Geometry Processing (601.458/658)

Misha Kazhdan

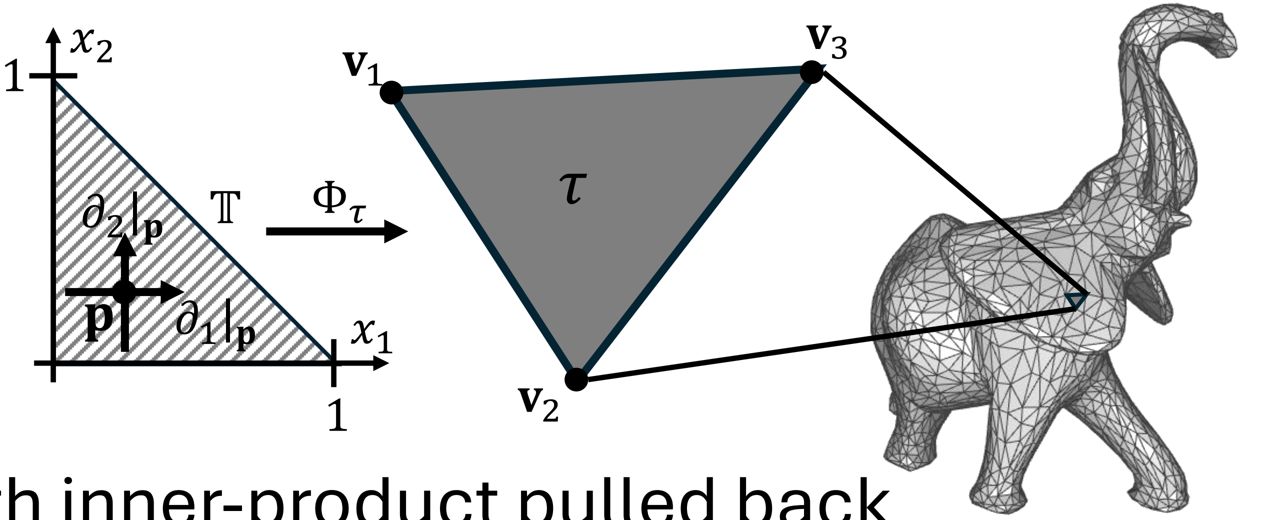
"Geodesics in Heat", Crane *et al.*, 2017

Outline

Recall

Geodesics in Heat

Recall



Given the unit-right triangle $\mathbb{T}$, with inner-product pulled back from a triangle $\tau \subset \mathbb{R}^3$ we define:

At each point $\mathbf{p} \in \mathbb{T}$ we have a tangent space $T_{\mathbf{p}}\mathbb{T}$ spanned by $\{\partial_1|_{\mathbf{p}}, \partial_2|_{\mathbf{p}}\}$.

The dual space $T_{\mathbf{p}}^*\mathbb{T}$ is spanned by the canonical dual basis $\{dx_1|_{\mathbf{p}}, dx_2|_{\mathbf{p}}\}$.

We denote the pulled back inner-product as $g_\tau: T_{\mathbf{p}}\mathbb{T} \to T_{\mathbf{p}}^*\mathbb{T}$.

The inner-product on the dual space is $g_\tau^{-1}: T_{\mathbf{p}}^*\mathbb{T} \to T_{\mathbf{p}}\mathbb{T}$.

Recall

$$V \xrightarrow{M} V^* \qquad \overline{V} \xrightarrow{\overline{M}} \overline{V}^*$$

Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we define:

A finite-dimensional space of scalar fields, spanned by the piecewise linear hat basis functions:

$$V = \mathrm{Span}(\{\phi_v\}_{v\in\mathcal{V}})$$

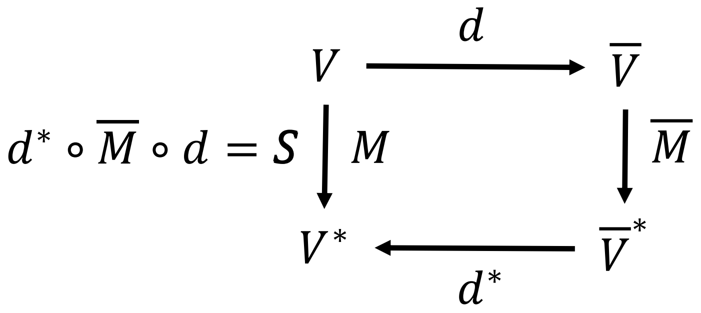
forming a partition of unity.

A finite-dimensional space of cotangent vector fields, spanned by the piecewise constant functions (one per cartesian direction):

$$\overline{V} = \mathrm{Span}(\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}})$$

Integrating with respect to the inner-product defined on the tangent space, these are also inner-product spaces $\{V, M: V \to V^*\}$ and $\left\{\overline{V}, \overline{M}: \overline{V} \to \overline{V}^*\right\}$.

Recall



Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we define:

We denote by $d: V \to \overline{V}$ the linear map giving the differential.

We define the stiffness operator $S: V \to V^*$ as the bilinear form giving the inner-product of the differential of scalar fields.

This is the pull-back of the inner-product $\overline{M}: \overline{V} \to \overline{V}^*$ via the differential:

$$S = d^* \circ \overline{M} \circ d$$

Recall

With respect to the bases $\{\phi_v\}_{v\in\mathcal{V}}$ and $\{\eta_\tau^1,\eta_\tau^2\}_{\tau\in\mathcal{T}}$:

- The scalar field mass matrix: $\mathbf{M}\in\mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$
- The scalar field stiffness matrix: $\mathbf{S}\in\mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$
- The cotangent vector field mass matrix: $\overline{\mathbf{M}}\in\mathbb{R}^{2|\mathcal{T}|\times 2|\mathcal{T}|}$
- The differential matrix: $\mathbf{D}\in\mathbb{R}^{2|\mathcal{T}|\times|\mathcal{V}|}$
- The factorization of the scalar field stiffness matrix: $\mathbf{S}=\mathbf{D}^\top\cdot\overline{\mathbf{M}}\cdot\mathbf{D}$

Recall

With respect to the bases $\{\phi_v\}_{v\in\mathcal{V}}$ and $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$:

Given functions $f, h \in V$, expressed w.r.t. the basis $\{\phi_v\}_{v\in\mathcal{V}}$ as $\mathbf{f}, \mathbf{h} \in \mathbb{R}^{|\mathcal{V}|}$:

$$f = \mathbf{f}_1 \cdot \phi_1 + \cdots + \mathbf{f}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$

$$h = \mathbf{h}_1 \cdot \phi_1 + \cdots + \mathbf{h}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$

the inner-product of f with h is given by:

$$\langle\langle f, h\rangle\rangle_{\mathcal{M}} = \mathbf{f}^\top \cdot \mathbf{M} \cdot \mathbf{h}$$

The inner-product of the differential of f with the differential of h is given by:

$$\langle\langle df, dh\rangle\rangle_{\mathcal{M}} = \mathbf{f}^\top \cdot \mathbf{S} \cdot \mathbf{h}$$

Recall

With respect to the bases $\{\phi_v\}_{v\in\mathcal{V}}$ and $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$:

Given a function $f \in V$, expressed w.r.t. the basis $\{\phi_v\}_{v\in\mathcal{V}}$ as $\mathbf{f} \in \mathbb{R}^{|\mathcal{V}|}$:

$$f = \mathbf{f}_1 \cdot \phi_1 + \cdots + \mathbf{f}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$

we can compute $\mathbf{w} = \mathbf{D} \cdot \mathbf{f} \in \mathbb{R}^{2|\mathcal{T}|}$.

This gives the expression of the differential of f w.r.t the basis $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$:

$$df = \mathbf{w}_1^1 \cdot \eta_1^1 + \mathbf{w}_1^2 \cdot \eta_1^2 + \cdots + \mathbf{w}_{|\mathcal{T}|}^1 \cdot \eta_{|\mathcal{T}|}^1 + \mathbf{w}_{|\mathcal{T}|}^2 \cdot \eta_{|\mathcal{T}|}^2$$

Recall

Gradient domain processing:

Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, given a target scalar field $h \in V$ and cotangent vector field $\zeta \in \overline{V}$, we seek the solution $f \in V$ minimizing the gradient domain energy:

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \zeta, df - \zeta \rangle\rangle_{\mathcal{M}}$$

Expressed in terms of the scalar field and cotangent vector field basis, the coefficients $\mathbf{f} \in \mathbb{R}^{|\mathcal{V}|}$ is:

$$\mathbf{f} = (\alpha \cdot \mathbf{M} + \beta \cdot \mathbf{S})^{-1} \cdot \left(\alpha \cdot \mathbf{M} \cdot \mathbf{h} + \beta \cdot \mathbf{D}^{\top} \cdot \overline{\mathbf{M}} \cdot \boldsymbol{\zeta}\right)$$

Diffusing/smoothing a signal $\mathbf{h} \in \mathbb{R}^{|\mathcal{V}|}$ for a time-step of ε by gives:

$$\mathbf{f} = (\mathbf{M} + \varepsilon \cdot \mathbf{S})^{-1} \cdot \mathbf{M} \cdot \mathbf{h}$$

Outline

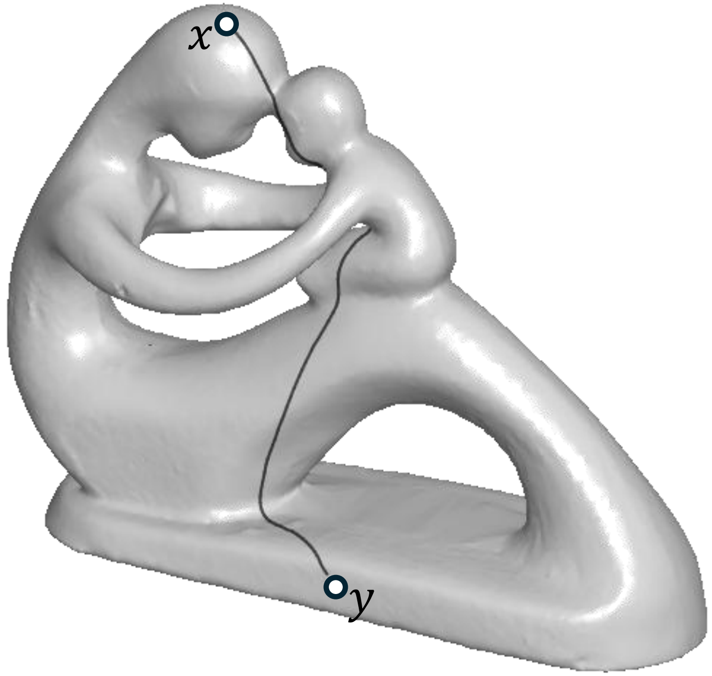
Recall

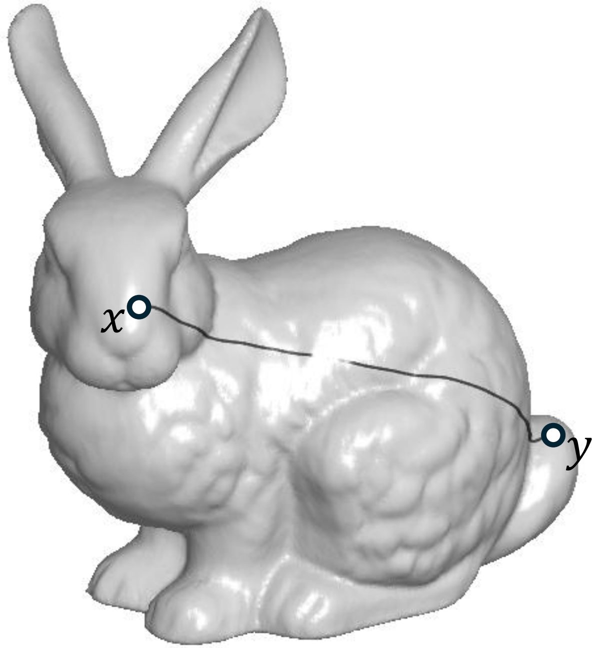
Geodesics in Heat

Geodesics in Heat

Definition:

Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, a *geodesic path* between two vertices $x, y \in \mathcal{V}$ is a path on $\mathcal{M}$ that is locally shortest (i.e. any local perturbation will increase the path length).





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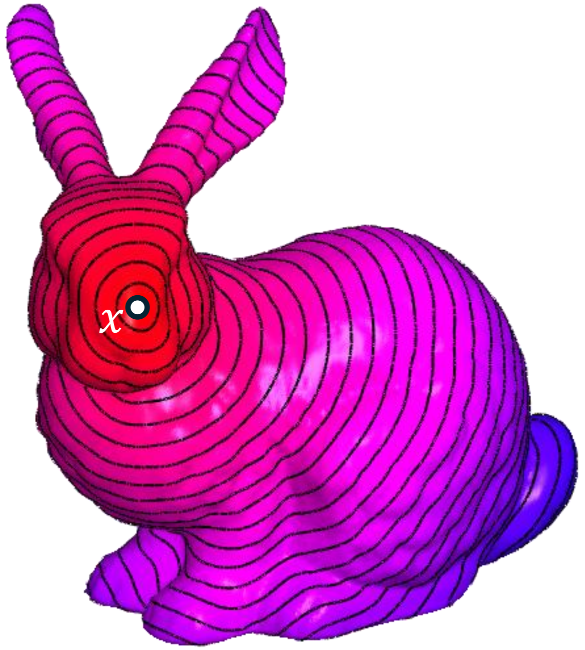
Goal:

Given a source vertex $x \in \mathcal{V}$, compute the function giving the (shortest) *geodesic distance* from x to every other vertex:

$$d_x : \mathcal{V} \to \mathbb{R}^{\geq 0}$$



d_x

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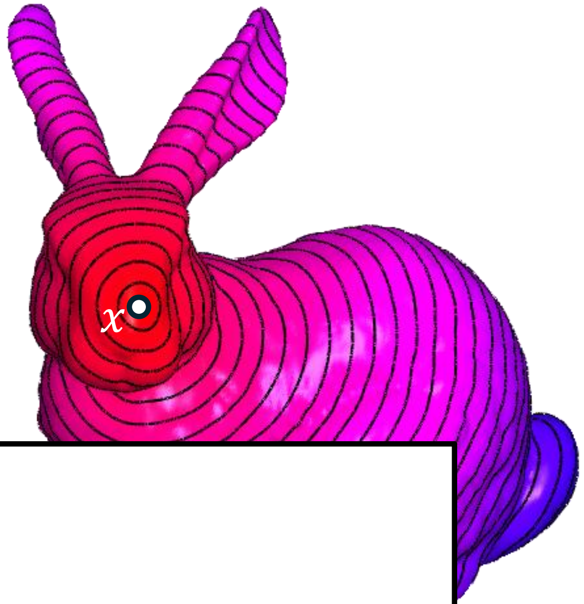


Note:
The shortest path need not lie along the edges of the mesh.
⇒ Computing geodesic distances is not a matter of applying Dijkstra's algorithm.

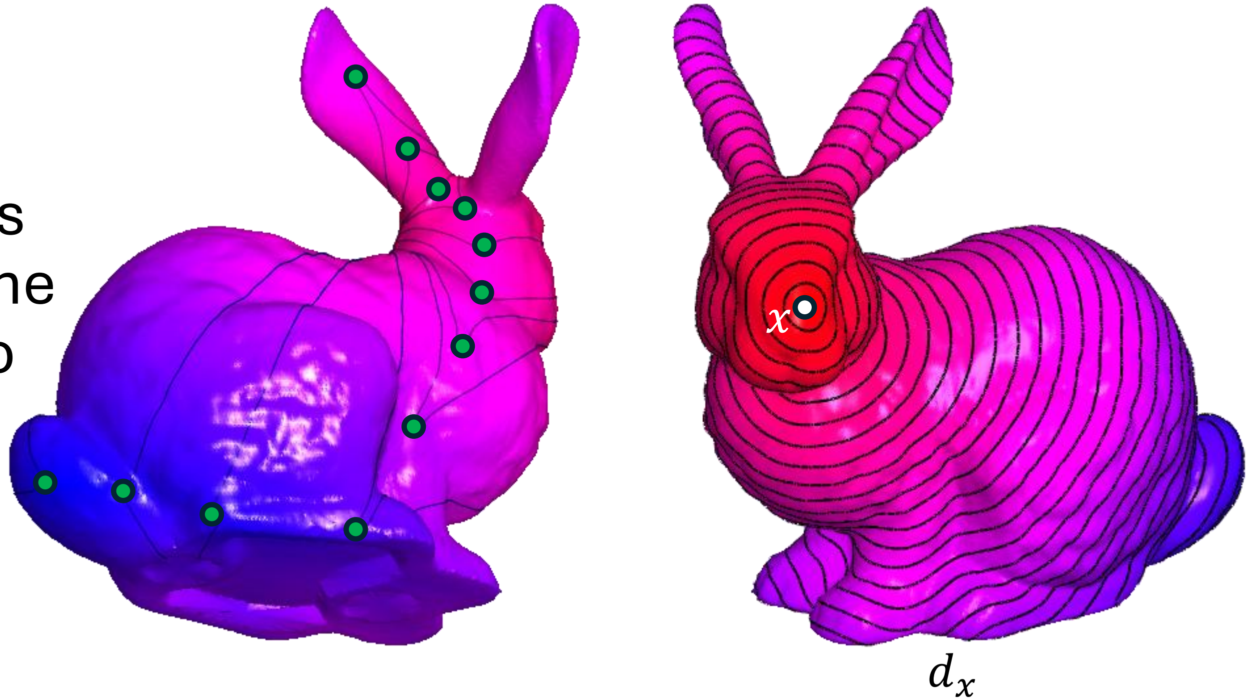
Geodesics in Heat

Definition:

Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, a *geodesic path* between two vertices $x, y \in \mathcal{V}$ is a path on $\mathcal{M}$ that is locally shortest (i.e. any local perturbation will increase the path length).

Definition:

Given a source $x \in \mathcal{V}$, the *cut locus* is (the closure of) the set of points on the mesh that have two shortest paths to the source.

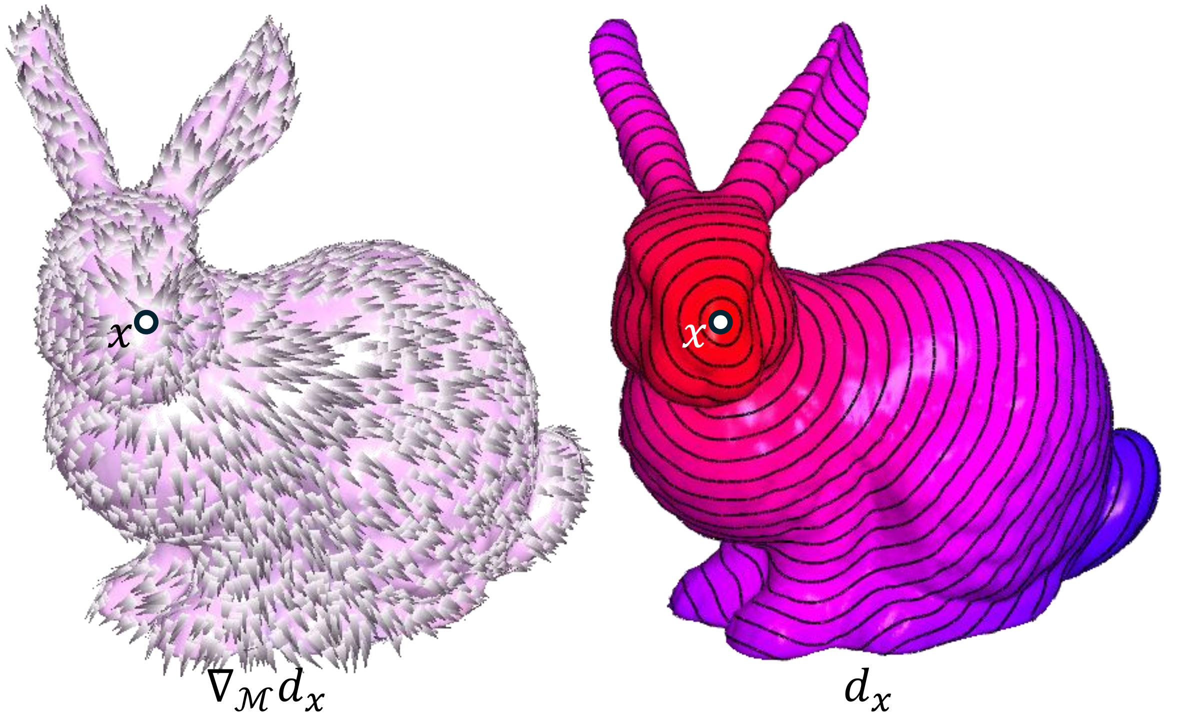


Geodesics in Heat

Properties:

Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, and given a source vertex $x \in \mathcal{V}$, the gradient $\nabla_{\mathcal{M}} d_x$ of the geodesic distance function:

1. Points away from the source, and
2. Has unit length (Eikonal equation)

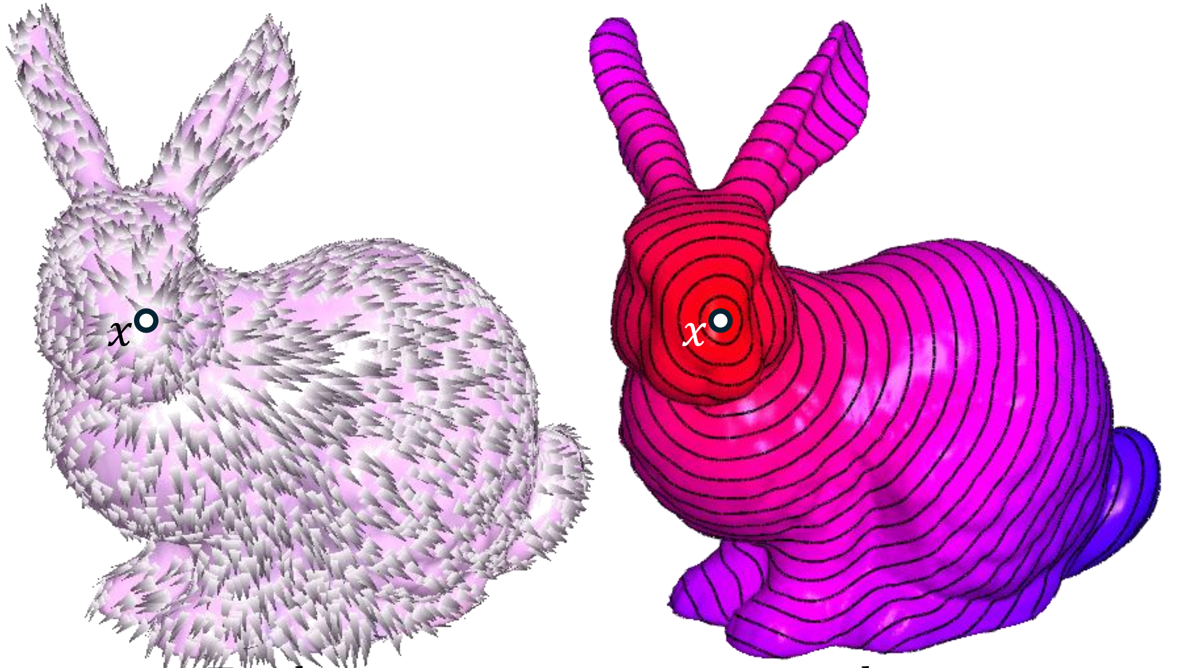


Geodesics in Heat

Approach:

1. Estimate the differential $\zeta \in \overline{V}$ of the geodesic distance function.
2. Solve the gradient domain problem for the scalar field whose differential matches the target cotangent vector field:

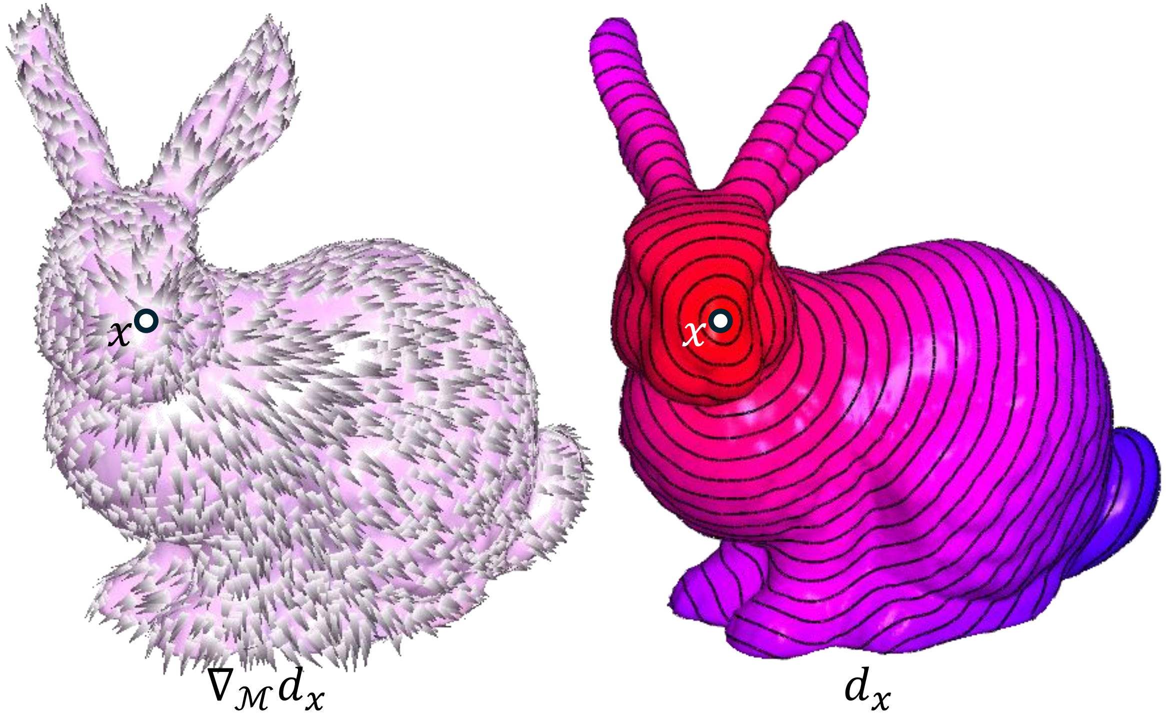
$$E(f) = \frac{1}{2}\langle\langle df - \zeta, df - \zeta\rangle\rangle_{\mathcal{M}}$$
$$\Downarrow$$
$$\mathbf{f} = \mathbf{S}^{-1} \cdot \mathbf{D}^\top \cdot \overline{\mathbf{M}} \cdot \boldsymbol{\zeta}$$



$\nabla_{\mathcal{M}} d_x$ d_x

Geodesics in Heat

How do we estimate the differential $\zeta \in \overline{V}$ of the geodesic distance function?



The Heat Kernel

Definition:

Given a point $x \in \mathcal{M}$, denote by $\delta_x \in V$ the *impulse function* at x.

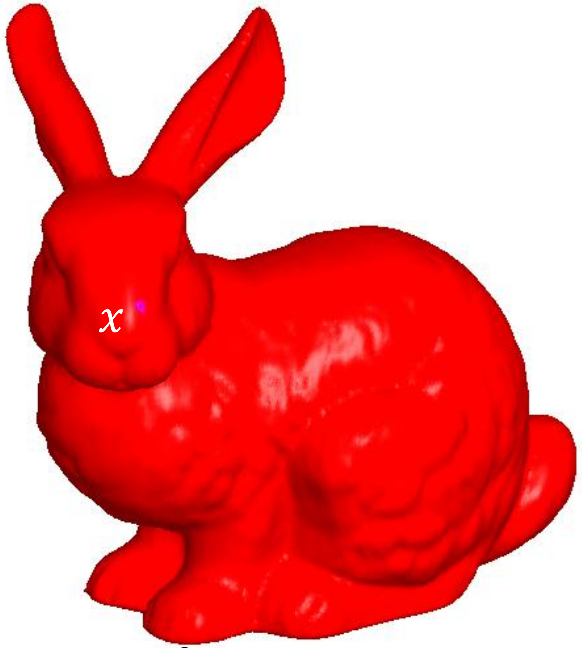
Informally, δ_x is a function that is:

Zero away from x

Infinite at x

And has unit mass:

$$\int_{\mathcal{M}} \delta_x \cdot \omega_E = 1$$



$\delta_x : \mathcal{M} \to \mathbb{R}$

The Heat Kernel

Definition:

Given a point $x \in \mathcal{M}$, denote by $\delta_x^* \in V^*$ the *impulse functional* at x.

Formally:

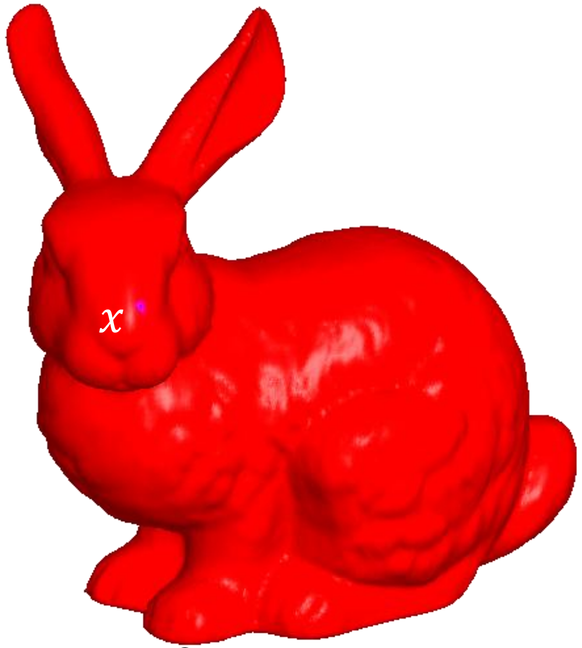
$\delta_x^* \in V^*$ is the vector in the dual space acting by evaluation:

$$\delta_x^*(f) = f(x)$$

for all $f \in V$.

We could turn this into primal vector in V by applying the inverse of the inner-product:

$$\delta_x = M^{-1}(\delta_x^*)$$
$$\Leftrightarrow \quad M(\delta_x) = \delta_x^*$$



$\delta_x : \mathcal{M} \to \mathbb{R}$

The Heat Kernel

Definition:

Given a point $x \in \mathcal{M}$, denote by $\delta_x^* \in V^*$ the *impulse functional* at x.

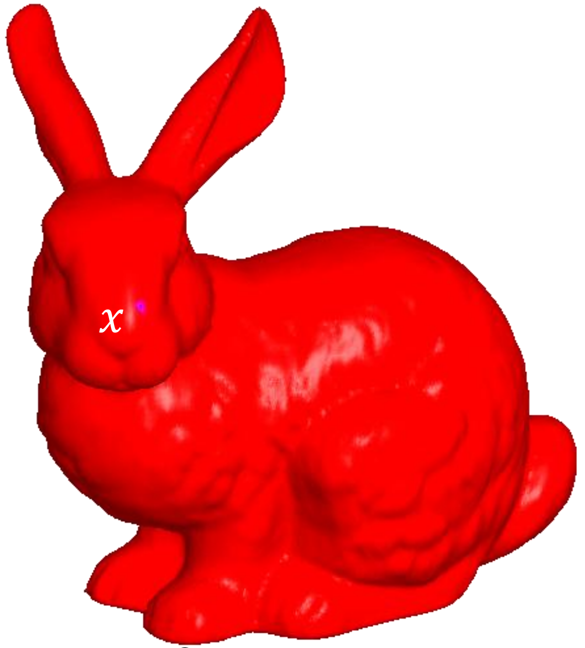
Formally:

For any $f \in V$:

$$\delta_x^*(f) = f(x)$$
$$M(\delta_x) = \delta_x^*$$

Equivalently:

$$\begin{aligned} f(x) &= \delta_x^*(f) \\ &= [M(\delta_x)](f) \\ &= \langle\langle f, \delta_x \rangle\rangle_{\mathcal{M}} \end{aligned}$$



$\delta_x: \mathcal{M} \to \mathbb{R}$

The Heat Kernel

Definition:

Thinking of δ_x as the initial distribution of heat on $\mathcal{M}$, we can diffuse heat from x by solving the heat equation:

$$\frac{\partial k_x^t}{\partial t} = \Delta k_x^t$$

with $k_x^0(y) = \delta_x(y)$.

The function giving the amount of heat at $y \in \mathcal{M}$ after diffusing from $x \in \mathcal{M}$ for time t is the *heat kernel*:

$$k_x^t : \mathcal{M} \to \mathbb{R}^{\geq 0}$$

Or:

$$k_t : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^{\geq 0}$$

Or:

$$k : \mathbb{R} \times \mathcal{M} \times \mathcal{M} \to \mathbb{R}^{\geq 0}$$

$k_x^t : \mathcal{M} \to \mathbb{R}$

Varadhan's Formula

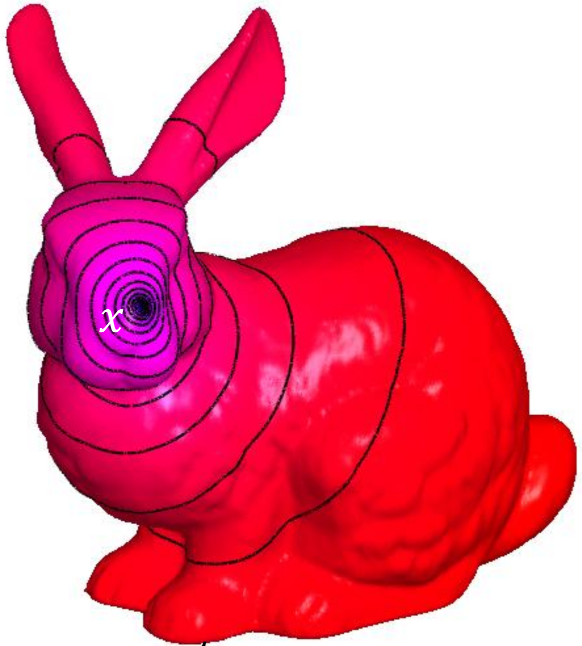
Informally:

We expect, at least initially, that heat will diffuse away from x as a function of geodesic distance from x.

Formally:

Varadhan's formula states that initially the heat decays exponentially with geodesic distance:

$$d_x(y) = \lim_{t \to 0} \sqrt{-4t \log k_x^t(y)}$$



$k_x^t : \mathcal{M} \to \mathbb{R}$

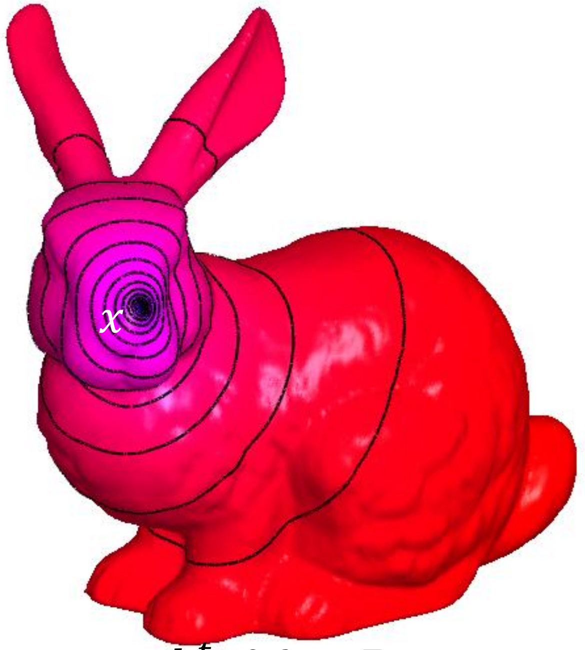
The Heat Kernel

$$d_x(y) = \lim_{t \to 0} \sqrt{-4t \log k_x^t(y)}$$

In Practice:

This is challenging to work with because:

1. It is only true in the limit as t goes to zero.
2. At small values of t the heat kernel decays exponentially with distance from x, and the RHS is numerically unstable.



$k_x^t: \mathcal{M} \to \mathbb{R}$

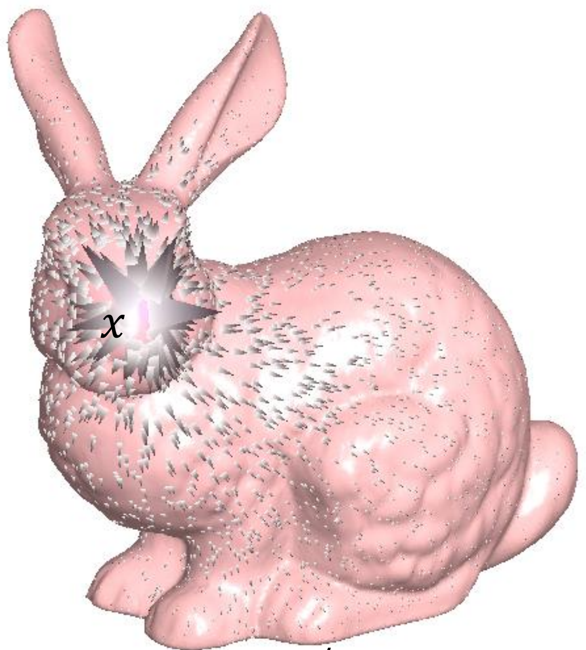
The Heat Kernel

Observation:

If we solve for the heat kernel for a small (but not tiny) value of t, the resulting function will have a gradient pointing **towards** x.

$\Rightarrow$ We can get the approximate gradient of the geodesic distance function, $d_x: \mathcal{M} \rightarrow \mathbb{R}^{\geq 0}$, by:

1. Negating the gradient (so that it points away from x).
2. Normalizing so that the gradient has unit length (to satisfy the Eikonal equation).



$\nabla_{\mathcal{M}} k_x^t$

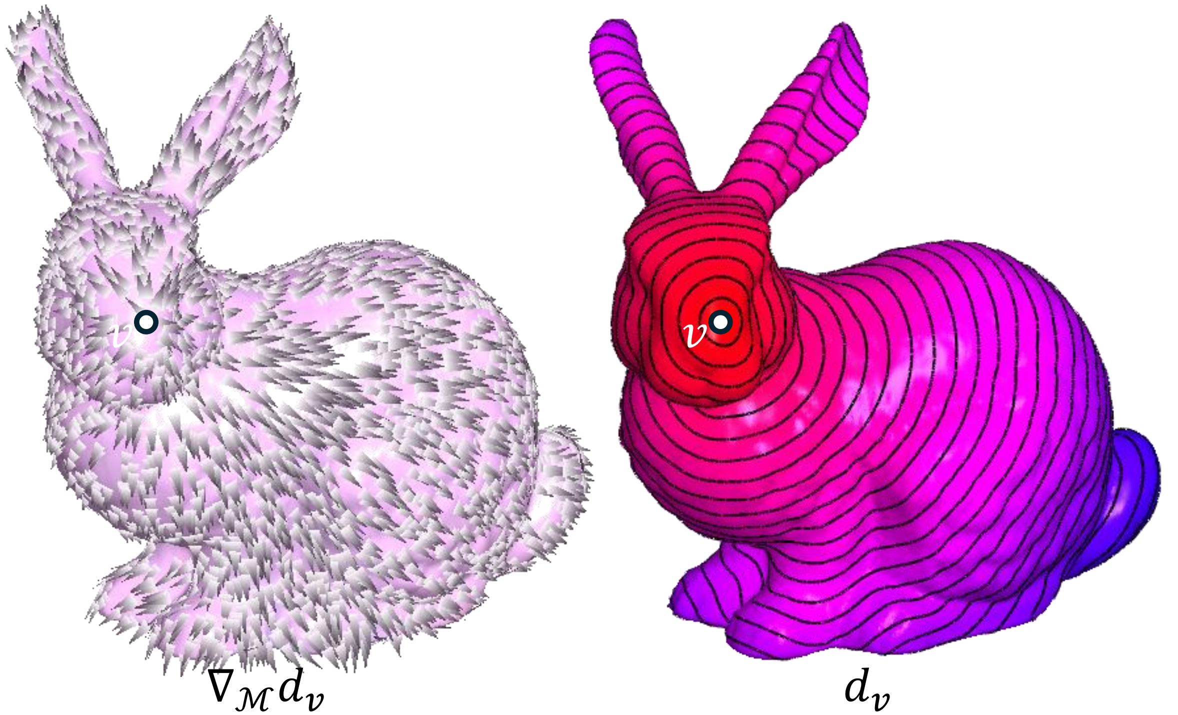
Equivalently:

Negate and normalize the differential of the heat kernel.

Geodesics in Heat

Approach:

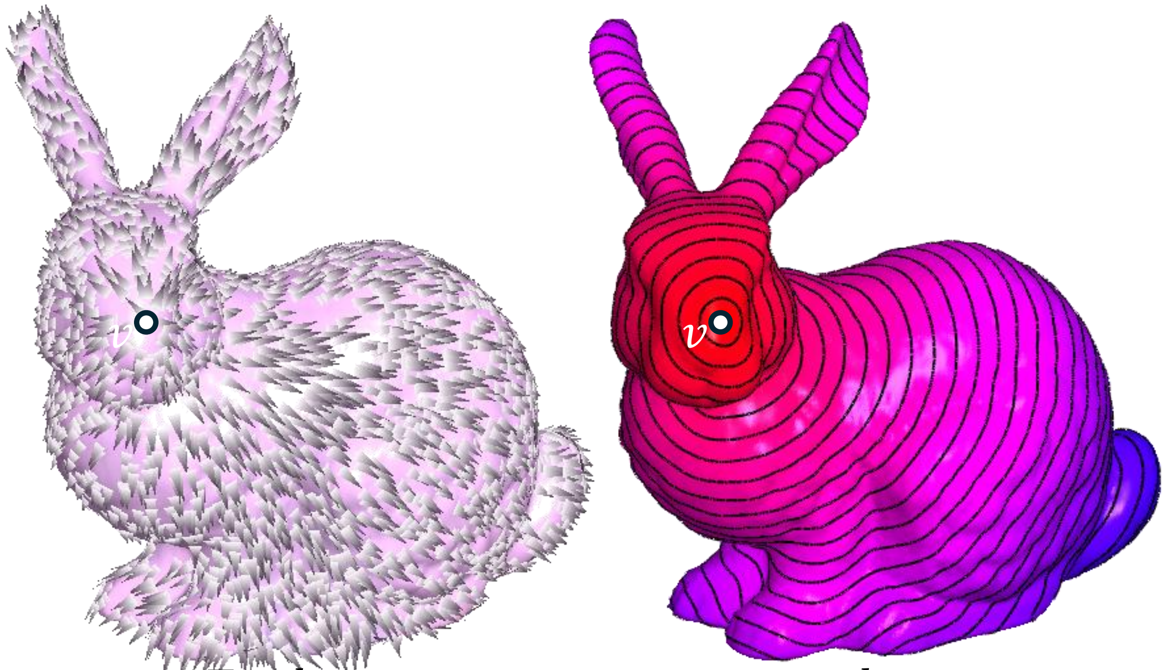
1. Estimate the differential $\zeta \in \overline{V}$.
2. Fit a scalar field to the target differentials ζ.



Geodesics in Heat

Given a vertex $x \in \mathcal{V}$:

1. Diffuse the delta function δ_x for a small time t.
2. Compute the differential.
3. Negate and normalize.



$\nabla_{\mathcal{M}} d_v$ d_v

Geodesics in Heat

Approach:

1. **Estimate the differential $\zeta \in \overline{V}$**
   a. **Diffuse the delta function**
   b. Compute the differential
   c. Negate and normalize
2. Fit a scalar field to the target differentials ζ

Diffuse the Delta Function:

Letting δ_x be the delta function at $x \in \mathcal{V}$, this amounts to solving:

$$E(f) = \frac{1}{2} \cdot \langle\langle f - \delta_x, f - \delta_x \rangle\rangle_{\mathcal{M}} + t \cdot \frac{1}{2} \cdot \langle\langle df, df \rangle\rangle_{\mathcal{M}}$$

$$\Downarrow$$

$$\mathbf{f} = (\mathbf{M} + t \cdot \mathbf{S})^{-1} \cdot \mathbf{M} \cdot \boldsymbol{\delta}$$

where $\boldsymbol{\delta} \in \mathbb{R}^{|\mathcal{V}|}$ are the coefficients of the delta function at x and t is the diffusion time-step.

Question:

What time-step t should we use?

How should we set the coefficients of the delta function, $\boldsymbol{\delta}$.

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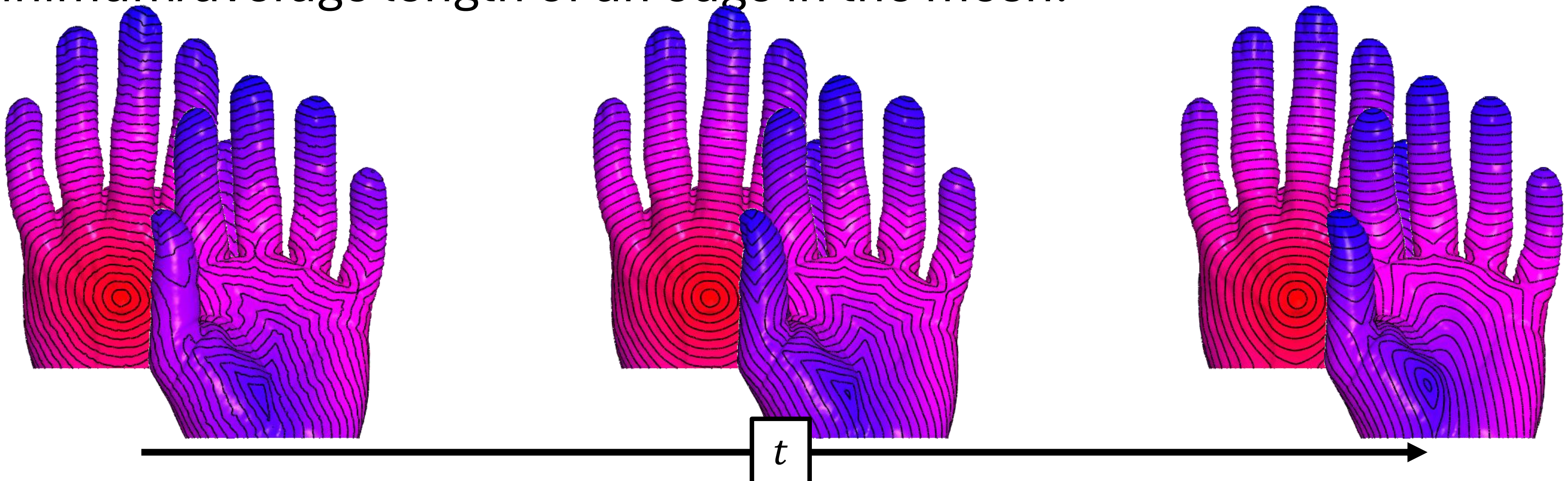
$$d_x(y) = \lim_{t \to 0} \sqrt{-4t \log k_x^t(y)}$$

Setting the time-step:

As Varadhan's formula is true in the limit as $t \to 0$, we set the time-step t to be "as small as possible" – typically as a function of the length of the minimum/average length of an edge in the mesh.



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Setting $\boldsymbol{\delta} \in \mathbb{R}^{|\mathcal{V}|}$ (naïve):

Approximate the delta function at vertex $x \in \mathcal{V}$ with a function that is 1 at x and 0 at all other vertices.

Recall that the hat basis satisfies the Lagrange property:

$$\phi_v(w) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

⇒ Set the x-th coefficient to 1 and all the others to zero:

$$\boldsymbol{\delta}_w = \begin{cases} 1 & \text{if } w = x \\ 0 & \text{otherwise} \end{cases}$$

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Approximate the delta function at vertex $x \in \mathcal{V}$ with a function that is 1 at x and 0 at all other vertices.

Recall tha

Though this function will not have unit mass, that doesn't matter since we normalize in any case.

$$\phi_v(w) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ Set the x-th coefficient to 1 and all the others to zero:

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Approach:

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Geodesics in Heat

Setting $\boldsymbol{\delta} \in \mathbb{R}^{|\mathcal{V}|}$:

In the formulation of the linear system, we don't need the coefficients of the delta function, we need its dual representation:

$$\mathbf{f} = (\mathbf{M} + t \cdot \mathbf{S})^{-1} \cdot \boxed{\mathbf{M} \cdot \boldsymbol{\delta}}$$

Recall:

$\mathbf{M} \cdot \boldsymbol{\delta}$ is the expression for the dual vector $\delta_x^* = M(\delta_x) \in V^*$ with respect to the dual basis $\{\phi_v^*\}_{v \in \mathcal{V}}$.

⇒ For all $v \in \mathcal{V}$, the v-th coefficient of $\mathbf{M} \cdot \boldsymbol{\delta}$ is obtained by evaluating the dual vector δ_x^* on the v-th basis function:

$$\begin{aligned}(\mathbf{M} \cdot \boldsymbol{\delta})_v &= \delta_x^*(\phi_v) \\ &= \phi_v(x)\end{aligned}$$

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$$\mathbf{f} = (\mathbf{M} + t \cdot \mathbf{S})^{-1} \cdot \mathbf{M} \cdot \boldsymbol{\delta}$$
$$(\mathbf{M} \cdot \boldsymbol{\delta})_v = \phi_v(x)$$

1. Compared to the naïve implementation, we do **not** apply the mass matrix to the vector with 1 in the v-th entry and 0 in all other entries.
2. In principle, this allows us to define the delta functional at any point on the mesh, not just at the vertices.

Geodesics in Heat

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$$\mathbf{f} = (\mathbf{M} + \varepsilon \cdot \mathbf{S})^{-1} \cdot \mathbf{M} \cdot \boldsymbol{\delta}$$

Compute the differential:

Given the solution $\mathbf{f} \in \mathbb{R}^{|\mathcal{V}|}$, the coefficients of the differential of the associated function in V, w.r.t. the cotangent vector field basis $\{\eta_\tau^1, \eta_\tau^2\}_{\tau \in \mathcal{T}}$ are given by:

$$\hat{\boldsymbol{\zeta}} = \mathbf{D} \cdot \mathbf{f} \in \mathbb{R}^{2|\mathcal{T}|}$$

We want the coefficients of the cotangent vector field $\zeta \in \overline{V}$ corresponding to the **negated** and **normalized** $\hat{\zeta}$**.**

Geodesics in Heat

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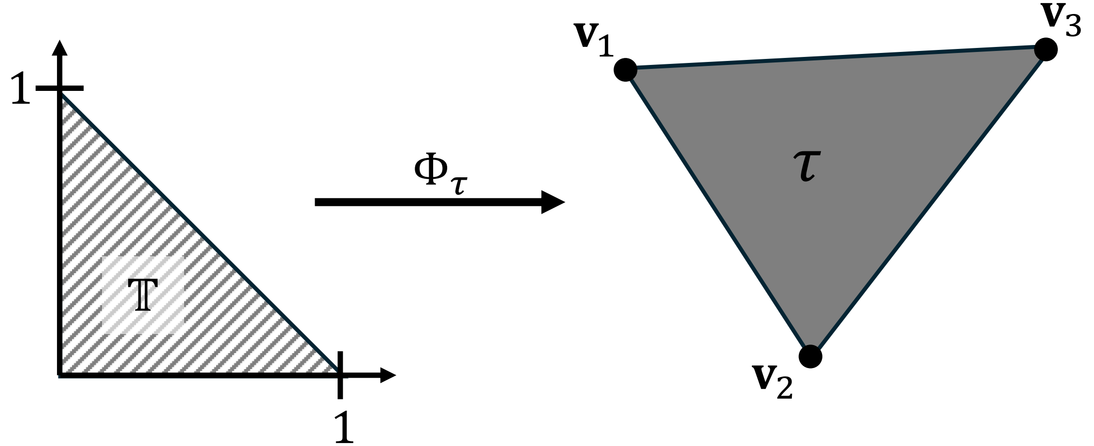
2. Fit a scalar field to the target differentials ζ

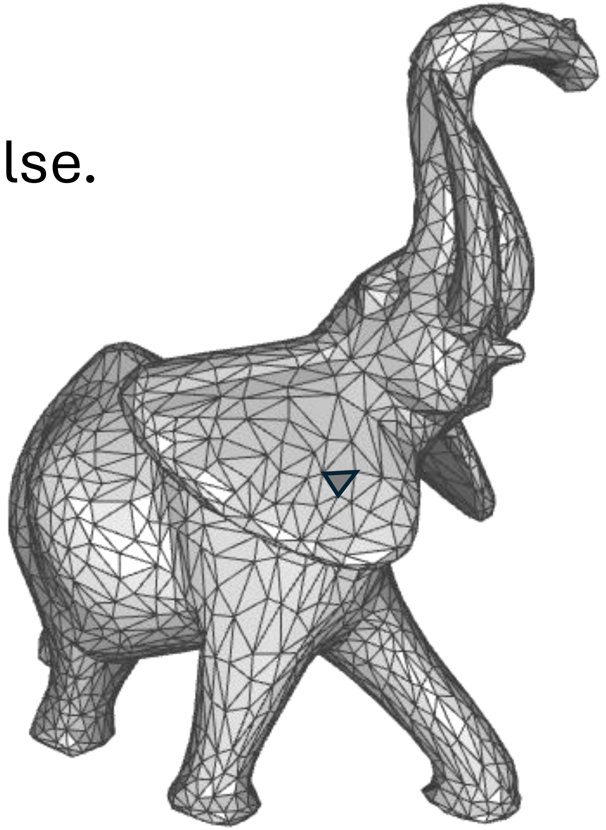
$$\mathbf{f} = (\mathbf{M} + \varepsilon \cdot \mathbf{S})^{-1} \cdot \mathbf{M} \cdot \boldsymbol{\delta}$$
$$\hat{\boldsymbol{\zeta}} = \mathbf{D} \cdot \mathbf{f} \in \mathbb{R}^{2|\mathcal{T}|}$$

Negating and normalizing:

Recall that the cotangent vector fields η_τ^1 and η_τ^2:

1. Are non-zero in triangle $\tau \in \mathcal{T}$ and zero everywhere else.
2. Correspond to the cotangent vectors $dx_1|_\mathbf{p}$ and $dx_2|_\mathbf{p}$ in $T_\mathbf{p}^*\mathbb{T}$, for $\mathbf{p} \in \mathbb{T}$.





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    b. Compute the differential
    c. **Negate and normalize**
2. Fit a scalar field to the target differentials ζ

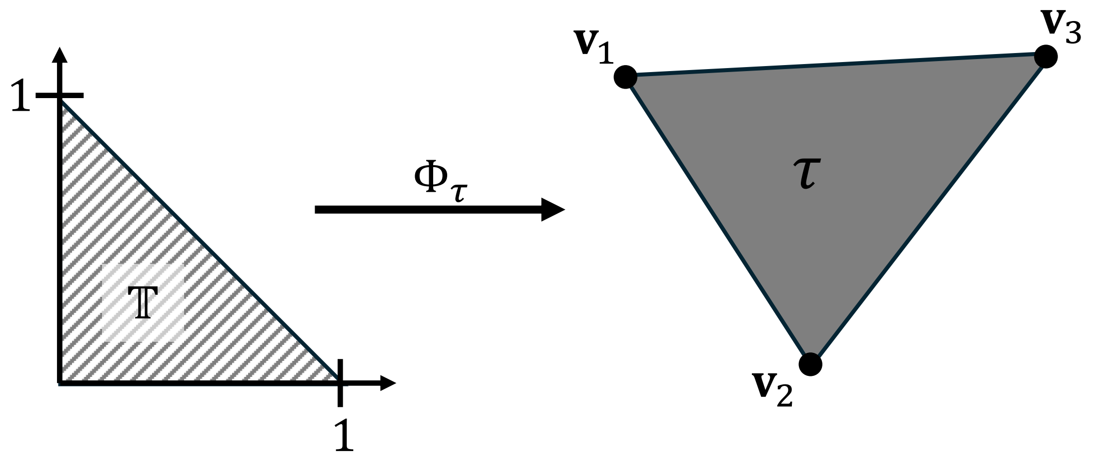
$$\mathbf{f} = (\mathbf{M} + \varepsilon \cdot \mathbf{S})^{-1} \cdot \mathbf{M} \cdot \boldsymbol{\delta}$$
$$\hat{\boldsymbol{\zeta}} = \mathbf{D} \cdot \mathbf{f} \in \mathbb{R}^{2|\mathcal{T}|}$$

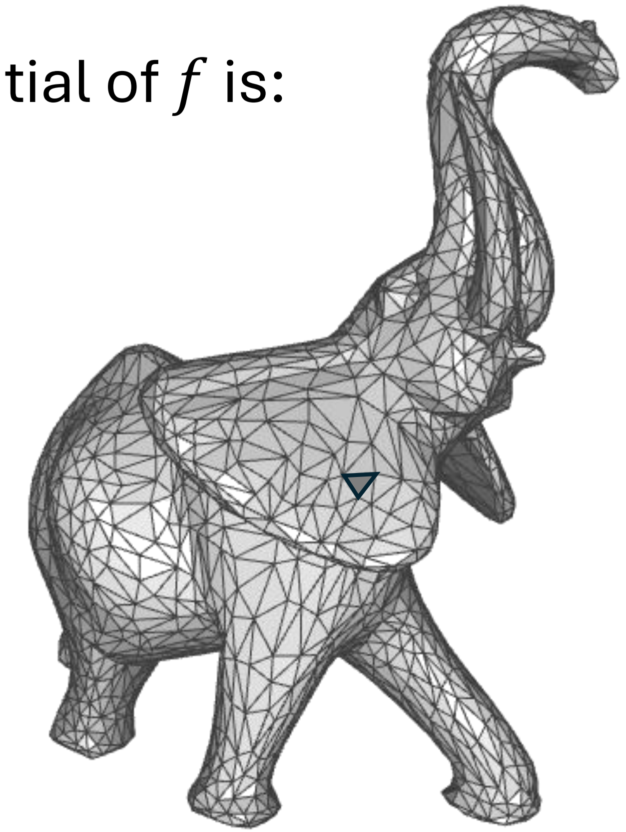
Negating and normalizing the differential:

Restricted to $\tau \in \mathcal{T}$ (and expressed over $\mathbb{T}$), the differential of f is:

$$df\,\Big|_{\tau} = \hat{\boldsymbol{\zeta}}_{2\tau+1} \cdot dx_1 + \hat{\boldsymbol{\zeta}}_{2\tau+2} \cdot dx_2$$

$\Rightarrow$ The norm of the differential is $\sqrt{\langle df, df \rangle}_{g_\tau^{-1}}$.





Geodesics in Heat

Approach:

1. **Estimate the differential $\zeta \in \overline{V}$**
   a. Diffuse the delta function
   b. Compute the differential
   c. **Negate and normalize**
2. Fit a scalar field to the target differentials ζ

$$\mathbf{f} = (\mathbf{M} + \varepsilon \cdot \mathbf{S})^{-1} \cdot \mathbf{M} \cdot \boldsymbol{\delta}$$
$$\hat{\boldsymbol{\zeta}} = \mathbf{D} \cdot \mathbf{f} \in \mathbb{R}^{2|\mathcal{T}|}$$

Negate and normalize the differential:

Restricted to $\tau \in \mathcal{T}$ (and expressed over $\mathbb{T}$), the differential of f is:

$$df\Big|_{\tau} = \hat{\zeta}_{2\tau+1} \cdot dx_1 + \hat{\zeta}_{2\tau+2} \cdot dx_2$$

⇒ The norm of the differential is $\sqrt{\langle df, df \rangle}_{g_\tau^{-1}}$.

⇒ The coefficients of the negated and normalized differential, $\boldsymbol{\zeta} \in \mathbb{R}^{2|\mathcal{T}|}$, are:

$$\begin{pmatrix} \zeta_{2\tau+1} \\ \zeta_{2\tau+2} \end{pmatrix} = -\frac{\begin{pmatrix} \hat{\zeta}_{2\tau+1} \\ \hat{\zeta}_{2\tau+2} \end{pmatrix}}{\sqrt{\begin{pmatrix} \hat{\zeta}_{2\tau+1} \\ \hat{\zeta}_{2\tau+2} \end{pmatrix}^{\top} \cdot \mathbf{g}_\tau^{-1} \cdot \begin{pmatrix} \hat{\zeta}_{2\tau+1} \\ \hat{\zeta}_{2\tau+2} \end{pmatrix}}}$$

Approach:

1. Estimate the differential $\zeta \in \overline{V}$
2. **Fit a scalar field to the target differentials ζ**

Geodesics in Heat

Given the target cotangent vector field $\zeta \in \overline{V}$, we want to solve for the scalar function $f \in V$, whose differential best matches ζ.

Naïvely:

We can formulate this as a gradient domain problem:

$$E(f) = \frac{1}{2}\langle\langle df - \zeta, df - \zeta\rangle\rangle_{\mathcal{M}}$$
$$\Downarrow$$
$$\mathbf{f} = \mathbf{S}^{-1} \cdot \mathbf{D}^{\top} \cdot \overline{\mathbf{M}} \cdot \boldsymbol{\zeta}$$

The stiffness matrix $\boldsymbol{S} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ represents a symmetric positive **semi**-definite bilinear form and is not invertible.

Approach:

1. Estimate the differential $\zeta \in \overline{V}$
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Geodesics in Heat

Given the target cotangent vector field $\zeta \in \overline{V}$, we want to solve for the scalar function $f \in V$, whose differential best matches ζ.

Regularization:

Modify the system to make it non-singular by adding a mass regularizer.

$$E(f) = \boxed{\varepsilon \cdot \frac{1}{2} \cdot \langle\langle f, f \rangle\rangle_{\mathcal{M}}} + \frac{1}{2} \cdot \langle\langle df - \zeta, df - \zeta \rangle\rangle_{\mathcal{M}}$$

$$\Downarrow$$

$$\mathbf{f} = (\varepsilon \cdot \mathbf{M} + \mathbf{S})^{-1} \cdot \mathbf{D}^{\top} \cdot \overline{\mathbf{M}} \cdot \boldsymbol{\zeta}$$

This biases the solver towards the least-squared-norm solution.

For tiny ε, this offsets the function by a constant to give a function with average value equal to zero.

Approach:

1. Estimate the differential $\zeta \in \overline{V}$
2. **Fit a scalar field to the target differentials ζ**

Geodesics in Heat

Given the target cotangent vector field $\zeta \in \overline{V}$, we want to solve for the scalar function $f \in V$, whose differential best matches ζ.

Regularization:

- ✓ The regularized system is positive definite
- ✗ We don't want the solution with average value equal to zero, we want the function equal to zero at the source vertex $x \in \mathcal{V}$.

$\Rightarrow$ We need to offset the function by a (different) constant term.

Approach:

1. Estimate the differential $\zeta \in \overline{V}$
2. **Fit a scalar field to the target differentials ζ**

Geodesics in Heat

Given the target cotangent vector field $\zeta \in \overline{V}$, we want to solve for the scalar function $f \in V$, whose differential best matches ζ.

Recall:

The hat basis forms a partition of unity.

$\Rightarrow$ Adding the same constant $\alpha \in \mathbb{R}$ to each coefficient of $\mathbf{f}$ is equivalent to offsetting the function f by a constant value of α.

$\Rightarrow$ Adjust for the constant term by evaluating the solution at x and subtracting that value from all the coefficients.

Approach:

1. Estimate the differential $\zeta \in \overline{V}$
2. **Fit a scalar field to the target differentials ζ**

Geodesics in Heat

Given the target cotangent vector field $\zeta \in \overline{V}$, we want to solve for the scalar function $f \in V$, whose differential best matches ζ.

We will see how to explicitly constrain the solution to evaluate to 0 at x next class:

✓ Forces the function to have the desired value at the source x.

✓ Removes the need for a regularizer.

✗ Results in a new system (i.e. requires re-factorization) when the source changes.

⇒ Adding the same constant $\alpha \in \mathbb{R}$ to each coefficient of $\mathbf{f}$ is equivalent to offsetting the function f by a constant value of α.

⇒ Adjust for the constant term by evaluating the solution at x and subtracting that value from all the coefficients.

Geodesics in Heat

Approach:

1. Estimate the differential $\zeta \in \overline{V}$
   a. Diffuse the delta function
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   c. Negate and **normalize**
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Bigger Perspective:

The underlying problem is inherently non-linear (normalization).

The algorithm is effective because it factors/parameterizes the problem into **local** and **global** components:

The **local** computation manages the non-linearity

The **global** computation makes the local computation consistent

Gradient Domain Processing

Note:

In solving for the scalar function that best fits target differentials:

$$E(f) = \frac{1}{2}\langle\langle df - \zeta, df - \zeta \rangle\rangle_{\mathcal{M}}$$

we are solving for the function giving the least-squares error.

Even ignoring issues of positive-definiteness, we are not guaranteed to get a solution with zero energy.

This is because the differential of a function is always **curl-free**, while the target differential field may not be:

$$\int_0^1 df\Big|_{c(t)}\big(c'(t)\big)dt = f\big(c(1)\big) - f\big(c(0)\big)$$

for all curves $c\colon [0,1] \to \mathcal{M}$.