Geometry Processing (601.458/658)

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[Vallet and Lévy, 2008] *Spectral Geometry Processing with Manifold Harmonics*
[Bhat *et al.*, 2008] *Fourier Analysis of the 2D Screened Poisson Equation for Gradient Domain Problems*

Outline

Recall

Spectral Geometry Processing

Gradient Modulation Revisited

Heat Diffusion Revisited

Recall

Given a vector space W, for any subspaces $U, V \subset W$, the intersection is also a subspace.

Proof:

For any $w_1, w_2 \in U \cap V$ and any $\alpha_1, \alpha_2 \in \mathbb{R}$, we have:

$$w_1, w_2 \in U \text{ and } w_1, w_2 \in V$$
$$\Downarrow$$
$$\alpha_1 \cdot w_1 + \alpha_2 \cdot w_2 \in U \text{ and } \alpha_1 \cdot w_2 + \alpha_2 \cdot W_2 \in V$$
$$\Downarrow$$
$$\alpha_1 \cdot w_1 + \alpha_2 \cdot w_2 \in U \cap V$$

Recall

Given a vector space W, for any subspaces $U, V \subset W$ let $\{w_1, \dots, w_k\}$ be a basis for $U \cap V$.

We can complete this to a basis $\{w_1, \dots, w_k, u_1, \dots, u_l\}$ for U and a basis basis $\{w_1, \dots, w_k, v_1, \dots, v_m\}$ for V.

Claim:

The vectors $\{w_1, \dots, w_k, u_1, \dots, u_l, v_1, \dots, v_m\}$ are linearly independent.

$U, V \subset W$
$\{w_1, \dots, w_k\}$ a basis for $U \cap V$
$\{w_1, \dots, w_k, u_1, \dots, u_l\}$ a basis for U
$\{w_1, \dots, w_k, v_1, \dots, v_m\}$ a basis for V

Recall

Claim:

The vectors $\{w_1, \dots, w_k, u_1, \dots, u_l, v_1, \dots, v_m\}$ are linearly independent.

Proof:

Suppose there exist $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l, \gamma_1, \dots, \gamma_m \in \mathbb{R}$ not all zero such that:

$$0 = \alpha_1 \cdot w_1 + \cdots + \alpha_k \cdot w_k + \beta_1 \cdot u_1 + \cdots + \beta_l \cdot u_l + \gamma_1 \cdot v_1 + \cdots + \gamma_m \cdot v_m$$

$\Rightarrow$ Some $\gamma_i \neq 0$, otherwise $\{w_1, \dots, w_k, u_1, \dots, u_l\}$ is linearly dependent.

$\Rightarrow$ Some $\alpha_i \neq 0$ or some $\beta_i \neq 0$, otherwise $\{v_1, \dots, v_m\}$ is linearly dependent.

$\Rightarrow$ There exist $\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l\}$ and $\{\gamma_1, \dots, \gamma_m\}$, neither all zero, with:

$$\alpha_1 \cdot w_1 + \cdots + \alpha_k \cdot w_k + \beta_1 \cdot u_1 + \cdots + \beta_l \cdot u_l = \gamma_1 \cdot v_1 + \cdots + \gamma_m \cdot v_m$$

Recall

$U, V \subset W$
$\{w_1, \ldots, w_k\}$ a basis for $U \cap V$
$\{w_1, \ldots, w_k, u_1, \ldots, u_l\}$ a basis for U
$\{w_1, \ldots, w_k, v_1, \ldots, v_m\}$ a basis for V

Claim:

The vectors $\{w_1, \ldots, w_k, u_1, \ldots, u_l, v_1, \ldots, v_m\}$ are linearly independent.

Proof:

There exist $\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l\}$ and $\{\gamma_1, \ldots, \gamma_m\}$ neither all zero, with:

$$\alpha_1 \cdot w_1 + \cdots + \alpha_k \cdot w_k + \beta_1 \cdot u_1 + \cdots + \beta_l \cdot u_l = \gamma_1 \cdot v_1 + \cdots + \gamma_m \cdot v_m$$

$\Rightarrow$ The left-hand-side is a non-zero vector in U.

$\Rightarrow$ The right-hand-side is a non-zero vector in V.

$\Rightarrow$ The right-hand-side is not in $U \cap V$
(otherwise $\{w_1, \ldots, w_k, v_1, \ldots, v_m\}$ would be linearly dependent.)

$\Rightarrow \Leftarrow$

$U, V \subset W$
$\{w_1, \dots, w_k\}$ a basis for $U \cap V$
$\{w_1, \dots, w_k, u_1, \dots, u_l\}$ a basis for U
$\{w_1, \dots, w_k, v_1, \dots, v_m\}$ a basis for V

Recall

Claim:

The vectors $\{w_1, \dots, w_k, u_1, \dots, u_l, v_1, \dots, v_m\}$ are linearly independent.

Corollary:

$$\begin{aligned}
\dim(W) &\geq k + l + m \\
&= k + (k + l) + (k + m) - 2k \\
&= (k + l) + (k + m) - k \\
&= \dim(U) + \dim(V) - \dim(U \cap V)
\end{aligned}$$

$$\Updownarrow$$

$$\dim(U \cap V) \geq \dim(U) + \dim(V) - \dim(W)$$

Recall

Given a vector space V, vectors $v_1, v_2 \in V$, and real values $\alpha_1, \alpha_2 \in \mathbb{R}$ which do not sum to zero, the *weighted average* of the two vectors is:

$$\frac{\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2}{\alpha_1 + \alpha_2} = \frac{\alpha_1}{\alpha_1 + \alpha_2} \cdot v_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} \cdot v_2$$

This is a weighted average* because the weights sum to 1.

*Possibly with negative weights

Recall

Unit Sphere:

Given an inner-product space $\{V, B: V \to V^*\}$ the *unit-sphere in* V is the zero level-set of the function:

$$
\begin{aligned}
F: V &\to \mathbb{R} \\
v &\mapsto \|v\|_B^2 - 1
\end{aligned}
$$

Recall

Spectral Decomposition:

Given an inner-product space $\{V, B: V \to V^*\}$ and a self-adjoint $A \in \mathrm{End}(V)$, solving the **eigenproblem**:

$$A(v_i) = \lambda_i \cdot v_i$$

gives an orthonormal eigen-basis $\{(v_1, \lambda_1), \dots, (v_n, \lambda_n)\} \subset V \times \mathbb{R}$.

An eigenvector must exist because for any self-adjoint $A \in \mathrm{End}(V)$, the extremum of the quadratic form:

$$Q_{B \circ A}: V \to \mathbb{R}$$

must be an eigenvector.

Other eigenvectors are obtained inductively by noting that self-adjoint operators take spaces orthogonal to an eigenvector back to themselves.

Recall

Spectral Decomposition:

Alternatively, given an inner-product space $\{V, B: V \to V^*\}$ and a symmetric operator $A \in \mathrm{Hom}(V, V^*)$, solving the **generalized eigenproblem**:

$$A(v_i) = \lambda_i \cdot B(v_i)$$

gives an orthonormal eigen-basis $\{(v_1, \lambda_1), \ldots, (v_n, \lambda_n)\} \subset V \times \mathbb{R}$.

This can be solved by solving the eigenproblem using the self-adjoint operator $B^{-1} \circ A: V \to V$.

Recall

Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we discretize the space of scalar functions, V, using the hat basis.

On the space of scalar functions, we have two bilinear forms:

$M \in \mathrm{Hom}(V, V^*)$: The inner-product on scalar functions:

$$[M(f)](g) = \langle f, g \rangle_{\mathcal{M}}$$

$S \in \mathrm{Hom}(V, V^*)$: The pull-back of the inner-product on cotangent vector-fields (a.k.a. the stiffness):

$$[S(f)](g) = \langle df, dg \rangle_{\mathcal{M}}$$

The stiffness gives (twice) the Dirichlet energy when both arguments are the same:

$$[S(f)](f) = \langle df, df \rangle_{\mathcal{M}}$$

Recall

Using the mass and stiffness, we can express the heat diffusion PDE:

$$\frac{\partial f^t}{\partial t} = -M^{-1}\big(S(f^t)\big)$$

with $f^0 \in V$ the initial signal, $f^t \in V$ the signal after diffusing for time t, and

$$-M^{-1} \circ S \in \mathrm{End}(V)$$

the Laplace operator.

We can solve using explicit time-stepping:

$$\frac{f^{t+\varepsilon} - f^t}{\varepsilon} = -M^{-1}\big(S(f^t)\big)$$

$$\Downarrow$$

$$M(f^{t+\varepsilon}) = (M - \varepsilon \cdot S)(f^t)$$

Recall

Using the mass and stiffness, we can express the heat diffusion PDE:

$$\frac{\partial f^t}{\partial t} = -M^{-1}\left(S(f^t)\right)$$

with $f^0 \in V$ the initial signal, $f^t \in V$ the signal after diffusing for time t, and

$$-M^{-1} \circ S \in \mathrm{End}(V)$$

the Laplace operator.

Or, we can solve using implicit time-stepping:

$$\frac{f^{t+\varepsilon} - f^t}{\varepsilon} = -M^{-1}\left(S(f^{\boxed{t+\varepsilon}})\right)$$

$$\Downarrow$$

$$(M + \varepsilon \cdot S)(f^{t+\varepsilon}) = M(f^t)$$

Outline

Recall

Spectral Geometry Processing

Gradient Modulation Revisited

Heat Diffusion Revisited

Geometry Processing

Solving the (generalized) stiffness* eigenproblem we get an orthonormal eigen-basis $\left\{(\psi_1,\lambda_1),\cdots,\left(\psi_{|\mathcal{V}|},\lambda_{|\mathcal{V}|}\right)\right\}$ for the space of scalar functions, V, with $0\leq\lambda_1\leq\cdots\leq\lambda_{|\mathcal{V}|}$, satisfying:

$$S(\psi_i)=\lambda_i\cdot M(\psi_i)$$

⇒ These are dual vectors in V^*, and evaluating on ψ_i gives:

$$\begin{aligned}[S(\psi_i)](\psi_i) &= [\lambda_i\cdot M(\psi_i)](\psi_i)\\ &\Updownarrow\\ \langle d\psi_i,d\psi_i\rangle_{\mathcal{M}} &= \lambda_i\cdot[M(\psi_i)](\psi_i)\\ &= \lambda_i\cdot\langle\psi_i,\psi_i\rangle_{\mathcal{M}}\\ &= \lambda_i\end{aligned}$$

⇒ The eigenvalues are (twice) the Dirichlet energies of the eigen-functions.

*This is the same as solving the Laplacian eigenproblem (and negating the eigenvalues).

Geometry Processing

Solving the (generalized) stiffness eigenproblem, the eigenvalues are (twice) the Dirichlet energies of the eigen-functions.

Interpretation:

The unit sphere in V is the set of scalar functions with unit-norm.

The minimizer $\psi \in V$ of the Dirichlet energy over the unit sphere is an eigenvector of the self-adjoint Laplace operator $M^{-1} \circ S: V \to V$.

$\Rightarrow$ The **global** minimizer $\psi \in V$ of the Dirichlet energy over the unit sphere is the eigenvector with the smallest eigenvalue.

$\Rightarrow$ The first eigen-function ψ_1 is the global minimizer of the Dirichlet energy on the unit sphere.

Geometry Processing

Solving the (generalized) stiffness eigenproblem, the eigenvalues are (twice) the Dirichlet energies of the eigen-functions.

The first eigen-function ψ_1 is the global minimizer of the Dirichlet energy on the unit sphere.

Interpretation :

The second eigen-function is the smallest eigen-function of the restriction of the Dirichlet energy to the space of functions orthogonal to ψ_1.

$\Rightarrow$ The second eigen-function is the global minimizer of the Dirichlet energy within the subset of unit-norm functions orthogonal to ψ_1.

Geometry Processing

Solving the (generalized) stiffness eigenproblem, the eigenvalues are (twice) the Dirichlet energies of the eigen-functions.

The first eigen-function ψ_1 is the global minimizer of the Dirichlet energy on the unit sphere.

The second eigen-function ψ_2 is the global minimizer of the Dirichlet energy within the subset of unit-norm functions orthogonal to ψ_1.

The third eigen-function ψ_3 is the global minimizer of the Dirichlet energy within the subset of unit-norm functions orthogonal to both ψ_1 and ψ_2.

…

Since constant functions have zero Dirichlet energy, the first eigen-function is always the constant function

Geometry Processing

Frequency Decomposition:

The solution to the stiffness eigenproblem defines function subspaces:

$$V_k = \mathrm{Span}(\psi_1, \dots, \psi_k) \subset V$$

with $\dim(V_k) = k$ and the function spaces nesting:

$$V_1 \subset \cdots \subset V_n = V$$

Given any k-dimensional subspace of V, we can compute the maximum Dirichlet energy over the subset of unit-norm functions.

Claim:

The minimum, over all k-dimensional subspaces of V, of the maximum Dirichlet energy is obtained on V_k (and it will be twice λ_k).

Geometry Processing

Frequency Decomposition:

The solution to the stiffness eigenproblem defines function subspaces:

$$V_k = \mathrm{Span}(\psi_1, \dots, \psi_k) \subset V$$

with d

In the context of looking for nesting subspaces consisting of the smoothest functions, we cannot do better.

$\Downarrow$

In this context we are interested in the **smoother** eigen-functions.
i.e. eigen-functions with **smaller** eigenvalues/Dirichlet energies

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Claim:

The minimum, over all k-dimensional subspaces of V, of the maximum Dirichlet energy is obtained on V_k (and it will be twice λ_k).

For all $U, V \subset W$:
$\dim(U \cap V) \geq \dim(U) + \dim(V) - \dim(W)$

Geometry Processing

Proof:

Let $\hat{V} \subset V$ be any k-dimensional subspace. We would like to show that there is some unit-norm function in $\hat{v} \in \hat{V}$ with stiffness at least λ_k.

Note that $W_k = \mathrm{Span}(\psi_k, \dots, \psi_n)$ is an $(n-k+1)$-dimensional subspace.

$\Rightarrow$ The dimension of the intersection satisfies:

$$\dim\left(\hat{V} \cap W_k\right) \geq 1$$

$\Rightarrow$ There exist $\alpha_k, \dots, \alpha_n \in \mathbb{R}$, not all zero, such that:

$$\hat{v} = \alpha_k \cdot \psi_k + \cdots + \alpha_n \cdot \psi_n \in \hat{V}$$

$\Rightarrow$ We can rescale, so that $\alpha_k^2 + \cdots \alpha_n^2 = 1$ (i.e. so that $\hat{v} \in \hat{V}$ is unit-norm).

Geometry Processing

Proof:

Let $\hat{V} \subset V$ be any k-dimensional subspace, there exist $\alpha_k, \ldots, \alpha_n \in \mathbb{R}$ with $\alpha_k^2 + \cdots + \alpha_n^2 = 1$ such that:

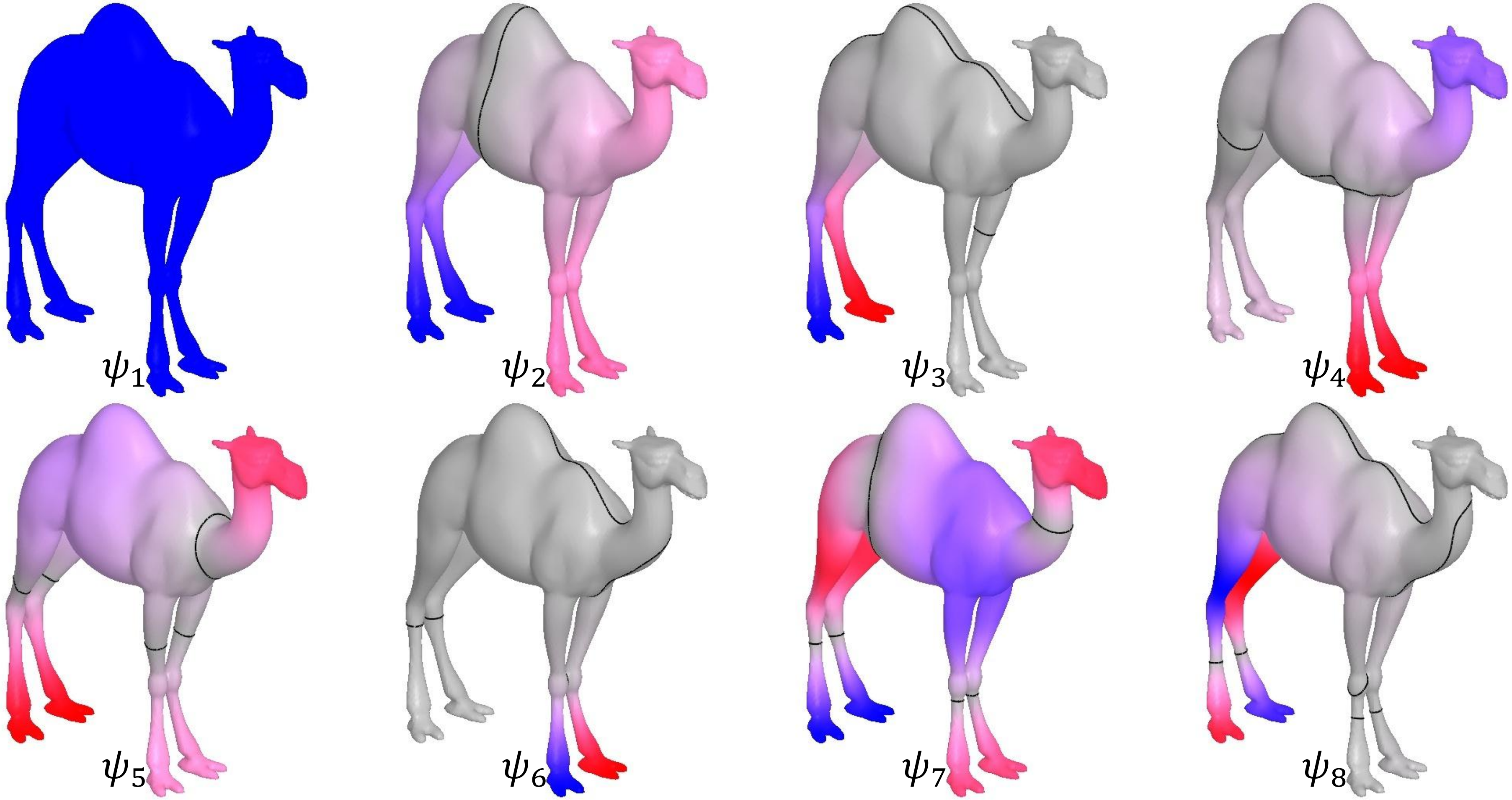
$$\hat{v} = \alpha_k \cdot \psi_k + \cdots + \alpha_n \cdot \psi_n \in \hat{V}$$

Taking the stiffness gives:

$$\begin{aligned}
[S(\hat{v})](\hat{v}) &= \left[M\big((M^{-1} \circ S)(\hat{v})\big)\right](\hat{v}) \\
&= \langle \hat{v}, (M^{-1} \circ S)(\hat{v}) \rangle_{\mathcal{M}} \\
&= \langle \alpha_k \cdot \psi_k + \cdots + \alpha_n \cdot \psi_n, \alpha_k \cdot \lambda_k \cdot \psi_k + \cdots + \alpha_n \cdot \lambda_n \cdot \psi_n \rangle_{\mathcal{M}} \\
&= \alpha_k^2 \cdot \lambda_k + \cdots + \alpha_n^2 \cdot \lambda_n \\
&\geq \alpha_k^2 \cdot \lambda_k + \cdots + \alpha_n^2 \cdot \lambda_k \\
&= \lambda_k \cdot \left(\alpha_k^2 + \cdots + \alpha_n^2\right) \\
&= \lambda_k
\end{aligned}$$

Geometry Processing

We can consider the eigen-basis itself.



Geometry Processing

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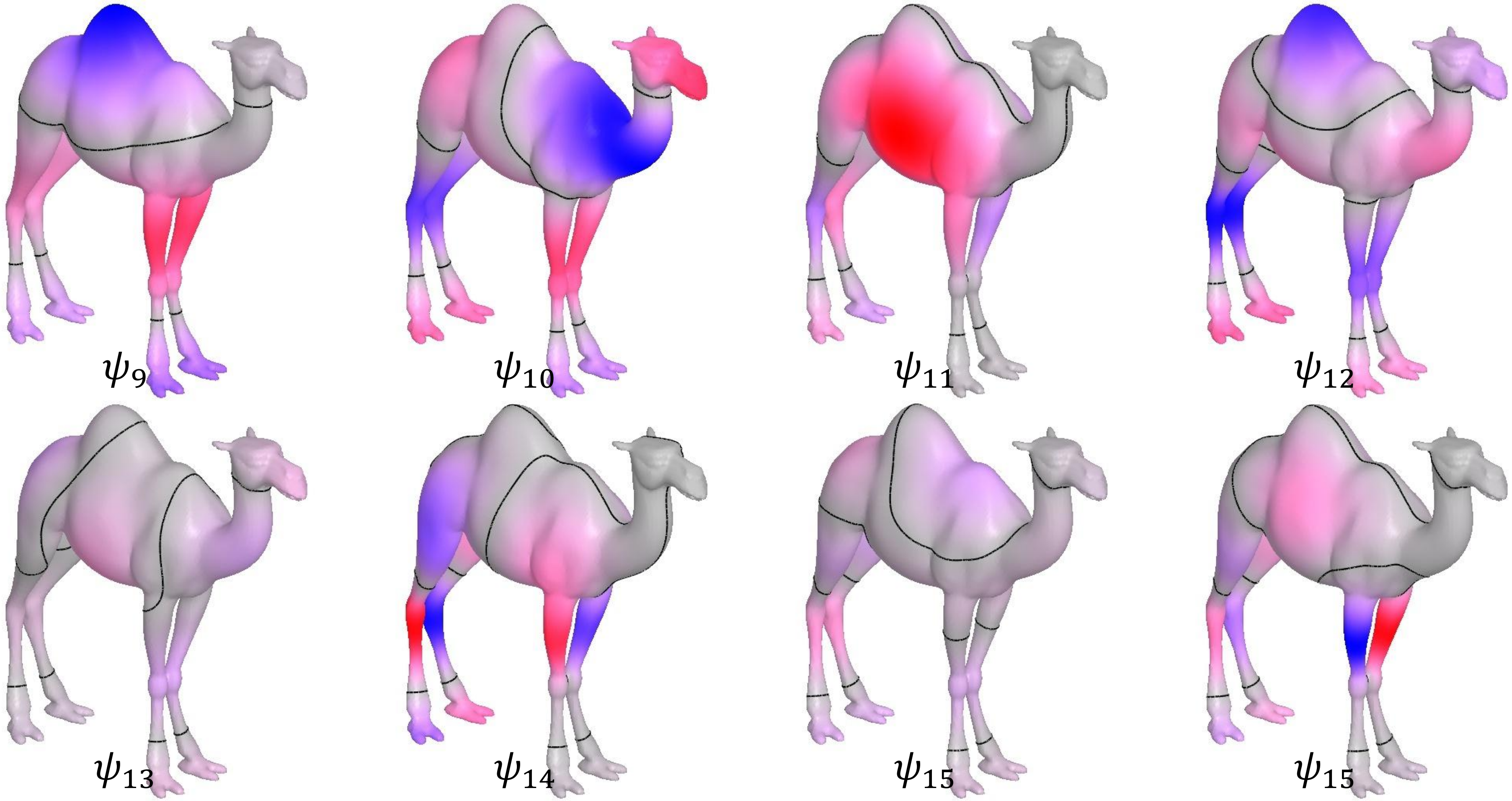


Geometry Processing

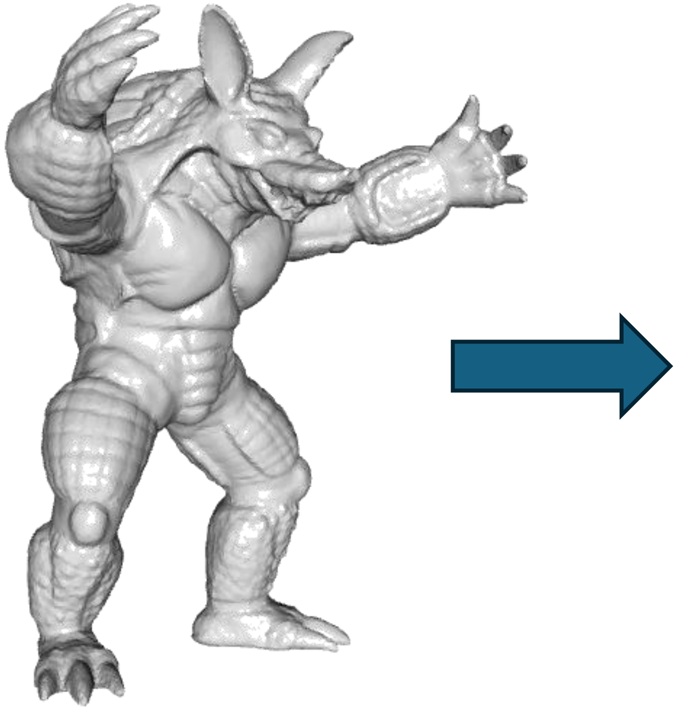
Or, we can consider the projection of a function defined over the mesh onto the subspaces $V_k = \text{Span}(\psi_1, \dots, \psi_k)$:

$$\begin{aligned}\pi_k \colon V &\to V_k \\ f &\mapsto \langle \psi_1, f\rangle_{\mathcal{M}} \cdot \psi_1 + \cdots + \langle \psi_k, f\rangle_{\mathcal{M}} \cdot \psi_k\end{aligned}$$

Notation:

For circular/spherical domains, the expression of a function in terms of the eigenvectors of the stiffness matrix is the *Fourier decomposition*.

Abusing terminology, the coefficient $\hat{\mathbf{f}}_i = \langle \psi_i, f\rangle_{\mathcal{M}}$ is called the i*-th Fourier coefficients* of f.



Geometry Processing

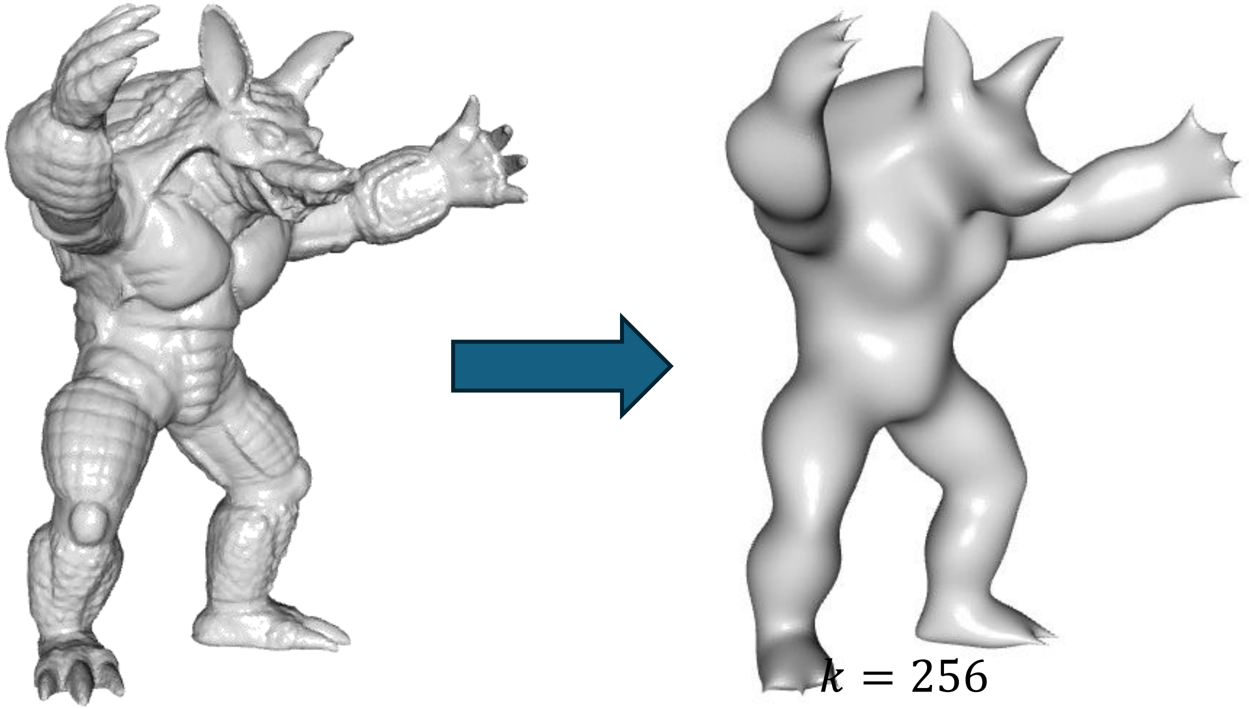
Or, we can consider the projection of a function defined over the mesh onto the subspaces $V_k = \mathrm{Span}(\psi_1, \dots, \psi_k)$:

$$\begin{aligned}\pi_k : V &\to V_k \\ f &\mapsto \langle \psi_1, f \rangle_{\mathcal{M}} \cdot \psi_1 + \cdots + \langle \psi_k, f \rangle_{\mathcal{M}} \cdot \psi_k\end{aligned}$$

In practice:

Need to use many eigenvectors to reproduce fine signal details.

- ✘ The Arnoldi method is quadratic in the number of eigenvectors (due to the orthogonalization).



Outline

Recall

Spectral Geometry Processing

Gradient Modulation Revisited

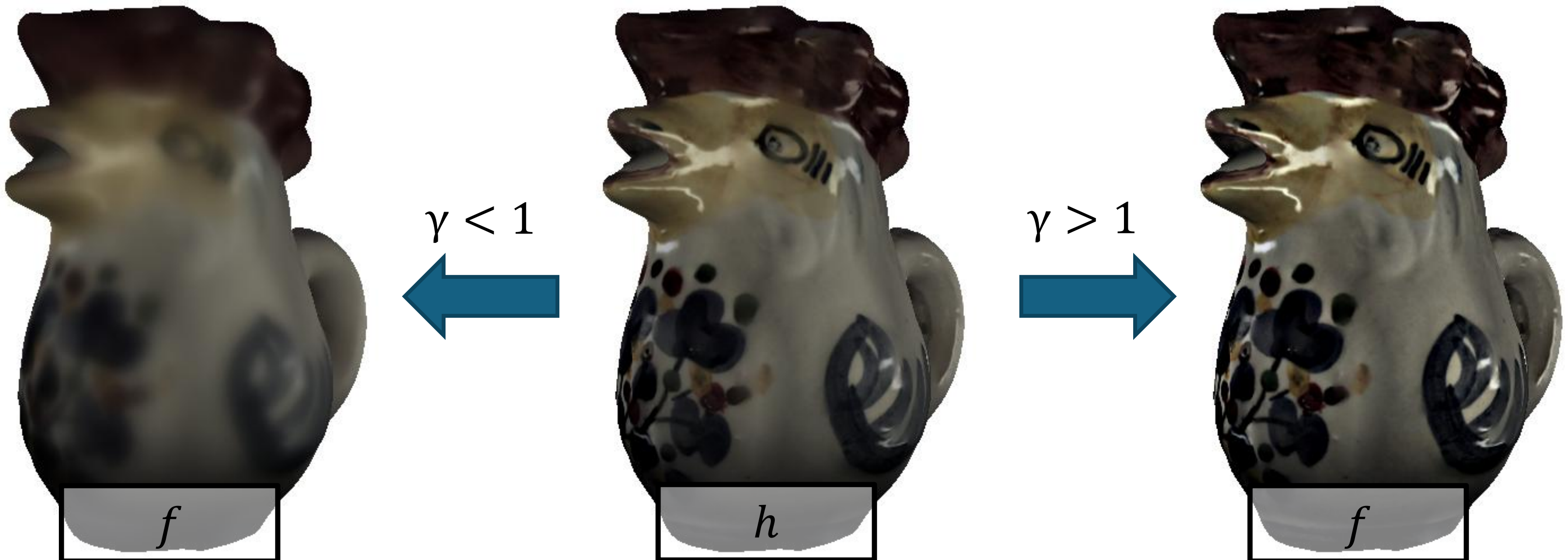
Heat Diffusion Revisited

Gradient Modulation Revisited

Recall:

Given an input scalar function $h \in V$ and a modulation factor $\gamma \geq 0$, we can solve the gradient modulation problem to smooth/sharpen detail:

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \gamma \cdot dh, df - \gamma \cdot dh \rangle\rangle_{\mathcal{M}}$$
$$\Downarrow$$
$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot \gamma \cdot S(h)$$



Gradient Modulation Revisited

Recall:

Given an input scalar function $h \in V$ and a modulation factor $\gamma \geq 0$, we can solve the gradient modulation problem to smooth/sharpen detail:

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \gamma \cdot dh, df - \gamma \cdot dh \rangle\rangle_{\mathcal{M}}$$
$$\Downarrow$$
$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot \gamma \cdot S(h)$$

Generalization:

Setting $g = \gamma \cdot h$, we can think of this as a more general expression of the *gradient domain blending* problem:

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - dg, df - dg \rangle\rangle_{\mathcal{M}}$$
$$\Downarrow$$
$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot S(g)$$

Gradient Modulation Revisited

$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot S(g)$$

Spectral Interpretation:

Let $\{(\psi_1, \lambda_1), \cdots, (\psi_n, \lambda_n)\}$ be the solution to the stiffness eigenproblem:

$$S(\psi_i) = \lambda_i \cdot M(\psi_i)$$

We can express the functions f, h, and g in terms of this basis:

$$f = \hat{\mathbf{f}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n \cdot \psi_n$$
$$h = \hat{\mathbf{h}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{h}}_n \cdot \psi_n$$
$$g = \hat{\mathbf{g}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{g}}_n \cdot \psi_n$$

Gradient Modulation Revisited

$$\boxed{\begin{aligned} S(\psi_i) &= \lambda_i \cdot M(\psi_i) \\ f &= \hat{\mathbf{f}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n \cdot \psi_n \\ h &= \hat{\mathbf{h}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{h}}_n \cdot \psi_n \\ g &= \hat{\mathbf{g}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{g}}_n \cdot \psi_n \end{aligned}}$$

$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot S(g)$$

Left-Hand-Side:

$$\begin{aligned} (\alpha \cdot M + \beta \cdot S)(f) &= (\alpha \cdot M + \beta \cdot S)\left(\hat{\mathbf{f}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n \cdot \psi_n\right) \\ &= \alpha \cdot \hat{\mathbf{f}}_1 \cdot M(\psi_1) + \beta \cdot \hat{\mathbf{f}}_1 \cdot S(\psi_1) + \cdots + \alpha \cdot \hat{\mathbf{f}}_n \cdot M(\psi_n) + \beta \cdot \hat{\mathbf{f}}_n \cdot S(\psi_n) \\ &= \alpha \cdot \hat{\mathbf{f}}_1 \cdot M(\psi_1) + \beta \cdot \hat{\mathbf{f}}_1 \cdot \lambda_1 \cdot M(\psi_1) + \cdots + \alpha \cdot \hat{\mathbf{f}}_n \cdot M(\psi_n) + \beta \cdot \hat{\mathbf{f}}_n \cdot \lambda_n \cdot M(\psi_n) \\ &= \hat{\mathbf{f}}_1 \cdot (\alpha + \beta \cdot \lambda_1) \cdot M(\psi_1) + \cdots + \hat{\mathbf{f}}_n \cdot (\alpha + \beta \cdot \lambda_n) \cdot M(\psi_n) \end{aligned}$$

This is a dual vector, and evaluating on ψ_i gives:

$$\begin{aligned} [(\alpha \cdot M + \beta \cdot S)(f)](\psi_i) &= \left[\hat{\mathbf{f}}_1 \cdot (\alpha + \beta \cdot \lambda_1) \cdot M(\psi_1) + \cdots + \hat{\mathbf{f}}_n \cdot (\alpha + \beta \cdot \lambda_n) \cdot M(\psi_n)\right](\psi_i) \\ &= \hat{\mathbf{f}}_1 \cdot (\alpha + \beta \cdot \lambda_1) \cdot [M(\psi_1)](\psi_i) + \cdots + \hat{\mathbf{f}}_n \cdot (\alpha + \beta \cdot \lambda_n) \cdot [M(\psi_n)](\psi_i) \\ &= \hat{\mathbf{f}}_1 \cdot (\alpha + \beta \cdot \lambda_1) \cdot \langle \psi_1, \psi_i \rangle_{\mathcal{M}} + \cdots + \hat{\mathbf{f}}_n \cdot (\alpha + \beta \cdot \lambda_n) \cdot \langle \psi_n, \psi_i \rangle_{\mathcal{M}} \\ &= \hat{\mathbf{f}}_i \cdot (\alpha + \beta \cdot \lambda_i) \end{aligned}$$

$$S(\psi_i) = \lambda_i \cdot M(\psi_i)$$
$$f = \hat{\mathbf{f}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n \cdot \psi_n$$
$$h = \hat{\mathbf{h}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{h}}_n \cdot \psi_n$$
$$g = \hat{\mathbf{g}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{g}}_n \cdot \psi_n$$

Gradient Modulation Revisited

$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot S(g)$$

Left-Hand-Side:

$$[(\alpha \cdot M + \beta \cdot S)(f)](\psi_i) = \hat{\mathbf{f}}_i \cdot (\alpha + \beta \cdot \lambda_i)$$

Right-Hand-Side:

$$\begin{aligned}
\alpha \cdot M(h) + \beta \cdot S(g) &= \alpha \cdot M\left(\hat{\mathbf{h}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{h}}_n \cdot \psi_n\right) + \beta \cdot S(\hat{\mathbf{g}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{g}}_n \cdot \psi_n) \\
&= \alpha \cdot \hat{\mathbf{h}}_1 \cdot M(\psi_1) + \cdots + \alpha \cdot \hat{\mathbf{h}}_n \cdot M(\psi_n) + \beta \cdot \hat{\mathbf{g}}_1 \cdot S(\psi_1) + \cdots + \beta \cdot \hat{\mathbf{g}}_n \cdot S(\psi_n) \\
&= \alpha \cdot \hat{\mathbf{h}}_1 \cdot M(\psi_1) + \cdots + \alpha \cdot \hat{\mathbf{h}}_n \cdot M(\psi_n) + \beta \cdot \hat{\mathbf{g}}_1 \cdot \lambda_1 \cdot M(\psi_1) + \cdots + \beta \cdot \hat{\mathbf{g}}_n \cdot \lambda_n \cdot M(\psi_n) \\
&= \left(\alpha \cdot \hat{\mathbf{h}}_1 + \beta \cdot \lambda_1 \cdot \hat{\mathbf{g}}_1\right) \cdot M(\psi_1) + \cdots + \left(\alpha \cdot \hat{\mathbf{h}}_n + \beta \cdot \lambda_n \cdot \hat{\mathbf{g}}_n\right) \cdot M(\psi_n)
\end{aligned}$$

This is also a dual vector, and evaluating on ψ_i gives:

$$\begin{aligned}
[\alpha \cdot M(h) + \beta \cdot S(g)](\psi_i) &= \left[\left(\alpha \cdot \hat{\mathbf{h}}_1 + \beta \cdot \hat{\mathbf{g}}_1 \cdot \lambda_1\right) \cdot M(\psi_1) + \cdots + \left(\alpha \cdot \hat{\mathbf{h}}_n + \beta \cdot \hat{\mathbf{g}}_n \cdot \lambda_n\right) \cdot M(\psi_n)\right](\psi_i) \\
&= \alpha \cdot \hat{\mathbf{h}}_i + \beta \cdot \lambda_i \cdot \hat{\mathbf{g}}_i
\end{aligned}$$

$$S(\psi_i) = \lambda_i \cdot M(\psi_i)$$
$$f = \hat{\mathbf{f}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n \cdot \psi_n$$
$$h = \hat{\mathbf{h}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{h}}_n \cdot \psi_n$$
$$g = \hat{\mathbf{g}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{g}}_n \cdot \psi_n$$

Gradient Modulation Revisited

$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot S(g)$$

$$[(\alpha \cdot M + \beta \cdot S)(f)](\psi_i) = \hat{\mathbf{f}}_i \cdot (\alpha + \beta \cdot \lambda_i)$$
$$[\alpha \cdot M(h) + \beta \cdot S(g)](\psi_i) = \alpha \cdot \hat{\mathbf{h}}_i + \beta \cdot \lambda_i \cdot \hat{\mathbf{g}}_i$$

Since the two sides are equal:

$$\hat{\mathbf{f}}_i \cdot (\alpha + \beta \cdot \lambda_i) = \alpha \cdot \hat{\mathbf{h}}_i + \beta \cdot \lambda_i \cdot \hat{\mathbf{g}}_i$$
$$\Updownarrow$$
$$\hat{\mathbf{f}}_i = \frac{\alpha}{\alpha + \beta \cdot \lambda_i} \cdot \hat{\mathbf{h}}_i + \frac{\beta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i} \cdot \hat{\mathbf{g}}_i$$

$$S(\psi_i) = \lambda_i \cdot M(\psi_i)$$
$$f = \hat{\mathbf{f}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n \cdot \psi_n$$
$$h = \hat{\mathbf{h}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{h}}_n \cdot \psi_n$$
$$g = \hat{\mathbf{g}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{g}}_n \cdot \psi_n$$

Gradient Modulation Revisited

$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot S(g)$$
$$\hat{\mathbf{f}}_i = \frac{\alpha}{\alpha + \beta \cdot \lambda_i} \cdot \hat{\mathbf{h}}_i + \frac{\beta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i} \cdot \hat{\mathbf{g}}_i$$

$\Rightarrow$ The solution is the function $f \in V$ whose Fourier coefficients are **frequency-weighted** averages of the Fourier coefficients of h and g.

- The Fourier coefficients of h are given preference when λ_i is small (low frequencies)
- The Fourier coefficients of g are given preference when λ_i is large (high frequencies)
- Larger relative values of α bias the interpolation towards h
- Larger relative values of β bias the interpolation towards g

$\Rightarrow$ Since the λ_i are non-negative, the weights are also non-negative, so we're *interpolating* Fourier coefficients.

$$S(\psi_i) = \lambda_i \cdot M(\psi_i)$$
$$f = \hat{\mathbf{f}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n \cdot \psi_n$$
$$h = \hat{\mathbf{h}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{h}}_n \cdot \psi_n$$
$$g = \hat{\mathbf{g}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{g}}_n \cdot \psi_n$$

Gradient Modulation Revisited

$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot S(g)$$
$$\hat{\mathbf{f}}_i = \frac{\alpha}{\alpha + \beta \cdot \lambda_i} \cdot \hat{\mathbf{h}}_i + \frac{\beta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i} \cdot \hat{\mathbf{g}}_i$$

For gradient domain modulation, we set $g = \gamma \cdot h$:

$$\begin{aligned} \hat{\mathbf{f}}_i &= \frac{\alpha}{\alpha + \beta \cdot \lambda_i} \cdot \hat{\mathbf{h}}_i + \frac{\beta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i} \cdot \gamma \cdot \hat{\mathbf{h}}_i \\ &= \left(\frac{\alpha}{\alpha + \beta \cdot \lambda_i} \cdot 1 + \frac{\beta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i} \cdot \gamma \right) \cdot \hat{\mathbf{h}}_i \end{aligned}$$

$\Rightarrow$ The Fourier coefficients of the solution are obtained by scaling the Fourier coefficients of the input.

As $\lambda_i \to 0$ (i.e. low frequencies) the scale factor is close to 1.

As $\lambda_i \to \infty$ (i.e. high frequencies) the scale factor is close to γ.

$$S(\psi_i) = \lambda_i \cdot M(\psi_i)$$
$$f = \hat{\mathbf{f}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n \cdot \psi_n$$
$$h = \hat{\mathbf{h}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{h}}_n \cdot \psi_n$$
$$g = \hat{\mathbf{g}}_1 \cdot \psi_1 + \cdots + \hat{\mathbf{g}}_n \cdot \psi_n$$

Gradient Modulation Revisited

$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot S(g)$$

⇒ Gradient domain modulation preserves lower frequencies and modulates the higher ones.

For gradient domain modulation, we set $g = \gamma \cdot h$:

$$\begin{aligned}\hat{\mathbf{f}}_i &= \frac{\alpha}{\alpha + \beta \cdot \lambda_i} \cdot \hat{\mathbf{h}}_i + \frac{\beta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i} \cdot \gamma \cdot \hat{\mathbf{h}}_i \\ &= \left(\frac{\alpha}{\alpha + \beta \cdot \lambda_i} \cdot 1 + \frac{\beta \cdot \lambda_i}{\alpha + \beta \cdot \lambda_i} \cdot \gamma \right) \cdot \hat{\mathbf{h}}_i\end{aligned}$$

⇒ The Fourier coefficients of the solution are obtained by scaling the Fourier coefficients of the input.

As $\lambda_i \to 0$ (i.e. low frequencies) the scale factor is close to 1.

As $\lambda_i \to \infty$ (i.e. high frequencies) the scale factor is close to γ.

Outline

Recall

Spectral Geometry Processing

Gradient Modulation Revisited

Heat Diffusion Revisited

Heat Diffusion Revisited

Recall:

Given an initial signal on a mesh, represented in terms of the hat basis as the vector $f^0 \in V$, the diffused signal at time t is the solution to the PDE:

$$\frac{\partial f^t}{\partial t} = -M^{-1}\big(S(f^t)\big)$$

Explicit time-stepping:

$$M(f^{t+\varepsilon}) = (M - \varepsilon \cdot S)(f^t)$$

Implicit time-stepping:

$$(M + \varepsilon \cdot S)(f^{t+\varepsilon}) = M(f^t)$$

Heat Diffusion Revisited

$$M(f^{t+\varepsilon}) \; = (M - \varepsilon \cdot S)(f^t)$$
$$(M + \varepsilon \cdot S)(f^{t+\varepsilon}) = M(f^t)$$

Let $\{(\psi_1, \lambda_1), \cdots, (\psi_n, \lambda_n)\}$ be the solution to the stiffness eigenproblem:
$$S(\psi_i) = \lambda_i \cdot M(\psi_i)$$

We can express the functions f^t in terms of its Fourier decomposition:
$$f^t = \hat{\mathbf{f}}_1^t \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n^t \cdot \psi_n$$

Applying to both sides and evaluating on ψ_i...

$$f^t = \hat{\mathbf{f}}_1^t \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n^t \cdot \psi_n$$

Heat Diffusion Revisited

Explicit Time-Stepping:

$$M(f^{t+\varepsilon}) = (M - \varepsilon \cdot S)(f^t)$$

Left-Hand-Side:

$$[M(f^{t+\varepsilon})](\psi_i) = \hat{\mathbf{f}}_i^{t+\varepsilon}$$

Right-Hand-Side:

$$[(M - \varepsilon \cdot S)(f^t)](\psi_i) = \hat{\mathbf{f}}_i^t \cdot (1 - \varepsilon \cdot \lambda_i)$$

Equating:

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \hat{\mathbf{f}}_i^t \cdot (1 - \varepsilon \cdot \lambda_i)$$

$$f^t = \hat{\mathbf{f}}_1^t \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n^t \cdot \psi_n$$

Explicit:

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = (1 - \varepsilon \cdot \lambda_i) \cdot \hat{\mathbf{f}}_i^t$$

Heat Diffusion Revisited

Implicit Time-Stepping:

$$(M + \varepsilon \cdot S)(f^{t+\varepsilon}) = M(f^t)$$

Left-Hand-Side:

$$[(M + \varepsilon \cdot S)(f^{t+\varepsilon})](\psi_i) = \hat{\mathbf{f}}_i^{t+\varepsilon} \cdot (1 + \varepsilon \cdot \lambda_i)$$

Right-Hand-Side:

$$[M(f^t)](\psi_i) = \hat{\mathbf{f}}_i^t$$

Equating:

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \frac{\hat{\mathbf{f}}_i^t}{1 + \varepsilon \cdot \lambda_i}$$

$$f^t = \hat{\mathbf{f}}_1^t \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n^t \cdot \psi_n$$

Explicit:

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = (1 - \varepsilon \cdot \lambda_i) \cdot \hat{\mathbf{f}}_i^t$$

Implicit:

$$\hat{\mathbf{f}}_i^{t+\varepsilon} = \frac{1}{1+\varepsilon \cdot \lambda_i} \cdot \hat{\mathbf{f}}_i^t$$

Heat Diffusion Revisited

Both methods obtain the values for the next time-step by scaling the Fourier coefficients, at the current time-step.

$\Rightarrow$ After k time-steps of size ε, the i-th Fourier coefficient of the solution is:

Explicit:

$$\hat{\mathbf{f}}_i^{k \cdot \varepsilon} = (1 - \varepsilon \cdot \lambda_i)^k \cdot \hat{\mathbf{f}}_i^0$$

Implicit:

$$\hat{\mathbf{f}}_i^{k \cdot \varepsilon} = \frac{1}{(1 + \varepsilon \cdot \lambda_i)^k} \cdot \hat{\mathbf{f}}_i^0$$

Stability:

We would like the scale factor to be in $(-1,1]$ for all i.

$f^t = \hat{\mathbf{f}}_1^t \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n^t \cdot \psi_n$

Explicit:

$$\hat{\mathbf{f}}_i^{k\cdot\varepsilon} = (1 - \varepsilon \cdot \lambda_i)^k \cdot \hat{\mathbf{f}}_i^0$$

Implicit:

$$\hat{\mathbf{f}}_i^{k\cdot\varepsilon} = \frac{1}{(1+\varepsilon\cdot\lambda_i)^k} \cdot \hat{\mathbf{f}}_i^0$$

Heat Diffusion Revisited

Explicit Time-Stepping:

$$-1 < 1 - \varepsilon \cdot \lambda_i \leq 1$$

The condition $1 - \varepsilon \cdot \lambda_i \leq 1$ is trivially satisfied since $\varepsilon > 0$ and $\lambda_i \geq 0$.

The condition $-1 < 1 - \varepsilon \cdot \lambda_i$ requires:

$$\varepsilon < \frac{2}{\lambda_i}$$

$\Rightarrow$ Since λ_i grows with frequency (and mesh resolution), stability demands the time-step to be small (and dependent on the mesh resolution).

$f^t = \hat{\mathbf{f}}_1^t \cdot \psi_1 + \cdots + \hat{\mathbf{f}}_n^t \cdot \psi_n$

Explicit:

$$\hat{\mathbf{f}}_i^{k\cdot\varepsilon} = (1 - \varepsilon \cdot \lambda_i)^k \cdot \hat{\mathbf{f}}_i^0$$

Implicit:

$$\hat{\mathbf{f}}_i^{k\cdot\varepsilon} = \frac{1}{(1+\varepsilon\cdot\lambda_i)^k} \cdot \hat{\mathbf{f}}_i^0$$

Heat Diffusion Revisited

Implicit Time-Stepping:

$$-1 < \frac{1}{1 + \varepsilon \cdot \lambda_i} \leq 1$$

$\Rightarrow$ Since $\lambda_i \geq 0$ this condition is satisfied for all $\varepsilon > 0$ (and the scale factor is strictly positive).

$\Rightarrow$ Implicit integration is *unconditionally stable*.

$\Rightarrow$ This is what you observed in assignment 2.