Geometry Processing (601.458/658)

Misha Kazhdan

Outline

Recall

Linear Algebra Review

Phong Rendering

Normal Smoothing

Recall

Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we discretize the space of scalar functions using the hat basis:

$$V = \mathrm{Span}(\{\phi_v\}_{v\in\mathcal{V}})$$

The hat-basis function $\phi_v\colon \mathcal{M} \to \mathbb{R}$ is the piecewise-linear interpolant:

$$\phi_v(w) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ For any function $f \in V$:

$$f(p) = \sum_{v\in\mathcal{V}} \alpha_v \cdot \phi_v(p)$$

$$\Updownarrow$$

$$f(v) = \alpha_v, \qquad \forall v \in \mathcal{V}$$

Recall

Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we discretize the space of scalar functions using the hat basis:

$$V = \mathrm{Span}(\{\phi_v\}_{v\in\mathcal{V}})$$

On the space of scalar functions, we have two bilinear forms:

$M \in \mathrm{Hom}(V, V^*)$: The inner-product on scalar functions:

$$[M(f)](g) = \langle f, g\rangle_{\mathcal{M}}$$

$S \in \mathrm{Hom}(V, V^*)$: The pull-back of the inner-product on cotangent vector-fields:

$$[S(f)](g) = \langle df, dg\rangle_{\mathcal{M}}$$

Combining the mass and stiffness we get the self-adjoint Laplace operator:

$$-M^{-1} \circ S \in \mathrm{End}(V)$$

Recall

Diffusion:

Using the Laplacian, we can express the heat diffusion PDE:

$$\frac{\partial f^t}{\partial t} = -M^{-1}\left(S(f^t)\right)$$

with $f^t \in V$ the signal after diffusing for time t.

We can solve this PDE using implicit time-stepping:

$$(M + \varepsilon \cdot S)(f^{t+\varepsilon}) = M(f^t)$$

Recall

Diffusion:

Alternatively, setting $f^{t+\varepsilon} = f^t + \delta$, we can solve for the **offset** $\delta \in V$:

$$\frac{(f^t + \delta) - f^t}{\varepsilon} = -M^{-1}\big(S(f^t + \delta)\big)$$
$$\Downarrow$$
$$\delta = -\varepsilon \cdot M^{-1}\big(S(f^t + \delta)\big)$$
$$\Downarrow$$
$$M(\delta) = -\varepsilon \cdot S(f^t + \delta)$$
$$\Downarrow$$
$$M(\delta) + \varepsilon \cdot S(\delta) = -\varepsilon \cdot S(f^t)$$
$$\Downarrow$$
$$(M + \varepsilon \cdot S)(\delta) = -\varepsilon \cdot S(f^t)$$

$f^{t+\varepsilon} = f^t + \delta$

Recall

Gradient-Domain Formulation:

Performing an implicit time-step of the diffusion PDE is equivalent to solving the gradient-domain smoothing minimization:

$$E(f^{t+\varepsilon}) = \|f^{t+\varepsilon} - f^t\|_{\mathcal{M}}^2 + \varepsilon \cdot \|d(f^{t+\varepsilon})\|_{\mathcal{M}}^2$$

This is equivalent to solving for the offset δ minimizing:

$$E(\delta) = \|\delta\|_{\mathcal{M}}^2 + \varepsilon \cdot \|d(f^t + \delta)\|_{\mathcal{M}}^2$$

Outline

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Normal Smoothing

$$V \xrightarrow{L} W$$

Linear Algebra Review

Recall:

Given vector spaces V and W, and given a linear map $L \in \mathrm{Hom}(V, W)$, the *kernel* of L is the subspace of V that gets mapped to zero:

$$\mathrm{Ker}(L) = \{v \in V | L(v) = 0\}$$

We say that the kernel is *non-trivial*, or that the linear operator is *singular*, if there exists a non-zero $v \in V$ in the kernel:

$$\mathrm{Ker}(L) \neq \{0\}$$

$$V \xrightarrow{L} W$$

Linear Algebra Review

Definition:

Given vector spaces V and W, and given a linear map $L \in \mathrm{Hom}(V, W)$, the *image* of L is the subset of W that is the output of L:

$$\mathrm{Im}(L) = \{w \in W | \exists v \in V \text{ with } w = L(v)\}$$

Note:

Given $w_1, w_2 \in \mathrm{Im}(L)$, there exist $v_1, v_2 \in L$ s.t. $L(v_1) = w_1$ and $L(v_2) = w_2$.

By linearity, for all $\alpha_1, \alpha_2 \in \mathbb{R}$ we have:

$$L(\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2) = \alpha_1 \cdot w_1 + \alpha_2 \cdot w_2$$
$$\Downarrow$$
$$\alpha_1 \cdot w_1 + \alpha_2 \cdot w_2 \in \mathrm{Im}(L)$$

$\Rightarrow$ The image of a linear map $L: V \to W$ is a subspace of W.

$$U \xrightarrow{L} V \xrightarrow{M} W$$

Linear Algebra Review

Given vector spaces U, V, and W, and given linear maps $L \in \mathrm{Hom}(U,V)$ and $M \in \mathrm{Hom}(V,W)$, a vector $u \in U$ is in the kernel of the composition:

$$u \in \mathrm{Ker}(M \circ L)$$
$$\Updownarrow$$
$$(M \circ L)(u) = 0$$
$$\Updownarrow$$
$$L(u) \in \mathrm{Ker}(M)$$
$$\Updownarrow$$
$$\mathrm{Ker}(M \circ L) = \mathrm{Im}(L) \cap \mathrm{Ker}(M)$$

$$V \xrightarrow{B} V^*$$

Linear Algebra Review

Recall:

Given a linear map $B: V \to V^*$ is *symmetric* if for all $u, v \in V$:

$$[B(u)](v) = [B(v)](u)$$

It is additionally *positive semi-definite* if for all $v \in V$:

$$[B(v)](v) \geq 0$$

$$V \xrightarrow{B} V^*$$

Linear Algebra Review

Claim:

Given a symmetric, positive semi-definite map $B: V \to V^*$, a vector $v \in V$ is in the kernel if and only if:

$$[B(v)](v) = 0$$

Equivalently, a symmetric, positive semi-definite map $B: V \to V^*$ is non-singular if and only $B: V \to V^*$ is definite.

$$V \xrightarrow{B} V^*$$

Linear Algebra Review

Proof ($v \in \mathrm{Ker}(B) \Rightarrow [B(v)](v) = 0$):

Since $v \in \mathrm{Ker}(B)$, we have:

$$\begin{aligned} [B(v)](v) &= 0(v) \\ &= 0 \end{aligned}$$

$$V \xrightarrow{B} V^*$$

Linear Algebra Review

Proof ($v \in \text{Ker}(B) \Leftarrow [B(v)](v) = 0$):

Assume $v \notin \text{Ker}(B)$.

$\Rightarrow$ There exists $w \in V$ such that $[B(v)](w) \neq 0$.

Since $B: V \to V^*$ is symmetric, positive semi-definite, for all $\alpha \in \mathbb{R}$ we have:

$$\begin{aligned} 0 &\leq [B(w + \alpha \cdot v)](w + \alpha \cdot v) \\ &= [B(w)](w) + [B(\alpha \cdot v)](\alpha \cdot v) + [B(\alpha \cdot v)](w) + [B(w)](\alpha \cdot v) \\ &= [B(w)](w) + \alpha^2 \cdot \cancel{[B(v)](v)} + \alpha \cdot [B(v)](w) + \alpha \cdot [B(w)](v) \\ &= [B(w)](w) + 2 \cdot \alpha \cdot [B(v)](w) \end{aligned}$$

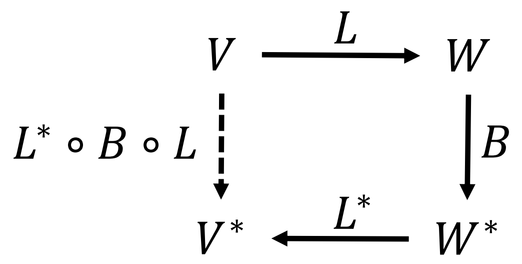
Since $[B(v)](w) \neq 0$, can choose $\alpha \in \mathbb{R}$ s.t. $\alpha \cdot [B(v)](w) \ll 0$.

$\Rightarrow$ We can choose $\alpha \in \mathbb{R}$ such that:

$$\begin{aligned} 0 &> [B(w)](w) + 2 \cdot \alpha \cdot [B(v)](w) \\ &= [B(w + \alpha \cdot v)](w + \alpha \cdot v) \end{aligned}$$

$\Rightarrow \Leftarrow$

Linear Algebra Review



Recall:

Given vector spaces V and W, a symmetric, positive **semi**-definite map $B: W \to W^*$, and a linear map $L \in \mathrm{Hom}(V, W)$, we can pull back B to get a symmetric, positive **semi**-definite bilinear map on V:

$$L^* \circ B \circ L: V \to V^*$$

Corollary:

If the image of L is not in the kernel of B:

$$\mathrm{Im}(L) \cap \mathrm{Ker}(B) = \{0\}$$
$$\Downarrow$$
$$\big[B\big(L(v)\big)\big]\big(L(v)\big) \neq 0, \qquad \forall v \in V, v \neq 0$$

$\Rightarrow$ The pull-back $L^* \circ B \circ L: V \to V^*$ is **strictly** positive definite.

Linear Algebra Review

Definition:

Given a vector space V, we denote by $V^{\oplus K}$ the space of K-tuples of vectors:

$$V^{\oplus K} \equiv \{[v_1, \dots, v_K] | v_i \in V\}$$

This is a vector space with:

$$\alpha \cdot [v_1, \dots, v_K] \equiv [\alpha \cdot v_1, \dots, \alpha \cdot v_K]$$

and:

$$[v_1, \dots, v_K] + [w_1, \dots, w_K] \equiv [v_1 + w_1, \dots, v_K + w_K]$$

Linear Algebra Review

Definition:

Given a vector space V, we denote by $V^{\oplus K}$ the space of K-tuples of vectors:

$$V^{\oplus K} \equiv \{[v_1, \dots, v_K] | v_i \in V\}$$

For example, we can define the vector space of K-tuples of dual vectors:

$$(V^*)^{\oplus K} \equiv \{[l_1, \dots, l_K] | l_k \in V^*\}$$

⇒ A vector $(l_1, \dots, l_K) \in (V^*)^{\oplus K}$ can be interpreted as a linear map on $V^{\otimes K}$:

$$[l_1, \dots, l_K]([v_1, \dots, v_K]) \equiv l_1(v_1) + \cdots + L_K(v_K)$$

This defines a (canonical) isomorphism:

$$(V^*)^{\oplus K} \simeq \left(V^{\oplus K}\right)^*$$

Linear Algebra Review

Definition:

Given vector spaces V and W, and a linear map $L \in \mathrm{Hom}(V, W)$, we have and induced map $L^{\oplus K} \in \mathrm{Hom}(V^{\oplus K}, W^{\oplus K})$:

$$L^{\oplus K}([v_1, \ldots, v_K]) \equiv [L(v_1), \ldots, L(v_K)]$$

$\Rightarrow$ In particular, given an inner-product space $\{V, B: V \to V^*\}$, we have an induced linear map:

$$B^{\oplus K}: V^{\oplus K} \to (V^*)^{\oplus} \simeq \left(V^{\oplus K}\right)^*$$

Claim:

This is symmetric and positive definite, defining an inner-product on $V^{\oplus K}$.

Linear Algebra Review

Definition:

More generally, given vector spaces V and W, and given linear maps $L_1, \dots, L_K \in \mathrm{Hom}(V, W)$, we have and induced map:

$$L_1 \oplus \cdots \oplus L_K : V^{\oplus K} \to W^{\oplus K}$$
$$[v_1, \dots, v_K] \mapsto [L_1(v_1), \dots, L_K(v_K)]$$

Still more generally, given vectors spaces $V_1, \dots, V_K$ we can define:

$$V_1 \oplus \cdots \oplus V_K \equiv \{[v_1, \dots, v_K] | v_i \in V_i\}$$

And still more generally, given vectors spaces $V_1, \dots, V_K$ and $W_1, \dots, W_K$, and given linear maps $L_i \in \mathrm{Hom}(V_i, W_i)$, we have the induced map:

$$L_1 \oplus \cdots \oplus L_K : V_1 \oplus \cdots \oplus V_K \to W_1 \oplus \cdots \oplus W_K$$
$$[v_1, \dots, v_K] \mapsto [L_1(v_1), \dots, L_K(v_K)]$$

Linear Algebra Review

Note:

Given a vector space V with basis $\{b_1, \dots, b_n\}$, we can define an associated basis for $V^{\oplus K}$:

$$
\begin{gathered}
[b_1, 0,0, \dots, 0,0] \\
[b_2, 0,0, \dots, 0,0] \\
\vdots \\
[b_n, 0,0, \dots, 0,0] \\
[0, b_1, 0, \dots, 0,0] \\
[0, b_2, 0, \dots, 0,0] \\
\vdots \\
[0, b_n, 0, \dots, 0,0] \\
\vdots \\
[0,0,0, \dots, 0, b_n]
\end{gathered}
$$

Linear Algebra Review

Recall:

Given a vector space V with basis $\{b_1, \ldots, b_n\}$, we can express a vector $v \in V$ as a column vector $\mathbf{v} \in \mathbb{R}^n$ w.r.t. the basis.

$\Rightarrow$ Given a vector $[v_1, \ldots, v_K] \in V^{\otimes K}$, we can represent each $v_i \in V$ by a column vector $\mathbf{v}_i \in \mathbb{R}^n$ w.r.t. the basis

The associated expression for the vector w.r.t. the associated basis for $V^{\oplus K}$ is the concatenated column vector $\mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_K \in \mathbb{R}^{n \cdot K}$:

$$\mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_K \equiv \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{K-1} \\ \mathbf{v}_K \end{pmatrix}$$

Linear Algebra Review

Recall:

Given vector spaces V and W with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$, we can express a map $L \in \mathrm{Hom}(V, W)$ as a matrix $\mathbf{L} \in \mathbb{R}^{m \times n}$ w.r.t. the bases.

$\Rightarrow$ With respect to the bases the linear map $L^{\oplus K} \in \mathrm{Hom}(V^{\oplus \mathrm{K}}, W^{\oplus K})$ is expressed by the block-diagonal matrix $\mathbf{L}^{\oplus K} \in \mathbb{R}^{m \cdot K \times n \cdot K}$:

$$\mathbf{L}^{\oplus K} = \underbrace{\begin{pmatrix} \mathbf{L} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{L} \end{pmatrix}}_{K \text{ times}}$$

Linear Algebra Review

Multi-Channel Diffusion:

If we are given a K-channel signal (e.g. color, with red, green, and blue components), we can diffuse the channels independently:

$$(\mathbf{M} + \varepsilon \cdot \mathbf{S})(\mathbf{f}_k^{t+\varepsilon}) = \mathbf{M}(\mathbf{f}_k^t) \quad \forall\, 1 \leq k \leq K$$

with $\mathbf{M}, \mathbf{S} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ the mass and stiffness matrices, and $\mathbf{f}_k^t \in \mathbb{R}^{|\mathcal{V}|}$ the k-th channel of the signal at time t, expressed in terms of the hat-basis.

Or, we can diffuse the channels simultaneously by concatenating:

$$\left(\mathbf{M}^{\otimes K} + \varepsilon \cdot \mathbf{S}^{\otimes K}\right)(\mathbf{f}_1^{t+\varepsilon} \oplus \cdots \oplus \mathbf{f}_K^{t+\varepsilon}) = \mathbf{M}^{\otimes K}(\mathbf{f}_1^t \oplus \cdots \oplus \mathbf{f}_k^t)$$

Note:
In practice, you do not want to solve simultaneously because the system matrix is larger (even if the symbolic factorization can figure out that it's K separate systems).

Outline

Recall

Linear Algebra Review

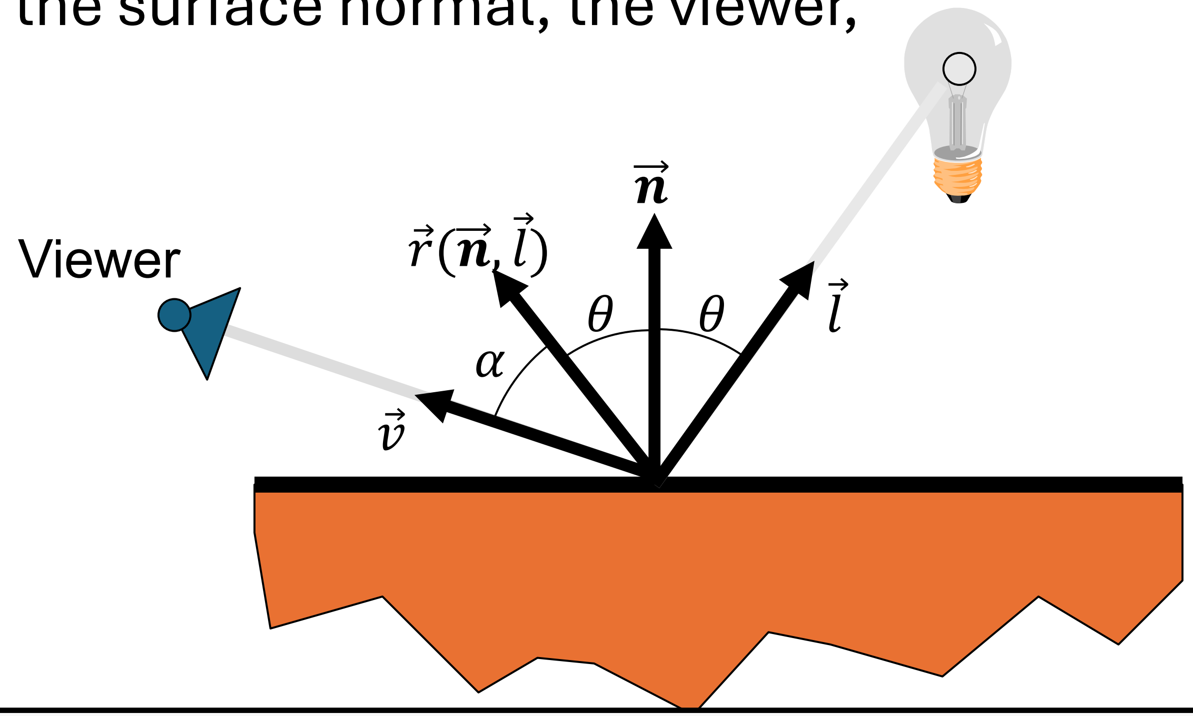
Phong Rendering

Normal Smoothing

Rendering

Shading:

When rendering triangle meshes, the shading at a point on a mesh is determined by the alignment of the surface normal, the viewer, and the incoming light.



$$I = K_E + K_A \cdot I_L^A + \left(K_D \cdot \langle \vec{\boldsymbol{n}}, \vec{l} \rangle + K_S \cdot \langle \vec{v}, \vec{r}(\vec{\boldsymbol{n}}, \vec{l}) \rangle^{K_n} \right) \cdot I_L$$

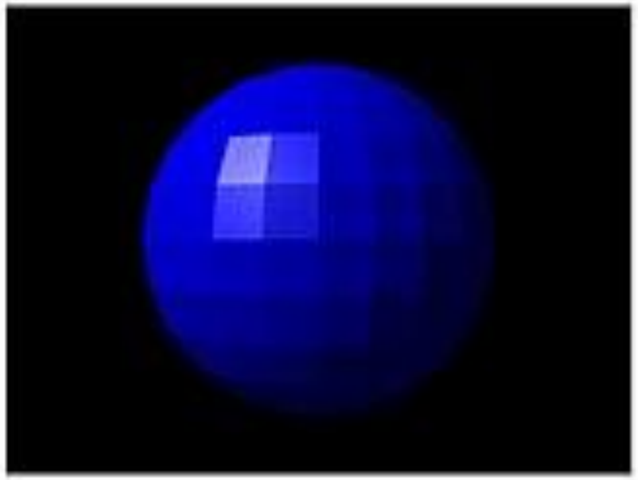
Rendering

Naively:

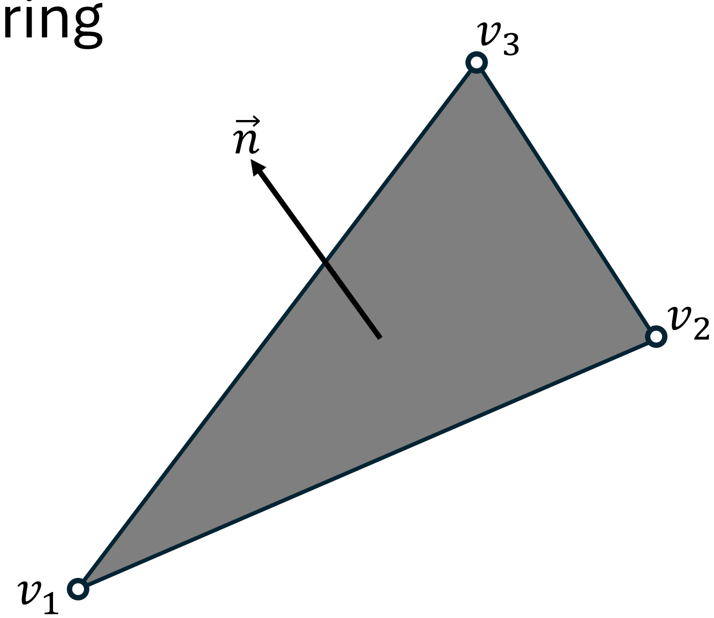
Given a triangle, $\{v_1, v_2, v_3\}$, assign the **same** normal to every point:

$$\vec{n} = \frac{(v_2 - v_1) \times (v_3 - v_1)}{\|(v_2 - v_1) \times (v_3 - v_1)\|}$$

- ✗ For coarse triangulations this results in a faceted rendering



https://en.wikipedia.org/wiki/Phong_shading



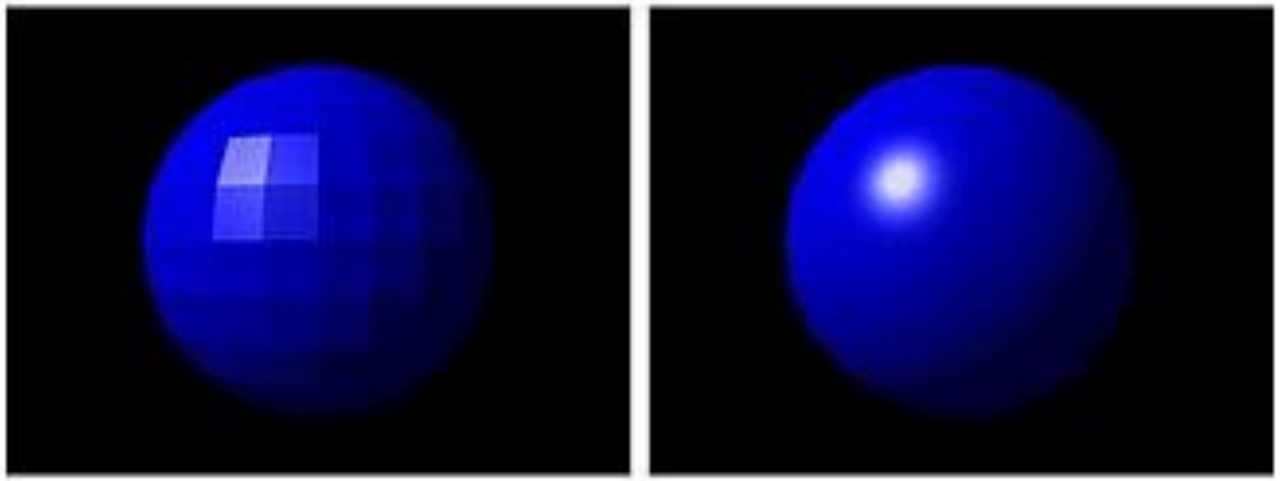
Rendering

Phong:

Given normals per-vertex, **interpolate** (and normalize) from the corners:

$$\vec{n}(s,t) = \frac{(1-s)\cdot\vec{n}_1 + s\cdot\vec{n}_2 + t\cdot\vec{n}_3}{\|(1-s)\cdot\vec{n}_1 + s\cdot\vec{n}_2 + t\cdot\vec{n}_3\|}$$

✓For coarse triangulations this creates a smooth rendering



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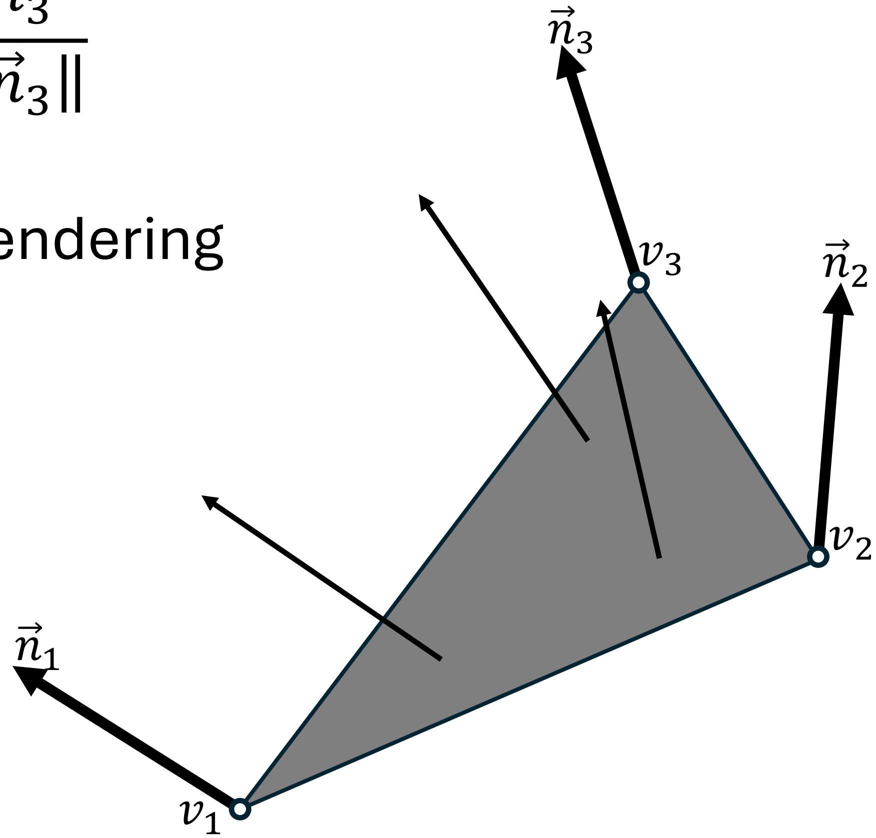

Rendering

Phong:

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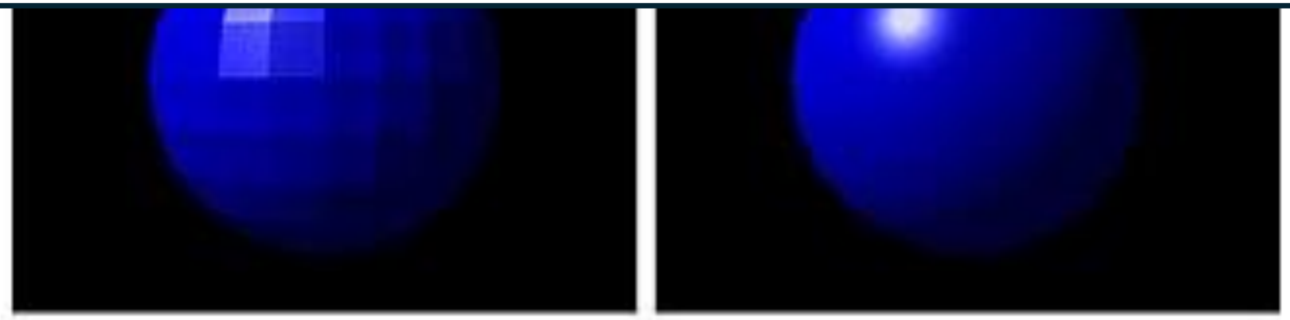
✓For coarse triangulations this creates a smooth rendering

If per-vertex normals are not given, we can use the (area-weighted) sum of the normals of incident triangles.



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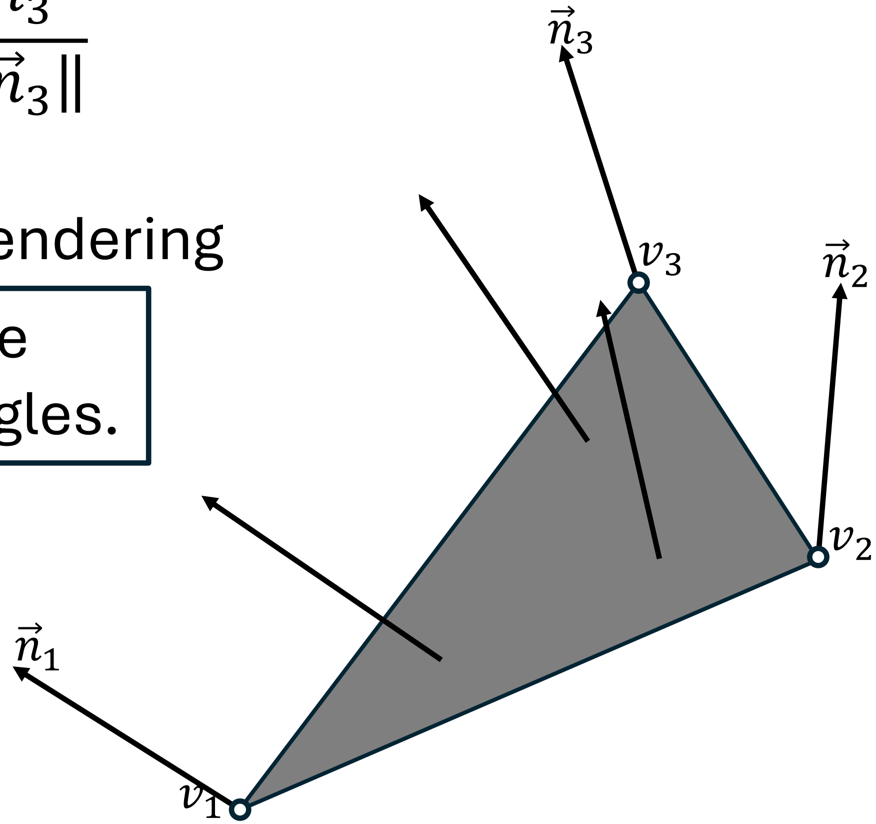


Outline

Recall

Linear Algebra Review

Phong Rendering

Normal Smoothing

Normal Smoothing

Challenge:

Given per-vertex normals, smooth the normals.

Notation:

We will represent normals and normal offsets as functions in $V^{\oplus 3}$ – the space of the 3-tuple-valued (a.k.a. $\mathbb{R}^3$-valued) functions on the mesh.

E.g. writing $N = [N_x, N_y, N_z] \in V^{\oplus 3}$, we refer to the function assigning a vector in $\mathbb{R}^3$ to every point on the mesh, having x-, y-, and z-coefficients $N_x \in V$, $N_y \in V$, and $N_z \in V$, respectively.

Normal Smoothing

Naïve:

At each vertex we have the x-, y-, and z-coefficients of the normals.

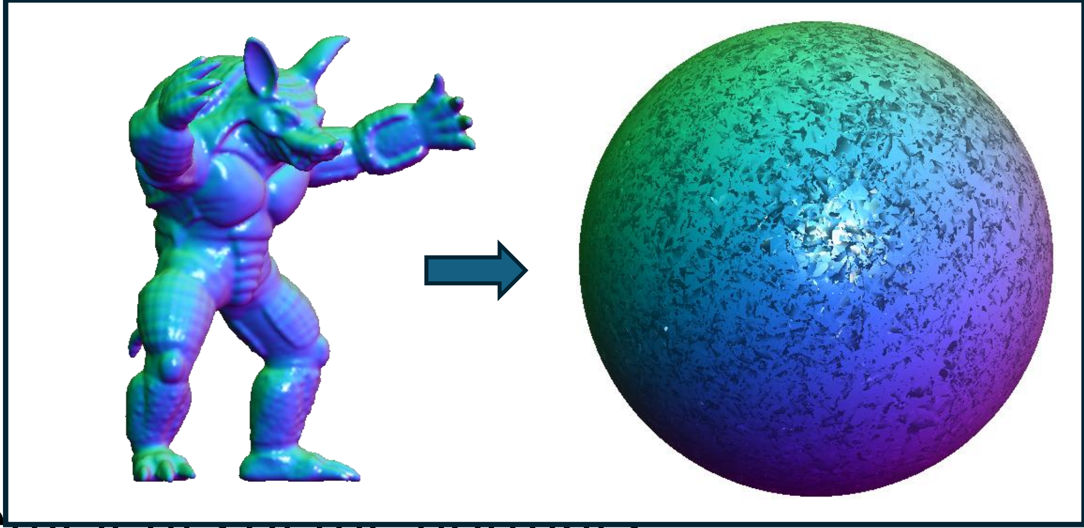
Smooth those independently.

✘ The result may no longer represent a unit-norm vector at each vertex

✓ Normalize

Note:
This is still not quite right, because the evaluation is only unit-norm at the vertices.

Normal Smoothing



Naïve:

At each vertex we have the x-, y-, and z-coefficients of the normals.

Smooth those independently.

✘ The result may no longer represent a unit-norm vector at each vertex

✓ Normalize

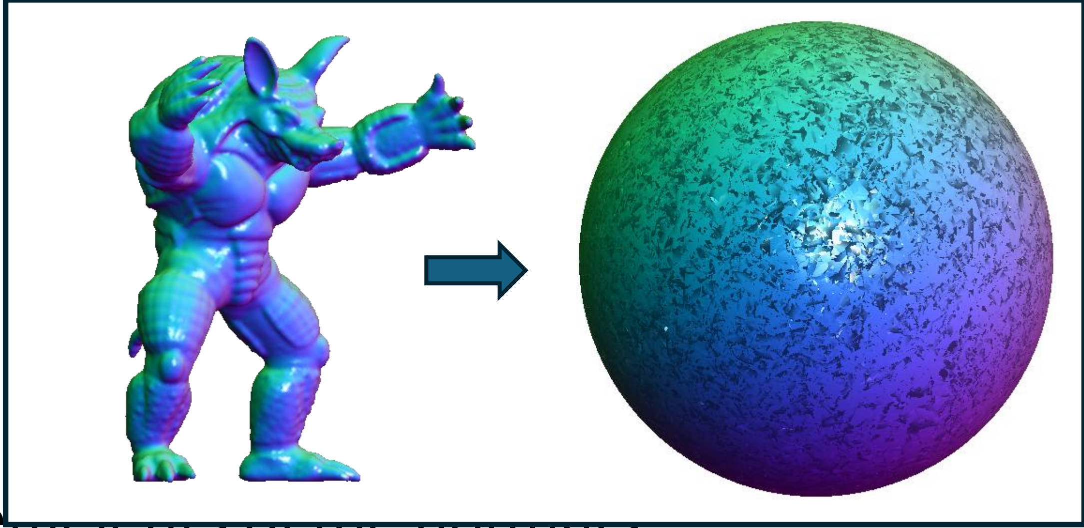
Visualization:

Assign a color to each vertex based on the normal direction

Render the sphere using

- Vertex positions defined by the (evolving) normals
- The triangulation of the input
- Colors defined by the initial assignment

Normal Smoothing



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Visualization:

Assign a color to each vertex based on the normal direction

Render the sphere using

Vertex posi

The triangu

Colors defi

Initially get a continuous color assignment
(speckling results from flipped triangles).

Normal Smoothing

Naïve:

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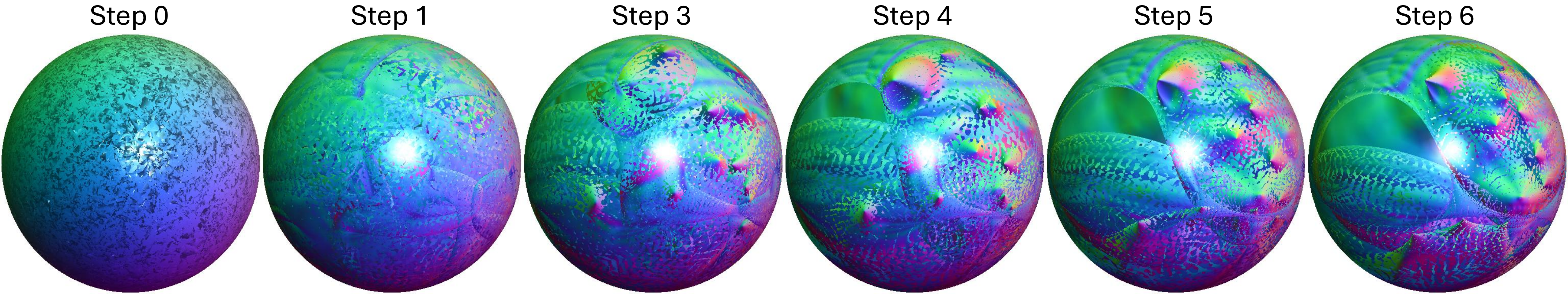

$\boldsymbol{\varepsilon = 10^{-3}}$

Normal Smoothing

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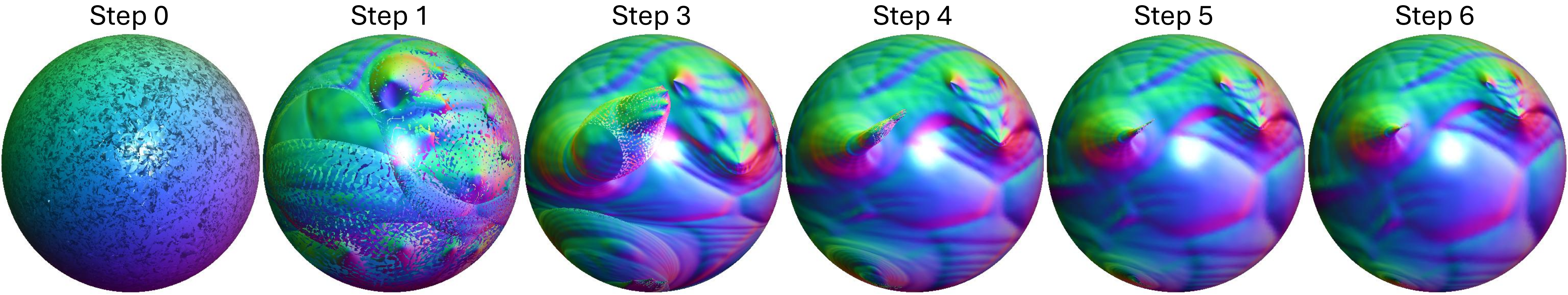
✓ Normalize



$\varepsilon = 10^{-2}$

Normal Smoothing

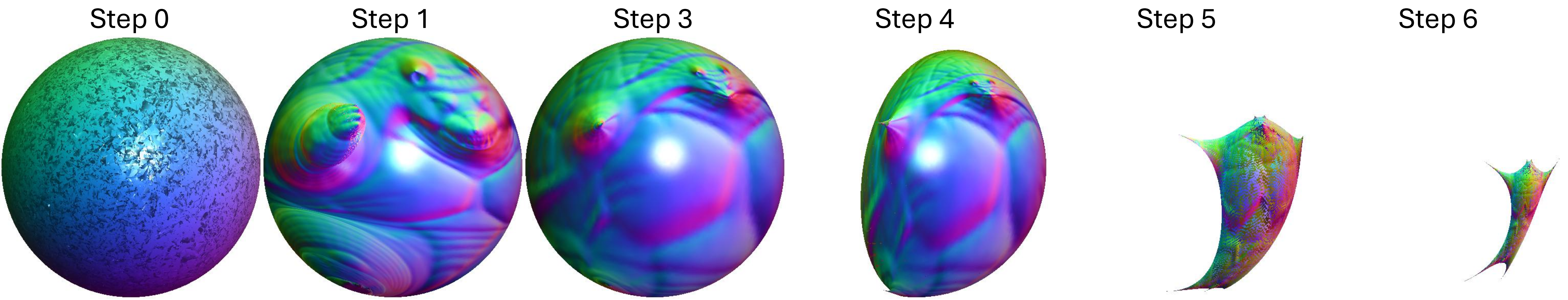
Naïve:

At each vertex we have the x-, y-, and z-coefficients of the normals.

Smooth those independently.

✗ The result may no longer represent a unit-norm vector at each vertex

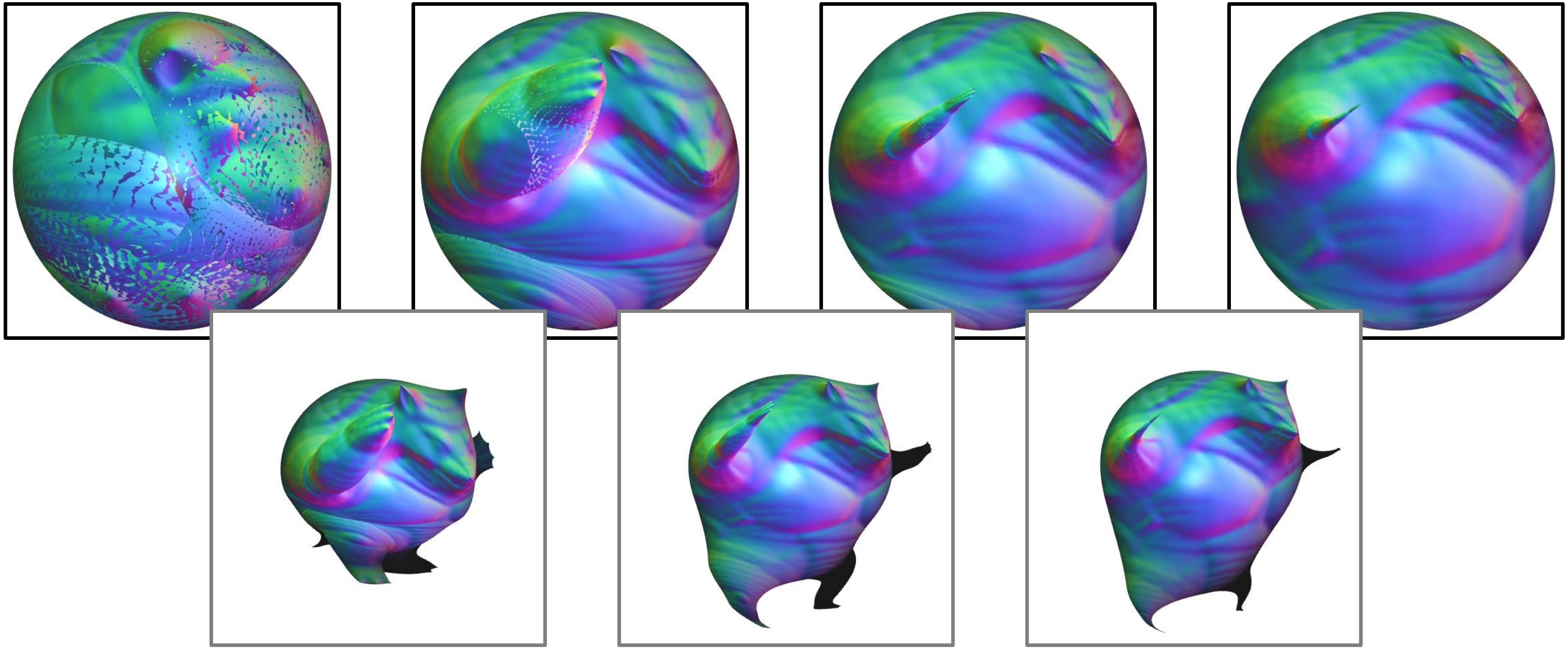
✓ Normalize



$$\boldsymbol{\varepsilon = 10^{-1}}$$

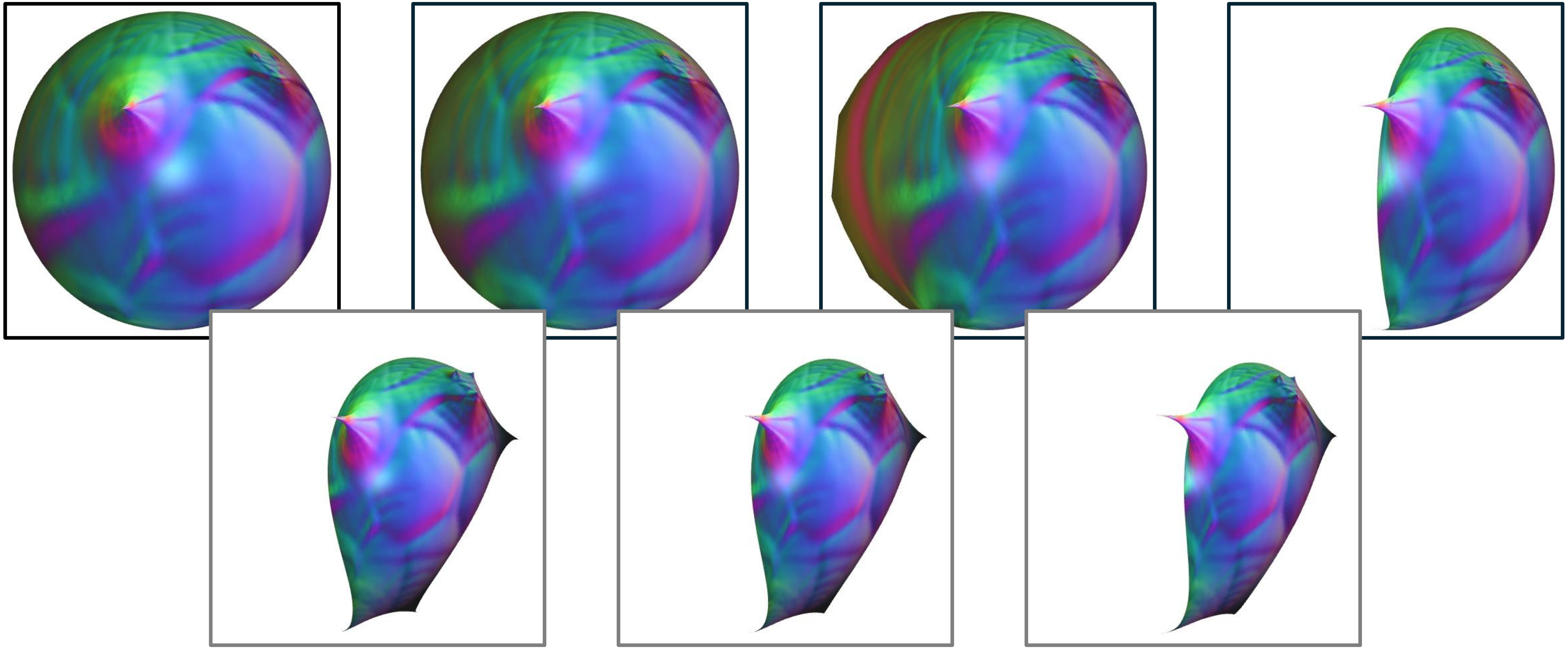
Normal Smoothing

Separating Smoothing and Normalization ($\varepsilon = 10^{-2}$; steps 1-4):



Normal Smoothing

Separating Smoothing and Normalization ($\varepsilon = 10^{-2}$; steps 10-13):



Normal Smoothing

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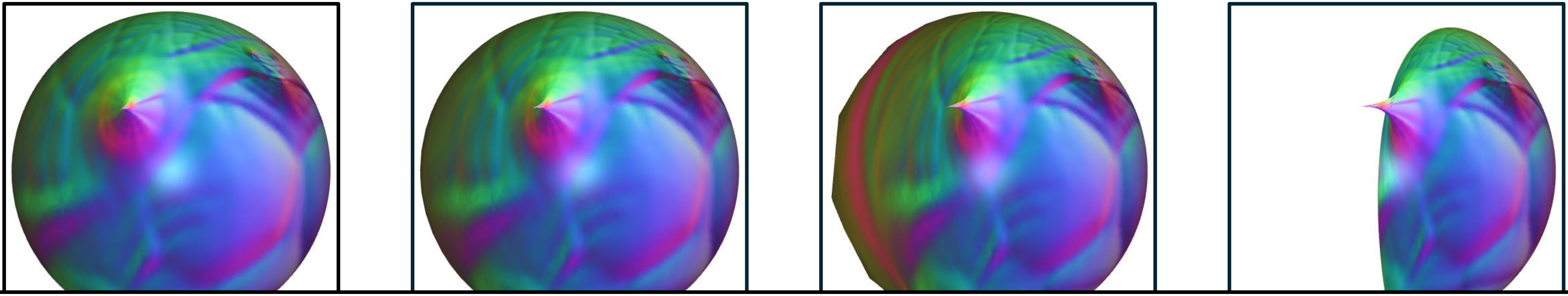
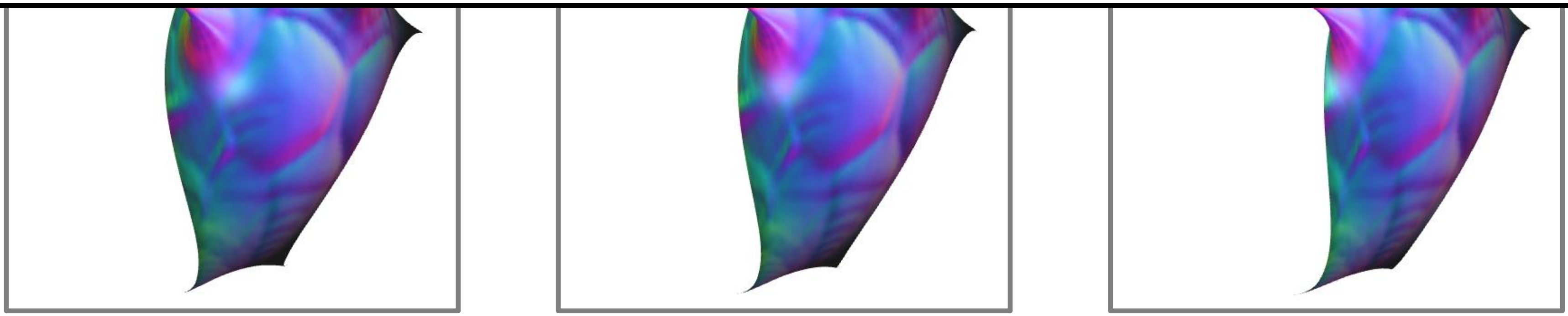


Smoothing coefficients moves normals off the unit-sphere, pulling them to the center.

⇒ Mitigate by restricting the smoothing to move normals **along** the sphere.

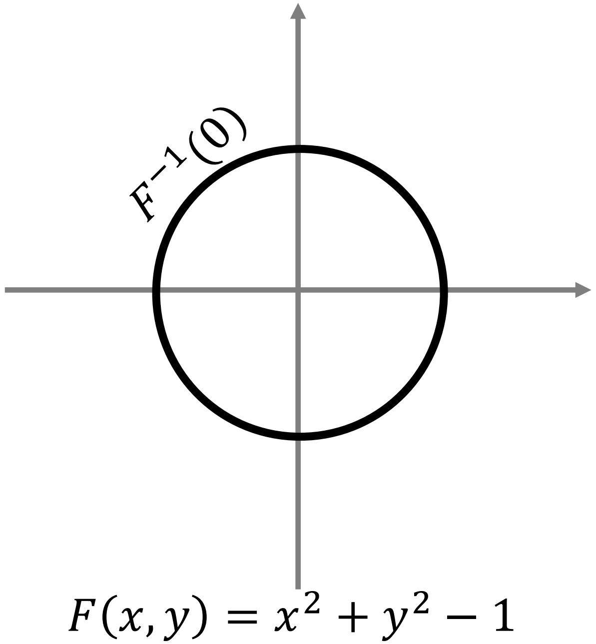


Normal Smoothing

Recall:

Given an inner-product space $\{V, B: V \to V^*\}$, the *unit-sphere in V* is the zero level-set of the function:

$$F: V \to \mathbb{R}$$
$$v \mapsto \|v\|_B^2 - 1$$

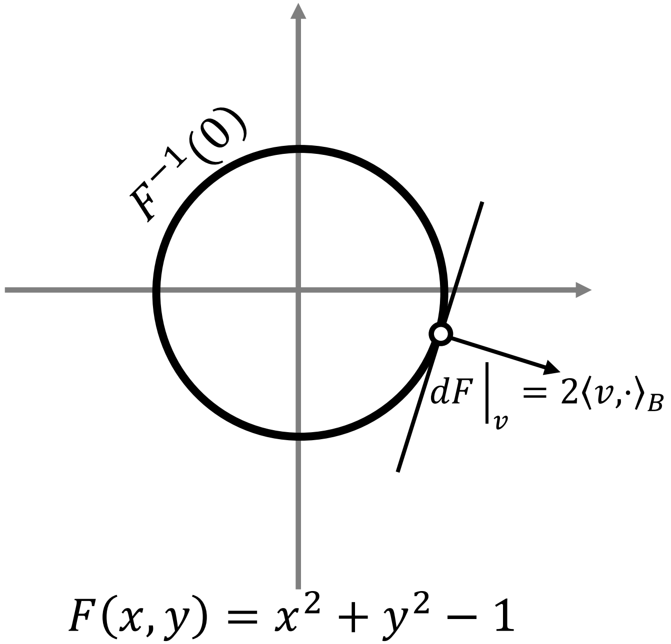


Normal Smoothing

Recall:

Given an inner-product space $\{V, B: V \to V^*\}$, the *tangent space at* a vector v on the unit-sphere in V is the subspace of V consisting of directions along which $F(v) = \|v\|_B^2 - 1$ does not change (locally):

$$\begin{aligned} T_v F^{-1}(0) &= \mathrm{Ker}\left(dF\Big|_v\right) \\ &= \{w \in V \mid \langle w, v\rangle_B = 0\} \end{aligned}$$



In Our Context:

The normals live on the unit-sphere in $\mathbb{R}^3$.

$\Rightarrow$ To have the normals evolve **along the unit-sphere**, we need to constrain the smoothing to move normals along the (two-dimensional) tangent space.

Constrained Normal Smoothing

Approach:

For $\iota \in \{x, y, z\}$, let $N_\iota^t \in V$ be the ι-coefficients of the normals at time t.

1. Solve for the offsets $\delta_\iota \in V$ giving the new normal coefficients in terms of the old:
$$N_\iota^{t+\varepsilon} = N_\iota^t + \delta_\iota$$
2. Let $\delta = [\delta_x, \delta_y, \delta_z] \in V^{\oplus 3}$ be the $\mathbb{R}^3$-valued function giving the offsets. For every $v \in \mathcal{V}$, constrain $\delta(v)$ to be in the tangent space of v.

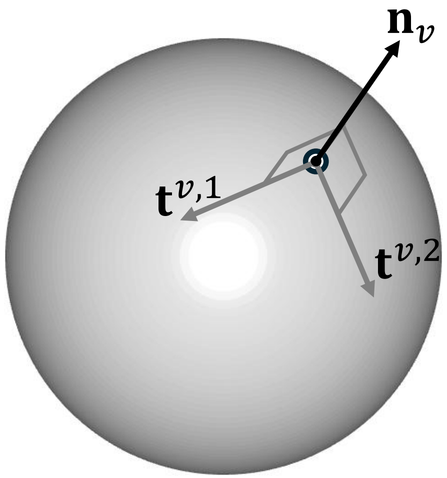
Constrained Normal Smoothing

Approach :

For every vertex $v \in \mathcal{V}$, let $\mathbf{n}_v$ be the (current) normal at v.

Choose two tangent vectors, $\mathbf{t}^{v,1}, \mathbf{t}^{v,2} \in \mathbb{R}^3$, perpendicular to normal $\mathbf{n}_v$.
(These are the directions along which we allow the normal at v to change.)

Constrain $\delta = [\delta_x, \delta_y, \delta_z] \in V^{\oplus 3}$ so that at vertex v, the cumulative offset $\delta(v) \in \mathbb{R}^3$ is in the space spanned by $\mathbf{t}^{v,1}$ and $\mathbf{t}^{v,2}$.



Constrained Normal Smoothing

Approach:

For a vertex $v \in \mathcal{V}$ we can define a map taking a pair of real values and returning a tangent vector at v:

$$\{\alpha_v^1, \alpha_v^2\} \mapsto \alpha_v^1 \cdot \mathbf{t}^{v,1} + \alpha_v^2 \cdot \mathbf{t}^{v,2}$$

Any tangent vector at v is the image of some pair of values $\{\alpha_v^1, \alpha_v^2\}$.

⇒ Defining a pair of values at each vertex $v \in \mathcal{V}$, is equivalent to assigning a tangent vector to every vertex.

⇒ Using the hat-basis, we can extend the per-vertex tangent vectors to a function over the mesh.

Constrained Normal Smoothing

Implementation:

Define the linear maps P_ι taking a pair of values at every vertex and returning the ι-th coefficient of the tangent vector:

$$P_\iota : (\mathbb{R}^2)^{\oplus|\mathcal{V}|} \to V$$

$$\left[\{\alpha_1^1, \alpha_1^2\}^\top, \dots, \{\alpha_{|\mathcal{V}|}^1, \alpha_{|\mathcal{V}|}^2\}^\top\right] \mapsto \sum_{v\in\mathcal{V}} (\alpha_v^1 \cdot \mathbf{t}_\iota^{v,1} + \alpha_v^2 \cdot \mathbf{t}_\iota^{v,2}) \cdot \phi_v$$

Combining, we get the piecewise linear function:

$$P = \left[P_x, P_y, P_z\right] \in V^{\oplus 3}$$

whose value at vertex $v \in \mathcal{V}$ is the tangent vector:

$$\alpha_v^1 \cdot \mathbf{t}^{v,1} + \alpha_v^2 \cdot \mathbf{t}^{v,2}$$

Constrained Normal Smoothing

Implementation:

With respect to the cartesian basis for $\mathbb{R}^2$ and the hat basis for V, the linear map $P \in \mathrm{Hom}\left((\mathbb{R}^2)^{\oplus|\mathcal{V}|}, V^{\oplus 3}\right)$ is represented by the matrix $\mathbf{P} \in \mathbb{R}^{3|\mathcal{V}|\times 2|\mathcal{V}|}$:

$$\mathbf{P} = \begin{pmatrix} \mathbf{t}_x^{1,1} & \mathbf{t}_x^{1,2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{t}_x^{2,1} & \mathbf{t}_x^{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{t}_x^{|\mathcal{V}|,1} & \mathbf{t}_x^{|\mathcal{V}|,2} \\ \mathbf{t}_y^{1,1} & \mathbf{t}_y^{1,2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{t}_y^{2,1} & \mathbf{t}_y^{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{t}_y^{|\mathcal{V}|,1} & \mathbf{t}_y^{|\mathcal{V}|,2} \\ \mathbf{t}_z^{1,1} & \mathbf{t}_z^{1,2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{t}_z^{2,1} & \mathbf{t}_z^{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{t}_z^{|\mathcal{V}|,1} & \mathbf{t}_z^{|\mathcal{V}|,2} \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} x\text{-coefficient block} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} y\text{-coefficient block} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} z\text{-coefficient block} \end{matrix}$$

Constrained Normal Smoothing

Implementation:

In the unconstrained formulation, given the normals $N^t \in V^{\oplus 3}$ at time t, we solve for the offsets $\delta \in V^{\oplus 3}$ minimizing:

$$E(\delta) = \|\delta\|_{\mathcal{M}}^2 + \varepsilon \cdot \left\| d^{\oplus 3}(N^t + \delta) \right\|_{\mathcal{M}}^2$$

In the constrained formulation, the offset $\delta \in V^{\oplus 3}$ is parameterized by the per-vertex pairs of offsets $\alpha \in (\mathbb{R}^2)^{\oplus |\mathcal{V}|}$:

$$\delta = P(\alpha)$$
$$\Downarrow$$
$$E(\alpha) = \|P(\alpha)\|_{\mathcal{M}}^2 + \varepsilon \cdot \left\| d^{\oplus 3}\big(N^t + P(\alpha)\big) \right\|_{\mathcal{M}}^2$$

Constrained Normal Smoothing

Implementation:

$$
\begin{aligned}
E(\alpha) &= \|P(\alpha)\|_{\mathcal{M}}^2 + \varepsilon\cdot\left\|d^{\oplus 3}\left(N^t+P(\alpha)\right)\right\|_{\mathcal{M}}^2\\
&= \left[M^{\oplus 3}\left(P(\alpha)\right)\right]\left(P(\alpha)\right) + \varepsilon\cdot\left[S^{\oplus 3}\left(N^t+P(\alpha)\right)\right]\left(N^t+P(\alpha)\right)\\
&= \left[M^{\oplus 3}\left(P(\alpha)\right)\right]\left(P(\alpha)\right) + \varepsilon\cdot\left(\left[S^{\oplus 3}\left(P(\alpha)\right)\right]\left(P(\alpha)\right) + 2\cdot\left[S^{\oplus 3}(N^t)\right]\left(P(\alpha)\right)+\cdots\right)\\
&= \left[(P^*\circ M^{\oplus 3}\circ P)(\alpha)\right](\alpha) + \varepsilon\cdot\left(\left[(P^*\circ S^{\oplus 3}\circ P)(\alpha)\right](\alpha) + 2\cdot\left[\left(P^*\circ S^{\oplus 3}\right)(N^t)\right](\alpha)+\cdots\right)\\
&= \left[\left(P^*\circ\left(M^{\oplus 3}+\varepsilon\cdot S^{\oplus 3}\right)\circ P\right)(\alpha)\right](\alpha) + 2\cdot\varepsilon\cdot\left[\left(P^*\circ S^{\oplus 3}\right)(N^t)\right](\alpha)+\cdots
\end{aligned}
$$

⇒ The parameters $\alpha\in(\mathbb{R}^2)^{\oplus|\mathcal{V}|}$ of the offset along the tangent directions giving the normals at the next time-step satisfy:

$$\left(P^*\circ\left(M^{\oplus 3}+\varepsilon\cdot S^{\oplus 3}\right)\circ P\right)(\alpha) = -\varepsilon\cdot\left(P^*\circ S^{\oplus 3}\right)(N^t)$$

Or, with respect to the cartesian basis for $\mathbb{R}^2$ and the hat basis for V, the coefficients of the parameters $\boldsymbol{\alpha}\in\mathbb{R}^{2|\mathcal{V}|}$ are:

$$\boldsymbol{\alpha} = -\varepsilon\cdot\left(\mathbf{P}^\top\cdot\left(\mathbf{M}^{\oplus 3}+\varepsilon\cdot\mathbf{S}^{\oplus 3}\right)\cdot\mathbf{P}\right)^{-1}\cdot\left(\mathbf{P}^\top\cdot\mathbf{S}^{\oplus 3}\right)\cdot\mathbf{N}^t$$

Constrained Normal Smoothing

Implementation:

$$\boldsymbol{\alpha} = -\varepsilon \cdot \left(\mathbf{P}^\top \cdot \left(\mathbf{M}^{\oplus 3} + \varepsilon \cdot \mathbf{S}^{\oplus 3}\right) \cdot \mathbf{P}\right)^{-1} \cdot \left(\mathbf{P}^\top \cdot \mathbf{S}^{\oplus 3}\right) \cdot \mathbf{N}^t$$

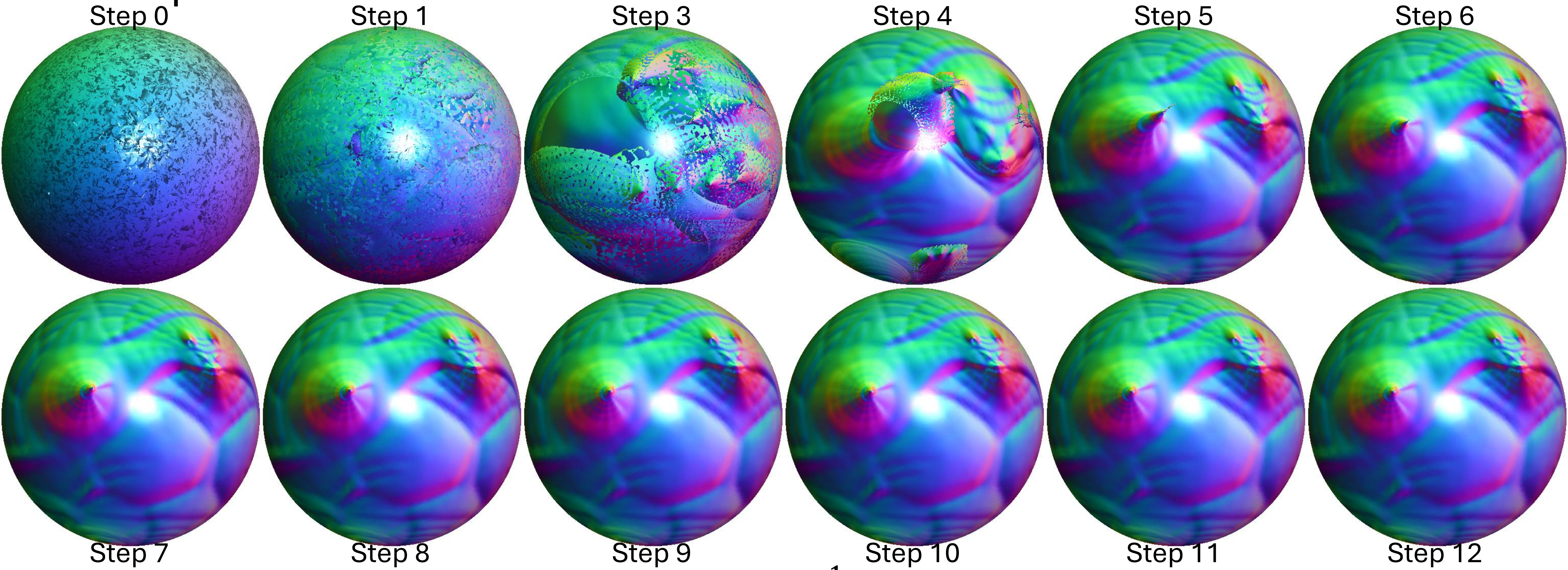
Given the coefficients of the tangential offset parameters $\boldsymbol{\alpha} \in \mathbb{R}^{2|\mathcal{V}|}$:

1. Get the coefficients of the offset $\boldsymbol{\delta} \in \mathbb{R}^{3|\mathcal{V}|}$:
$$\boldsymbol{\delta} = \mathbf{P} \cdot \boldsymbol{\alpha}$$
2. Get the (raw) normals, $\mathbf{N}^{t+\varepsilon} \in \mathbb{R}^{3|\mathcal{V}|}$ at the next time-step:
$$\mathbf{N}^{t+\varepsilon} = \mathbf{N}^t + \boldsymbol{\delta}$$
3. Rescale the per-vertex normals to have unit norm.*

*Could try to do something more fancy, like using the exponential map.

Constrained Normal Smoothing

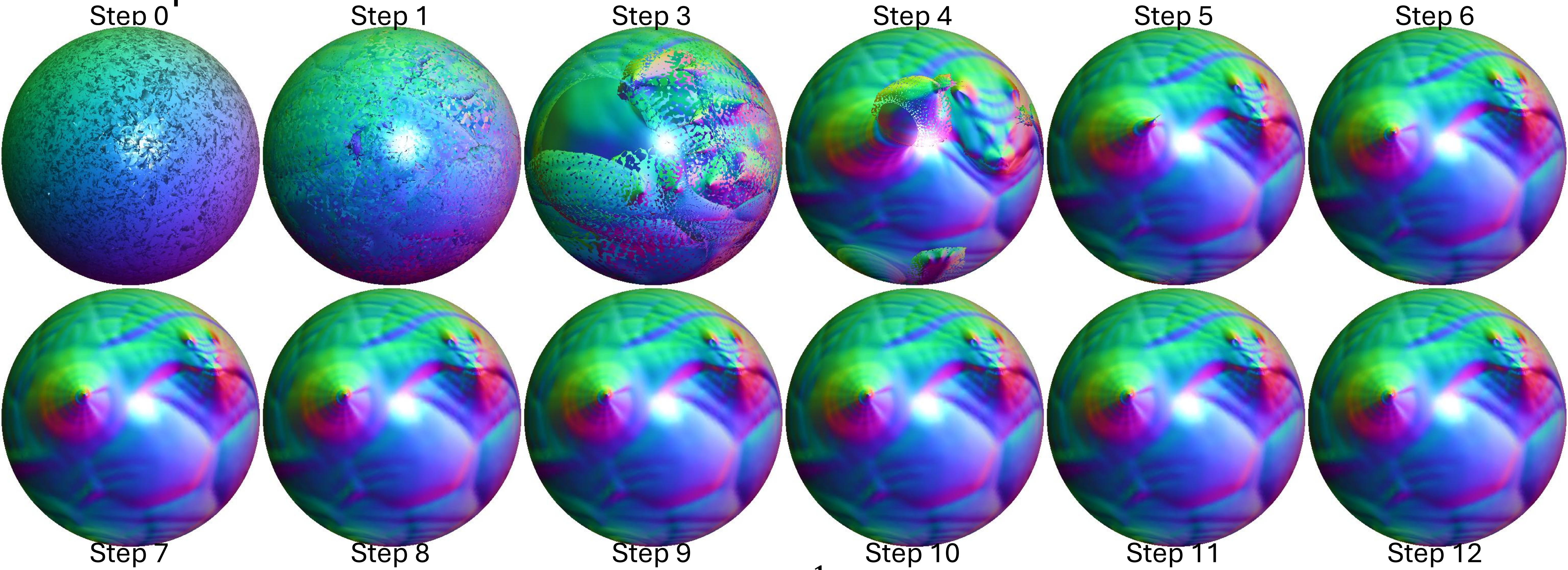
Implementation:



$\boldsymbol{\varepsilon} = \mathbf{10^{-1}}$

Constrained Normal Smoothing

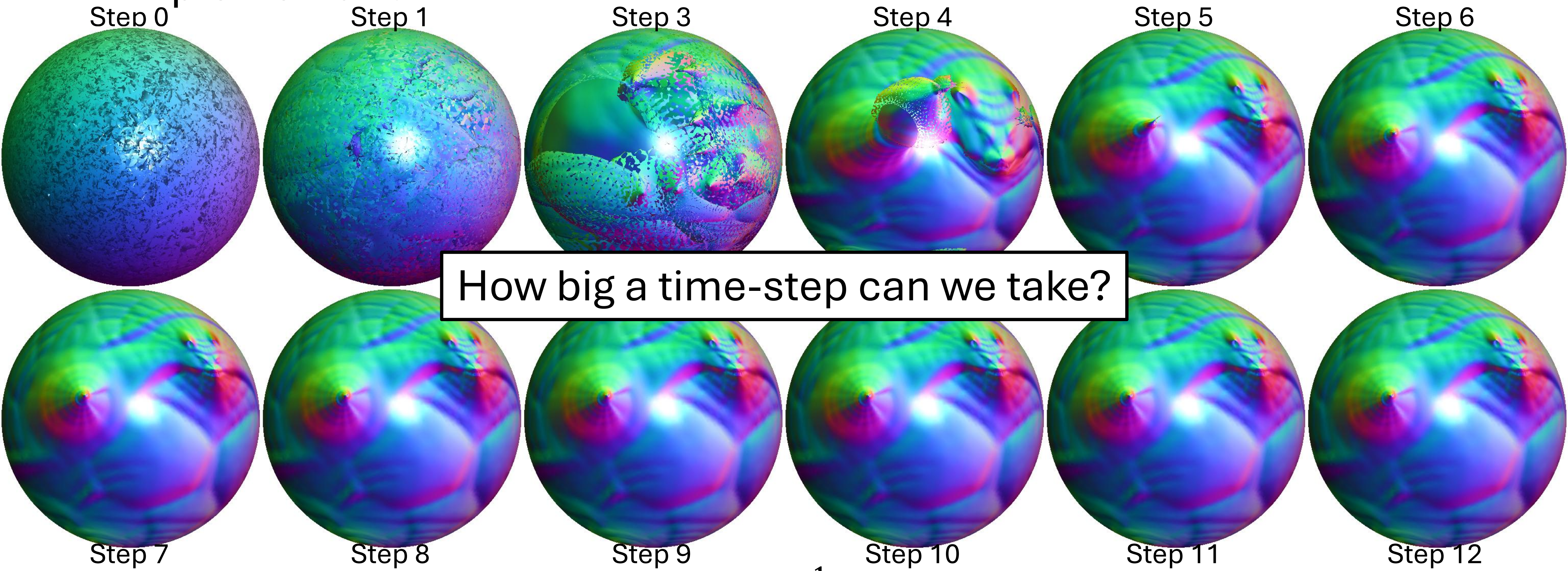
Implementation:



$\boldsymbol{\varepsilon = 10^1}$

Constrained Normal Smoothing

Implementation:



How big a time-step can we take?

$\boldsymbol{\varepsilon = 10^1}$

Unconstrained vs. Constrained Smoothing

Unconstrained Smoothing:

Denoting $\delta = [\delta_x, \delta_y, \delta_z] \in V^{\oplus 3}$ we get the system:

$$\left(M^{\oplus 3} + \varepsilon \cdot S^{\oplus 3}\right)(\delta) = -\varepsilon \cdot S^{\oplus 3}(N^t)$$

Taking the limit as $\varepsilon \to \infty$ gives:

$$S^{\oplus 3}(\delta) = -S^{\oplus 3}(N^t)$$

This is singular because:

$$\mathrm{Ker}\left(S^{\oplus 3}\right) = \left\{[\delta_x, \delta_y, \delta_z] \in V^{\oplus 3} | \delta_\iota = c_\iota, \forall \iota \in \{x, y, z\}\right\}$$

Unconstrained vs. Constrained Smoothing

Constrained Smoothing:

Denoting $\delta = [\delta_x, \delta_y, \delta_z] \in V^{\oplus 3}$ we get the system:

$$\left(P^* \circ \left(M^{\oplus 3} + \varepsilon \cdot S^{\oplus 3}\right) \circ P\right)(\alpha) = -\varepsilon \cdot \left(P^* \circ S^{\oplus 3}\right)(N^t)$$

Taking the limit as $\varepsilon \to \infty$ gives:

$$\left(P^* \circ S^{\oplus 3} \circ P\right)(\alpha) = -\left(P^* \circ S^{\oplus 3}\right)(N^t)$$

Claim:

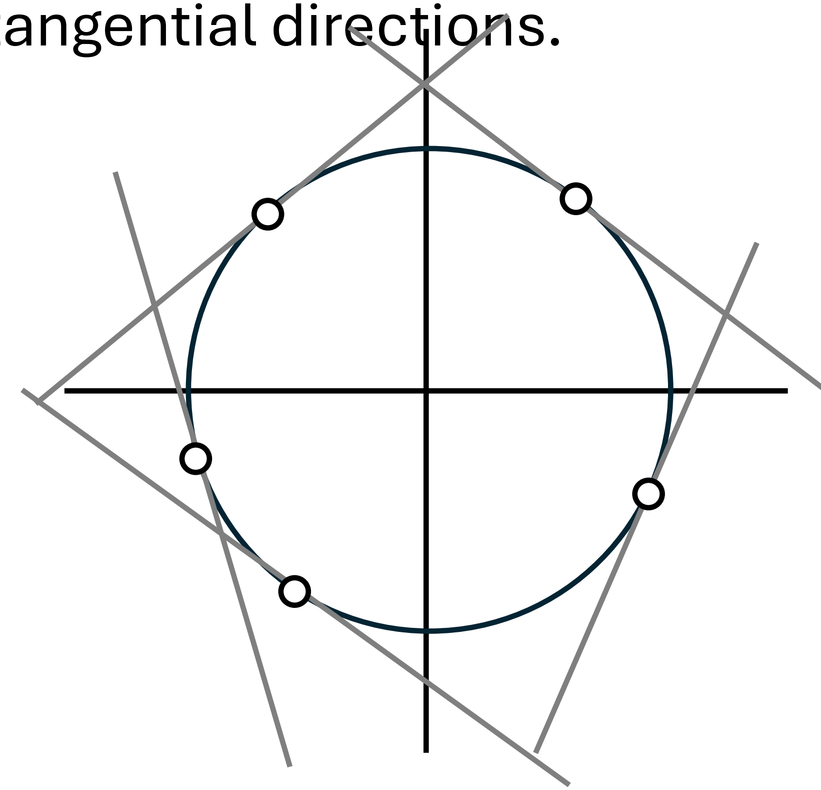
This is not a singular system.

Unconstrained vs. Constrained Smoothing

$$\left(P^* \circ S^{\oplus 3} \circ P\right)(\alpha) = -\left(P^* \circ S^{\oplus 3}\right)(N^t)$$

Proof:

In the constrained system, vertices only move along tangential directions.



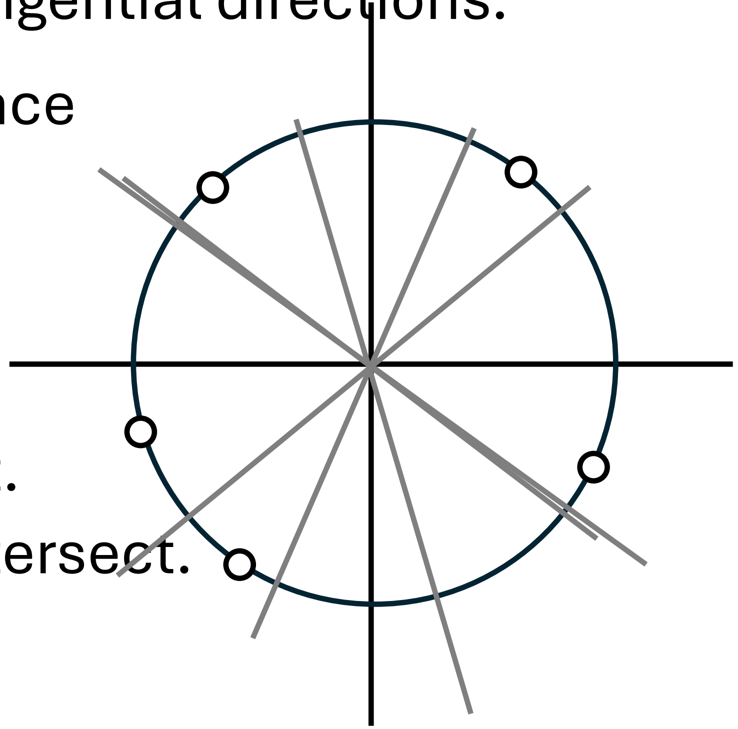
Unconstrained vs. Constrained Smoothing

$$\left(P^* \circ S^{\oplus 3} \circ P\right)(\alpha) = -\left(P^* \circ S^{\oplus 3}\right)(N^t)$$

Proof:

In the constrained system, vertices only move along tangential directions.

⇒ A vertex's offset is restricted to be within the 2D subspace obtained by shifting its tangent plane to the origin.



Note:

An offset is constant if all vertices have the same offset.

A constant offset is a point where the 2D subspaces intersect.

But the planes only intersect at the origin.

⇒ The only constant offset representable as $\delta = P(\alpha)$ is zero.

Unconstrained vs. Constrained Smoothing

$$\left(P^* \circ S^{\oplus 3} \circ P\right)(\alpha) = -\left(P^* \circ S^{\oplus 3}\right)(N^t)$$

Proof:

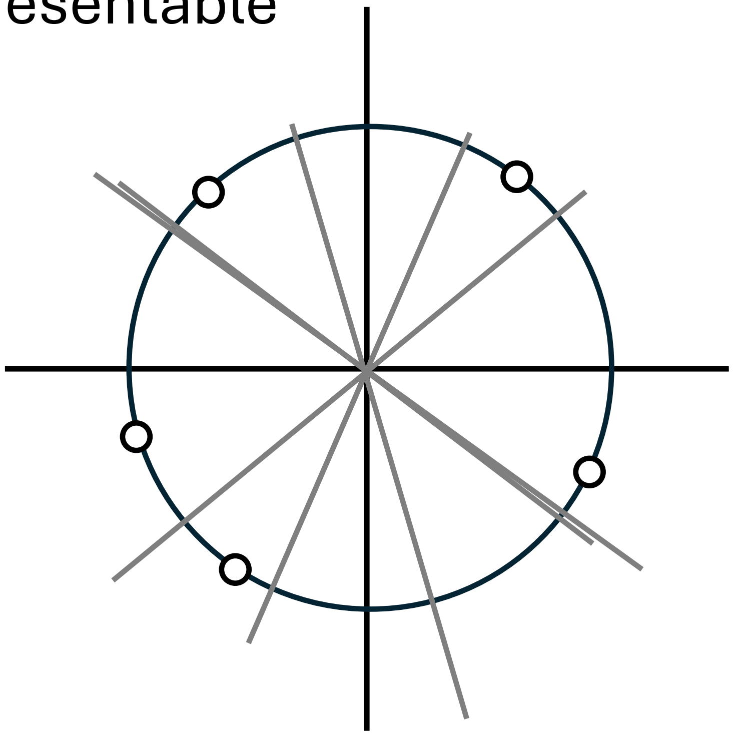
In the constrained system, the only constant offset representable as $\delta = P(\alpha)$ is zero.

⇒ Since the kernel of $S^{\oplus 3}$ are the constant functions:

$$\mathrm{Im}(P) \cap \mathrm{Ker}\left(S^{\oplus 3}\right) = \{0\}$$

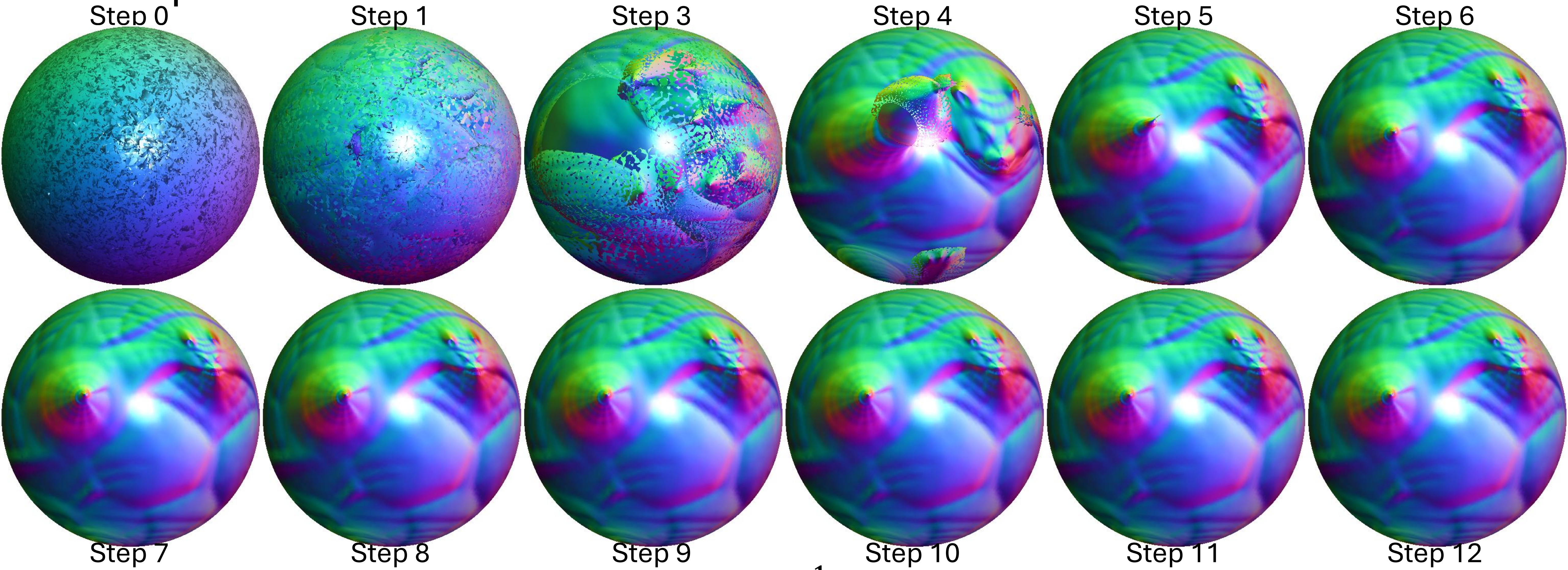
⇒ The bilinear map $P^* \circ S^{\oplus 3} \circ P$ is symmetric, positive definite.

⇒ The system is non-singular.



Constrained Normal Smoothing

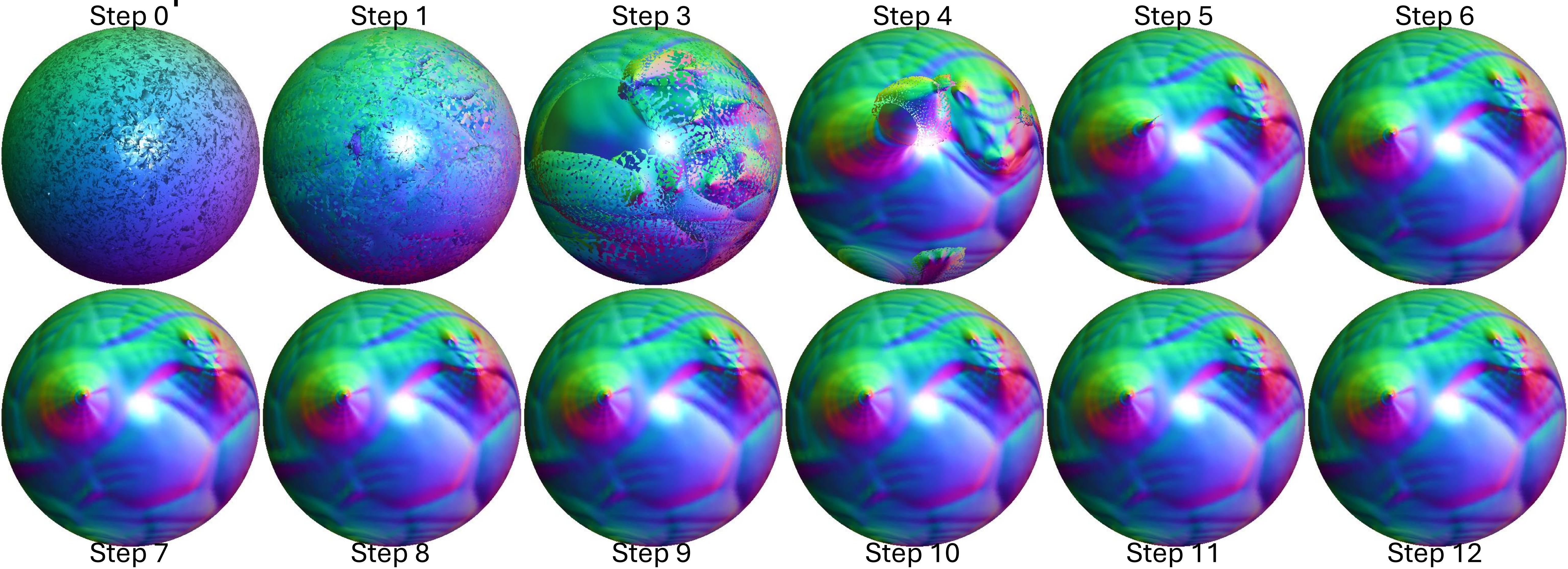
Implementation:



$\boldsymbol{\varepsilon = 10^1}$

Constrained Normal Smoothing

Implementation:



$\boldsymbol{\varepsilon} = \infty$