Geometry Processing (601.458/658)

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Outline

Recall

Size of an Endomorphism

The Wave Equation

Recall

Complex Numbers:

We represent a complex number $c \in \mathbb{C}$ as the sum of real and imaginary components:

$$c = a + i \cdot b$$

with $a \in \mathbb{R}$ the *real* component of c and $b \in \mathbb{R}$ the *imaginary* component.

Complex numbers support arithmetic operations:

$$a_1 + i \cdot b_1 + a_2 + i \cdot b_2 = (a_1 + a_2) + i \cdot (b_1 + b_2)$$

$$(a_1 + i \cdot b_1) \cdot (a_2 + i \cdot b_2) = (a_1 \cdot a_2 - b_1 \cdot b_2) + i \cdot (a_1 \cdot b_2 + a_2 \cdot b_1)$$

$$\frac{1}{a+i\cdot b} = \frac{a}{a^2+b^2} - i \cdot \frac{b}{a^2+b^2}$$

We can also compute the (squared) norm of a complex number:

$$|a + i \cdot b|^2 = a^2 + b^2$$

Recall

Complex Numbers:

The complex numbers are *algebraically closed* – any polynomial with complex coefficients will have at least one root.

$\Rightarrow$ Every complex polynomial factors as the product of linear factors (possibly with multiplicity)

Recall

Given inner-product spaces $\{V, B_V: V \to V^*\}$ and $\{W, B_W: W \to W^*\}$ and given a linear map $M \in \mathrm{Hom}(V, W)$, the adjoint $M^\dagger \in \mathrm{Hom}(W, V)$ satisfies:

$$\left\langle v, M^\dagger(w) \right\rangle_{B_V} = \langle M(v), w \rangle_{B_W}$$

$$\Updownarrow$$

$$M^\dagger = B_V^{-1} \circ M^* \circ B_W$$

The operator $M^\dagger \circ M \in \mathrm{End}(V)$ is self-adjoint.

Recall

Given an inner-product space $\{V, B_V: V \to V^*\}$ and a self-adjoint operator $M \in \mathrm{End}(V)$, we can define the quadratic form:

$$\begin{aligned} Q_{B\circ M}: V &\to \mathbb{R} \\ v &\mapsto \tfrac{1}{2}\langle v, M(v)\rangle_B \end{aligned}$$

On the unit sphere in V, a vector v is an extrema of the quadratic form if it is an eigenvector of M with eigenvalue:

$$\lambda = 2 \cdot Q_{B\circ M}(v)$$

Recall

Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we discretize the space of scalar functions, V, using the hat basis.

On the space of scalar functions, we have two bilinear forms:

$M \in \mathrm{Hom}(V, V^*)$: The inner-product on scalar functions:

$$[M(f)](g) = \langle f, g \rangle_{\mathcal{M}}$$

$S \in \mathrm{Hom}(V, V^*)$: The pull-back of the inner-product on cotangent vector-fields (a.k.a. the stiffness):

$$[S(f)](g) = \langle df, dg \rangle_{\mathcal{M}}$$

As a quadratic form, the stiffness defines the Dirichlet energy:

$$\frac{1}{2}[S(f)](f) = \frac{1}{2}\langle df, df \rangle_{\mathcal{M}}$$

Recall

Given the mass and stiffness operators, $M, S \in \mathrm{Hom}(V, V^*)$ we can solve the (generalized) stiffness problem:

$$S(\psi) = \lambda \cdot M(\psi)$$

to get an orthonormal eigen-basis $\{\psi_1, \ldots, \psi_{|\mathcal{V}|}\}$ with associated eigenvalues $\{\lambda_1, \ldots, \lambda_{|\mathcal{V}|}\}$.

With respect to this basis, we can compute the Fourier decomposition of any scalar function $f \in V$:

$$f = \hat{\mathbf{f}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_{|\mathcal{V}|} \cdot \psi_{|\mathcal{V}|}$$

The first eigen-function corresponds to constant functions, with $\lambda_1 = 0$.
(If the mesh is connected, this is the only eigenvector with eigenvalue 0.)

Recall

Given the mass and stiffness operators, $M, S \in \mathrm{Hom}(V, V^*)$, we can express the heat diffusion PDE:

$$\frac{\partial f^t}{\partial t} = -M^{-1}\big(S(f^t)\big)$$

with $f^0 \in V$ the initial signal, $f^t \in V$ the signal after diffusing for time t, and

$$-M^{-1} \circ S \in \mathrm{End}(V)$$

the Laplace operator.

We can solve this PDE using implicit time-stepping:

$$(M + \varepsilon \cdot S)(f^{t+\varepsilon}) = M(f^t)$$

Recall

$$(M + \varepsilon \cdot S)(f^{t+\varepsilon}) = M(f^{t})$$

Using the Fourier decomposition:

$$f^{t} = \hat{\mathbf{f}}_1^t \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_{|\mathcal{V}|}^t \cdot \psi_{|\mathcal{V}|}$$

implicit time-stepping acts as scalar multiplication within a frequency:

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \frac{1}{(1 + \varepsilon \cdot \lambda_i)} \cdot \hat{\mathbf{f}}_i^t$$

Outline

Recall

Size of an Endomorphism

The Wave Equation

Size of an Endomorphism

$$(M + \varepsilon \cdot S)(f^{t+\varepsilon}) = M(f^t)$$

Motivation:

We consider two questions regarding the temporal discretization:

1. **Step-Size Stability**: What happens after a single time-step as $\varepsilon \to \infty$?
2. **Iterative Behavior**: What happens after multiple time-steps?

Using the spectral interpretation of the heat diffusion PDE, we showed that implicit time-stepping amounted to scaling of the Fourier coefficients:

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \frac{1}{(1 + \varepsilon \cdot \lambda_i)} \cdot \hat{\mathbf{f}}_i^t$$

1. The scale factor $1/(1 + \varepsilon \cdot \lambda_i)$ is bounded regardless of the step-size ε.
2. The scale factor is in the range $(0,1]$ so that Fourier coefficients are either reduced or kept the same, in the course of time-stepping.

Size of an Endomorphism

$$(M + \varepsilon \cdot S)(f^{t+\varepsilon}) = M(f^t)$$

Motivation:

W

1.

2.

For each frequency, we have a 1×1 linear system relating the previous Fourier coefficients to the next one.

What happens when the relationship is multi-dimensional?

Using the spe … we showed that implicit time-stepping amounted to scaling of the Fourier coefficients:

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \frac{1}{(1 + \varepsilon \cdot \lambda_i)} \cdot \hat{\mathbf{f}}_i^t$$

1. The scale factor $1/(1 + \varepsilon \cdot \lambda_i)$ is bounded regardless of the step-size ε.
2. The scale factor is in the range $(0,1]$ so that Fourier coefficients are either reduced or kept the same, in the course of time-stepping.

Size of an Endomorphism

Characteristic Polynomial:

Given a vector space V and endomorphism $A \in \mathrm{End}(V)$, the *characteristic polynomial* is the polynomial of degree $\dim(V)$ obtained by taking the determinant of the endomorphism, offset by a multiple of the diagonal:

$$\chi_A(\lambda) = \det(A - \lambda \cdot \mathrm{Id}_V)$$

The eigenvalues of A are the roots of the characteristic polynomial.

⇒ Since every polynomial has at least one complex root, every endomorphism has at least one (possibly) complex eigenvalue.

Note:
Since the roots can have multiplicity, this does **not** imply that every matrix diagonalizes over the complex numbers.

Size of an Endomorphism: Frobenius Norm

Recall:

Given an inner-product space $\{V, B\colon V \to V^*\}$, we can define an inner-product on the space of endomorphisms in terms of the trace:

$$\begin{aligned}\langle\cdot,\cdot\rangle_B\colon \mathrm{End}(V)\times\mathrm{End}(V) &\to \mathbb{R}\\ (M,N) &\mapsto \mathrm{tr}(B^{-1}\circ M^*\circ B\circ N)\end{aligned}$$

Using this we define the *Frobenius norm* of a linear operator $M \in \mathrm{End}(V)$:

$$\begin{aligned}\|M\|_{F,B}^2 &= \mathrm{tr}(B^{-1}\circ M^*\circ B\circ M)\\ &= \mathrm{tr}\left(M^\dagger\circ M\right)\end{aligned}$$

Size of an Endomorphism: Frobenius Norm

Recall:

Given an inner-product space $\{V, B: V \to V^*\}$, we define a Frobenius norm on endomorphisms:

$$\|M\|_{F,B}^2 = \mathrm{tr}\left(M^\dagger \circ M\right)$$

Since $M^\dagger \circ M$ is self-adjoint, there is an orthonormal eigen-basis $\{v_1, \ldots, v_n\}$ with associated eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$:

$$\left(M^\dagger \circ M\right)(v_i) = \lambda_i \cdot v_i$$

⇒ In the basis $\{v_1, \ldots, v_n\}$, the matrix expression for $M^\dagger \circ M$ is diagonal, with the λ_i along the diagonal.

⇒ The squared Frobenius norm of the linear operator M is:

$$\|M\|_{F,B}^2 = \mathrm{tr}\left(M^\dagger \circ M\right) = \lambda_1 + \cdots + \lambda_n$$

Size of an Endomorphism: Frobenius Norm

$$\|M\|_{F,B}^2 = \mathrm{tr}\left(M^\dagger \circ M\right) = \lambda_1 + \cdots + \lambda_n$$

Note:

Since v_i is a (unit) eigenvector of $M^\dagger \circ M$ with eigenvalue λ_i, we have:

$$\left(M^\dagger \circ M\right)(v_i) = \lambda_i \cdot v_i$$

Taking the inner-product of both sides with v_i, we have:

$$\left\langle \left(M^\dagger \circ M\right)(v_i), v_i \right\rangle_B = \langle \lambda_i \cdot v_i, v_i \rangle_B$$

$$\Updownarrow$$

$$\langle M(v_i), M(v_i) \rangle_B = \lambda_i \cdot \langle v_i, v_i \rangle_B$$

$$\Updownarrow$$

$$\|M(v_i)\|_B^2 = \lambda_i$$

⇒ The eigenvalues of $M^\dagger \circ M$ are strictly non-negative.

Size of an Endomorphism: Spectral Norm

While the Frobenius norm measures the size of an endomorphism:

$$\|M\|_{F,B}^2 = \mathrm{tr}\left(M^\dagger \circ M\right)$$

it describes cumulative information, and fails to capture how the endomorphism acts on individual vectors.

We would like a have a notion of the “size” of a linear operator that describes how “extremely” it scales vectors.

Size of an Endomorphism: Spectral Norm

Definition:

Given an inner-product space $\{V, B: V \to V^*\}$ and an endomorphism $M \in \mathrm{End}(V)$, the *spectral norm* of M is the maximum, over all vectors $v \in V$ on the unit sphere, of the scale of the linear operator applied to v:

$$\begin{aligned}\|M\|_B^2 &\equiv \max_{\|v\|_B^2=1} \|M(v)\|_B^2 \\ &= \max_{\|v\|_B^2=1} \langle M(v), M(v)\rangle_B \\ &= \max_{\|v\|_B^2=1} \left\langle (M^\dagger \circ M)(v), v\right\rangle_B \\ &= \max_{\|v\|_B^2=1} 2 \cdot Q_{B\circ M^\dagger \circ M}\end{aligned}$$

This is extremized at the eigenvectors of $M^\dagger \circ M$, at which (twice) the quadratic form evaluates to the associated eigenvalue.

⇒ The squared spectral norm of M is the largest eigenvalue of the $M^\dagger \circ M$.

Size of an Endomorphism: Spectral Norm

Definition:

Given an inner-product space $\{V, B\colon V \to V^*\}$ and an endomorphism $M \in \mathrm{End}(V)$, the *spectral norm* of M is the maximum, over all vectors $v \in V$ on the unit sphere, of the scale of the linear operator applied to v:

$$\|M\|_B^2 \equiv \max_{\|v\|_B^2=1} \|M(v)\|_B^2$$

Since the squared Frobenius norm is the sum of **all** the eigenvalues of $M^\dagger \circ M$ and since the eigenvalues are non-negative:

$$\|M\|_B^2 \leq \|M\|_{F,B}^2$$

$B\circ M^\dagger\circ M$

$\|v\|_B^2=1$

This is extremized at the eigenvectors of $M^\dagger \circ M$, at which (twice) the quadratic form evaluates to the associated eigenvalue.

⇒ The squared spectral norm of M is the largest eigenvalue of the $M^\dagger \circ M$.

Size of an Endomorphism: Spectral Radius

For an inner-product space $\{V, B: V \to V^*\}$ and endomorphism $M \in \mathrm{End}(V)$, the spectral norm bounds the extent by which a vector can be scaled by M:

$$\|M(v)\|_B^2 \leq \|M\|_B^2 \cdot \|v\|_B^2$$

Inductively, we can bound the exponent:

$$\left\|M^k\right\|_B^2 \leq \left(\|M\|_B^2\right)^k$$

This can overestimate – consider the matrix representing the operator rotating the x- and y-axes, scaling one by $1/4$, and the other by 2:

$$\mathbf{M} = \begin{pmatrix} 0 & 2 \\ -\frac{1}{4} & 0 \end{pmatrix}$$

⇒ Since the y-axes is scaled by 2 when being mapped to the x-axis:

$$\|\mathbf{M}\|_B^2 = 4$$

Size of an Endomorphism: Spectral Radius

$$\mathbf{M} = \begin{pmatrix} 0 & 2 \\ -\frac{1}{4} & 0 \end{pmatrix}$$

Using the spectral norm:

$$\|\mathbf{M}\|_B^2 = 4$$

⇒ Raising $\mathbf{M}$ to the k-th power, we would get the bound:

$$\left\|\mathbf{M}^k\right\|_B^2 \leq (\|\mathbf{M}\|_B^2)^k = 4^k$$

⇒ $\mathbf{M}^k$ *appears* to become unbounded as $k \to \infty$

Size of an Endomorphism: Spectral Radius

$$\mathbf{M} = \begin{pmatrix} 0 & 2 \\ -\frac{1}{4} & 0 \end{pmatrix}$$

Using the spectral norm:

$\mathbf{M}^k$ *appears* to become unbounded as $k \to \infty$

Squaring $\mathbf{M}$ gives:

$$\begin{aligned} \mathbf{M}^2 &= \begin{pmatrix} 0 & 2 \\ -\frac{1}{4} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 \\ -\frac{1}{4} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \\ &= -\frac{1}{2} \cdot \mathbf{Id} \end{aligned}$$

Size of an Endomorphism: Spectral Radius

$$\mathbf{M} = \begin{pmatrix} 0 & 2 \\ -\frac{1}{4} & 0 \end{pmatrix}$$

Using the spectral norm:

$\mathbf{M}^k$ appears to become unbounded as $k \to \infty$

Squaring $\mathbf{M}$ gives:

$$\mathbf{M}^2 = -\tfrac{1}{2} \cdot \mathbf{Id}$$

$\Rightarrow$ Raising $\mathbf{M}$ to the $2k$-th power:

$$\mathbf{M}^{2k} = \left(-\tfrac{1}{2}\right)^k \cdot \mathbf{Id}$$

$\Rightarrow$ $\mathbf{M}^k$ goes to zero as $k \to \infty$.

Size of an Endomorphism: Spectral Radius

Definition:

Given a vector space V the *spectral radius* of $M \in \mathrm{End}(V)$ is the largest norm of the (possibly complex) eigenvalues of M:

$$\sigma(M) = \max_i |\lambda_i|$$

Since the eigenvalues are roots of the characteristic polynomial:

$$\sigma(M) = \max\{|\lambda| \,\big|\, \chi_M(\lambda) = 0\}$$

Size of an Endomorphism: Spectral Radius

Definition:

Given a vector space V the *spectral radius* of $M \in \mathrm{End}(V)$ is the largest magnitude of the (possibly complex) eigenvalues of M:

$$\sigma(M) = \max\left\{|\lambda| \,\middle|\, \chi_M(\lambda) = 0\right\}$$

Properties:

- The spectral radius does **not** depend on the definition of an inner-product
- The spectral radius of the k-th power is the k-th power of the spectral radius:

$$\sigma\left(M^k\right) = \sigma^k(M)$$

- The spectral radius is bounded above by the spectral norm:

$$\sigma^2(M) \leq \|M\|_B^2 \leq \|M\|_{F,B}^2$$

- The spectral radius is less than one if and only if $\lim_{k\to\infty} M^k \to 0$

Size of an Endomorphism: Spectral Radius

$$\mathbf{M} = \begin{pmatrix} 0 & 2 \\ -\frac{1}{4} & 0 \end{pmatrix}$$

Example:

Computing the characteristic polynomial:

$$\begin{aligned} \chi_{\mathbf{M}}(\lambda) &= \det \begin{pmatrix} -\lambda & 2 \\ -\frac{1}{4} & -\lambda \end{pmatrix} \\ &= \lambda^2 + \frac{1}{2} \end{aligned}$$

Solving, we get two (imaginary) roots:

$$\lambda = \pm i \sqrt{\frac{1}{2}}$$

$\Rightarrow$ The matrix $\mathbf{M}^k \to 0$ as $k \to \infty$.

Outline

- Recall
- Size of an Endomorphism
- The Wave Equation

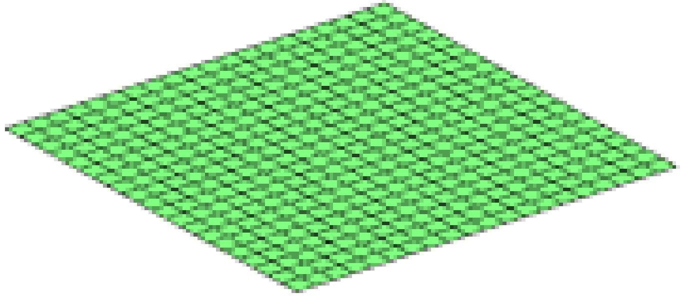
The Wave Equation

The *wave equation* describes the propagation of waves through a medium – the **force** acting on a point in space is proportional to the difference between the displacement of the point from the average displacement of its neighbors:

$$\Delta f^t = \frac{\partial^2 f^t}{\partial t^2}$$

$$\Downarrow$$

$$(-M^{-1} \circ S)(f^t) = \frac{\partial^2 f^t}{\partial t^2}$$



https://en.wikipedia.org/wiki/Wave_equation

The Wave Equation

$$\Delta f^t = \frac{\partial^2 f^t}{\partial t^2}$$

Property:

For a solution to the wave equation, the total potential energy is:

$$PE(t) = \frac{1}{2}\langle df^t, df^t \rangle_{\mathcal{M}}$$

And the total kinetic energy is:

$$KE(t) = \frac{1}{2}\left\langle \frac{\partial f^t}{\partial t}, \frac{\partial f^t}{\partial t} \right\rangle_{\mathcal{M}}$$

Preservation of energy says that, in an ideal system, the sum is constant:

$$PE(t) + KE(t) = c$$

The Wave Equation

$$(-M^{-1} \circ S)(f^{t}) = \frac{\partial f^{t}}{\partial t^{2}}$$

Discretizing Time:

The second partial derivative can be approximated by the difference of first partial derivatives:[*]

$$(-M^{-1} \circ S)\left(f^{\bar{t}}\right) = \frac{\frac{\partial f^{t+\frac{\varepsilon}{2}}}{\partial t} - \frac{\partial f^{t-\frac{\varepsilon}{2}}}{\partial t}}{\varepsilon}$$

[*] Will choose $\bar{t} \in \{t-\varepsilon, t, t+\varepsilon\}$.

The Wave Equation

$$(-M^{-1} \circ S)\left(f^{\bar{t}}\right) = \frac{\frac{\partial f^{t+\frac{\varepsilon}{2}}}{\partial t} - \frac{\partial f^{t-\frac{\varepsilon}{2}}}{\partial t}}{\varepsilon}$$

Discretizing Time:

The first partial derivative can be approximated by the difference of the function's values:

$$\begin{aligned}(-M^{-1} \circ S)\left(f^{\bar{t}}\right) &= \frac{\frac{f^{t+\varepsilon} - f^{t}}{\varepsilon} - \frac{f^{t} - f^{t-\varepsilon}}{\varepsilon}}{\varepsilon} \\ &= \frac{(f^{t+\varepsilon} - f^{t}) - (f^{t} - f^{t-\varepsilon})}{\varepsilon^2} \\ &= \frac{f^{t+\varepsilon} - 2f^{t} + f^{t-\varepsilon}}{\varepsilon^2}\end{aligned}$$

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*Will choose $\bar{t} \in \{t-\varepsilon, t, t+\varepsilon\}$.

The Wave Equation

$$\frac{f^{t+\varepsilon}-2f^{t}+f^{t-\varepsilon}}{\varepsilon^2}=(-M^{-1}\circ S)\left(f^{\bar{t}}\right)$$

Solving:

As with the heat equation, we can use **implicit** time-stepping:

$$\frac{f^{t+\varepsilon}-2f^{t}+f^{t-\varepsilon}}{\varepsilon^2}=(-M^{-1}\circ S)(f^{t+\varepsilon})$$

$$\Downarrow$$

$$f^{t+\varepsilon}-2f^{t}+f^{t-\varepsilon}=\varepsilon^2\cdot(-M^{-1}\circ S)(f^{t+\varepsilon})$$

$$\Downarrow$$

$$f^{t+\varepsilon}+\varepsilon^2\cdot(M^{-1}\circ S)(f^{t+\varepsilon})=2f^{t}-f^{t-\varepsilon}$$

$$\Downarrow$$

$$M(f^{t+\varepsilon})+\varepsilon^2\cdot S(f^{t+\varepsilon})=2\cdot M(f^{t})-M(f^{t-\varepsilon})$$

$$\Downarrow$$

$$(M+\varepsilon^2\cdot S)(f^{t+\varepsilon})=2\cdot M(f^{t})-M(f^{t-\varepsilon})$$

The Wave Equation (Implicit)

$$(M + \varepsilon^2 \cdot S)(f^{t+\varepsilon}) = 2 \cdot M(f^t) - M(f^{t-\varepsilon})$$

Initial Conditions:

Since the PDE is second order, getting the displacement at time t requires knowing the displacements at the **two** previous time-steps.

In particular, to get the displacement f^ε, we need both:

The initial displacement: f^0

The pre-initial displacement: $f^{-\varepsilon}$

Or, equivalently:

The initial displacement: f^0

The initial velocity $(f^0 - f^{-\varepsilon})/\varepsilon$

The Wave Equation (Implicit)

$$(M + \varepsilon^2 \cdot S)(f^{t+\varepsilon}) = 2 \cdot M(f^t) - M(f^{t-\varepsilon})$$

Initial Conditions:

Since the PDE is second order, getting the displacement at time t requires knowing the displacements at the **two** previous time-steps.

Often, we set $f^{-\varepsilon} = f^0$, corresponding to zero initial velocity.

In particular, to get the displacement f^ε, we need both:

The initial displacement: f^0

The pre-initial displacement: $f^{-\varepsilon}$

Or, equivalently:

The initial displacement: f^0

The initial velocity $(f^0 - f^{-\varepsilon})/\varepsilon$

The Wave Equation (Implicit)

$$(M+\varepsilon^2\cdot S)(f^{t+\varepsilon}) = 2\cdot M(f^t) - M(f^{t-\varepsilon})$$

Spectral Interpretation:

As with the heat equation, we let $\left\{(\psi_1,\lambda_1),\ldots,\left(\psi_{|\mathcal{V}|},\lambda_{|\mathcal{V}|}\right)\right\}$ be the solution to the generalized stiffness eigenvalue problem:

$$S(\psi) = \lambda\cdot M(\psi)$$

with the $\{\psi_i\}$ orthonormal and the $\{\lambda_i\}$ non-negative and monotonically non-decreasing.

This gives the Fourier decomposition of the displacements:

$$f^t = \hat{\mathbf{f}}_1^t\cdot\psi_1 + \cdots + \hat{\mathbf{f}}_{|\mathcal{V}|}^t\cdot\psi_{|\mathcal{V}|}$$

and we focus on the individual Fourier coefficients by evaluating on the ψ_i.

The Wave Equation (Implicit)

$$(M+\varepsilon^2\cdot S)(f^{t+\varepsilon}) = 2\cdot M(f^t) - M(f^{t-\varepsilon})$$
$$f^t = \hat{\mathbf{f}}_1^t\cdot\psi_1 + \cdots + \hat{\mathbf{f}}_{|\mathcal{V}|}^t\cdot\psi_{|\mathcal{V}|}$$

Left-Hand-Side:

$$\begin{aligned}(M+\varepsilon^2\cdot S)(f^{t+\varepsilon}) &= (M+\varepsilon^2\cdot S)\left(\hat{\mathbf{f}}_1^{t+\varepsilon}\cdot\psi_1 + \cdots + \hat{\mathbf{f}}_{|\mathcal{V}|}^{t+\varepsilon}\cdot\psi_{|\mathcal{V}|}\right)\\ &= \hat{\mathbf{f}}_1^{t+\varepsilon}\cdot\left(M(\psi_1)+\varepsilon^2\cdot S(\psi_1)\right) + \cdots + \hat{\mathbf{f}}_{|\mathcal{V}|}^{t+\varepsilon}\cdot\left(M\left(\psi_{|\mathcal{V}|}\right)+\varepsilon^2\cdot S\left(\psi_{|\mathcal{V}|}\right)\right)\\ &= \hat{\mathbf{f}}_1^{t+\varepsilon}\cdot\left(M(\psi_1)+\varepsilon^2\cdot\lambda_1\cdot M(\psi_1)\right) + \cdots + \hat{\mathbf{f}}_{|\mathcal{V}|}^{t+\varepsilon}\cdot\left(M\left(\psi_{|\mathcal{V}|}\right)+\varepsilon^2\cdot\lambda_{|\mathcal{V}|}\cdot M\left(\psi_{|\mathcal{V}|}\right)\right)\\ &= \hat{\mathbf{f}}_1^{t+\varepsilon}\cdot(1+\varepsilon^2\cdot\lambda_1)\cdot M(\psi_1) + \cdots + \hat{\mathbf{f}}_{|\mathcal{V}|}^{t+\varepsilon}\cdot\left(1+\varepsilon^2\cdot\lambda_{|\mathcal{V}|}\right)\cdot M\left(\psi_{|\mathcal{V}|}\right)\end{aligned}$$

Evaluating at ψ_i:

$$[(M+\varepsilon^2\cdot S)(f^{t+\varepsilon})](\psi_i) = \hat{\mathbf{f}}_i^{t+\varepsilon}\cdot(1+\varepsilon^2\cdot\lambda_i)$$

The Wave Equation (Implicit)

$$(M+\varepsilon^2\cdot S)(f^{t+\varepsilon}) = 2\cdot M(f^t) - M(f^{t-\varepsilon})$$
$$f^t = \hat{\mathbf{f}}_1^t\cdot\psi_1+\cdots+\hat{\mathbf{f}}_{|\mathcal{V}|}^t\cdot\psi_{|\mathcal{V}|}$$

Left-Hand-Side:

$$[(M+\varepsilon^2\cdot S)(f^{t+\varepsilon})](\psi_i) = \hat{\mathbf{f}}_i^{t+\varepsilon}\cdot(1+\varepsilon^2\cdot\lambda_i)$$

Right-Hand-Side:

$$\begin{aligned}2\cdot M(f^t)-M(f^{t-\varepsilon}) &= 2\cdot M\left(\hat{\mathbf{f}}_1^t\cdot\psi_1+\cdots+\hat{\mathbf{f}}_{|\mathcal{V}|}^t\cdot\psi_{|\mathcal{V}|}\right) - M\left(\hat{\mathbf{f}}_1^{t-\varepsilon}\cdot\psi_1+\cdots+\hat{\mathbf{f}}_{|\mathcal{V}|}^{t-\varepsilon}\cdot\psi_{|\mathcal{V}|}\right)\\ &= \left(2\cdot\hat{\mathbf{f}}_1^t-\hat{\mathbf{f}}_1^{t-\varepsilon}\right)\cdot M(\psi_1)+\cdots+\left(2\cdot\hat{\mathbf{f}}_{|\mathcal{V}|}^t-\hat{\mathbf{f}}_{|\mathcal{V}|}^{t-\varepsilon}\right)\cdot M\left(\psi_{|\mathcal{V}|}\right)\end{aligned}$$

Evaluating at ψ_i:

$$[2\cdot M(f^t)-M(f^{t-\varepsilon})](\psi_i) = 2\cdot\hat{\mathbf{f}}_i^t-\hat{\mathbf{f}}_i^{t-\varepsilon}$$

The Wave Equation (Implicit)

$$(M + \varepsilon^2 \cdot S)(f^{t+\varepsilon}) = 2 \cdot M(f^t) - M(f^{t-\varepsilon})$$
$$f^t = \hat{\mathbf{f}}_1^t \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_{|\mathcal{V}|}^t \cdot \psi_{|\mathcal{V}|}$$

Left-Hand-Side:

$$[(M + \varepsilon^2 \cdot S)(f^{t+\varepsilon})](\psi_i) = \hat{\mathbf{f}}_i^{t+\varepsilon} \cdot (1 + \varepsilon^2 \cdot \lambda_i)$$

Right-Hand-Side:

$$[2 \cdot M(f^t) - M(f^{t-\varepsilon})](\psi_i) = 2 \cdot \hat{\mathbf{f}}_i^t - \hat{\mathbf{f}}_i^{t-\varepsilon}$$

Since the two sides are equal:

$$\hat{\mathbf{f}}_i^{t+\varepsilon} \cdot (1 + \varepsilon^2 \cdot \lambda_i) = 2 \cdot \hat{\mathbf{f}}_i^t - \hat{\mathbf{f}}_i^{t-\varepsilon}$$
$$\Downarrow$$
$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \frac{2 \cdot \hat{\mathbf{f}}_i^t - \hat{\mathbf{f}}_i^{t-\varepsilon}}{1 + \varepsilon^2 \cdot \lambda_i}$$

The Wave Equation (Implicit)

$$(M+\varepsilon^2\cdot S)(f^{t+\varepsilon}) = 2\cdot M(f^t) - M(f^{t-\varepsilon})$$
$$f^t = \hat{\mathbf{f}}_1^t\cdot\psi_1 + \cdots + \hat{\mathbf{f}}_{|\mathcal{V}|}^t\cdot\psi_{|\mathcal{V}|}$$

Left-Hand-Side:

$$[(M+\varepsilon^2\cdot S)(f^{t+\varepsilon})](\psi_i) = \hat{\mathbf{f}}_i^{t+\varepsilon}\cdot(1+\varepsilon^2\cdot\lambda_i)$$

Right-Hand-Side:

Q1: What happens as we increase the step-size?

Q2: What happens as we repeatedly time-step?

Since the two sides are equal:

$$\hat{\mathbf{f}}_i^{t+\varepsilon}\cdot(1+\varepsilon^2\cdot\lambda_i) = 2\cdot\hat{\mathbf{f}}_i^t - \hat{\mathbf{f}}_i^{t-\varepsilon}$$
$$\Downarrow$$
$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \frac{2\cdot\hat{\mathbf{f}}_i^t - \hat{\mathbf{f}}_i^{t-\varepsilon}}{1+\varepsilon^2\cdot\lambda_i}$$

Behavior of the Implicit Solution

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \frac{2 \cdot \hat{\mathbf{f}}_i^t - \hat{\mathbf{f}}_i^{t-\varepsilon}}{1 + \varepsilon^2 \cdot \lambda_i}$$

At t, the next displacements are related to the current displacements as:

$$\begin{pmatrix} \hat{\mathbf{f}}_i^{t+\varepsilon} \\ \hat{\mathbf{f}}_i^t \end{pmatrix} = \begin{pmatrix} \frac{2}{1 + \varepsilon^2 \cdot \lambda_i} & \frac{-1}{1 + \varepsilon^2 \cdot \lambda_i} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \hat{\mathbf{f}}_i^t \\ \hat{\mathbf{f}}_i^{t-\varepsilon} \end{pmatrix}$$

Or, setting $\alpha_i(\varepsilon) = \frac{1}{1+\varepsilon^2 \cdot \lambda_i} \in (0,1]$, we have:

$$\begin{pmatrix} \hat{\mathbf{f}}_i^{t+\varepsilon} \\ \hat{\mathbf{f}}_i^t \end{pmatrix} = \begin{pmatrix} 2 \cdot \alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \hat{\mathbf{f}}_i^t \\ \hat{\mathbf{f}}_i^{t-\varepsilon} \end{pmatrix}$$

Behavior of the Implicit Solution

$$\begin{pmatrix} \hat{\mathbf{f}}_i^{t+\varepsilon} \\ \hat{\mathbf{f}}_i^{t} \end{pmatrix} = \begin{pmatrix} 2 \cdot \alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \hat{\mathbf{f}}_i^{t} \\ \hat{\mathbf{f}}_i^{t-\varepsilon} \end{pmatrix}$$

$$0 < \alpha_i(\varepsilon) = \frac{1}{1 + \varepsilon^2 \cdot \lambda_i} \leq 1$$

Goal:

Setting $\mathbf{A}_i(\varepsilon) \in \mathbb{R}^{2\times 2}$ to be the matrix:

$$\mathbf{A}_i(\varepsilon) = \begin{pmatrix} 2 \cdot \alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix}$$

with $0 < \alpha_i(\varepsilon) \leq 1$, we would like to understand:

- **Step-Size Stability**: What happens to the size of $\mathbf{A}_i(\varepsilon)$ as $\varepsilon \to \infty$
- **Iterative Behavior**: What happens to the size of $\left(\mathbf{A}_i(\varepsilon)\right)^k$ as $k \to \infty$

Behavior of the Implicit Solution

$$\begin{pmatrix} \hat{\mathbf{f}}_i^{t+\varepsilon} \\ \hat{\mathbf{f}}_i^{t} \end{pmatrix} = \begin{pmatrix} 2\cdot\alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \hat{\mathbf{f}}_i^{t} \\ \hat{\mathbf{f}}_i^{t-\varepsilon} \end{pmatrix}$$

$$0 < \alpha_i(\varepsilon) = \frac{1}{1+\varepsilon^2\cdot\lambda_i} \leq 1$$

$$\mathbf{A}_i(\varepsilon) = \begin{pmatrix} 2\cdot\alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix}$$

Step-Size Stability:

Taking the Frobenius norm of $\mathbf{A}_i(\varepsilon)$ w.r.t. to the standard inner-product, we get the sum of the squares of the entries:

$$\begin{aligned} \|\mathbf{A}_i(\varepsilon)\|_{F,B}^2 &= 4\cdot\alpha_i^2(\varepsilon) + \alpha_i^2(\varepsilon) + 1 \\ &= 5\cdot\alpha_i^2(\varepsilon) + 1 \\ &\leq 6 \end{aligned}$$

$\Rightarrow$ The solver is unconditionally stable for all frequencies.

$$\boxed{0 < \alpha_i(\varepsilon) = \frac{1}{1 + \varepsilon^2 \cdot \lambda_i} \leq 1}$$

Behavior of the Implicit Solution

$$\mathbf{A}_i(\varepsilon) = \begin{pmatrix} 2 \cdot \alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix}$$

Iterative Behavior:

- Use the spectral radius to estimate the size of $\mathbf{A}_i(\varepsilon)$.
  - First compute the characteristic polynomial of $\mathbf{A}_i(\varepsilon)$,
  - Then compute the magnitude of the largest root of the characteristic polynomial.

$$\boxed{0 < \alpha_i(\varepsilon) = \frac{1}{1 + \varepsilon^2 \cdot \lambda_i} \leq 1}$$

Behavior of the Implicit Solution

$$\mathbf{A}_i(\varepsilon) = \begin{pmatrix} 2 \cdot \alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix}$$

Characteristic polynomial:

$$\begin{aligned} \chi_{\mathbf{A}_i(\varepsilon)}(\lambda) &= \det \begin{pmatrix} 2 \cdot \alpha_i(\varepsilon) - \lambda & -\alpha_i(\varepsilon) \\ 1 & -\lambda \end{pmatrix} \\ &= \lambda^2 - 2 \cdot \alpha_i(\varepsilon) \cdot \lambda + \alpha_i(\varepsilon) \end{aligned}$$

Roots:

$$\begin{aligned} \lambda &= \frac{2 \cdot \alpha_i(\varepsilon) \pm \sqrt{4 \cdot \alpha_i^2(\varepsilon) - 4 \cdot \alpha_i(\varepsilon)}}{2} \\ &= \alpha_i(\varepsilon) \pm \sqrt{\alpha_i^2(\varepsilon) - \alpha_i(\varepsilon)} \\ &= \alpha_i(\varepsilon) \pm i \cdot \sqrt{\alpha_i(\varepsilon) - \alpha_i^2(\varepsilon)} \end{aligned}$$

$$\boxed{0 < \alpha_i(\varepsilon) = \frac{1}{1 + \varepsilon^2 \cdot \lambda_i} \leq 1}$$

Behavior of the Implicit Solution

$$\mathbf{A}_i(\varepsilon) = \begin{pmatrix} 2 \cdot \alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix}$$

Eigenvalue magnitude:

$$\lambda = \alpha_i(\varepsilon) \pm i \cdot \sqrt{\alpha_i(\varepsilon) - \alpha_i^2(\varepsilon)}$$

Since $\alpha_i(\varepsilon) - \alpha_i^2(\varepsilon)$ is non-negative and the square norm is the sum of the squares of the real and imaginary:

$$\begin{aligned} |\lambda|^2 &= \alpha_i^2(\varepsilon) + \left(\sqrt{\alpha_i(\varepsilon) - \alpha_i^2(\varepsilon)}\right)^2 \\ &= \alpha_i^2(\varepsilon) + \left(\alpha_i(\varepsilon) - \alpha_i^2(\varepsilon)\right) \\ &= \alpha_i(\varepsilon) \end{aligned}$$

⇒ The squared spectral radius of $\mathbf{A}_i(\varepsilon)$ is $\alpha_i(\varepsilon)$.

$$\boxed{0 < \alpha_i(\varepsilon) = \frac{1}{1 + \varepsilon^2 \cdot \lambda_i} \leq 1}$$

Behavior of the Implicit Solution

$$\mathbf{A}_i(\varepsilon) = \begin{pmatrix} 2 \cdot \alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix}$$

Spectral radius:

$$\sigma^2\big(\mathbf{A}_i(\varepsilon)\big) = \frac{1}{1 + \varepsilon^2 \cdot \lambda_i}$$

For $\lambda_i > 0$, the spectral radius is less than one:

$$\lim_{k\to\infty} \sigma^2\left(\mathbf{A}_i^k(\varepsilon)\right) = \lim_{k\to\infty} \frac{1}{(1 + \varepsilon^2 \cdot \lambda_i)^k} = 0$$

For $\lambda_i = 0$, the spectral radius is one:

$$\lim_{k\to\infty} \sigma^2\left(\mathbf{A}_i^k(\varepsilon)\right) = \lim_{k\to\infty} \frac{1}{(1 + \varepsilon^2 \cdot \lambda_i)^k} = 1$$

$$\boxed{0 < \alpha_i(\varepsilon) = \frac{1}{1 + \varepsilon^2 \cdot \lambda_i} \leq 1}$$

Behavior of the Implicit Solution

$$\mathbf{A}_i(\varepsilon) = \begin{pmatrix} 2 \cdot \alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix}$$

For the non-constant eigen-functions, with $\lambda_i > 0$:

$$\lim_{k \to \infty} \sigma\left(\mathbf{A}_i^k(\varepsilon)\right) = 0$$

✓ For these frequencies, the method remains bounded over time.

✗ For these frequencies, displacement converges to zero.

⇒ Implicit time-stepping **does not** preserve the total energy of the system (numerical dissipation)

$$\boxed{0 < \alpha_i(\varepsilon) = \frac{1}{1+\varepsilon^2\cdot\lambda_i} \leq 1}$$

Behavior of the Implicit Solution

$$\mathbf{A}_i(\varepsilon) = \begin{pmatrix} 2\cdot\alpha_i(\varepsilon) & -\alpha_i(\varepsilon) \\ 1 & 0 \end{pmatrix}$$

For the constant eigen-function(s), with $\lambda_i = 0$:

$$\lim_{k\to\infty} \sigma\left(\mathbf{A}_i^k(\varepsilon)\right) = 1$$

⇒ For these frequencies, the method may not be bounded over time.

Note:

If initially the initial displacement is zero ($f^0 = 0$), and the initial velocity is constant and non-zero ($f^{-\varepsilon} = -c\cdot\varepsilon \neq 0$), we get the unbounded solution:

$$f^t = c\cdot t$$