

Geometry Processing (601.458/658)

Misha Kazhdan

Outline

Recall

Separation of Variables

The Heat and Wave Equation Revisited

Recall

Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we discretize the space of scalar functions, $V = \text{Span}(\{\phi_v\}_{v \in \mathcal{V}})$, using the hat basis.

On the space of scalar functions, we have two bilinear forms:

$M \in \text{Hom}(V, V^*)$: The inner-product on scalar functions:

$$[M(f)](g) = \langle f, g \rangle_{\mathcal{M}}$$

$S \in \text{Hom}(V, V^*)$: The pull-back of the inner-product on cotangent vector-fields:

$$[S(f)](g) = \langle df, dg \rangle_{\mathcal{M}}$$

Combining we get the self-adjoint Laplace operator:

$$-M^{-1} \circ S \in \text{End}(V)$$

Recall

The (negative) Laplacian is self-adjoint, so there is an orthonormal eigenbasis $\{\psi_1, \dots, \psi_{|\mathcal{V}|}\}$, with associated eigenvalues $\{\lambda_1, \dots, \lambda_{|\mathcal{V}|}\}$ satisfying:

$$(M^{-1} \circ S)(\psi_i) = \lambda_i \cdot \psi_i$$
$$\Updownarrow$$
$$S(\psi_i) = \lambda \cdot M(\psi_i)$$

The (non-negative) eigenvalues are (twice) the Dirichlet energy of the associated eigen-functions.

The first eigen-function ψ_1 is the constant function, with eigenvalue $\lambda_1 = 0$.

If the mesh is connected, there are no other eigen-functions with eigenvalue 0.

Recall

Given scalar functions $f, g \in V$, we can express them in terms of their spectral/Fourier decomposition:

$$f(p) = \sum_{i=1}^{|\mathcal{V}|} \hat{\mathbf{f}}_i \cdot \psi_i(p), \quad g(p) = \sum_{i=1}^{|\mathcal{V}|} \hat{\mathbf{g}}_i \cdot \psi_i(p)$$

Because the basis is orthonormal:

$$\begin{aligned} \langle f, g \rangle_{\mathcal{M}} &= \sum_{i=1}^{|\mathcal{V}|} \hat{\mathbf{f}}_i \cdot \hat{\mathbf{g}}_i \\ &\Downarrow \\ \|f\|_{\mathcal{M}}^2 &= \sum_{i=1}^{|\mathcal{V}|} (\hat{\mathbf{f}}_i)^2 \end{aligned}$$

Recall

Using the Laplacian, we have the **heat** diffusion PDE:

$$\frac{\partial f^t}{\partial t} = -M^{-1}(S(f^t))$$

with $f^t \in V$ the signal after diffusing for time t .

We can solve this PDE using implicit time-stepping:

$$(M + \varepsilon \cdot S)(f^{t+\varepsilon}) = M(f^t)$$

Using the spectral/Fourier decomposition:

$$f^t = \hat{\mathbf{f}}_1^t \cdot \psi_1 + \dots + \hat{\mathbf{f}}_{|\mathcal{V}|}^t \cdot \psi_{|\mathcal{V}|}$$

implicit time-stepping acts per-frequency:

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \frac{1}{1 + \varepsilon \cdot \lambda_i} \cdot \hat{\mathbf{f}}_i^t$$

Recall

Using the Laplacian, we have the **wave** equation PDE:

$$\frac{\partial^2 f^t}{\partial t^2} = -M^{-1}(S(f^t))$$

with $f^t \in V$ the signal after propagating for time t .

We can solve this PDE using implicit time-stepping:

$$(M + \varepsilon^2 \cdot S)(f^{t+\varepsilon}) = 2 \cdot M(f^t) - M(f^{t-\varepsilon})$$

Using the spectral/Fourier decomposition:

$$f^t = \hat{\mathbf{f}}_1^t \cdot \psi_1 + \dots + \hat{\mathbf{f}}_{|\mathcal{V}|}^t \cdot \psi_{|\mathcal{V}|}$$

implicit time-stepping acts per-frequency :

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \frac{2 \cdot \hat{\mathbf{f}}_i^t - \hat{\mathbf{f}}_i^{t-\varepsilon}}{1 + \varepsilon^2 \cdot \lambda_i}$$

Outline

Recall

Separation of Variables

The Heat and Wave Equation Revisited

Simple ODEs

Recall:

Given the homogenous, linear, first-order ODE:

$$f' = \lambda \cdot f$$

For every $\lambda \in \mathbb{R}$, the solutions are functions of the form:

$$f(t) = a \cdot e^{\lambda \cdot t}$$

Simple ODEs

Recall:

Given the homogenous, linear, second-order ODE :

$$f'' = \lambda \cdot f$$

For $\lambda > 0$, the solutions are functions of the form:

$$f(t) = a \cdot e^{\sqrt{\lambda} \cdot t} + b \cdot e^{-\sqrt{\lambda} \cdot t}$$

For $\lambda < 0$, the solutions are functions of the form:

$$f(t) = a \cdot \cos(\sqrt{-\lambda} \cdot t) + b \cdot \sin(\sqrt{-\lambda} \cdot t)$$

For $\lambda = 0$, the solutions are functions of the form:

$$f(t) = a + b \cdot t$$

Separation of Variables

Goal:

Solve the homogenous, linear, PDE for $f: \mathbb{R}^{\geq 0} \times \mathbb{R}^d \rightarrow \mathbb{R}$:

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = b_0 \cdot f + \sum_{i=1}^d b_{1,i} \cdot \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^d b_{1,i,j} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} + \dots$$

satisfying initial conditions.

Superposition:

Because the system is linear and homogenous, if $f(t, x_1, \dots, x_d)$ and $g(t, x_1, \dots, x_d)$ are solutions, so is any linear combinations $\alpha \cdot f + \beta \cdot g$.

Separation of Variables

Goal:

Solve the homogenous, linear, PDE for $f: \mathbb{R}^{\geq 0} \times \mathbb{R}^d \rightarrow \mathbb{R}$:

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = b_0 \cdot f + \sum_{i=1}^d b_{1,i} \cdot \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^d b_{1,i,j} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} + \dots$$

satisfying initial conditions.

Denote by D the spatial derivatives on the right-hand-side:

$$D(f) = b_0 \cdot f + \sum_{i=1}^d b_{1,i} \cdot \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^d b_{1,i,j} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} + \dots$$

This is an endomorphism on the/a space of functions.

Separation of Variables

$$D(f) = b_0 \cdot f + \sum_{i=1}^d b_{1,i} \cdot \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^d b_{1,i,j} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} + \dots$$

Goal:

Solve the homogenous, linear, PDE for $f: \mathbb{R}^{\geq 0} \times \mathbb{R}^d \rightarrow \mathbb{R}$:

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

satisfying initial conditions.

Suppose f is the product of temporal and spatial functions:

$$f(t, x_1, \dots, x_d) = g(t) \cdot h(x_1, \dots, x_d)$$

Then the PDE becomes:

$$a_1 \cdot \frac{\partial(g \cdot h)}{\partial t} + a_2 \cdot \frac{\partial^2(g \cdot h)}{\partial t^2} + \dots = D(g \cdot h)$$

$$\left(a_1 \cdot \frac{\partial g}{\partial t} + a_2 \cdot \frac{\partial^2 g}{\partial t^2} + \dots \right) \cdot h = g \cdot D(h)$$

Separation of Variables

$$D(f) = b_0 \cdot f + \sum_{i=1}^d b_{1,i} \cdot \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^d b_{1,i,j} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} + \dots$$
$$f(t, x_1, \dots, x_d) = g(t) \cdot h(x_1, \dots, x_d)$$

Goal:

Solve the homogenous, linear, PDE for $g: \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ and $h: \mathbb{R}^d \rightarrow \mathbb{R}$:

$$\left(a_1 \cdot \frac{\partial g}{\partial t} + a_2 \cdot \frac{\partial^2 g}{\partial t^2} + \dots \right) \cdot h = g \cdot D(h)$$

satisfying initial conditions.

If $h_\lambda(x_1, \dots, x_d)$ is an eigenvector of D with eigenvalue λ , the PDE becomes:

$$\left(a_1 \cdot \frac{\partial g}{\partial t} + a_2 \cdot \frac{\partial^2 g}{\partial t^2} + \dots \right) \cdot h_\lambda = g \cdot \lambda \cdot h_\lambda$$

$$a_1 \cdot \frac{\partial g}{\partial t} + a_2 \cdot \frac{\partial^2 g}{\partial t^2} + \dots = \lambda \cdot g$$

\Rightarrow Solving the PDE reduces to solving an ODE.

Separation of Variables

$$D(f) = b_0 \cdot f + \sum_{i=1}^d b_{1,i} \cdot \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^d b_{1,i,j} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} + \dots$$
$$f(t, x_1, \dots, x_d) = g(t) \cdot h(x_1, \dots, x_d)$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/eigenvalues $\{h_\lambda, \lambda\}$ of D .
- For each eigenvalue λ , solve the ODE for g_λ :

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set f_λ to be the associated solution(s):

$$f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$$

- Find the linear combination of the f_λ satisfying the initial conditions.

Separation of Variables

$$D(f) = b_0 \cdot f + \sum_{i=1}^d b_{1,i} \cdot \frac{\partial f}{\partial x_i} + \sum_{i,j=1}^d b_{1,i,j} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} + \dots$$
$$f(t, x_1, \dots, x_d) = g(t) \cdot h(x_1, \dots, x_d)$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/eigenvalues $\{h_\lambda, \lambda\}$ of D .
- If the endomorphism D is self-adjoint, we can find an eigen-basis.

↓

There are many eigenvectors/eigenvalues to work with.

- Set f_λ to be the associated solution(s).

$$f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$$

- Find the linear combination of the f_λ satisfying the initial conditions.

Outline

Recall

Separation of Variables

The Heat and Wave Equation Revisited

The Heat and Wave Equation Revisited

Recall:

The (generalized) stiffness eigen-problem:

$$(M^{-1} \circ S)(\psi) = \lambda \cdot \psi$$

$$\Updownarrow$$

$$S(\psi) = \lambda \cdot M(\psi)$$

has an orthonormal eigen-basis $\{\psi_1, \dots, \psi_{|\mathcal{V}|}\}$ with associated non-negative eigenvalues $\{\lambda_1, \dots, \lambda_{|\mathcal{V}|}\}$.

$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

Goal:

Given an initial distribution of heat, $f^0: \mathcal{M} \rightarrow \mathbb{R}$, solve for the function $f: \mathbb{R}^{\geq 0} \times \mathcal{M} \rightarrow \mathbb{R}$ satisfying the heat diffusion PDE:

$$\frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

with initial condition:

$$f(0, p) = f^0(p)$$

$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

⇒ The spatial derivative is:

$$D = -M^{-1} \circ S$$

⇒ The generalized eigen-basis of the stiffness operator $\{\psi_1, \dots, \psi_{|\nu|}\}$ are eigenvectors of D with non-positive eigenvalues $\{-\lambda_1, \dots, -\lambda_{|\nu|}\}$:

$$D(\psi_i) = -\lambda_i \cdot \psi_i$$

$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- **Solve the ODE:**

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

⇒ The spatial derivative is:

$$D = -M^{-1} \circ S$$

⇒ The generalized eigen-basis of the stiffness operator $\{\psi_1, \dots, \psi_{|\mathcal{V}|}\}$ are eigenvectors of D with non-positive eigenvalues $\{-\lambda_1, \dots, -\lambda_{|\mathcal{V}|}\}$:

$$D(\psi_i) = -\lambda_i \cdot \psi_i$$

⇒ For each λ_i , this gives the ODE:

$$\frac{\partial g_i}{\partial t} = -\lambda_i \cdot g_i$$

⇓

$$g_i(t) = a_i \cdot e^{-\lambda_i \cdot t}$$

$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

⇒ For every eigenvector $\psi_i: \mathcal{M} \rightarrow \mathbb{R}$ and associated eigenvalue $\lambda_i \in \mathbb{R}$ of the stiffness eigenproblem, we get a one-dimensional space of solutions to the heat diffusion PDE:

$$f_i(t, p) = a_i \cdot e^{-\lambda_i \cdot t} \cdot \psi_i(p)$$

⇒ Any function of the form:

$$f(t, p) = \sum_{i=1}^{|\mathcal{V}|} a_i \cdot e^{-\lambda_i \cdot t} \cdot \psi_i(p)$$

is a solution to the heat diffusion PDE.

$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

$$f(t, p) = \sum_{i=1}^{|\mathcal{V}|} a_i \cdot e^{-\lambda_i \cdot t} \cdot \psi_i(p)$$

Initial Condition:

We want the coefficients a_i to satisfy:

$$\begin{aligned} f^0(p) &= f(0, p) \\ &= \sum_{i=1}^{|\mathcal{V}|} a_i \cdot e_i^{-\lambda \cdot 0} \cdot \psi_i(p) \\ &= \sum_{i=1}^{|\mathcal{V}|} a_i \cdot \psi_i(p) \end{aligned}$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

$$f(t, p) = \sum_{i=1}^{|\mathcal{V}|} a_i \cdot e^{-\lambda_i \cdot t} \cdot \psi_i(p)$$

Initial Condition:

We want the coefficients a_i to satisfy:

$$f^0(p) = \sum_{i=1}^{|\mathcal{V}|} a_i \cdot \psi_i(p)$$

\Rightarrow The a_i are the Fourier coefficients of the initial heat distribution f^0 :

$$a_i = \hat{\mathbf{f}}_i^0 = \langle f^0, \psi_i \rangle_{\mathcal{M}}$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

Solving the PDE:

Given an initial distribution of heat:

$$f^0(p) = \hat{\mathbf{f}}_1^0 \cdot \psi_1(p) + \dots + \hat{\mathbf{f}}_{|\mathcal{V}|}^0 \cdot \psi_{|\mathcal{V}|}(p)$$

the solution to the heat PDE is:

$$f(t, p) = \sum_{i=1}^{|\mathcal{V}|} \hat{\mathbf{f}}_i^0 \cdot e^{-\lambda_i \cdot t} \cdot \psi_i(p)$$

⇒ Solving the heat PDE amounts to having the Fourier coefficients **decay** as a function of time, with the decay rate increasing with frequency:

$$\begin{aligned} \hat{\mathbf{f}}_i(t + \varepsilon) &= \hat{\mathbf{f}}_i^0 \cdot e^{-\lambda_i \cdot (t + \varepsilon)} \\ &= \hat{\mathbf{f}}_i^0 \cdot e^{-\lambda_i \cdot t} \cdot e^{-\lambda_i \cdot \varepsilon} \\ &= \hat{\mathbf{f}}_i(t) \cdot e^{-\lambda_i \cdot \varepsilon} \end{aligned}$$

$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

The Fourier coefficients evolve as:

Continuous time-stepping:

$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot e^{-\lambda_i \cdot \varepsilon}$$

Explicit time-stepping:

$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot (1 - \varepsilon \cdot \lambda_i)$$

Implicit time-stepping:

$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot \frac{1}{1 + \varepsilon \cdot \lambda_i}$$

Continuous ($\lambda = 0$) ε

Explicit ($\lambda = 0$) ε

Implicit ($\lambda = 0$) ε

$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

The Fourier coefficients evolve as:

Continuous time-stepping:

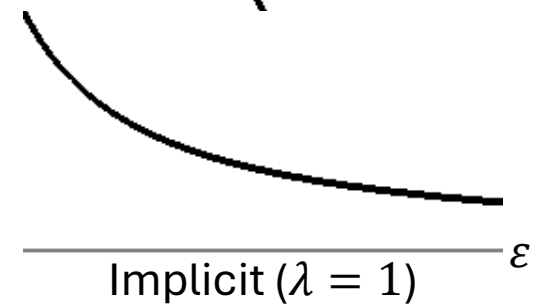
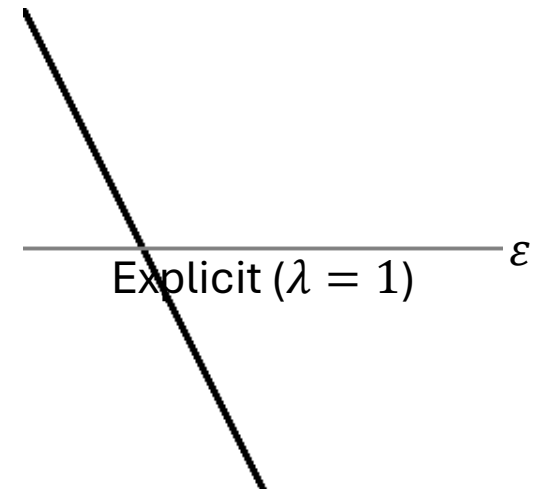
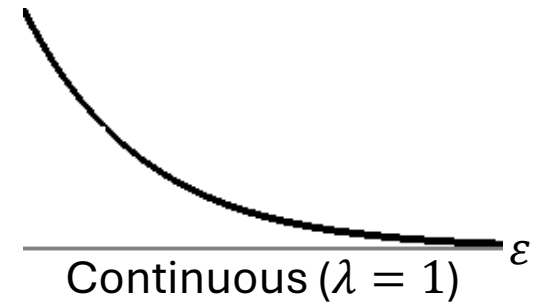
$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot e^{-\lambda_i \cdot \varepsilon}$$

Explicit time-stepping:

$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot (1 - \varepsilon \cdot \lambda_i)$$

Implicit time-stepping:

$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot \frac{1}{1 + \varepsilon \cdot \lambda_i}$$



$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

The Fourier coefficients evolve as:

Continuous time-stepping:

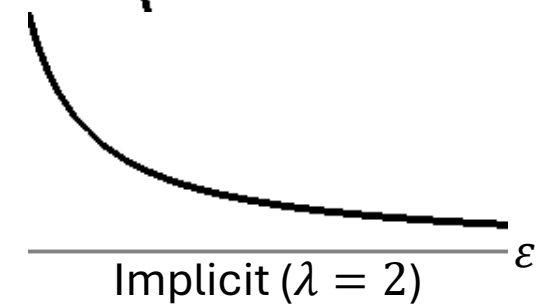
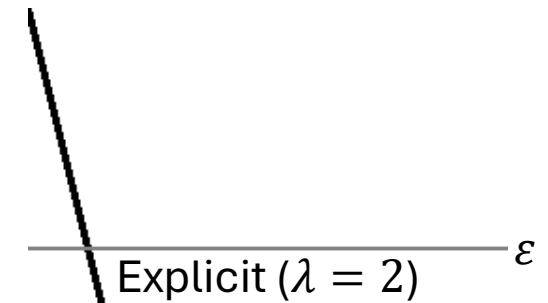
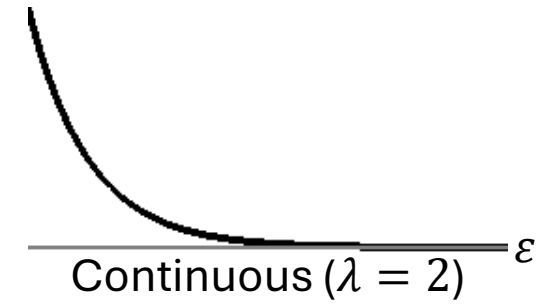
$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot e^{-\lambda_i \cdot \varepsilon}$$

Explicit time-stepping:

$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot (1 - \varepsilon \cdot \lambda_i)$$

Implicit time-stepping:

$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot \frac{1}{1 + \varepsilon \cdot \lambda_i}$$



$$\text{Heat: } \frac{\partial f}{\partial t} = -M^{-1}(S(f))$$

The Fourier coefficients evolve as:

Continuous time-stepping:

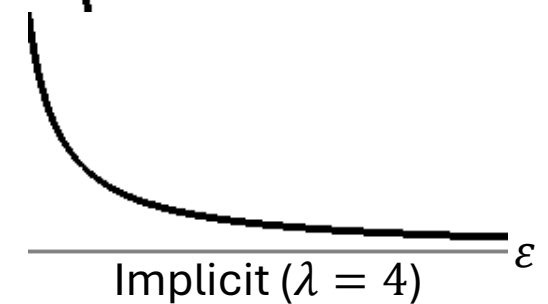
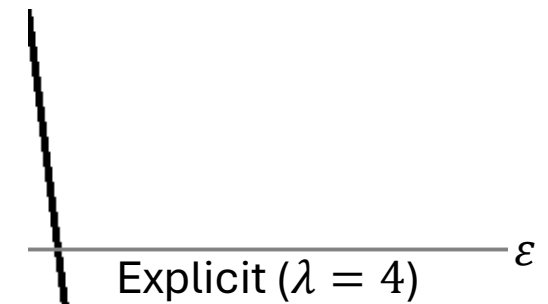
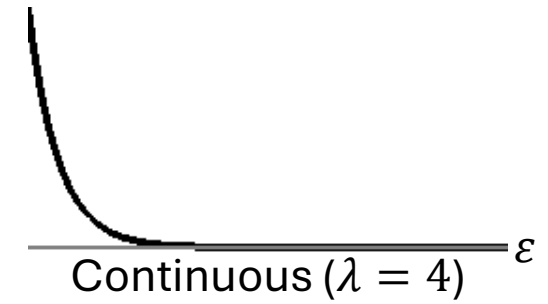
$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot e^{-\lambda_i \cdot \varepsilon}$$

Explicit time-stepping:

$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot (1 - \varepsilon \cdot \lambda_i)$$

Implicit time-stepping:

$$\hat{\mathbf{f}}_i(t + \varepsilon) = \hat{\mathbf{f}}_i(t) \cdot \frac{1}{1 + \varepsilon \cdot \lambda_i}$$



$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

Goal:

Given initial displacement values, $f^0: \mathcal{M} \rightarrow \mathbb{R}$, and velocities $v^0: \mathcal{M} \rightarrow \mathbb{R}$, solve for the function $f: \mathbb{R}^{\geq 0} \times \mathcal{M} \rightarrow \mathbb{R}$ satisfying the wave equation PDE:

$$\frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

with initial conditions:

$$\begin{aligned} f(0, p) &= f^0(p) \\ \frac{\partial f}{\partial t} \Big|_{(0, p)} &= v^0(p) \end{aligned}$$

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

⇒ The spatial derivative is:

$$D = -M^{-1} \circ S$$

⇒ The generalized eigen-basis of the stiffness operator $\{\psi_1, \dots, \psi_{|\mathcal{V}|}\}$ are eigenvectors of D with non-positive eigenvalues $\{-\lambda_1, \dots, -\lambda_{|\mathcal{V}|}\}$:

$$D(\psi_i) = -\lambda_i \cdot \psi_i$$

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- **Solve the ODE:**

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

⇒ The spatial derivative is:

$$D = -M^{-1} \circ S$$

⇒ The generalized eigen-basis of the stiffness operator $\{\psi_1, \dots, \psi_{|\nu|}\}$ are eigenvectors of D with non-positive eigenvalues $\{-\lambda_1, \dots, -\lambda_{|\nu|}\}$:

$$D(\psi_i) = -\lambda_i \cdot \psi_i$$

⇒ For each λ_i , this gives the ODE:

$$\frac{\partial^2 g_i}{\partial t^2} = -\lambda_i \cdot g_i$$

⇓

$$g_i(t) = a_i + b_i \cdot t, \quad \lambda_i = 0$$

$$g_i(t) = a_i \cdot \cos(\sqrt{\lambda_i} \cdot t) + b_i \cdot \sin(\sqrt{\lambda_i} \cdot t), \quad \lambda_i > 0$$

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

⇒ For every eigenvector $\psi_i: \mathcal{M} \rightarrow \mathbb{R}$ and associated eigenvalue $\lambda_i \in \mathbb{R}$ of the stiffness eigenproblem, we get a two-dimensional space of solutions to the heat diffusion PDE:

$$f_i(t, p) = (a_i + b_i \cdot t) \cdot \psi_i(p), \quad \lambda_i = 0$$

$$f_i(t, p) = \left(a_i \cdot \cos(\sqrt{\lambda_i} \cdot t) + b_i \cdot \sin(\sqrt{\lambda_i} \cdot t) \right) \cdot \psi_i(p), \quad \lambda_i > 0$$

⇒ Any function of the form:

$$f(t, p) = (a_1 + b_1 \cdot t) \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \left(a_i \cdot \cos(\sqrt{\lambda_i} \cdot t) + b_i \cdot \sin(\sqrt{\lambda_i} \cdot t) \right) \cdot \psi_i(p)$$

is a solution to the wave equation PDE.*

*Assuming the mesh is connected.

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

$$f(t, p) = (a_1 + b_1 \cdot t) \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \left(a_i \cdot \cos(\sqrt{\lambda_i} \cdot t) + b_i \cdot \sin(\sqrt{\lambda_i} \cdot t) \right) \cdot \psi_i(p)$$

Initial Conditions (values at $t = 0$):

$$f^0(p) = f(0, p)$$

$$= (a_1 + b_1 \cdot 0) \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \left(a_i \cdot \cos(\sqrt{\lambda_i} \cdot 0) + b_i \cdot \sin(\sqrt{\lambda_i} \cdot 0) \right) \cdot \psi_i(p)$$

$$= a_1 \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} a_i \cdot \psi_i(p)$$

$$= \sum_{i=1}^{|\mathcal{V}|} a_i \cdot \psi_i(p)$$

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

$$f(t, p) = (a_1 + b_1 \cdot t) \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \left(a_i \cdot \cos(\sqrt{\lambda_i} \cdot t) + b_i \cdot \sin(\sqrt{\lambda_i} \cdot t) \right) \cdot \psi_i(p)$$

Initial Conditions (values at $t = 0$):

$$f^0(p) = \sum_{i=1}^{|\mathcal{V}|} a_i \cdot \psi_i(p)$$

\Rightarrow The a_i are the Fourier coefficients of $f^0(p)$:

$$a_i = \hat{\mathbf{f}}_i^0 = \langle f^0, \psi_i \rangle_{\mathcal{M}}$$

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

$$f(t, p) = (a_1 + b_1 \cdot t) \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \left(a_i \cdot \cos(\sqrt{\lambda_i} \cdot t) + b_i \cdot \sin(\sqrt{\lambda_i} \cdot t) \right) \cdot \psi_i(p)$$

Initial Conditions (velocities at $t = 0$):

$$\begin{aligned} v^0(p) &= \left. \frac{\partial f}{\partial t} \right|_{(0,p)} \\ &= b_1 \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \left(-a_i \cdot \sqrt{\lambda_i} \cdot \sin(\sqrt{\lambda_i} \cdot 0) + b_i \cdot \sqrt{\lambda_i} \cdot \cos(\sqrt{\lambda_i} \cdot 0) \right) \cdot \psi_i(p) \\ &= b_1 \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} b_i \cdot \sqrt{\lambda_i} \cdot \psi_i(p) \end{aligned}$$

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

$$f(t, p) = (a_1 + b_1 \cdot t) \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \left(a_i \cdot \cos(\sqrt{\lambda_i} \cdot t) + b_i \cdot \sin(\sqrt{\lambda_i} \cdot t) \right) \cdot \psi_i(p)$$

Initial Conditions (velocities at $t = 0$):

$$v^0(p) = b_1 \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} b_i \cdot \sqrt{\lambda_i} \cdot \psi_i(p)$$

⇒ The b_i are the modulated Fourier coefficients of $v^0(p)$:

$$b_i = \hat{v}_i^0 = \langle v^0, \psi_i \rangle_{\mathcal{M}}, \quad i = 1$$

$$b_i = \frac{\hat{v}_i^0}{\sqrt{\lambda_i}} = \frac{\langle v^0, \psi_i \rangle_{\mathcal{M}}}{\sqrt{\lambda_i}}, \quad i \neq 1$$

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$a_1 \cdot \frac{\partial f}{\partial t} + a_2 \cdot \frac{\partial^2 f}{\partial t^2} + \dots = D(f)$$

Approach:

- Compute the eigenvectors/values $\{h_\lambda, \lambda\}$ of D
- Solve the ODE:

$$a_1 \cdot \frac{\partial g_\lambda}{\partial t} + a_2 \cdot \frac{\partial^2 g_\lambda}{\partial t^2} + \dots = \lambda \cdot g_\lambda$$

- Set the $f_\lambda(t, x_1, \dots, x_d) = g_\lambda(t) \cdot h_\lambda(x_1, \dots, x_d)$
- Satisfy the initial conditions

Solving the PDE:

Given an initial displacement values and velocities:

$$f^0(p) = \hat{\mathbf{f}}_1^0 \cdot \psi_1(p) + \dots + \hat{\mathbf{f}}_{|\mathcal{V}|}^0 \cdot \psi_{|\mathcal{V}|}(p)$$

$$v^0(p) = \hat{\mathbf{v}}_1^0 \cdot \psi_1(p) + \dots + \hat{\mathbf{v}}_{|\mathcal{V}|}^0 \cdot \psi_{|\mathcal{V}|}(p)$$

the solution to the heat PDE is:

$$f(t, p) = (\hat{\mathbf{f}}_1^0 + \hat{\mathbf{v}}_1^0 \cdot t) \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \left(\hat{\mathbf{f}}_i^0 \cdot \cos(\sqrt{\lambda_i} \cdot t) + \hat{\mathbf{v}}_i^0 \cdot \frac{\sin(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right) \cdot \psi_i(p)$$

⇒ Solving the wave PDE amounts to trigonometrically recombining the Fourier coefficients of the initial values and velocity.

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$f(t, p) = (\hat{\mathbf{f}}_1^0 + \hat{\mathbf{v}}_1^0 \cdot t) \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \left(\hat{\mathbf{f}}_i^0 \cdot \cos(\sqrt{\lambda_i} \cdot t) + \hat{\mathbf{v}}_i^0 \cdot \frac{\sin(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right) \cdot \psi_i(p)$$

Potential Energy (stiffness of the values at time t):

$$\begin{aligned} PE(t) &= [S(f^t)](f^t) \\ &= \langle f^t, (M^{-1} \circ S)(f^t) \rangle_{\mathcal{M}} \\ &= \left\langle f^t, \sum_{i=2}^{|\mathcal{V}|} \lambda_i \cdot \left(\hat{\mathbf{f}}_i^0 \cdot \cos(\sqrt{\lambda_i} \cdot t) + \hat{\mathbf{v}}_i^0 \cdot \frac{\sin(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right) \cdot \psi_i(p) \right\rangle_{\mathcal{M}} \\ &= \sum_{i=2}^{|\mathcal{V}|} \lambda_i \cdot \left(\hat{\mathbf{f}}_i^0 \cdot \cos(\sqrt{\lambda_i} \cdot t) + \hat{\mathbf{v}}_i^0 \cdot \frac{\sin(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right)^2 \\ &= \sum_{i=2}^{|\mathcal{V}|} \lambda_i \cdot \left((\hat{\mathbf{f}}_i^0)^2 \cdot \cos^2(\sqrt{\lambda_i} \cdot t) + (\hat{\mathbf{v}}_i^0)^2 \cdot \frac{\sin^2(\sqrt{\lambda_i} \cdot t)}{\lambda_i} + 2 \cdot \hat{\mathbf{f}}_i^0 \cdot \hat{\mathbf{v}}_i^0 \cdot \frac{\cos(\sqrt{\lambda_i} \cdot t) \cdot \sin(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right) \end{aligned}$$

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$f(t, p) = (\hat{\mathbf{f}}_1^0 + \hat{\mathbf{v}}_1^0 \cdot t) \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \left(\hat{\mathbf{f}}_i^0 \cdot \cos(\sqrt{\lambda_i} \cdot t) + \hat{\mathbf{v}}_i^0 \cdot \frac{\sin(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right) \cdot \psi_i(p)$$

Kinetic Energy (mass of the velocities at time t):

$$\begin{aligned} KE(t) &= [M(\partial f^t / \partial t)](\partial f^t / \partial t) \\ &= \langle \partial f^t / \partial t, \partial f^t / \partial t \rangle_{\mathcal{M}} \\ &= \left\| \hat{\mathbf{v}}_1^0 \cdot \psi_1(p) + \sum_{i=2}^{|\mathcal{V}|} \sqrt{\lambda_i} \cdot \left(-\hat{\mathbf{f}}_i^0 \cdot \sin(\sqrt{\lambda_i} \cdot t) + \hat{\mathbf{v}}_i^0 \cdot \frac{\cos(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right) \cdot \psi_i(p) \right\|_{\mathcal{M}}^2 \\ &= (\hat{\mathbf{v}}_1^0)^2 + \sum_{i=2}^{|\mathcal{V}|} \lambda_i \cdot \left(-\hat{\mathbf{f}}_i^0 \cdot \sin(\sqrt{\lambda_i} \cdot t) + \hat{\mathbf{v}}_i^0 \cdot \frac{\cos(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right)^2 \\ &= (\hat{\mathbf{v}}_1^0)^2 + \sum_{i=2}^{|\mathcal{V}|} \lambda_i \cdot \left((\hat{\mathbf{f}}_i^0)^2 \cdot \sin^2(\sqrt{\lambda_i} \cdot t) + (\hat{\mathbf{v}}_i^0)^2 \cdot \frac{\cos^2(\sqrt{\lambda_i} \cdot t)}{\lambda_i} - 2 \cdot \hat{\mathbf{f}}_i^0 \cdot \hat{\mathbf{v}}_i^0 \cdot \frac{\cos(\sqrt{\lambda_i} \cdot t) \cdot \sin(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right) \end{aligned}$$

$$\text{Wave: } \frac{\partial^2 f}{\partial t^2} = -M^{-1}(S(f))$$

$$PE(t) = \sum_{i=2}^{|\mathcal{V}|} \lambda_i \cdot \left((\hat{\mathbf{f}}_i^0)^2 \cdot \cos^2(\sqrt{\lambda_i} \cdot t) + (\hat{\mathbf{v}}_i^0)^2 \cdot \frac{\sin^2(\sqrt{\lambda_i} \cdot t)}{\lambda_i} + 2 \cdot \hat{\mathbf{f}}_i^0 \cdot \hat{\mathbf{v}}_i^0 \cdot \frac{\cos(\sqrt{\lambda_i} \cdot t) \cdot \sin(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right)$$

$$KE(t) = (\hat{\mathbf{v}}_1^0)^2 + \sum_{i=2}^{|\mathcal{V}|} \lambda_i \cdot \left((\hat{\mathbf{f}}_i^0)^2 \cdot \sin^2(\sqrt{\lambda_i} \cdot t) + (\hat{\mathbf{v}}_i^0)^2 \cdot \frac{\cos^2(\sqrt{\lambda_i} \cdot t)}{\lambda_i} - 2 \cdot \hat{\mathbf{f}}_i^0 \cdot \hat{\mathbf{v}}_i^0 \cdot \frac{\cos(\sqrt{\lambda_i} \cdot t) \cdot \sin(\sqrt{\lambda_i} \cdot t)}{\sqrt{\lambda_i}} \right)$$

Total Energy:

$$PE(t) + KE(t) = (\hat{\mathbf{v}}_1^0)^2 + \sum_{i=2}^{|\mathcal{V}|} \lambda_i \cdot \left((\hat{\mathbf{f}}_i^0)^2 \cdot (\cos^2(\sqrt{\lambda_i} \cdot t) + \sin^2(\sqrt{\lambda_i} \cdot t)) + (\hat{\mathbf{v}}_i^0)^2 \cdot \frac{\sin^2(\sqrt{\lambda_i} \cdot t) + \cos^2(\sqrt{\lambda_i} \cdot t)}{\lambda_i} \right)$$

$$= (\hat{\mathbf{v}}_1^0)^2 + \sum_{i=2}^{|\mathcal{V}|} \lambda_i \cdot (\hat{\mathbf{f}}_i^0)^2 + (\hat{\mathbf{v}}_i^0)^2$$

⇒ The total energy is constant in time.