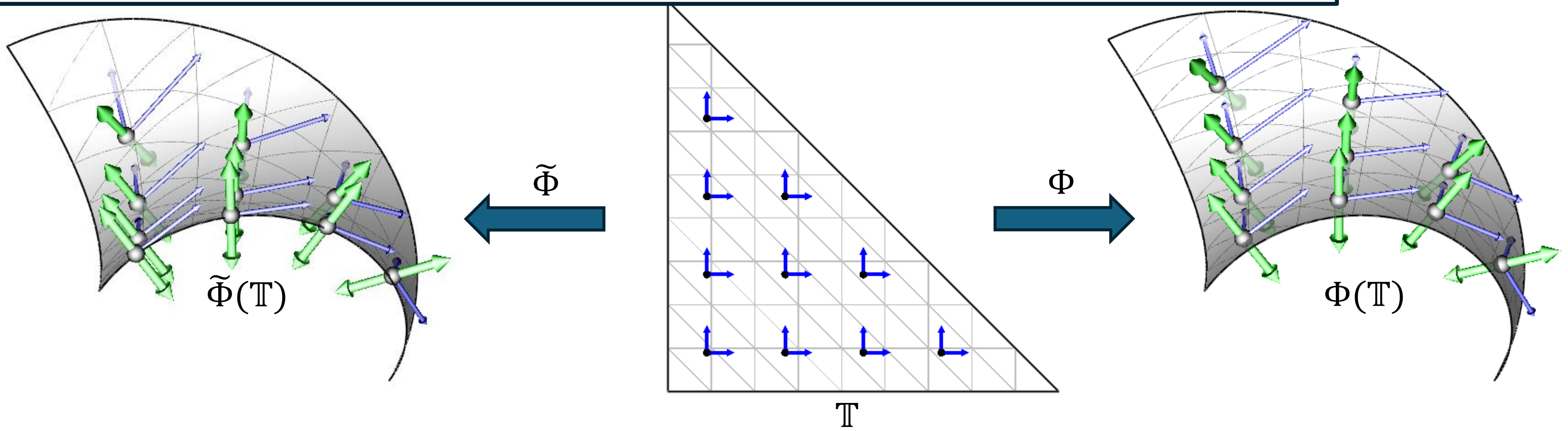
the inner-product $\mathbf{g}_\tau : \mathbb{T} \to \mathbb{R}^{2\times 2}$ and Gauss map $N : \mathbb{T} \to \mathbb{R}^3$ won't be either.

We can get a different mapping from $\mathbb{T}$ to the surface by changing the parameterization.

Using the dif… …irections $\{\partial_1|_p$ …

The inner-product and Gauss map, as maps on $\mathbb{T}$, will change.

Up to sign, the normal over a point on the surface will stay the same.



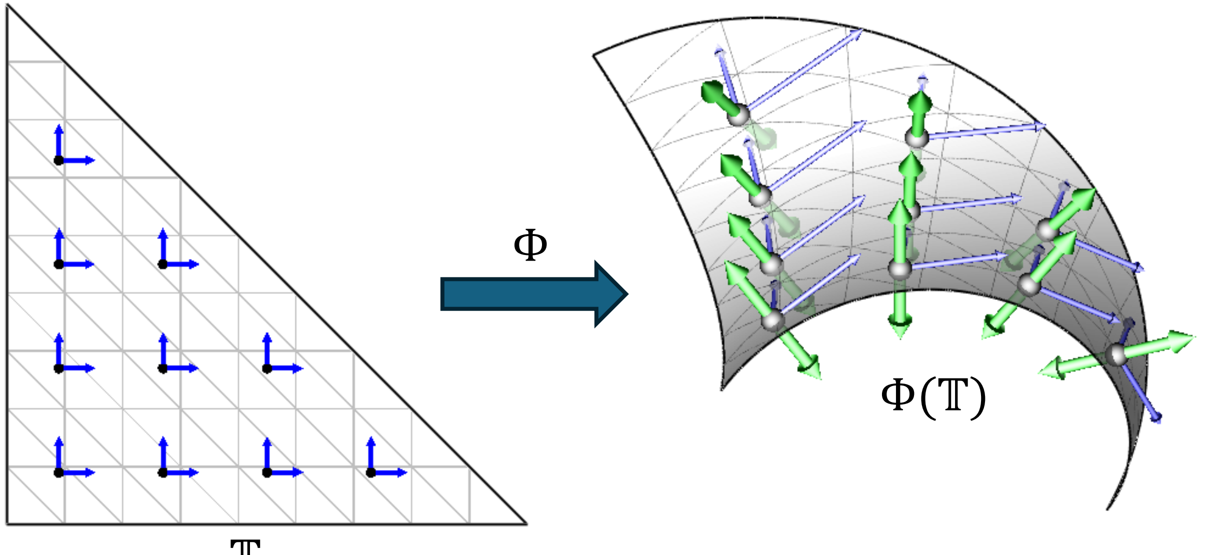
Differential Geometry

Definition:

The *first fundamental form at* $\mathbf{p}$, denoted $I|_{\mathbf{p}} \in \mathrm{Hom}(T_{\mathbf{p}}\mathbb{T}, T_{\mathbf{p}}^*\mathbb{T})$, is the bilinear map:

$$\left[I \Big|_{\mathbf{p}} (u) \right](v) = \left\langle d\Phi \Big|_{\mathbf{p}} (u), d\Phi \Big|_{\mathbf{p}} (v) \right\rangle_{\mathbb{R}^3}, \qquad u, v \in T_{\mathbf{p}}\mathbb{T}$$

This is the pulled-back inner-product $g_\tau|_{\mathbf{p}}: T_{\mathbf{p}}\mathbb{T} \to T_{\mathbf{p}}^*\mathbb{T}$ and is symmetric and positive definite.



$$\left[I\Big|_{\mathbf{p}}(u)\right](v) = \left\langle d\Phi\Big|_{\mathbf{p}}(u), d\Phi\Big|_{\mathbf{p}}(v)\right\rangle_{\mathbb{R}^3}$$

Differential Geometry

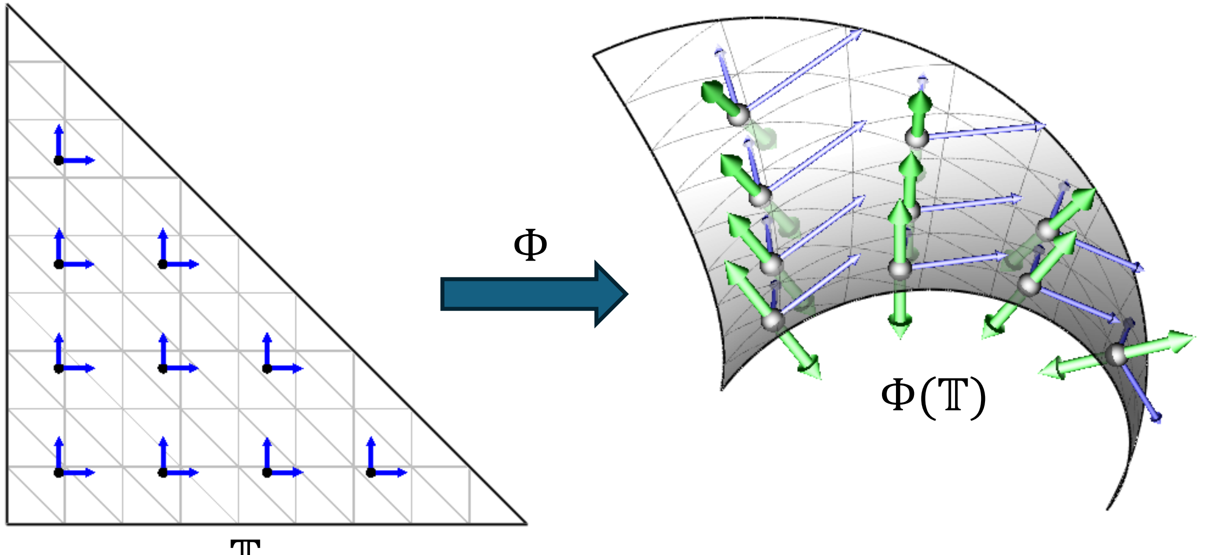
Definition:

Letting $N\colon \mathbb{T} \to S^2$ be the associated Gauss map, the *second fundamental form at* $\mathbf{p}$, denoted $II|_{\mathbf{p}} \in \mathrm{Hom}(T_{\mathbf{p}}\mathbb{T}, T_{\mathbf{p}}^*\mathbb{T})$, is the bilinear map:

$$\left[II\Big|_{\mathbf{p}}(u)\right](v) = \left\langle d\Phi\Big|_{\mathbf{p}}(u), dN\Big|_{\mathbf{p}}(v)\right\rangle_{\mathbb{R}^3}, \qquad u, v \in T_{\mathbf{p}}\mathbb{T}$$

Claim:

The second fundamental form is symmetric.



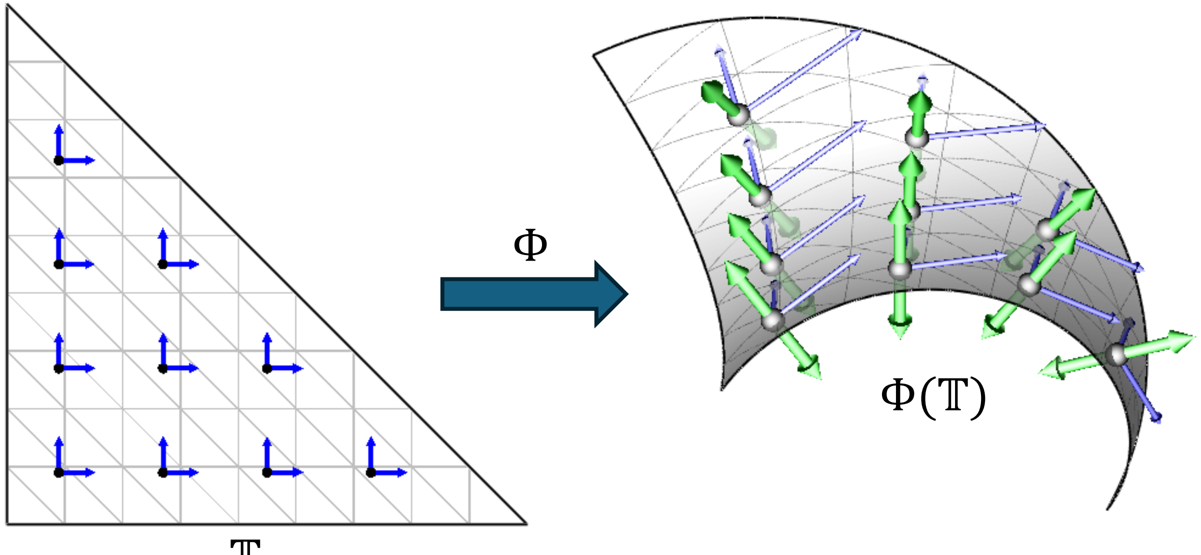
$$\left[II\Big|_{\mathbf{p}}(u) \right](v) = \left\langle d\Phi\Big|_{\mathbf{p}}(u), dN\Big|_{\mathbf{p}}(v) \right\rangle_{\mathbb{R}^3}$$

Differential Geometry

Proof:

We can express the second fundamental form in terms of partial derivatives:

$$\left[II\Big|_{\mathbf{p}}(u) \right](v) = \left\langle \frac{\partial \Phi}{\partial u}\Big|_{\mathbf{p}}, \frac{\partial N}{\partial v}\Big|_{\mathbf{p}} \right\rangle_{\mathbb{R}^3}$$



$$\left[II\big|_{\mathbf{p}}(u)\right](v) = \left\langle \frac{\partial \Phi}{\partial u}\bigg|_{\mathbf{p}}, \frac{\partial N}{\partial v}\bigg|_{\mathbf{p}} \right\rangle_{\mathbb{R}^3}$$

Differential Geometry

Proof:

For $w \in T_{\mathbf{p}}\mathbb{T}$, consider the function $f_w\colon \mathbb{T} \to \mathbb{R}$:

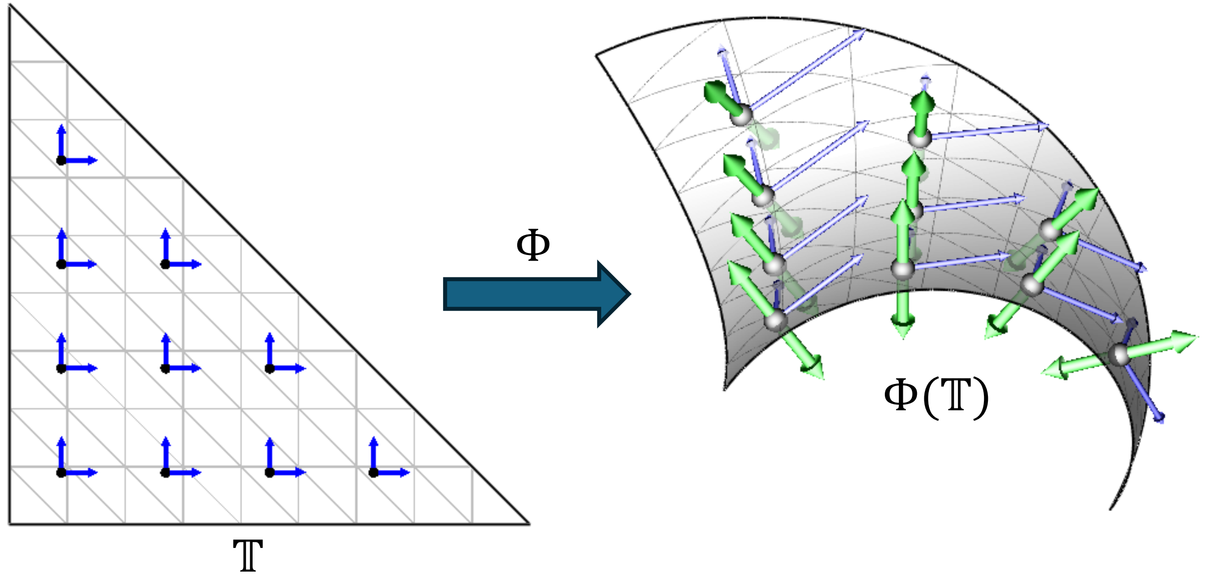
$$f_w(\mathbf{p}) = \left\langle \frac{\partial \Phi}{\partial w}\bigg|_{\mathbf{p}}, N(\mathbf{p}) \right\rangle_{\mathbb{R}^3}$$

Note:

At $\mathbf{p} \in \mathbb{T}$, the differential $d\Phi|_{\mathbf{p}}$ maps vectors in $T_{\mathbf{p}}\mathbb{T}$ to directions tangent to the surface at $\Phi(\mathbf{p})$.

The normal $N(\mathbf{p})$ is perpendicular to the surface at $\Phi(\mathbf{p})$.

$\Rightarrow$ The function $f_w\colon \mathbb{T} \to \mathbb{R}$ is constantly zero for all $w \in T_{\mathbf{p}}\mathbb{T}$.



$$\left[II\big|_{\mathbf{p}}(u)\right](v) = \left\langle \frac{\partial\Phi}{\partial u}\bigg|_{\mathbf{p}}, \frac{\partial N}{\partial v}\bigg|_{\mathbf{p}} \right\rangle_{\mathbb{R}^3}$$

Differential Geometry

$$\frac{\partial f}{\partial \mathbf{v}}\bigg|_{\mathbf{p}} \langle \Phi(\mathbf{p}), \Psi(\mathbf{p})\rangle = \left\langle \frac{\partial\Phi}{\partial \mathbf{v}}\bigg|_{\mathbf{p}}, \Psi(\mathbf{p}) \right\rangle + \left\langle \Phi(\mathbf{p}), \frac{\partial\Psi}{\partial \mathbf{v}}\bigg|_{\mathbf{p}} \right\rangle$$

Proof:

For $w \in T_{\mathbf{p}}\mathbb{T}$, consider the function $f_w\colon \mathbb{T} \to \mathbb{R}$:

$$0 = f_w(\mathbf{p}) = \left\langle \frac{\partial\Phi}{\partial w}\bigg|_{\mathbf{p}}, N(\mathbf{p}) \right\rangle_{\mathbb{R}^3}$$

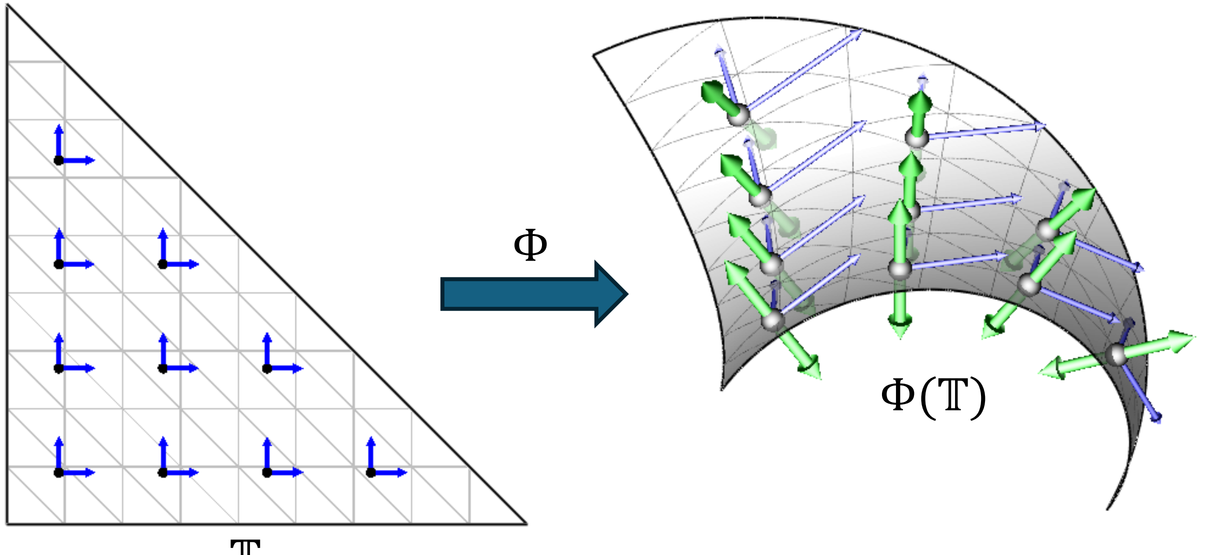
$\Rightarrow$ For $u, v \in V$, consider the partial derivative of f_u along direction v:

$$0 = \frac{\partial f_u}{\partial v}\bigg|_{\mathbf{p}} = \frac{\partial}{\partial v}\left\langle \frac{\partial\Phi}{\partial u}\bigg|_{\mathbf{p}}, N(\mathbf{p}) \right\rangle$$

$$= \left\langle \frac{\partial^2\Phi}{\partial v \partial u}\bigg|_{\mathbf{p}}, N(\mathbf{p}) \right\rangle + \left\langle \frac{\partial\Phi}{\partial u}\bigg|_{\mathbf{p}}, \frac{\partial N}{\partial v}\bigg|_{\mathbf{p}} \right\rangle$$

$$\Downarrow$$

$$\left\langle \frac{\partial\Phi}{\partial u}\bigg|_{\mathbf{p}}, \frac{\partial N}{\partial v}\bigg|_{\mathbf{p}} \right\rangle = -\left\langle \frac{\partial^2\Phi}{\partial v \partial u}\bigg|_{\mathbf{p}}, N(\mathbf{p}) \right\rangle$$



$$\left[II\Big|_{\mathbf{p}}(u)\right](v) = \left\langle \frac{\partial \Phi}{\partial u}\Big|_{\mathbf{p}}, \frac{\partial N}{\partial v}\Big|_{\mathbf{p}} \right\rangle_{\mathbb{R}^3}$$

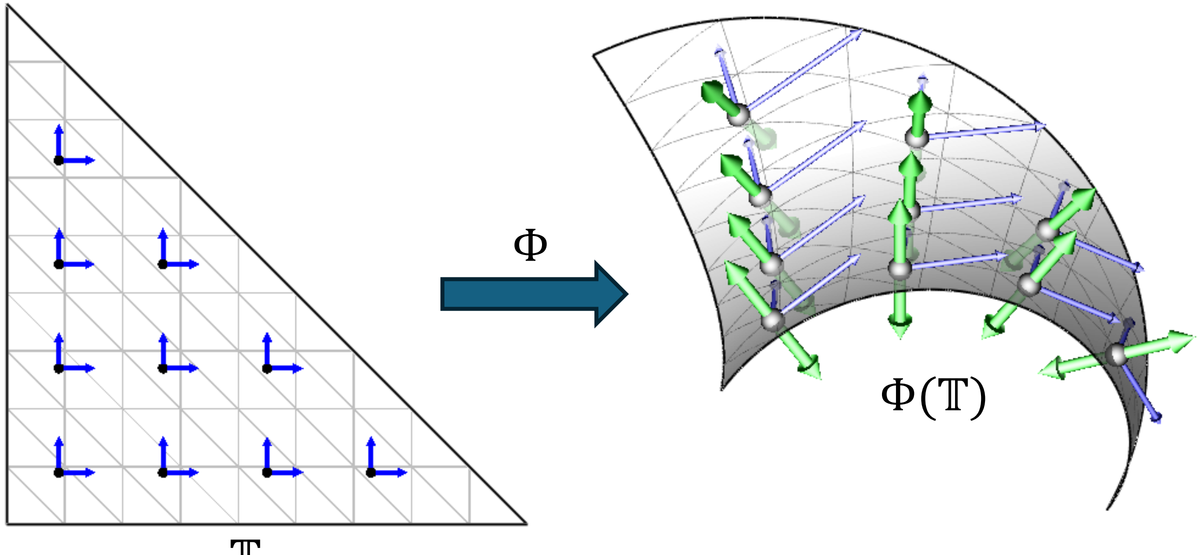
Differential Geometry

Proof:

$$\left\langle \frac{\partial \Phi}{\partial u}\Big|_{\mathbf{p}}, \frac{\partial N}{\partial v}\Big|_{\mathbf{p}} \right\rangle = -\left\langle \frac{\partial^2 \Phi}{\partial v \partial u}\Big|_{\mathbf{p}}, N(\mathbf{p}) \right\rangle$$

Using the commutativity of partial derivatives:

$$\begin{aligned}\left[II\Big|_{\mathbf{p}}(u)\right](v) &= \left\langle \frac{\partial \Phi}{\partial u}\Big|_{\mathbf{p}}, \frac{\partial N}{\partial v}\Big|_{\mathbf{p}} \right\rangle_{\mathbb{R}^3} \\ &= -\left\langle \frac{\partial^2 \Phi}{\partial v \partial u}\Big|_{\mathbf{p}}, N(\mathbf{p}) \right\rangle \\ &= -\left\langle \frac{\partial^2 \Phi}{\partial u \partial v}\Big|_{\mathbf{p}}, N(\mathbf{p}) \right\rangle \\ &= \left[II\Big|_{\mathbf{p}}(v)\right](u)\end{aligned}$$



$\Rightarrow\ II|_{\mathbf{p}}$ is symmetric.

Differential Geometry

Summarizing:

For a parameterization $\Phi:\mathbb{T}\to\mathbb{R}^3$, we can define the *first and second fundamental forms at* $\mathbf{p}$ – $I|_{\mathbf{p}}, II|_{\mathbf{p}} \in \mathrm{Hom}(T_{\mathbf{p}}\mathbb{T}, T^*_{\mathbf{p}}\mathbb{T})$:

$$\left[I\Big|_{\mathbf{p}}(u)\right](v) = \left\langle \frac{\partial\Phi}{\partial u}\Big|_{\mathbf{p}}, \frac{\partial\Phi}{\partial v}\Big|_{\mathbf{p}} \right\rangle_{\mathbb{R}^3}$$

$$\left[II\Big|_{\mathbf{p}}(u)\right](v) = \left\langle \frac{\partial\Phi}{\partial u}\Big|_{\mathbf{p}}, \frac{\partial N}{\partial v}\Big|_{\mathbf{p}} \right\rangle_{\mathbb{R}^3}$$

The first is the symmetric positive definite inner-product.

The second is a symmetric bilinear form.

$\Rightarrow$ The composition $S_{\mathbf{p}} = I|_{\mathbf{p}}^{-1} \circ II|_{\mathbf{p}}$ is a self-adjoint map on $T_{\mathbf{p}}\mathbb{T}$.

This is called the *shape operator* or *Weingarten map*.

Differential Geometry

Summarizing:

For a parameterization $\Phi\colon \mathbb{T} \to \mathbb{R}^3$, we can define the *first and second fundamental forms at* $\mathbf{p}$ – $I|_{\mathbf{p}}, II|_{\mathbf{p}} \in \mathrm{Hom}(T_{\mathbf{p}}\mathbb{T}, T_{\mathbf{p}}^*\mathbb{T})$:

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The first is the symmetric positive definite inner-product.

The second is a symmetric bilinear form.

⇒ At every $\mathbf{p} \in \mathbb{T}$ there are orthonormal vectors $v_1, v_2 \in T_{\mathbf{p}}\mathbb{T}$, with associated eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ such that:

$$II\Big|_{\mathbf{p}}(v_i) = \lambda_i \cdot I\Big|_{\mathbf{p}}(v_i)$$

Differential Geometry

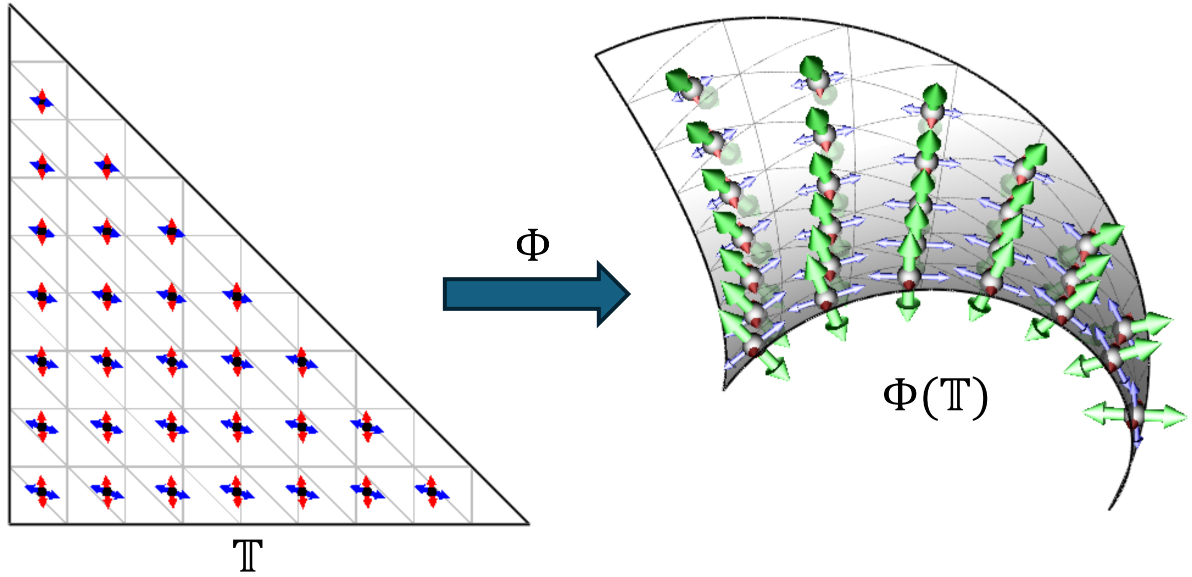
Definition:

Letting $\{(v_1, \kappa_1), (v_2, \kappa_2)\}$ be the eigenvectors and eigenvalues of the generalized eigen-problem at the point $\mathbf{p} \in \mathbb{T}$:

$$II\Big|_{\mathbf{p}}(v_i) = \kappa_i \cdot I\Big|_{\mathbf{p}}(v_i)$$

the eigenvalues at $\mathbf{p}$ are the *principal curvatures* , and the eigenvectors are the *principal curvature directions*.

The vectors give the directions along which the surface curves most.



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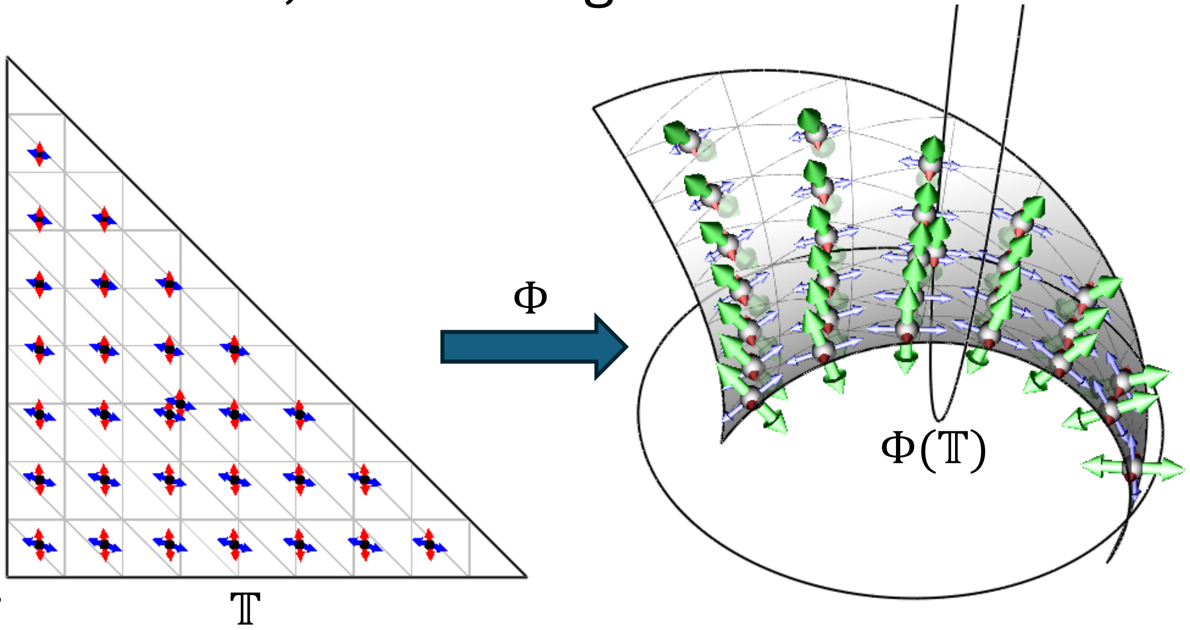
the eigenvalues at $\mathbf{p}$ are the *principal curvatures* , and the eigenvectors are the *principal curvature directions*.

The vectors give the directions along which the surface curves most.

The reciprocal of the values is the radius of the best-fit tangential circle.

The sign describes if the surface curves towards/away from the normal.

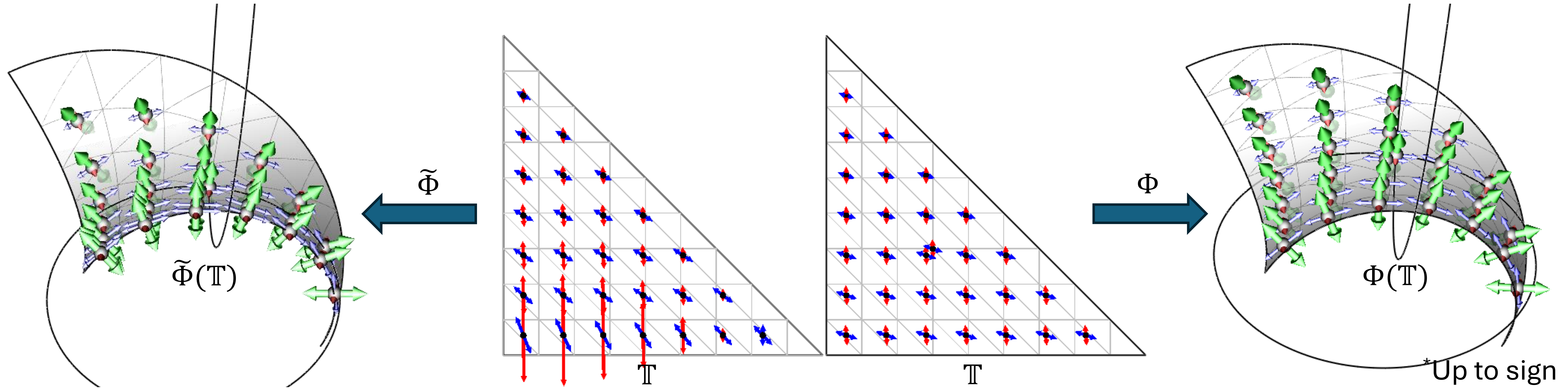


Differential Geometry

Note:

Curvature directions (in $\mathbb{R}^3$) and values **do not** depend on the surface parameterization.*

Curvature values can have different signs.



*Up to sign

Differential Geometry

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- Curvature directions (in $\mathbb{R}^3$) and values **do not** depend on the surface parameterization.*
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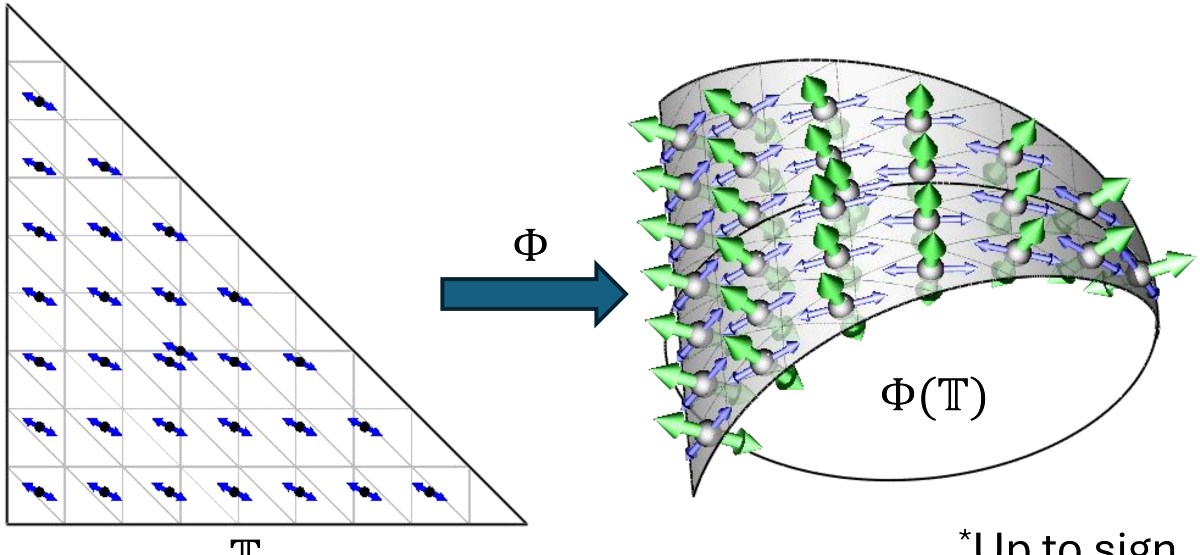
- Curvature values can be zero.



*Up to sign

Differential Geometry

Note:

Curvature directions (in $\mathbb{R}^3$) and values **do not** depend on the surface parameterization.*

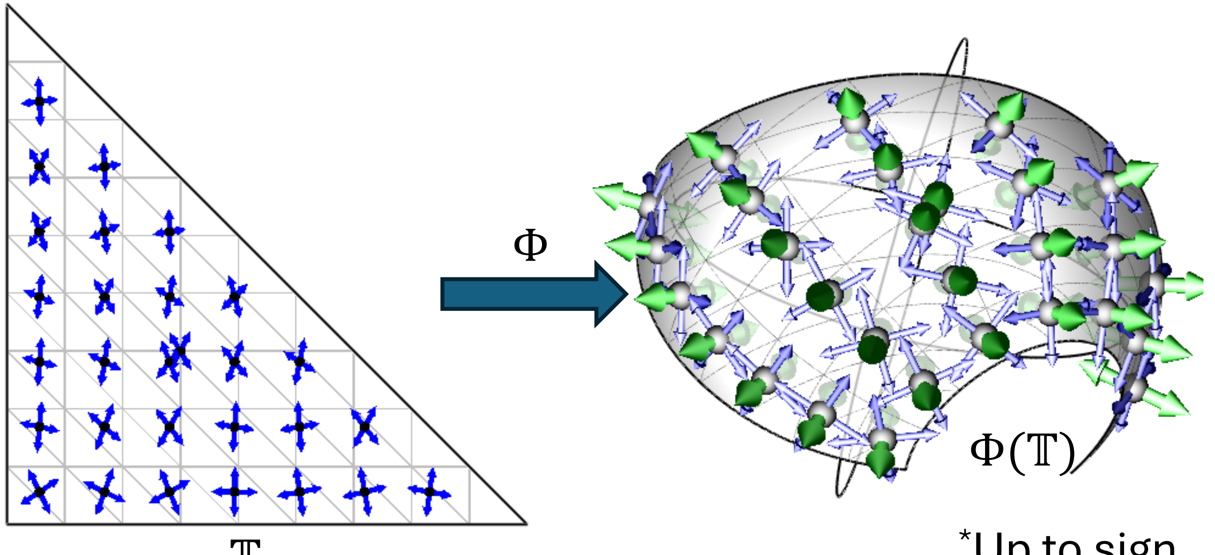
Curvature values can have different signs.

Curvature values can be zero.

Curvature values can be the same.

$\Rightarrow$ The eigenspace is two-dimensional and **any** tangent vector is an eigenvector.

Such a point is called *umbilic*.



*Up to sign

Differential Geometry

Terminology:

The sum of the eigenvalues, $\kappa_1 + \kappa_2$, is the *mean curvature*.
This is the trace of the shape operator:

$$I\Big|_{\mathbf{p}}^{-1} \circ II\Big|_{\mathbf{p}} : T_{\mathbf{p}}\mathbb{T} \to T_{\mathbf{p}}\mathbb{T}$$

The product of the eigenvalues , $\kappa_1 \cdot \kappa_2$, is the *Gaussian curvature*.
This is determinant of the shape operator.

Both give a **parameterization-independent*** characterization of how the surface curves.

The Gaussian curvature is **isometry-invariant**.

*Up to sign for mean curvature

Differential Geometry

Terminology:

The sum of the squares of the eigenvalues, $\kappa_1^2 + \kappa_2^2$, is the *total curvature*.

This is the square-norm of the shape operator:

$$I\Big|_{\mathbf{p}}^{-1} \circ II\Big|_{\mathbf{p}} : T_{\mathbf{p}}\mathbb{T} \to T_{\mathbf{p}}\mathbb{T}$$

It gives a **parameterization-independent**[*] characterization of how the surface curves with larger values corresponding to more curved regions and smaller values corresponding to flatter regions.

[*]Up to sign for mean curvature

Outline

Recall

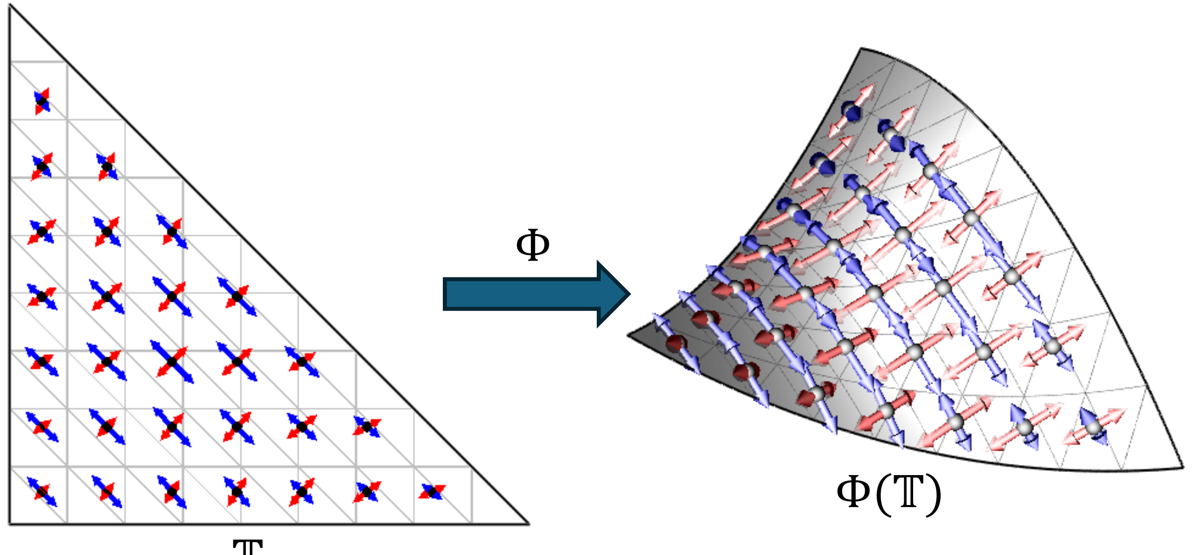
Differential Geometry Review

Approximating Curvature

$$\left[II\Big|_{\mathbf{p}}(u)\right](v) = \left\langle dN\Big|_{\mathbf{p}}(u), d\Phi\Big|_{\mathbf{p}}(v)\right\rangle_{\mathbb{R}^3}$$

Approximating the 2nd Fundamental Form

Given smooth geometry, we can get the normals and curvature.



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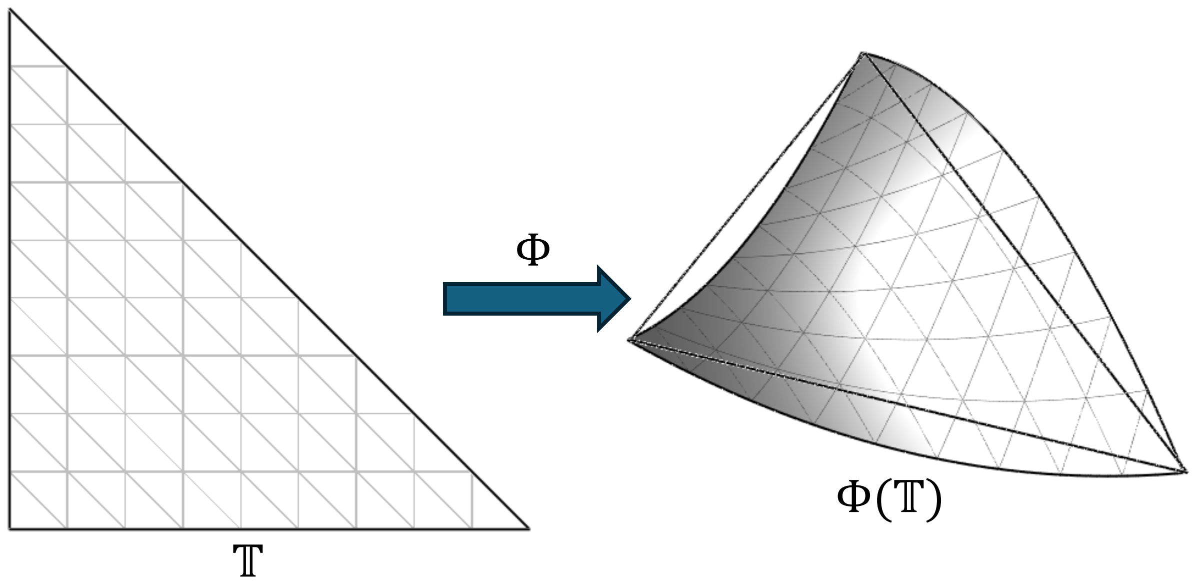
Approximating the 2nd Fundamental Form

Given smooth geometry, we can get the normals and curvature.

Q: What if we use a discrete mesh to sample the geometry?



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Approximating the 2nd Fundamental Form

Naïve:

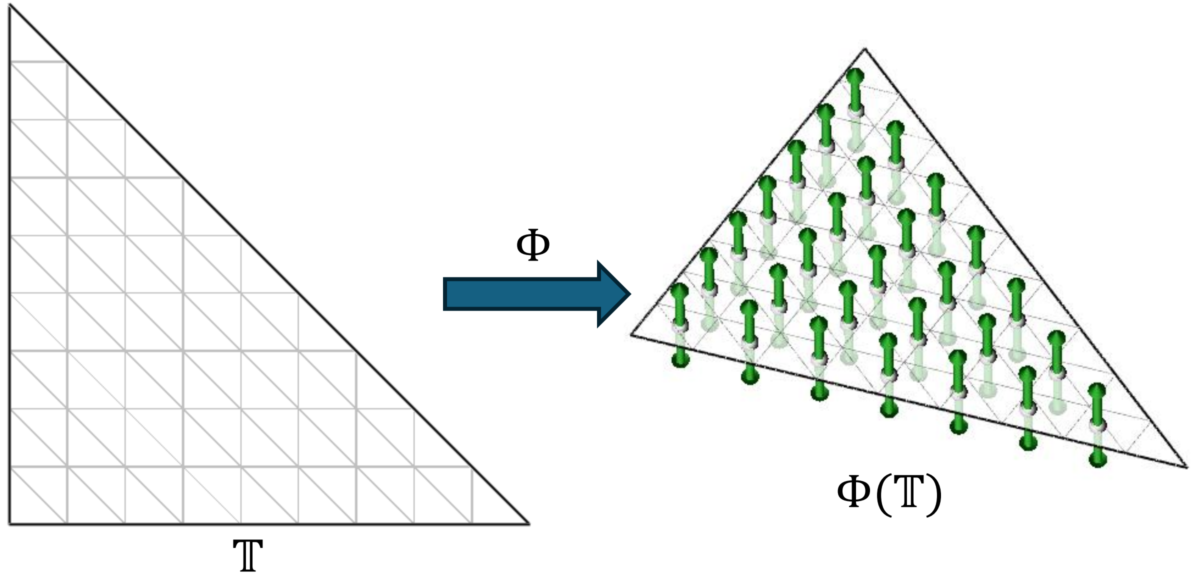
We can assign normals using the embedding of the triangle.

$\Rightarrow$ Normals will be constant

$\Rightarrow$ Normal differential will be zero

$\Rightarrow$ 2nd fundamental form will be zero

$\Rightarrow$ No useful curvature information!



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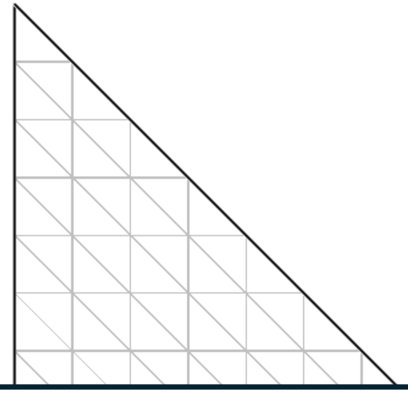
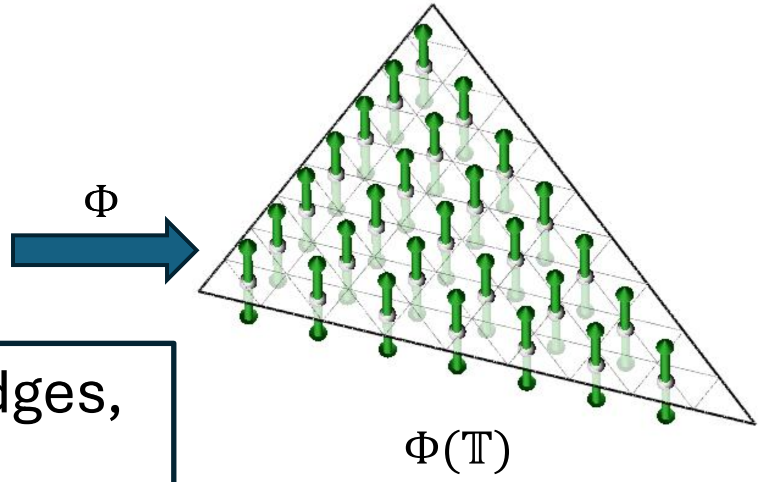




The normal will also be **discontinuous** across edges, so the differential will be unbounded there.

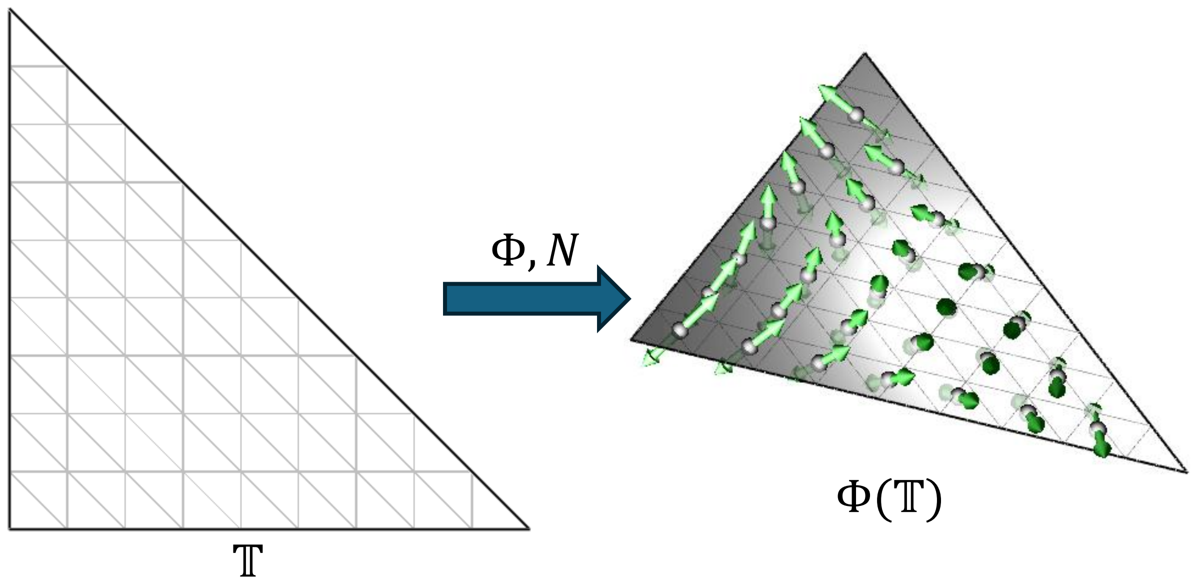
$$\left[II\Big|_{\mathbf{p}}(u)\right](v) = \left\langle dN\Big|_{\mathbf{p}}(u), d\Phi\Big|_{\mathbf{p}}(v)\right\rangle_{\mathbb{R}^3}$$

Approximating the 2nd Fundamental Form

Phong:

We can define/sample normals at the corners and interpolate

⇒ Normals will vary over the triangle



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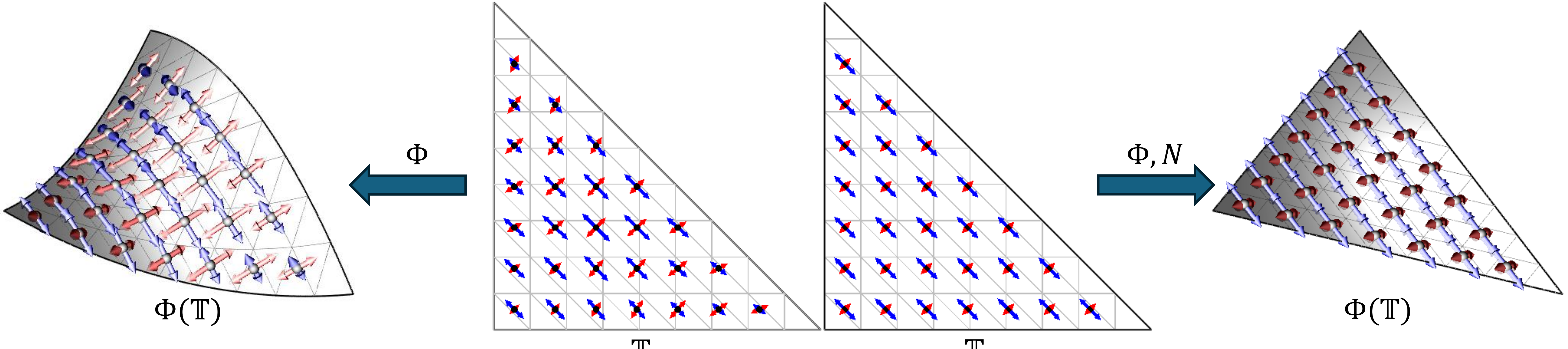
Approximating the 2nd Fundamental Form

Phong:

We can define/sample normals at the corners and interpolate

$\Rightarrow$ Normals will vary over the triangle

$\Rightarrow$ Can differentiate the embedding and the normals to get curvature

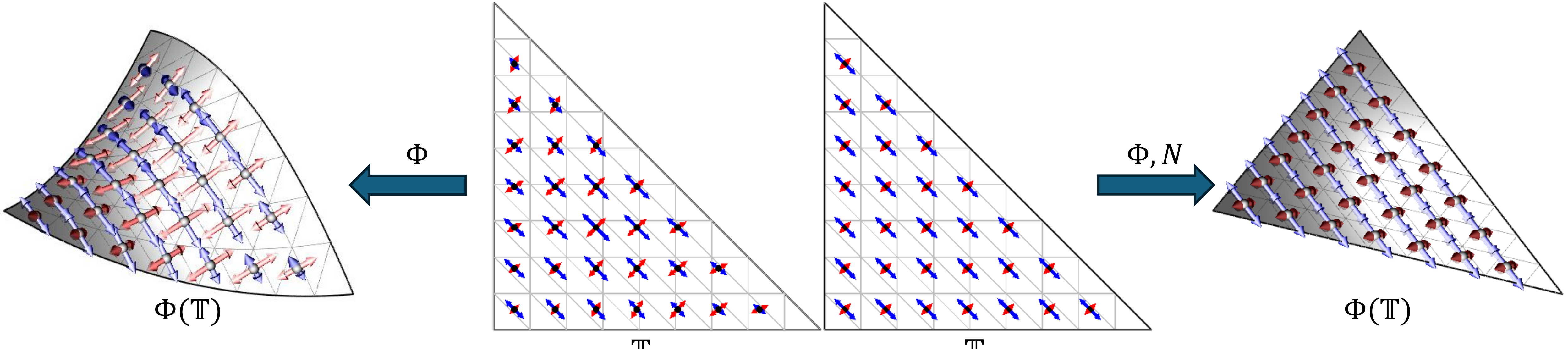


$$\left[II\Big|_{\mathbf{p}}(u) \right](v) = \left\langle dN\Big|_{\mathbf{p}}(u), d\Phi\Big|_{\mathbf{p}}(v) \right\rangle_{\mathbb{R}^3}$$

Approximating the 2nd Fundamental Form

Note:

- ✗ If we rescale interpolated normals, the interpolant won't be linear. But if we don't, the normals won't be unit-length.



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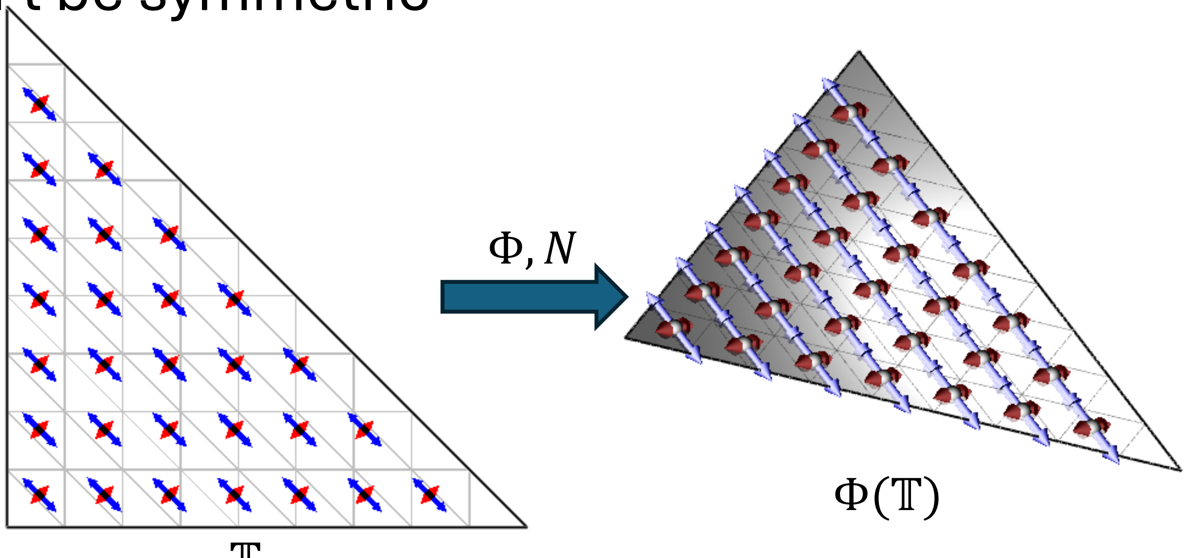
Approximating the 2nd Fundamental Form

Note:

- ✗ If we rescale interpolated normals, the interpolant won't be linear. But if we don't, the normals won't be unit-length.
- ✗ Either way, the geometry doesn't match the normals
- ⇒ The image of $d\Phi|_{\mathbf{p}}$ will not be perpendicular to $N(\mathbf{p})$
- ⇒ The second fundamental form won't be symmetric
- ⇒ Need explicit symmetrization to ensure orthonormal eigenvectors.



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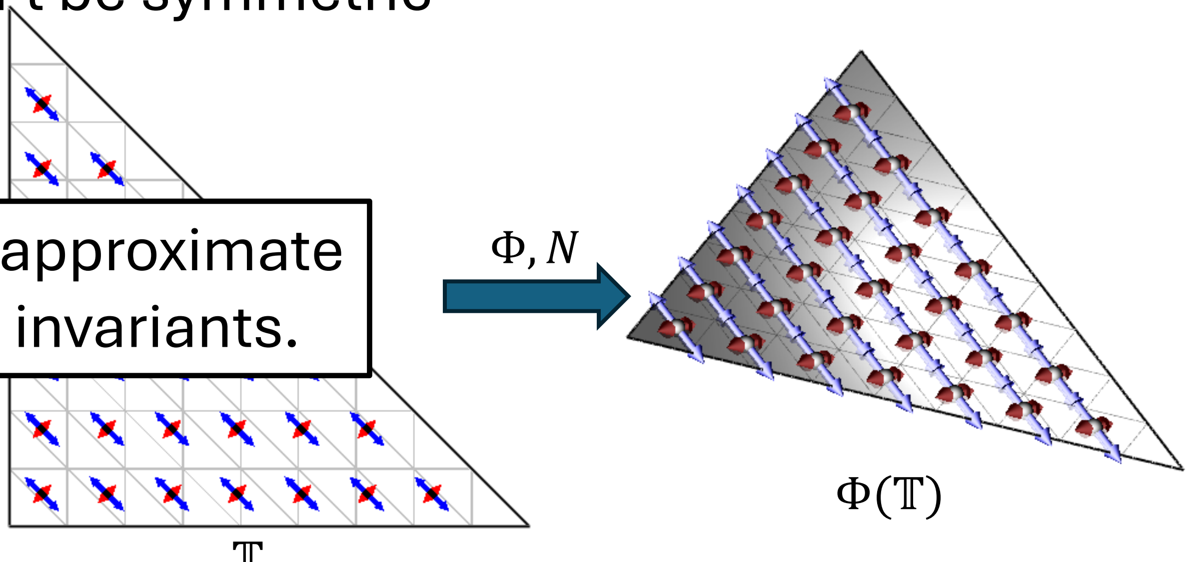
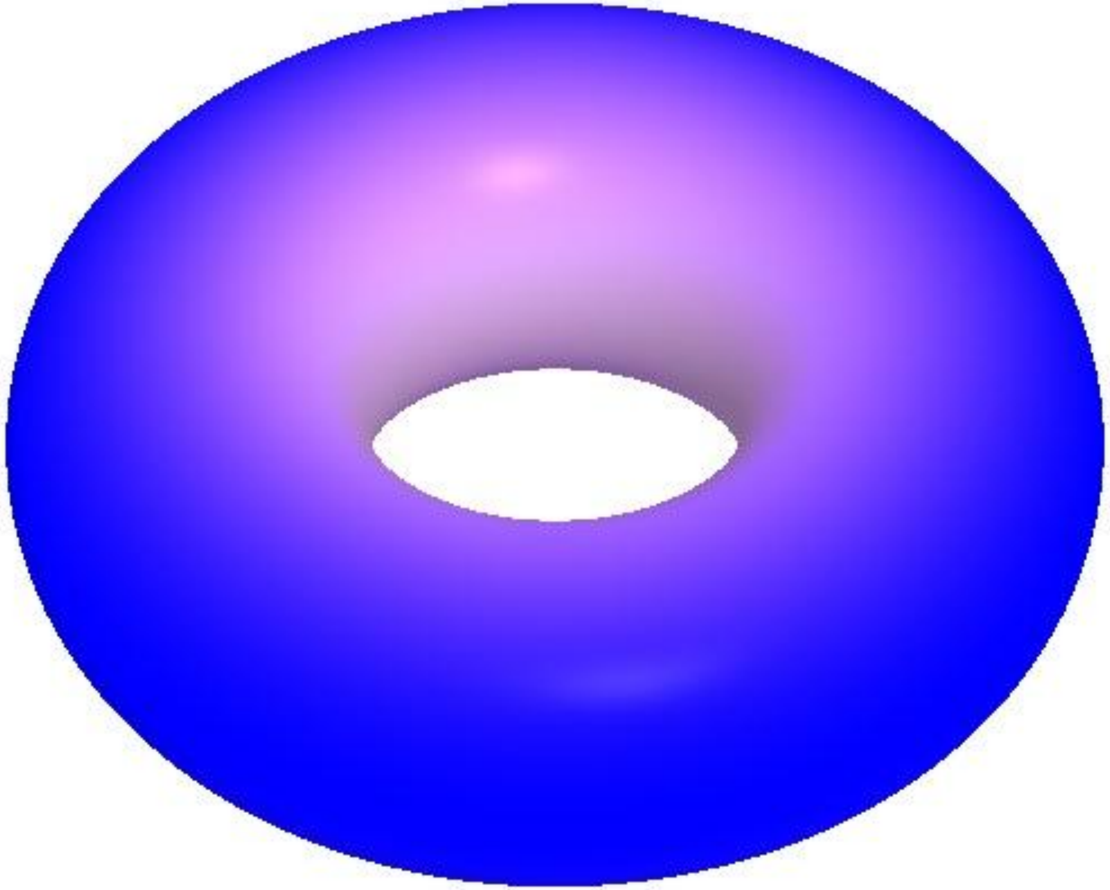
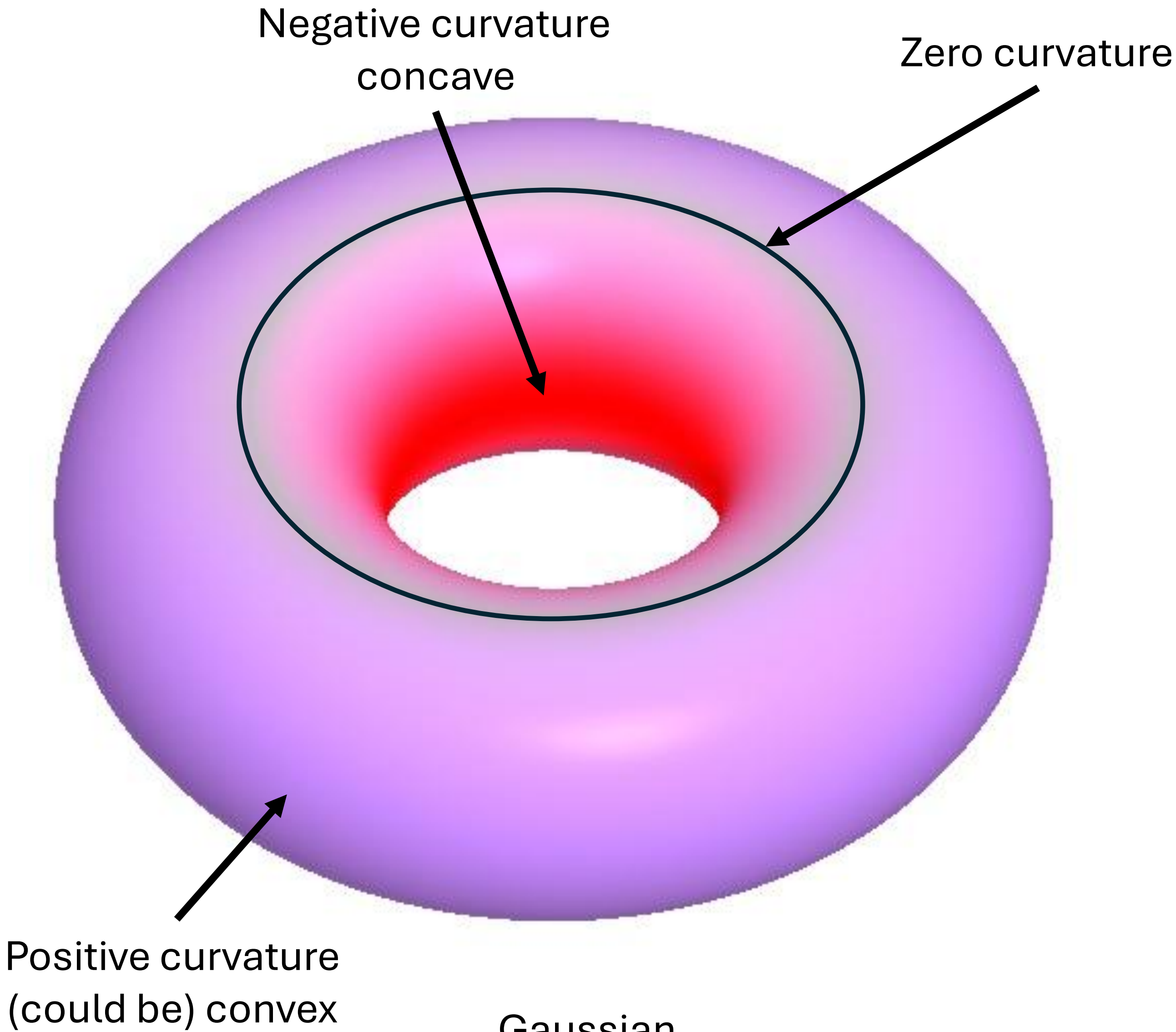
The curvature estimates are approximate and don't preserve discrete invariants.



Approximating the 2nd Fundamental Form



mean



Gaussian

Approximating the 2nd Fundamental Form

mean

Gaussian