Geometry Processing (601.458/658)

Misha Kazhdan

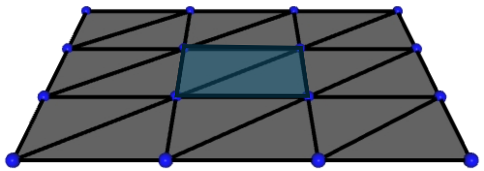
Outline

Recall

Scalar Bases

Whitney (1-form) Basis

Recall



Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$:

We have a space of scalar fields, spanned by the **per-vertex** functions:

$$V = \mathrm{Span}(\{\phi_v\}_{v \in \mathcal{V}})$$

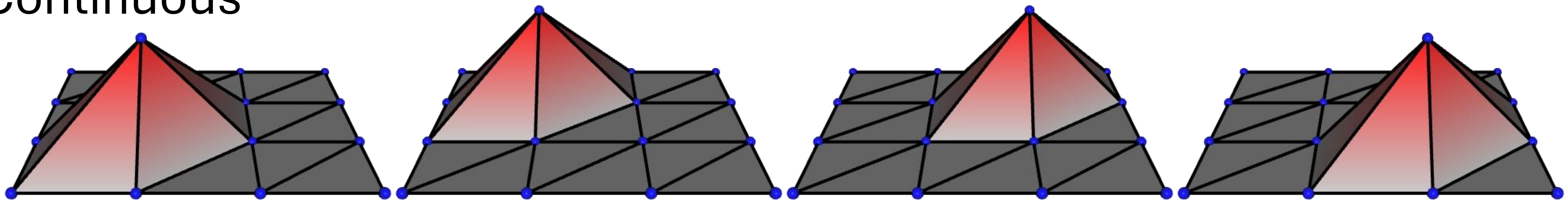
Properties:

Vertex interpolating (1 at its own vertex, 0 at all others)

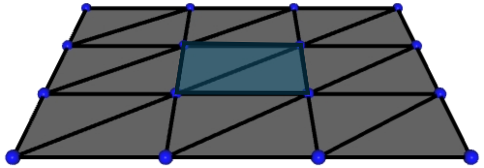
Linear within a triangle

✓ Form a partition of unity

✓ Continuous



Recall



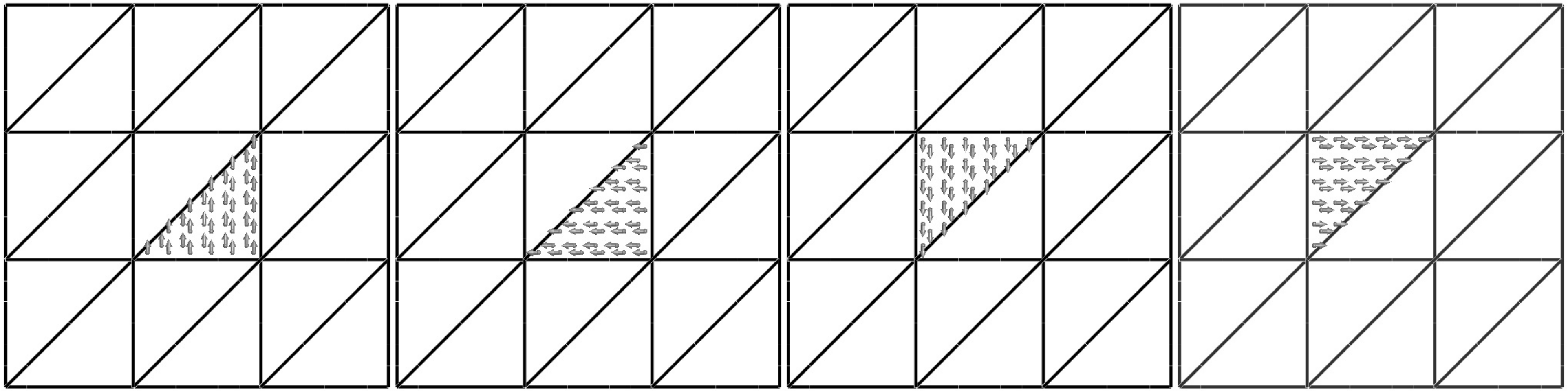
Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$:

We have a space of cotangent vector fields, spanned by the **per-triangle** functions:

$$\overline{V} = \mathrm{Span}(\{\eta_\tau^1, \eta_\tau^2\}_{\tau \in \mathcal{T}})$$

Properties:

Constant within a triangle



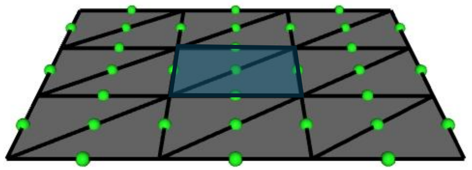
Outline

Recall

Scalar Bases

Whitney (1-form) Basis

Crouzeix-Raviart Bases



Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{E}, \mathcal{T}\}$:

One can define a space of scalar fields, spanned by the **per-edge** *Crouzeix-Raviart* functions.

Properties:

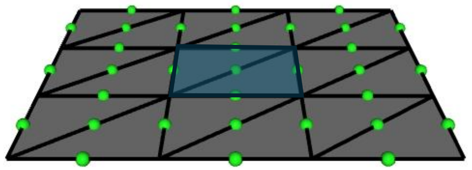
Edge mid-point interpolating (1 at its own edge mid-point, 0 at all others)

Linear within a triangle

✓ Form a partition of unity

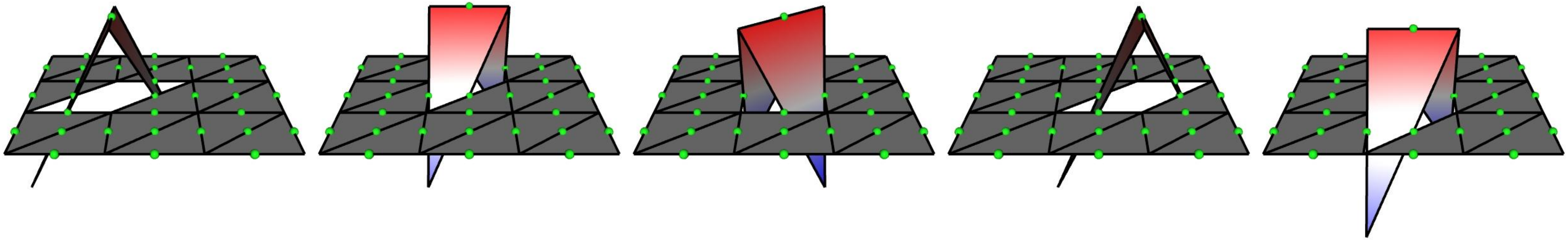
✗ Discontinuous

Crouzeix-Raviart Bases

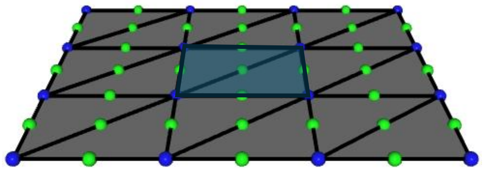


Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{E}, \mathcal{T}\}$:

One can define a space of scalar fields, spanned by the **per-edge** *Crouzeix-Raviart* functions.



Second-Order Lagrange Bases



Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{E}, \mathcal{T}\}$:

One can define a space of scalar fields, spanned by the **per-vertex** and **per-edge** *second-order Lagrange interpolants*.

Properties:

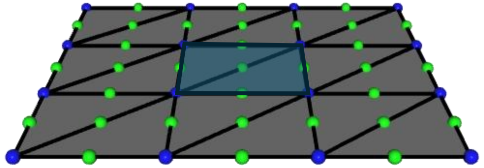
Vertex and edge mid-point interpolating (1 at its own node, 0 at all others)

Quadratic within a triangle

✓ Form a partition of unity

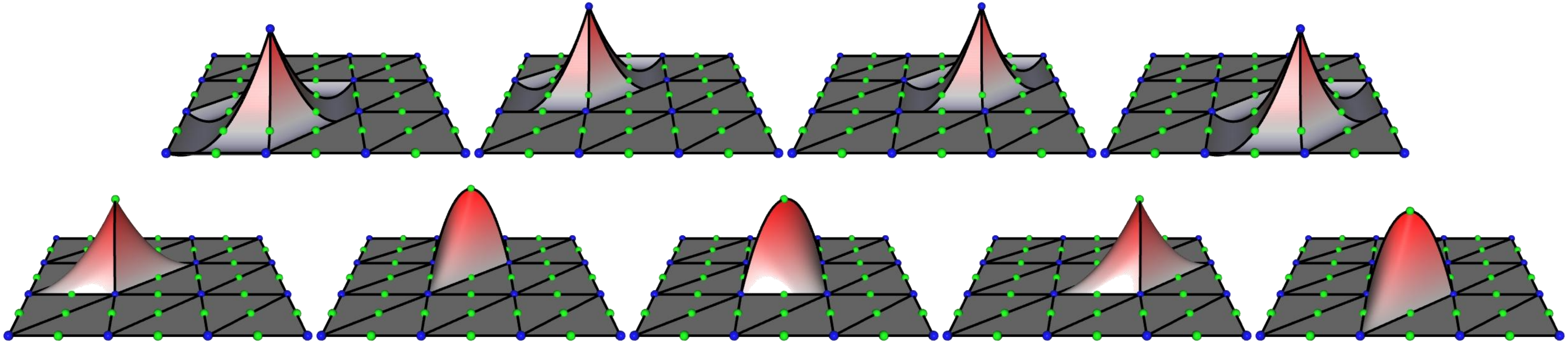
✓ Continuous

Second-Order Lagrange Bases

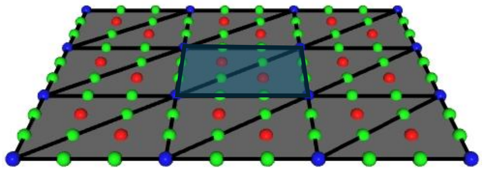


Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{E}, \mathcal{T}\}$:

One can define a space of scalar fields, spanned by the **per-vertex** and **per-edge** *second-order Lagrange interpolants*.

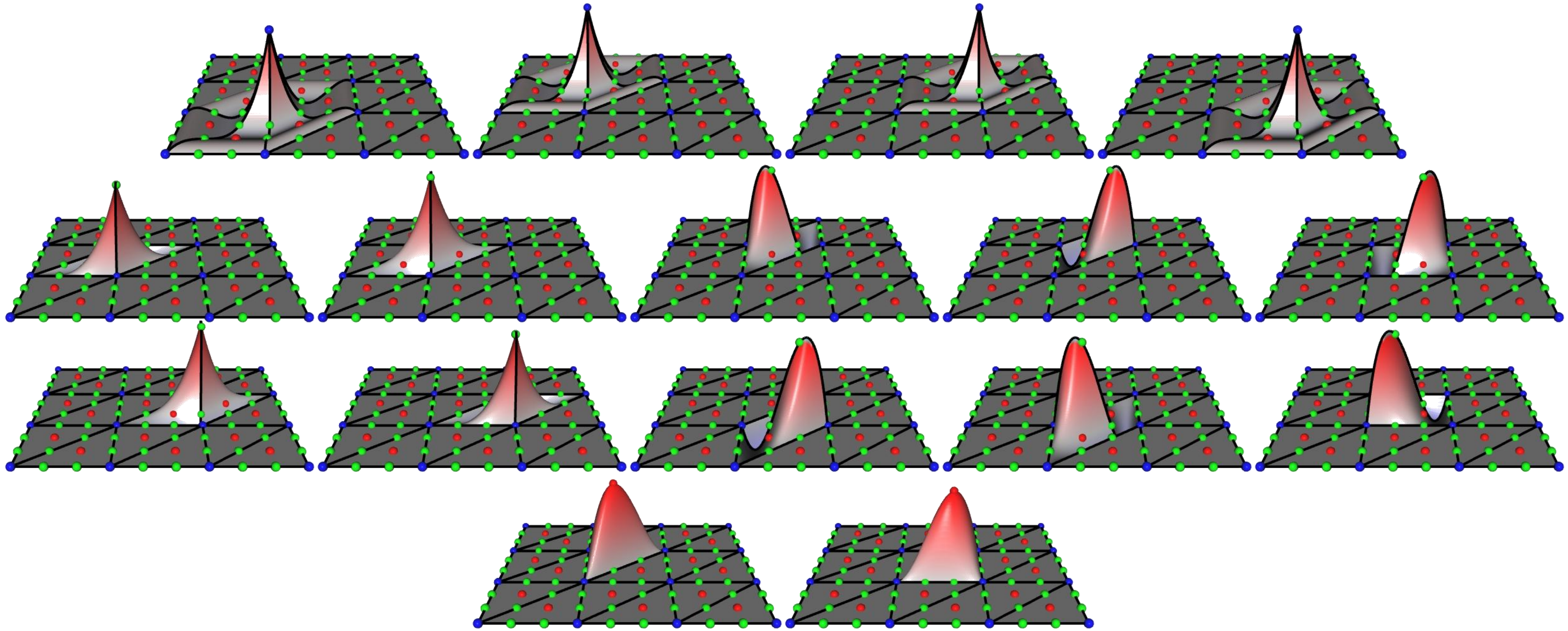


Higher-Order Lagrange Bases

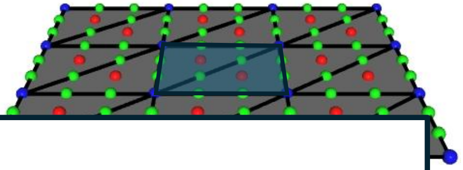


Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{E}, \mathcal{T}\}$:

One can define a space of scalar fields, spanned by *higher-order Lagrange interpolants* centered at vertices, along edges, and inside triangles.



Lagrange Bases



For all of these:

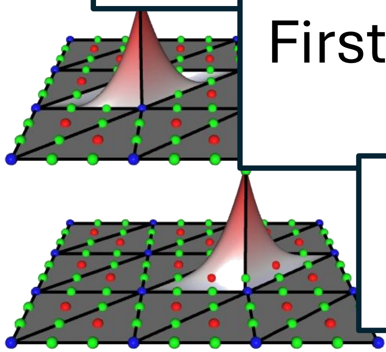
- Functions centered at vertices are supported in the one-ring of triangles
- Functions centered along edges are supported on the (at most) two incident triangles
- Functions centered in triangles are supported on the single triangle.

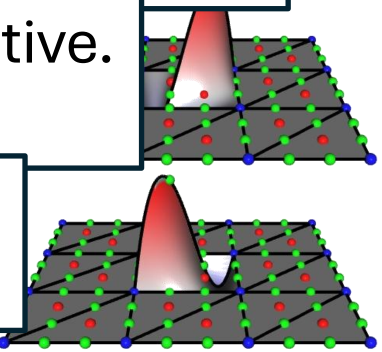
The functions are continuous across edges but their derivatives are not.*

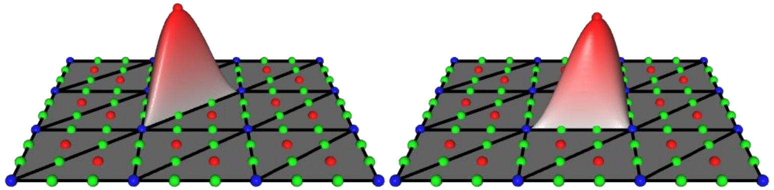
At every point on the triangle mesh, the functions sum to one.
⇒ Evaluation gives the weighted average of node values.

First-order (a.k.a "hat") elements are non-negative.
⇒ Interpolate the values at the nodes

Higher-order elements can be negative.
⇒ May extrapolate the values at the nodes







*Not counting zero-th order.

Lagrange Bases

In two-dimensions:

The space of linear polynomials is **three**-dimensional $\{x, y, 1\}$.

The space of quadratic polynomials is **six**-dimensional $\{x^2, y^2, xy, x, y, 1\}$.

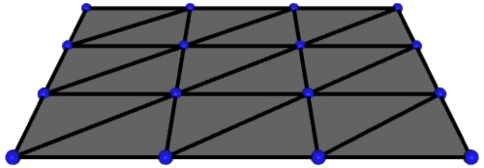
The space of cubic polynomials is **ten**-dimensional $\{x^3, x^2y, xy^2, y^3, x^2, y^2, xy, x, y, 1\}$.

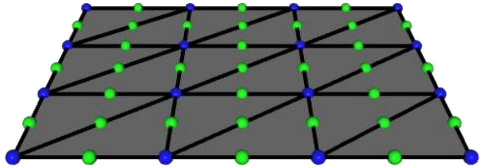
First-order Lagrange elements define **three** nodes per triangle.

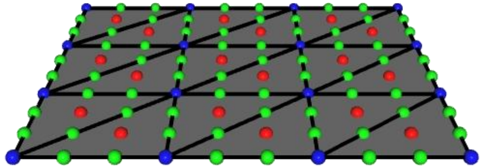
Second-order Lagrange elements define **six** nodes per triangle

Third-order Lagrange elements defined **ten** nodes per triangle.

Lagrange elements of order d reproduce polynomials of degree d from their values at the nodes.



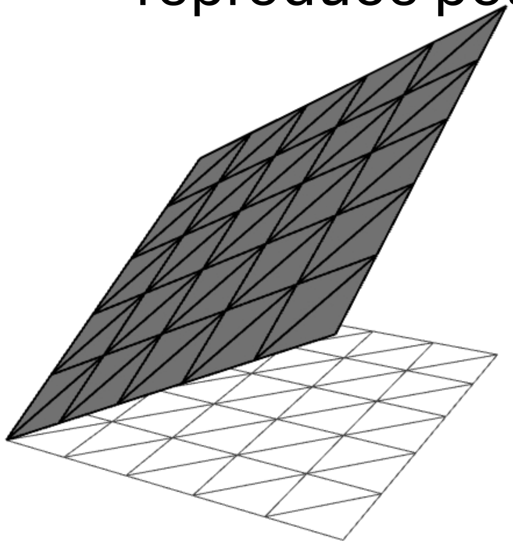




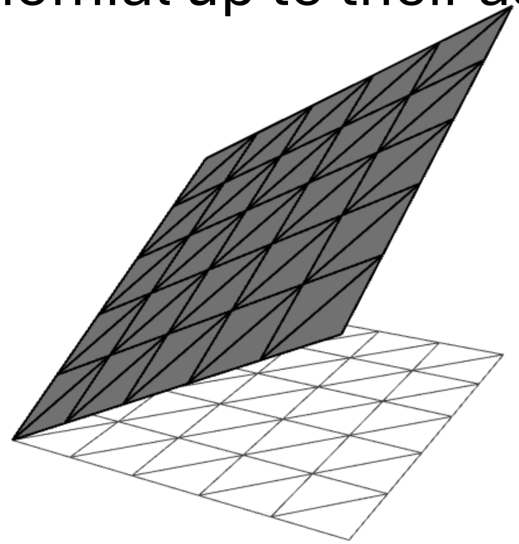
Scalar Bases

Polynomial Reproduction

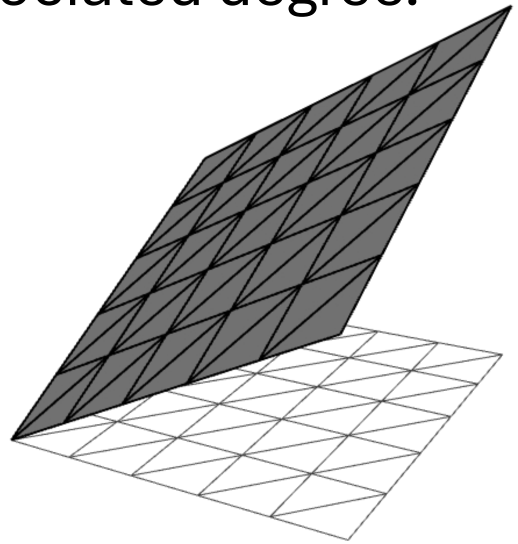
Because they are interpolatory and piecewise polynomial, all the elements reproduce polynomial up to their associated degree.



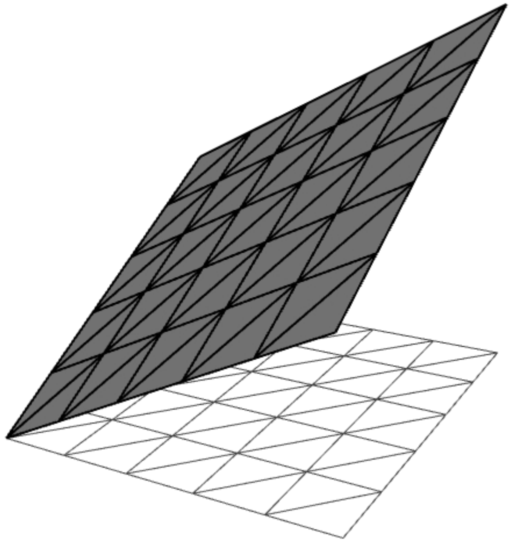
Crouzeix-Raviart



1st-order Lagrange



2nd-order Lagrange



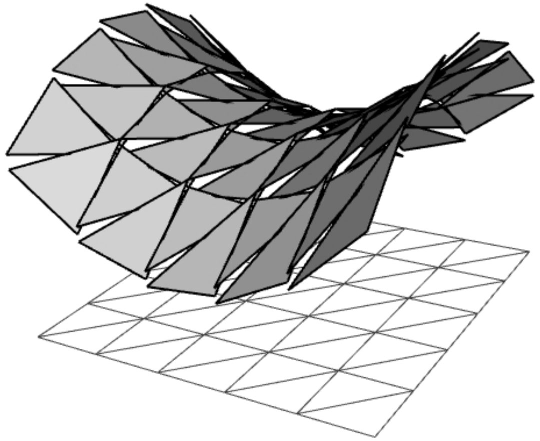
3rd-order Lagrange

$$P(x, y) = x - y$$

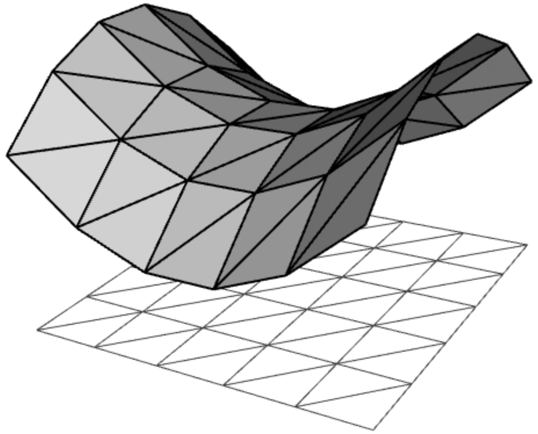
Scalar Bases

Polynomial Reproduction

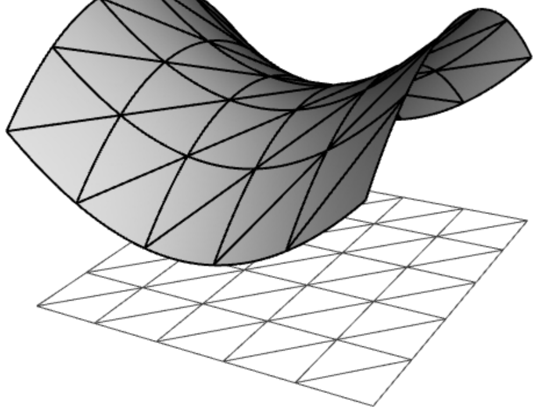
Because they are interpolatory and piecewise polynomial, all the elements reproduce polynomial up to their associated degree.



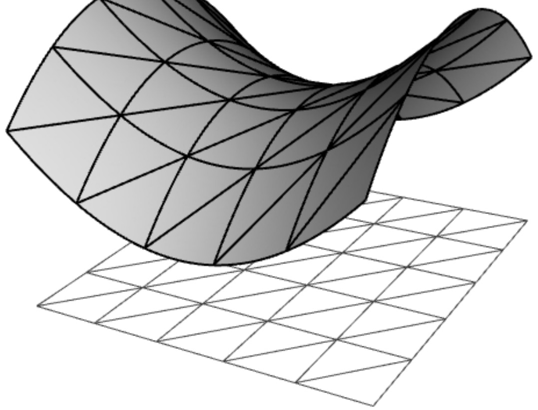
Crouzeix-Raviart



1st-order Lagrange



2nd-order Lagrange



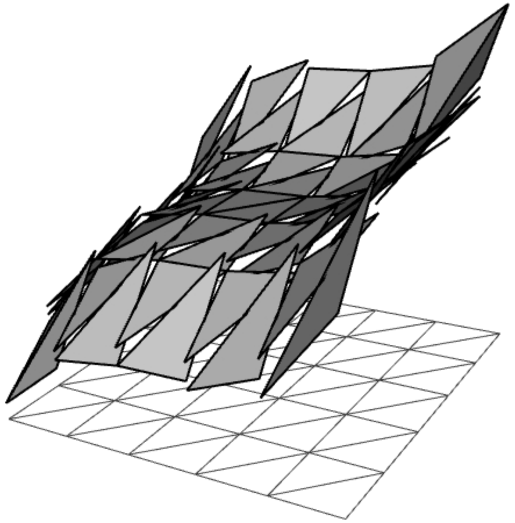
3rd-order Lagrange

$$P(x, y) = x^2 - y^2$$

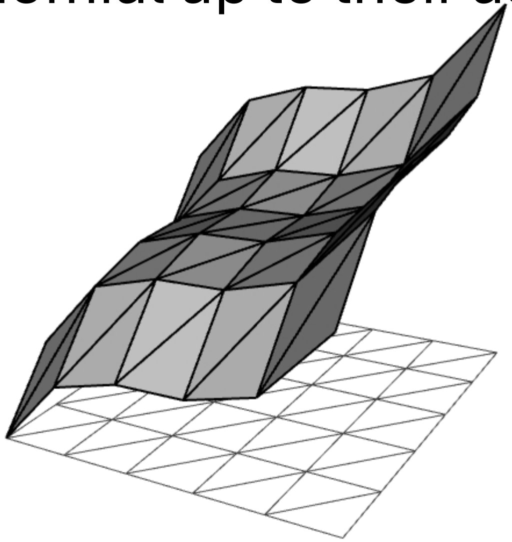
Scalar Bases

Polynomial Reproduction

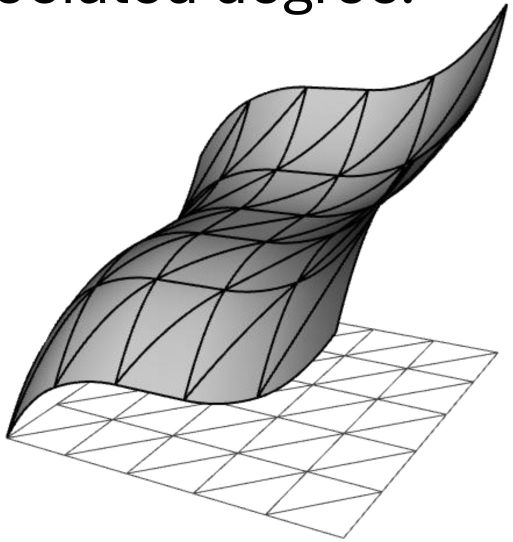
Because they are interpolatory and piecewise polynomial, all the elements reproduce polynomial up to their associated degree.



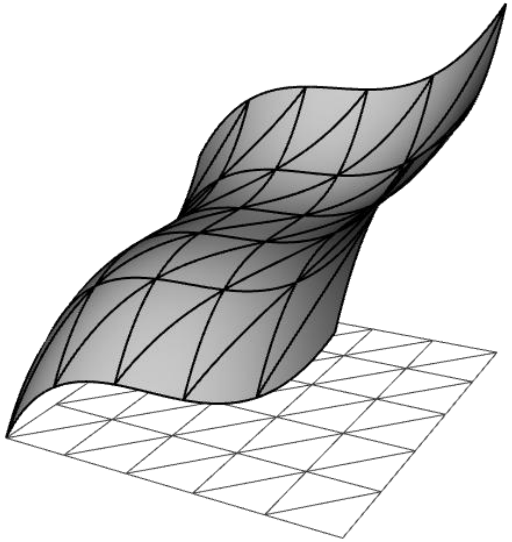
Crouzeix-Raviart



1st-order Lagrange



2nd-order Lagrange



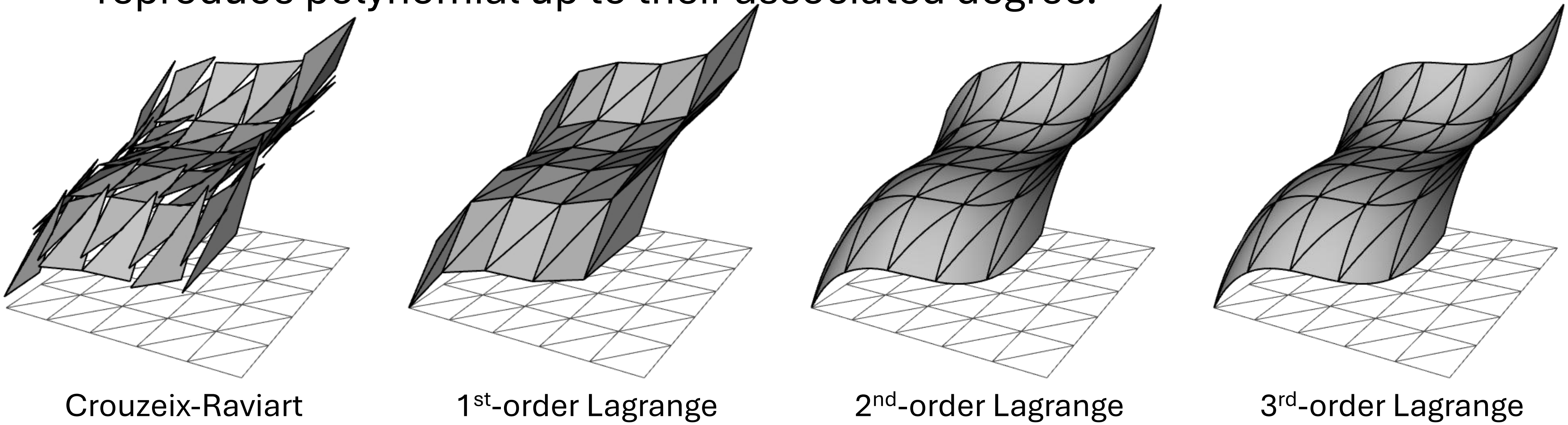
3rd-order Lagrange

$$P(x, y) = x^3 - y^3$$

Scalar Bases

Polynomial Reproduction

Because they are interpolatory and piecewise polynomial, all the elements reproduce polynomial up to their associated degree.



Though Crouzeix-Raviart elements are not continuous across edges, they are continuous (on average) at the edge mid-point.

Scalar Bases

Polynomial Reproduction

Because they are interpolatory and piecewise polynomial, all the elements reproduce polynomial up to their associated degree.

Being ineterpolatory also makes them “easy” to work with because the coefficients are the values of the function at the associated nodes.

Outline

Recall

Scalar Bases

Whitney (1-form) Basis

Support Graph

Definition:

Given a functions $\phi: \mathcal{M} \to \mathbb{R}$, the *support* of ϕ, denoted $\mathrm{Supp}(\phi)$ is the (closure of the) set of points on which the function is non-zero:

$$\mathrm{Supp}(\phi) = \overline{\{p \in \mathcal{M} \mid \phi(p) \neq 0\}}$$

Definition:

Given two functions $\phi, \psi: \mathcal{M} \to \mathbb{R}$, we say the functions have *overlapping support* if the intersections of their supports is not empty:

$$\mathrm{Supp}(\phi) \cap \mathrm{Supp}(\psi) \neq \emptyset$$

Support Graph

Definition:

Given a functions $\phi: \mathcal{M} \to \mathbb{R}$, the *support* of ϕ, denoted $\text{Supp}(\phi)$ is the (closure of the) set of points on which the function is non-zero:

$$\text{Supp}(\phi) = \overline{\{p \in \mathcal{M} \mid \phi(p) \neq 0\}}$$

Note:

For a differentiable function $\phi: \mathcal{M} \to \mathbb{R}$, the differential is zero wherever the function is zero.

$\Rightarrow$ The differential is zero in the interior of the support.

$\Rightarrow$ The support of the differential contains the support of the function:

$$\text{Supp}(\phi) \subset \text{Supp}(d\phi)$$

Support Graph

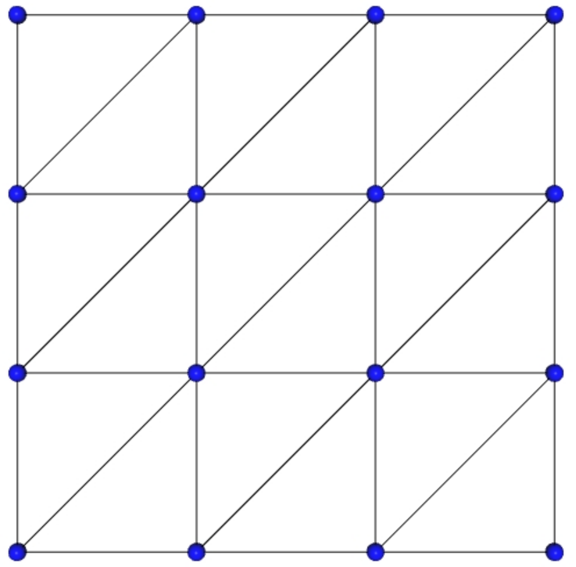
Definition:

Given a set of functions $\{\psi_i : \mathcal{M} \to \mathbb{R}\}_{i \in \mathcal{N}}$, we define the *support graph* to be the **undirected** graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}\}$ such that:

There is a distinct node for every function,

There is an edge between nodes if the associated functions have overlapping support.

Support Graph

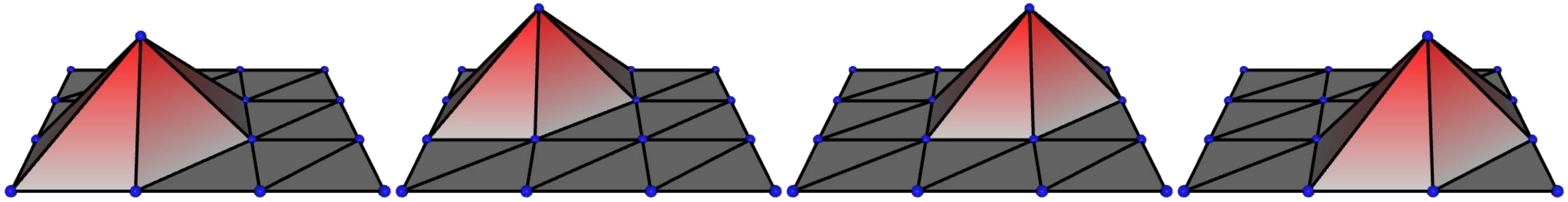


Hat Basis – Vertices:

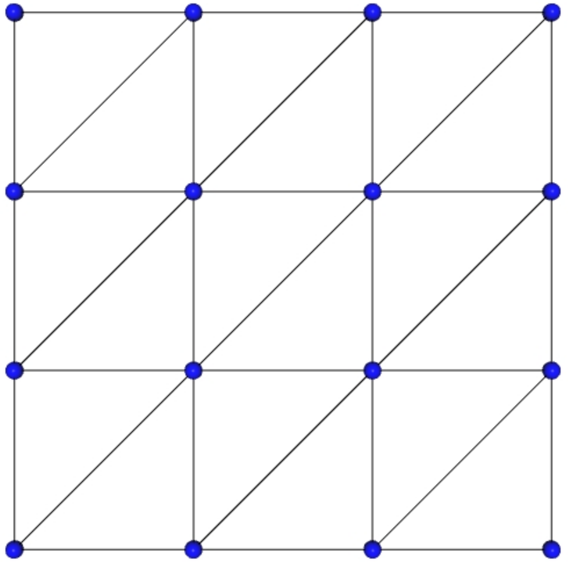
There is a function associated with every vertex.

⇒ The nodes of the support graph are the same as the vertices of the triangle mesh.

⇒ There are three support graph nodes associated with every mesh triangle.



Support Graph



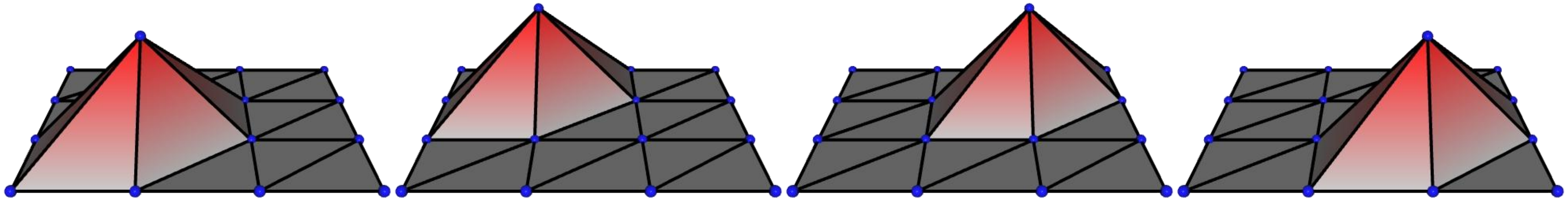
Hat Basis – Edges:

Functions have overlapping support if they are supported on the same triangle.

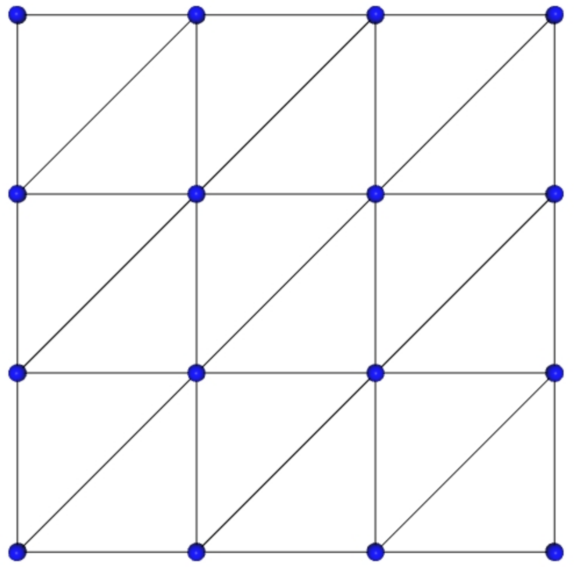
⇒ Functions have overlapping support if they are both vertices of the same triangle.

⇒ Functions have overlapping support if they are end-points of an edge in the triangle mesh.

⇒ There are three support graph edges associated with every mesh triangle.

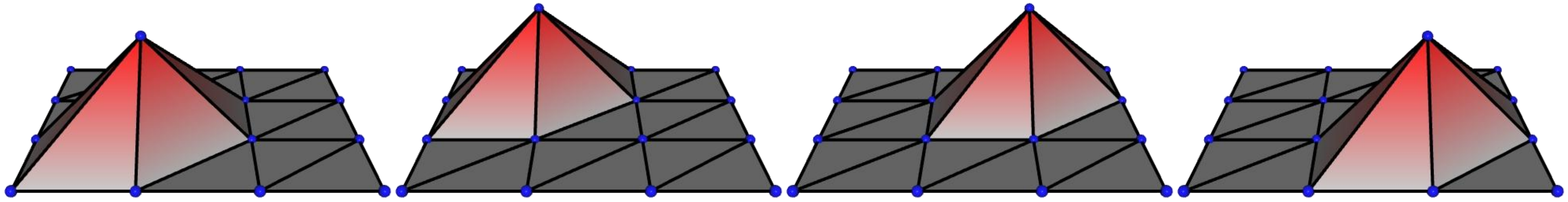


Support Graph

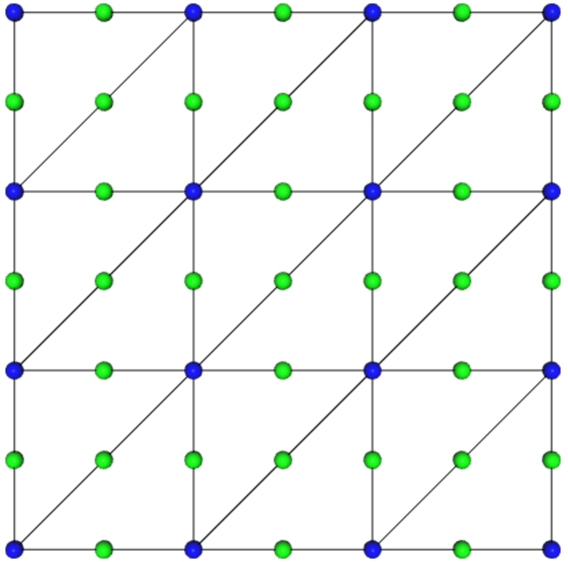


Hat Basis:

The support graph is **identical** to the mesh's edge graph.



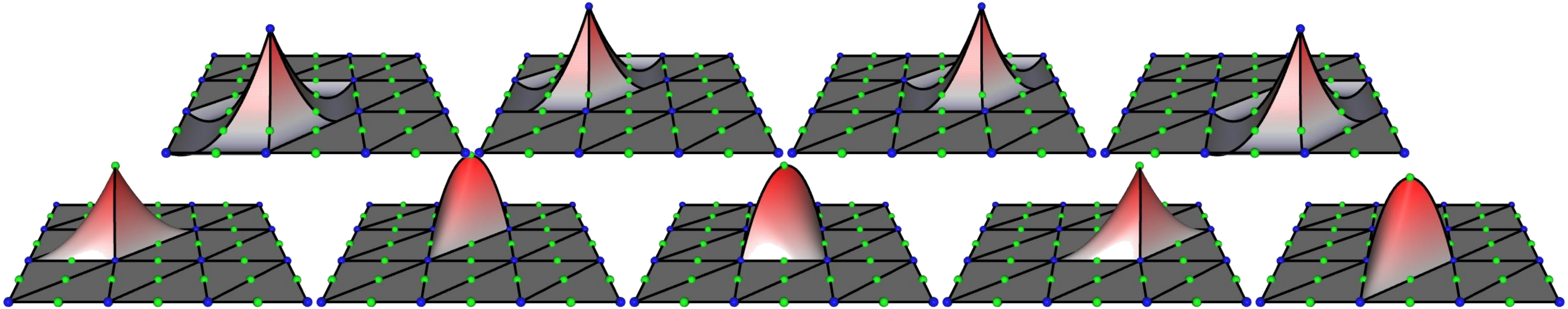
Support Graph



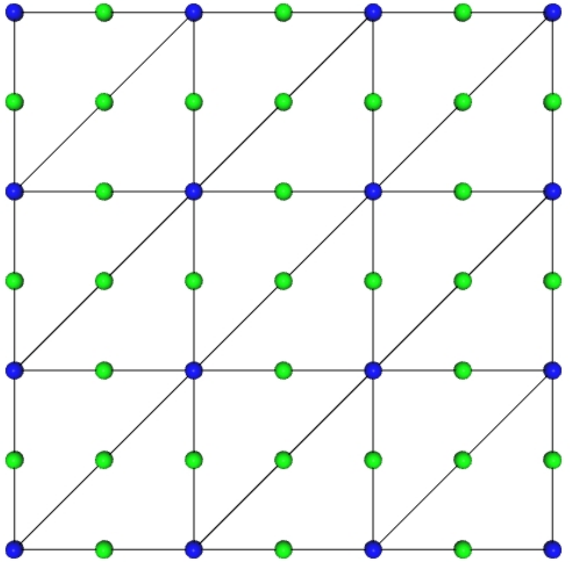
Second-Order Lagrange – Vertices:

There is a function associated with every vertex and every edge of the triangle mesh.

⇒ There are six support graph nodes associated with every mesh triangle.



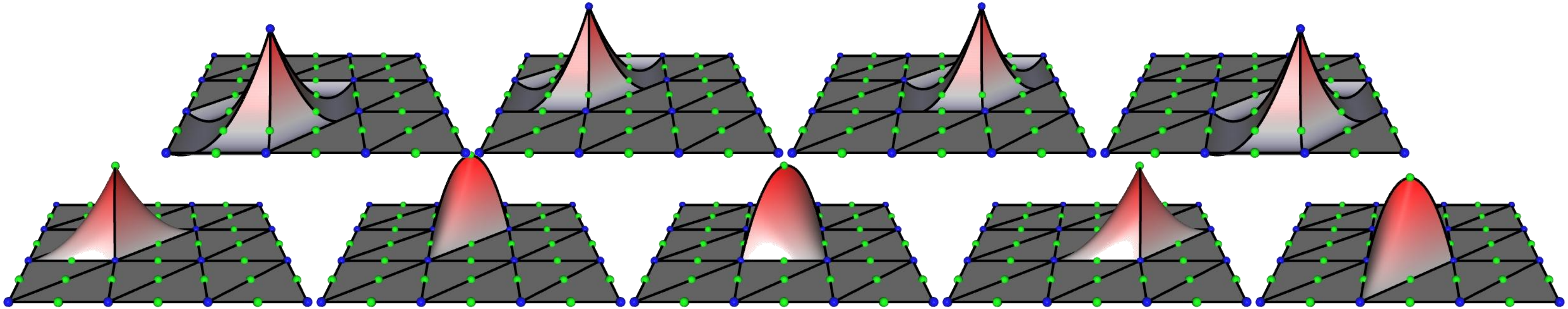
Support Graph



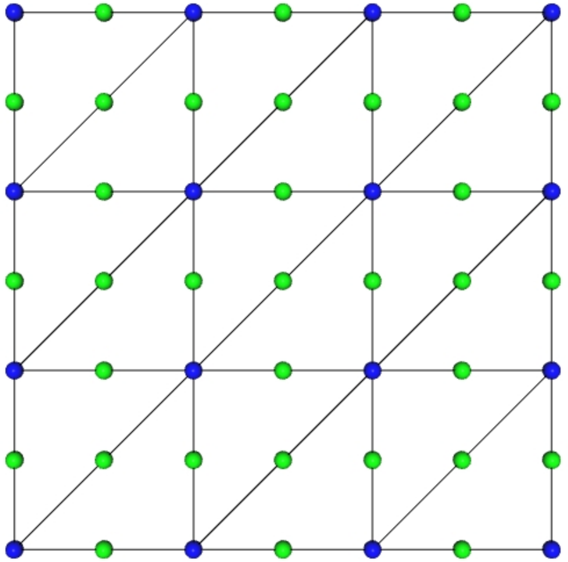
Second-Order Lagrange – Edges:

Functions have overlapping support if they are supported on the same triangle.

⇒ There are fifteen support graph edges associated with every mesh triangle.

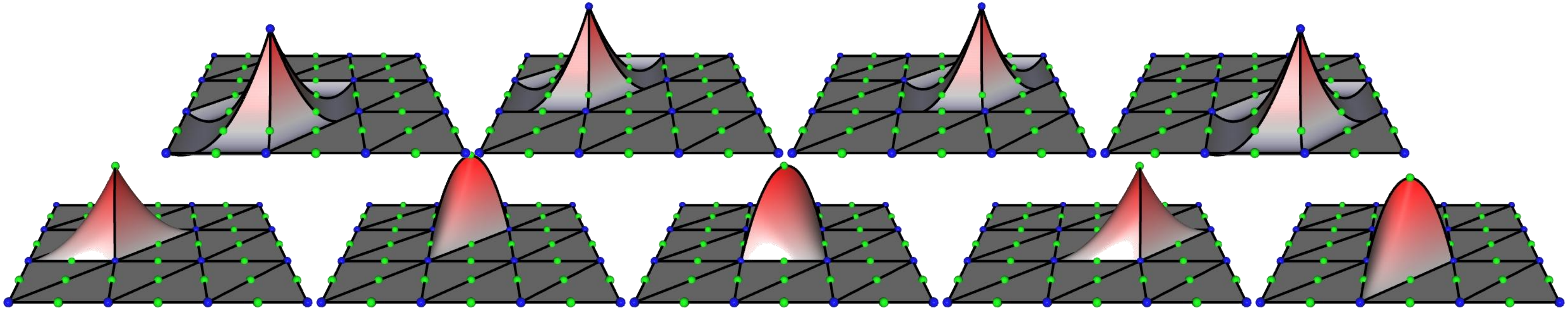


Support Graph



Second-Order Lagrange:

The support graph is **different** from the mesh's edge graph.



Whitney (1-form) Basis

Recall:

We say that a basis for scalar fields $\{\psi_i : \mathcal{M} \to \mathbb{R}\}_{i\in\mathcal{N}}$ *satisfies the partition of unity (PoU) property* if, for every point $p \in \mathcal{M}$:

$$\sum_{i\in\mathcal{N}} \psi_i(p) = 1$$

Property:

If the basis satisfies the PoU property, the sum of the differentials is zero:

$$\begin{aligned}\sum_{i\in\mathcal{N}} d\psi_i &= d\left(\sum_{i\in\mathcal{N}} \psi_i\right)\\ &= d(1)\\ &= 0\end{aligned}$$

Whitney (1-form) Basis

Goal:

Given a basis for scalar fields $\{\psi_i : \mathcal{M} \to \mathbb{R}\}_{i\in\mathcal{N}}$ we would like to define a basis for cotangent vector fields containing the differentials of the ψ_i.

This will make it possible to define a differential matrix.

Combined with mass matrices for scalar fields and cotangent vector fields this gives the components required for gradient-domain processing.

Whitney (1-form) Basis

Naïve Approach:

Given a basis for scalar fields $\{\psi_i: \mathcal{M} \to \mathbb{R}\}_{i\in\mathcal{N}}$ define a basis for cotangent vector fields by taking the differentials of the scalar fields:

$$\{d\psi_i\}_{i\in\mathcal{N}}$$

✓Simple to define

✘ Hard to formulate target vector fields for gradient-domain processing

✘ Only represents curl-free vector-fields*

*Not actually a limitation for gradient-domain processing.

$\text{Supp}(\phi) \subset \text{Supp}(d\phi)$

Whitney (1-form) Basis

Approach:

Given a basis for scalar fields $\{\psi_i : \mathcal{M} \to \mathbb{R}\}_{i \in \mathcal{N}}$ **satisfying the PoU property** define a spanning set for cotangent vector fields as:

$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

Note:

If ψ_i and ψ_j do not have overlapping support, the cotangent vector field ω_{ij} will be everywhere zero.

$\Rightarrow$ We can ignore it.

Whitney (1-form) Basis

Approach:

Given a basis for scalar fields $\{\psi_i : \mathcal{M} \to \mathbb{R}\}_{i \in \mathcal{N}}$ **satisfying the PoU property** define a spanning set for cotangent vector fields as:

$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

Note:

When the two indices are the same, we have:

$$\begin{aligned}\omega_{ii} &= \psi_i \cdot d\psi_i - \psi_i \cdot d\psi_j \\ &= 0\end{aligned}$$

⇒ We can ignore it.

Whitney (1-form) Basis

Approach:

Given a basis for scalar fields $\{\psi_i: \mathcal{M} \to \mathbb{R}\}_{i\in\mathcal{N}}$ **satisfying the PoU property** define a spanning set for cotangent vector fields as:

$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

Note:

More generally, for any pair of indices $i, j \in \mathcal{N}$, we have:

$$\begin{aligned}\omega_{ij} &= \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j \\ &= -\psi_i \cdot d\psi_j + \psi_j \cdot d\psi_i \\ &= -(\psi_i \cdot d\psi_j - \psi_j \cdot d\psi_i) \\ &= -\omega_{ji}\end{aligned}$$

$\Rightarrow$ We only need to consider one of the two pairs.

Whitney (1-form) Basis

Approach:

Given a basis for scalar fields $\{\psi_i : \mathcal{M} \to \mathbb{R}\}_{i \in \mathcal{N}}$ **satisfying the PoU property** define a spanning set for cotangent vector fields as:

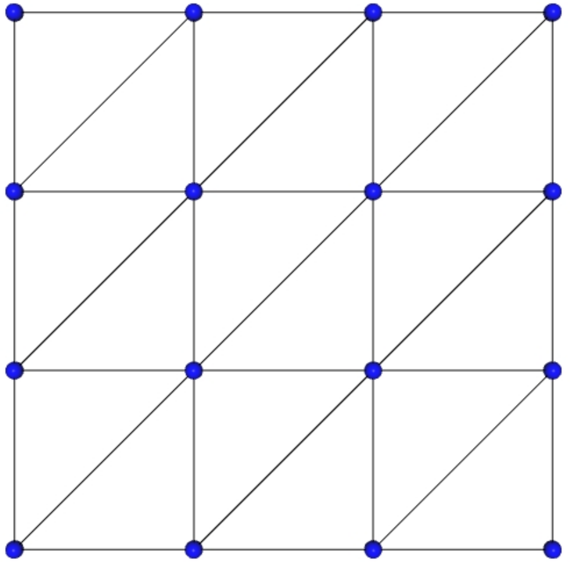
$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

Formally:

We can index the cotangent vector fields using the edges of the support graph (with edge orientation chosen arbitrarily).*

*Still not guaranteed that these will be linearly independent.

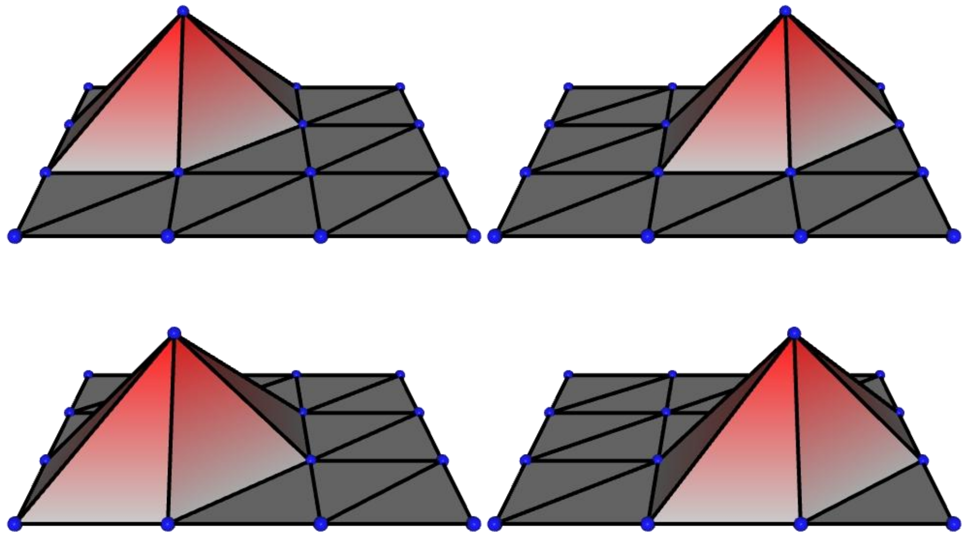
Whitney (1-form) Basis



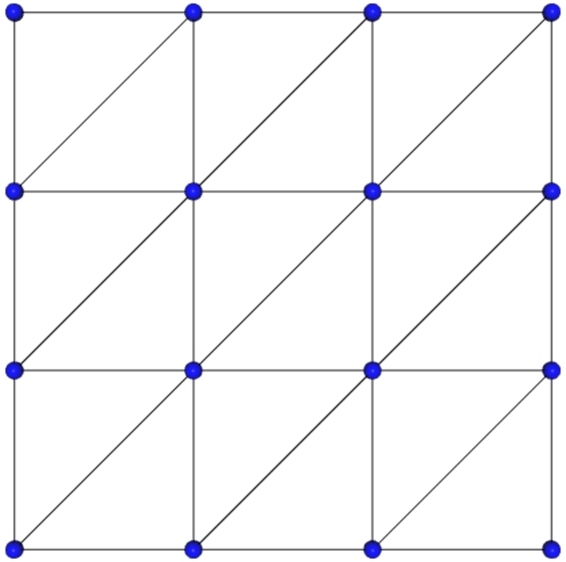
$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

For the hat basis:

There is a vector field associated to every edge of the triangle mesh (with edge orientation chosen arbitrarily).

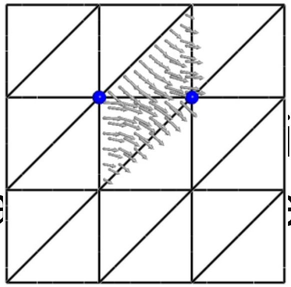


Whitney (1-form) Basis



$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

For the hat basis:



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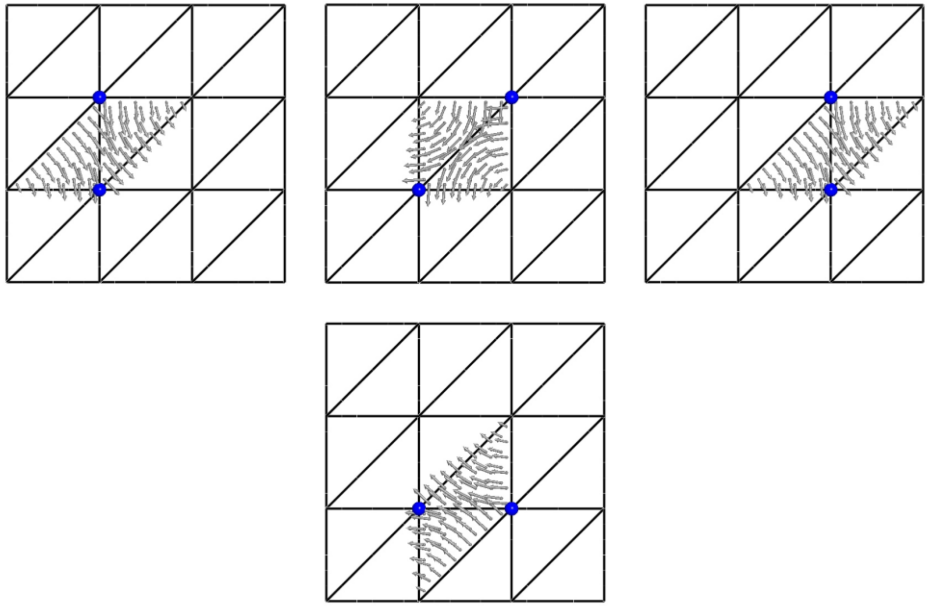
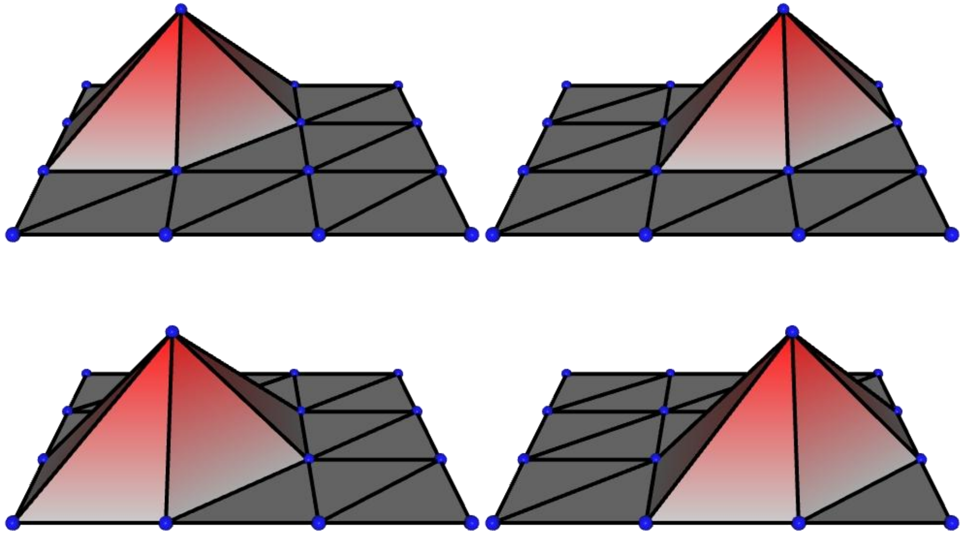
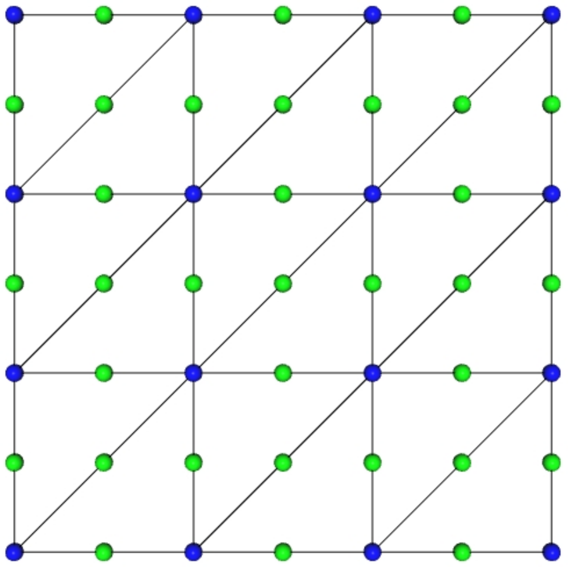




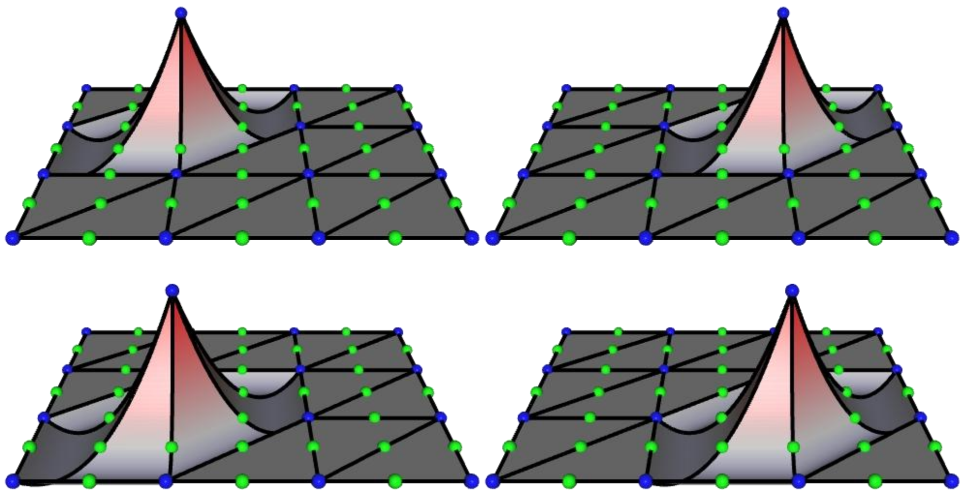


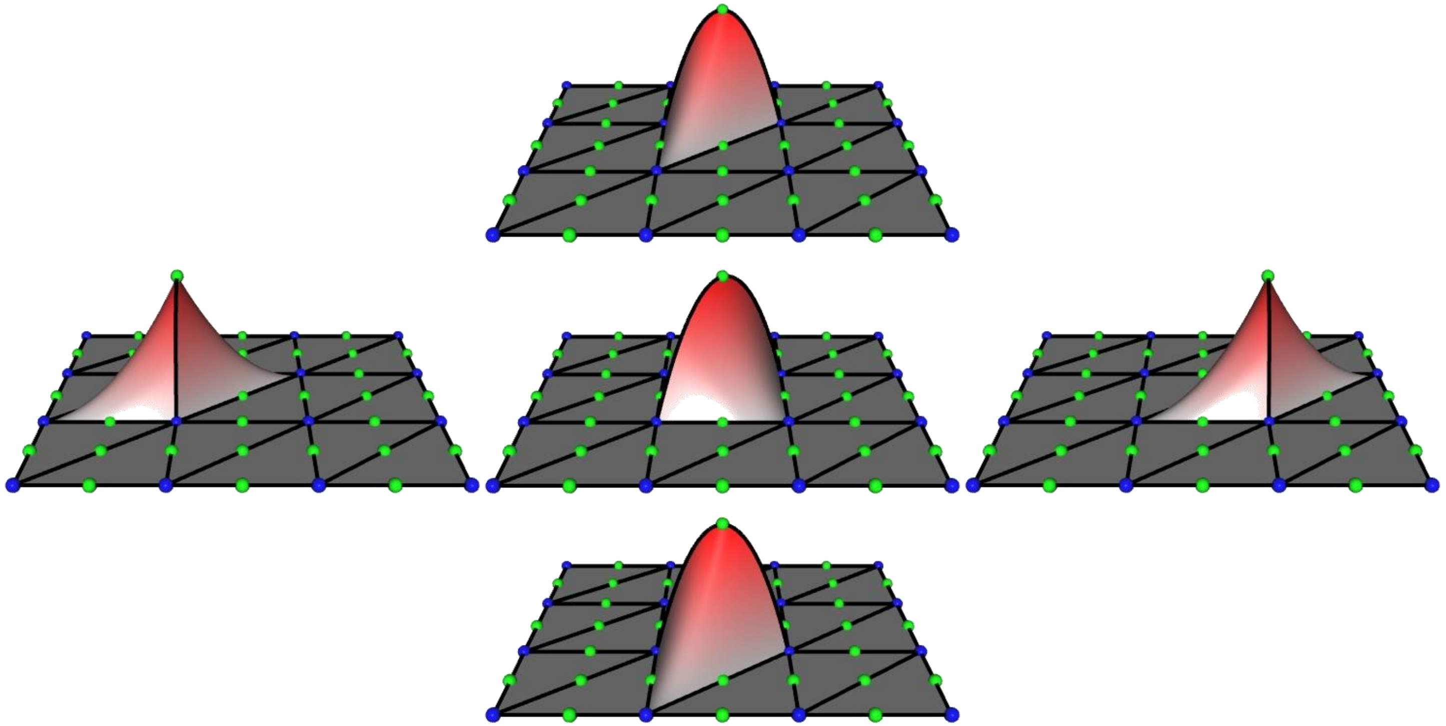
Whitney (1-form) Basis

$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

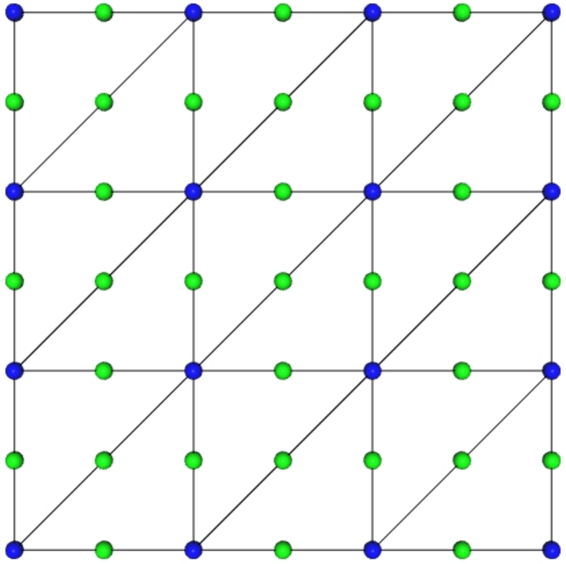
For second-order Lagrange:

Things are more complicated





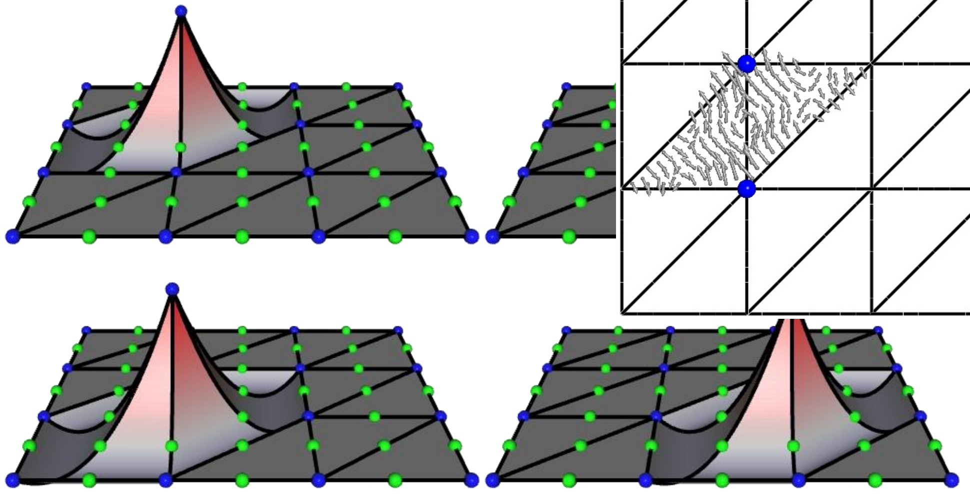
Whitney (1-form) Basis

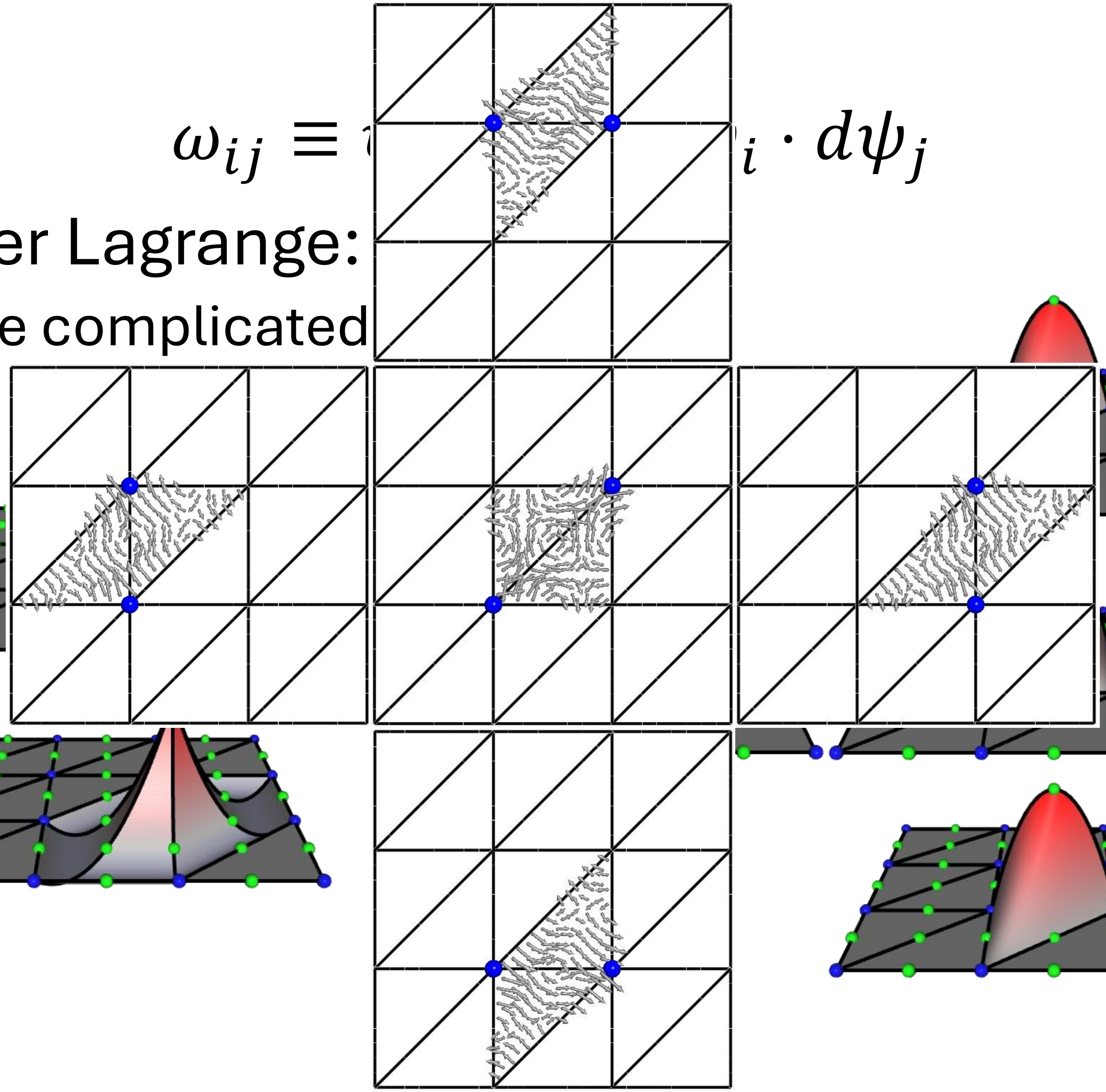


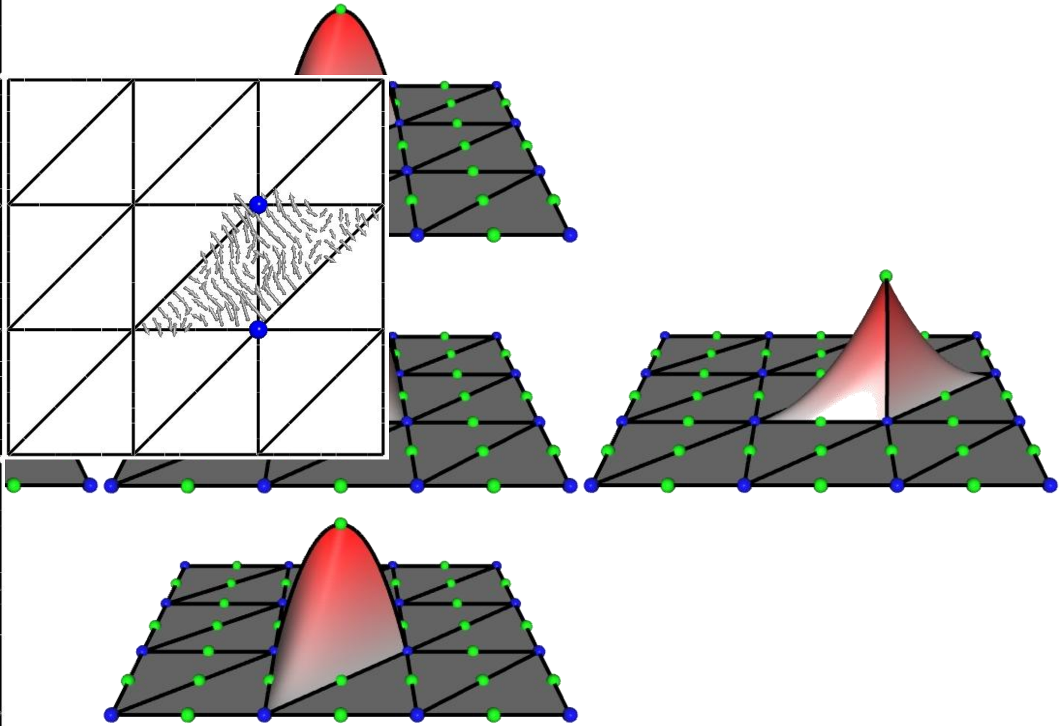
$$\omega_{ij} \equiv \ldots {}_i \cdot d\psi_j$$

For second-order Lagrange:

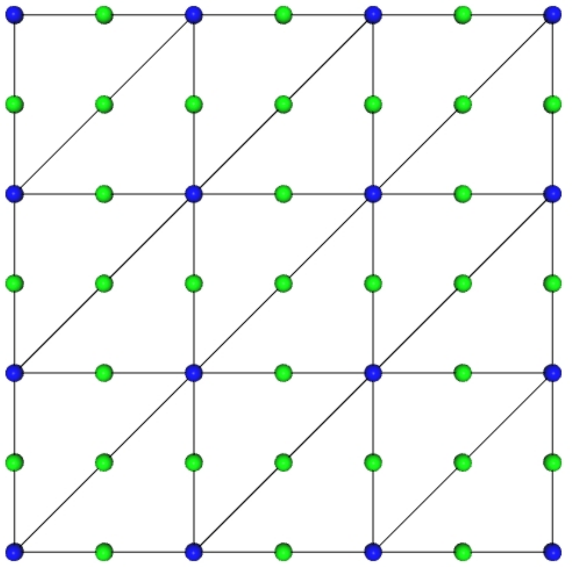
Things are more complicated

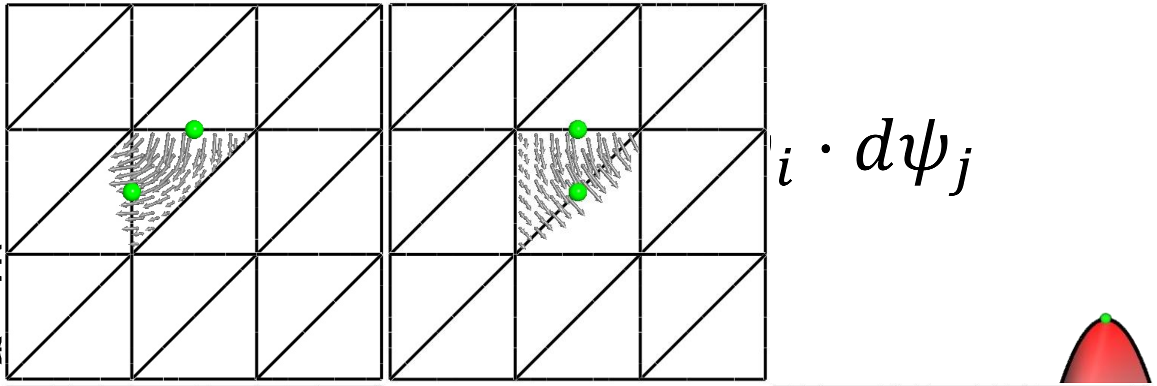






Whitney (1-form) Basis

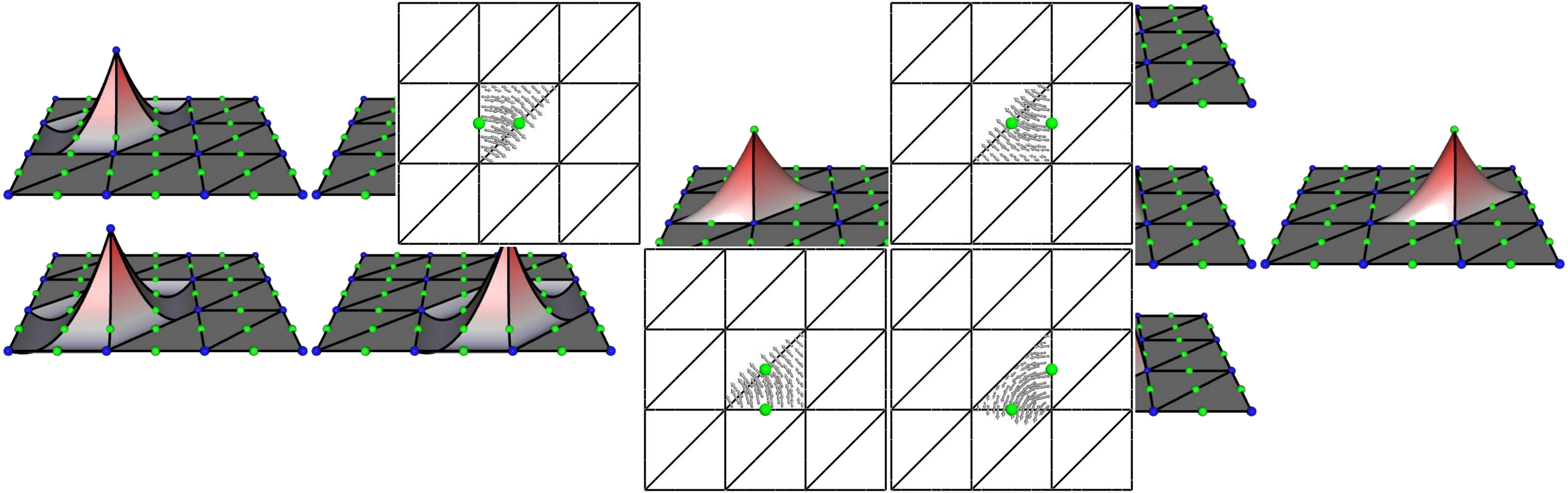




$_i \cdot d\psi_j$

For second-orde

Things are more



Whitney (1-fo sis

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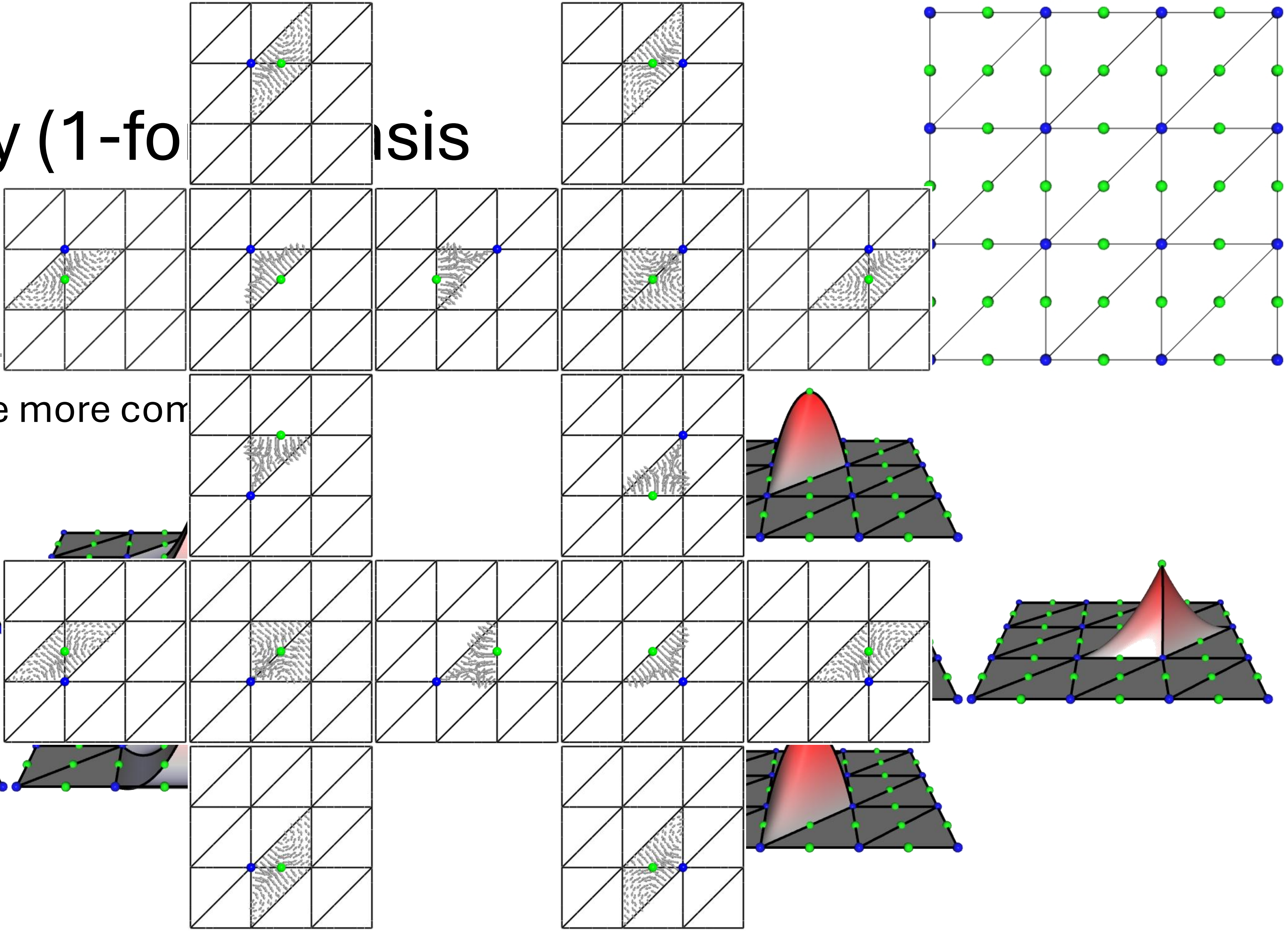

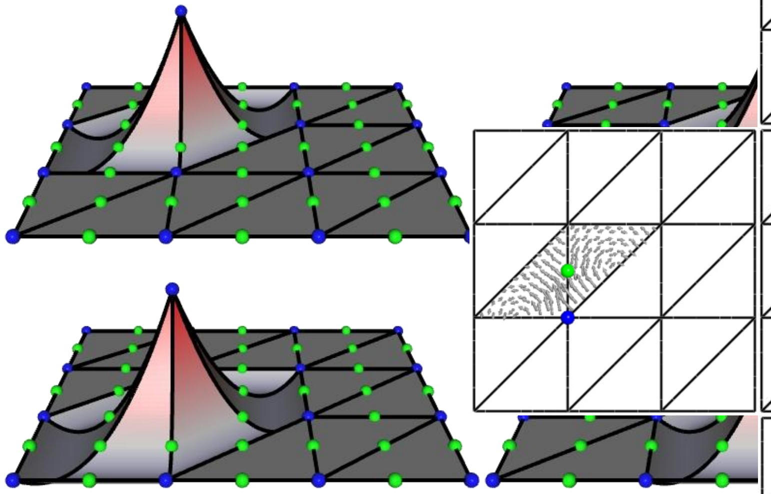

Whitney (1-form) Basis

$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

For k-th Order Lagrange:

- The ω_{ij} are not continuous across edges of the triangle mesh (because the scalar functions are not derivative continuous).
- Within a triangle, the ω_{ij} are polynomials of degree $2k-1$.

Whitney (1-form) Basis

$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

Properties:

For any scalar basis function ψ_i, the differential is:

$$\begin{aligned} d\psi_i &= d\psi_i \cdot 1 - \psi_i \cdot 0 \\ &= d\psi_i \cdot \left(\sum_j \psi_j\right) - \psi_i \cdot \left(\sum_j d\psi_j\right) \\ &= \sum_j d\psi_i \cdot \psi_j - \psi_i \cdot d\psi_j \\ &= \sum_j \omega_{ji} \end{aligned}$$

⇒ For any function in the span of the scalar basis,
its differential is in the span of the associated Whitney 1-forms.

Whitney (1-form) Basis

$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

Properties:

For any scalar basis ψ_i, the differential is:

$$d\psi_i = \sum_j \omega_{ji}$$

Using the undirected edges of the support graph to index:

$$d\psi_i = \sum_{(j,i)\in\mathcal{E}} \omega_{ji} - \sum_{(i,j)\in\mathcal{E}} \omega_{ij}$$

Whitney (1-form) Basis

$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$

Properties:

Let f be a scalar function:

$$f = \sum_{i \in \mathcal{N}} \mathbf{f}_i \cdot \psi_i$$

The differential of f is:

$$\begin{aligned} df &= \sum_{i \in \mathcal{N}} \mathbf{f}_i \cdot \left(\sum_{(j,i) \in \mathcal{E}} \omega_{ji} - \sum_{(i,j) \in \mathcal{E}} \omega_{ij} \right) \\ &= \sum_{(i,j) \in \mathcal{E}} \left(\mathbf{f}_j - \mathbf{f}_i \right) \cdot \omega_{ij} \end{aligned}$$

Whitney (1-form) Basis

$$\omega_{ij} \equiv \psi_j \cdot d\psi_i - \psi_i \cdot d\psi_j$$
$$df = \sum_{(i,j)\in\mathcal{E}} \left(\mathbf{f}_j - \mathbf{f}_i\right) \cdot \omega_{ij}$$

Properties:

The coefficients of the differential are the differences in the function's values across support graph edges (signed based on edge orientation).

$\Rightarrow$ With respect to the bases $\{\psi_i\}_{i\in\mathcal{N}}$ and $\{\omega_{ij}\}_{(i,j)\in\mathcal{E}}$, the differential is expressed as the matrix $\mathbf{D} \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{N}|}$ with:

$$\mathbf{D}_{(i,j),k} = \begin{cases} 1 & \text{if } j = k \\ -1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Whitney (1-form) Basis

$$\mathbf{D}_{(i,j),k} = \begin{cases} 1 & \text{if } j = k \\ -1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Observation:

Earlier on, we defined the *combinatorial Laplacian* giving the sum of squared differences along triangle mesh edges:

$$\mathbf{L}_{ij} = \begin{cases} |n(i)| & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } i \in n(j) \\ 0 & \text{otherwise} \end{cases}$$

In the context of the hat basis, this is equivalent to:

$$\mathbf{L} = \mathbf{D}^\top \cdot \mathbf{D}$$

Whitney (1-form) Basis

$$\mathbf{D}_{(i,j),k} = \begin{cases} 1 & \text{if } j = k \\ -1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Observation:

For the combinatorial Laplacian, we have:

$$\mathbf{L} = \mathbf{D}^\top \cdot \mathbf{D}$$

For the geometry-aware formulation, we define the stiffness matrix by computing the mass matrix for cotangent vector fields:

$$\overline{\mathbf{M}}_{(i,j),(k,l)} = \langle\langle \omega_{ij}, \omega_{kl} \rangle\rangle_{\mathcal{M}}$$

$\Rightarrow$ The stiffness matrix is:

$$\mathbf{S} = \mathbf{D}^\top \cdot \overline{\mathbf{M}} \cdot \mathbf{D}$$