Geometry Processing (601.458/658)

Misha Kazhdan

Outline

Recall

Quadratic Optimization

Gradient Domain Processing

Recall

Quadratic Energies:

Give a non-negative real value $q \geq 0$, the quadratic polynomial

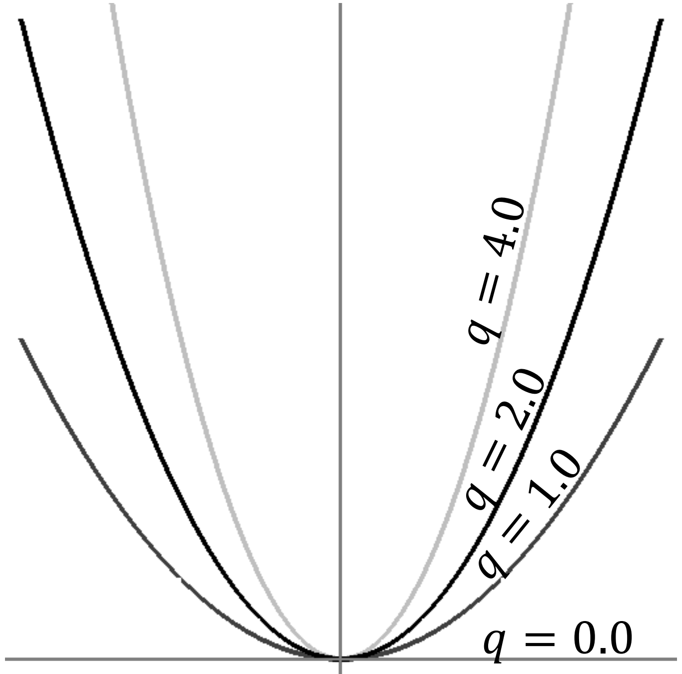
$$Q(x) = \frac{1}{2} \cdot q \cdot x^2$$

is bounded below by $Q(x) \geq 0$ and has derivative:

$$Q'(x) = q \cdot x$$

⇒ If $q > 0$, the polynomial has a unique extremum at $x = 0$.

⇒ If $q > 0$, the polynomial has a global minimum at $x = 0$.



Recall

Quadratic Energies:

Give a non-negative real value $q \geq 0$, and a real values l and c, the quadratic polynomial

$$Q(x) = \frac{1}{2} \cdot q \cdot x^2 - l \cdot x + c$$

has derivative:

$$Q'(x) = q \cdot x - l$$

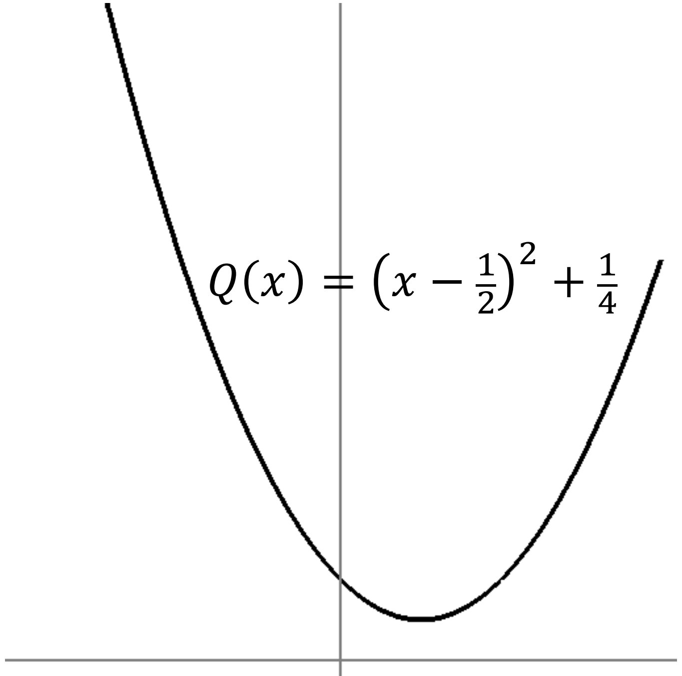
$\Rightarrow$ If $q > 0$, the polynomial has a global minimum at $x = l/q$.

Recall

Completing the square:

Give a non-negative real value $q \geq 0$, and a real values l and c, the quadratic polynomial

$$\begin{aligned} Q(x) &= \tfrac{1}{2} \cdot q \cdot x^2 - l \cdot x + c \\ &= \tfrac{1}{2} \cdot q \cdot \left(x - \tfrac{l}{q}\right)^2 - \tfrac{l^2}{2q} + c \end{aligned}$$



$\Rightarrow$ If $q > 0$, the polynomial $Q(x)$ is bounded below (by $c - \frac{l^2}{2q}$) and has an extremum at $x = \frac{l}{q}$.

$\Rightarrow$ If $q > 0$, the polynomial has a global minimum at $x = \frac{l}{q}$.

Recall

Quadratic Energies:

Given a collection of k quadratic polynomials

$$\left\{Q_1(x) = \tfrac{1}{2} \cdot q_1 \cdot x^2 - l_1 \cdot x + c_1, \ldots, Q_k(x) = \tfrac{1}{2} \cdot q_k \cdot x^2 - l_k \cdot x + c_k\right\}$$

the sum:

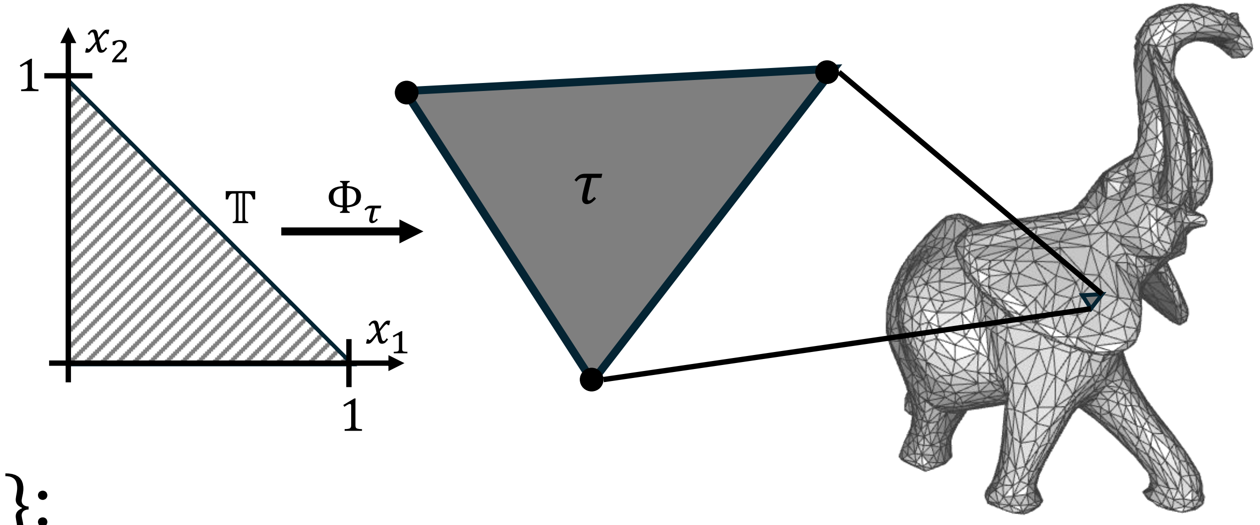
$$Q(x) = Q_1(x) + \cdots + Q_k(x)$$

is a quadratic polynomial.

If the sum of quadratic coefficients is positive (e.g. they are all non-negative and one is positive) the polynomial $Q(x)$ will have global minimum at:

$$x = \frac{l_1 + \cdots + l_k}{q_1 + \cdots + q_k}$$

Recall



Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$:

We can define a finite-dimensional space of scalar fields, spanned by the piecewise linear hat basis functions:

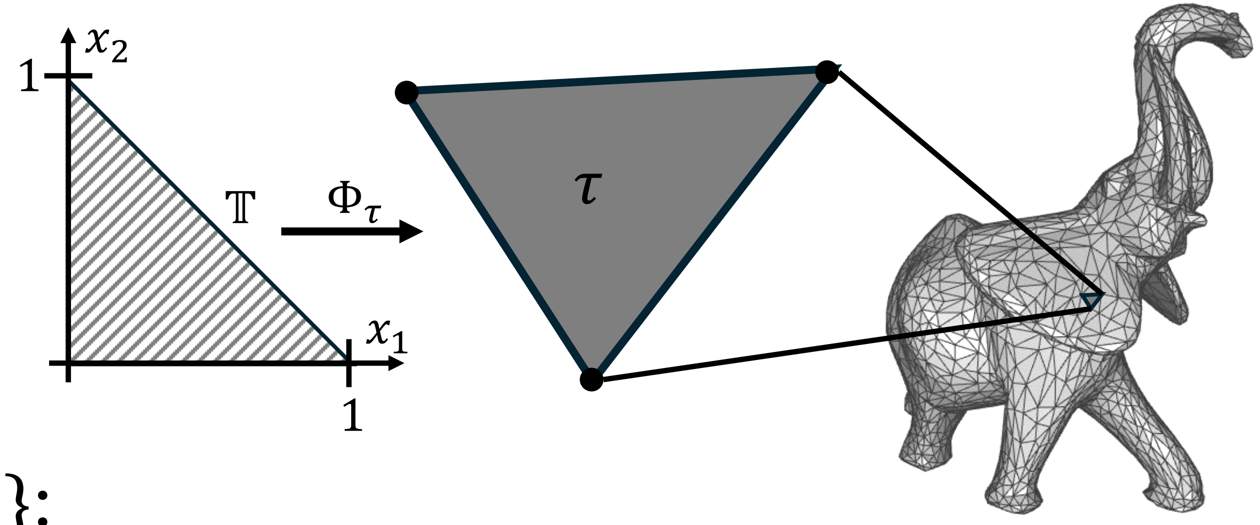
$$V = \mathrm{Span}(\{\phi_v\}_{v\in\mathcal{V}})$$

And a finite-dimensional space of cotangent vector fields, spanned by the piecewise constant functions (one per cartesian direction):

$$\overline{V} = \mathrm{Span}(\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}})$$

Integrating with respect to the inner-product defined on the tangent space, these are also inner-product spaces $\{V, M: V \to V^*\}$ and $\left\{\overline{V}, \overline{M}: \overline{V} \to \overline{V}^*\right\}$.

Recall



Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$:

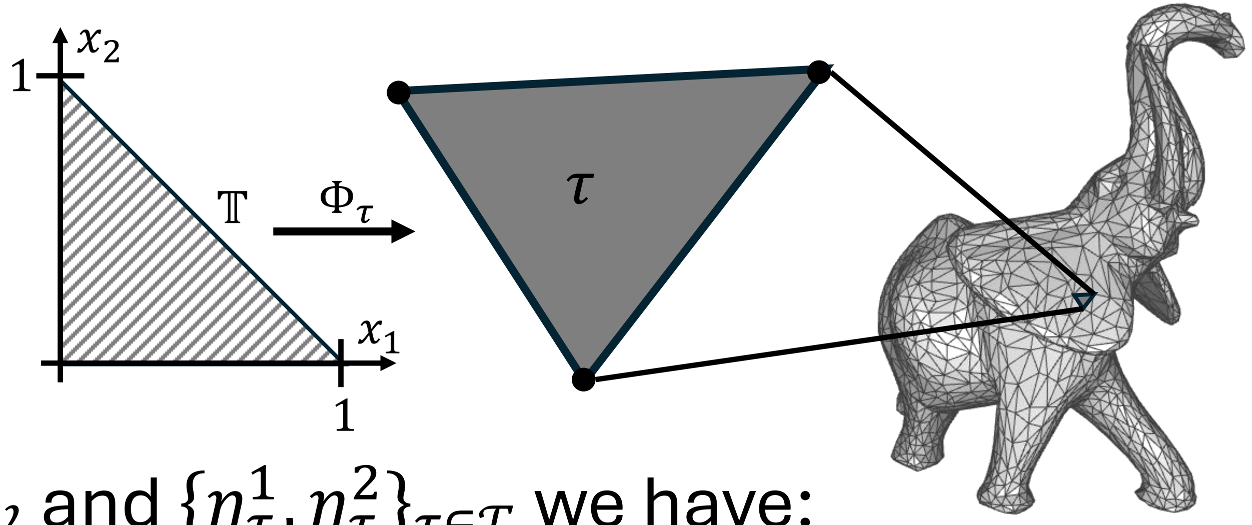
We denote by $d: V \to \overline{V}$ the linear map giving the differential.

We define the stiffness operator $S: V \to V^*$ as the bilinear form giving the inner-product of the differential of scalar fields.

This is the pull-back of the inner-product $\overline{M}: \overline{V} \to \overline{V}^*$ via the differential:

$$S = d^* \circ \overline{M} \circ d$$

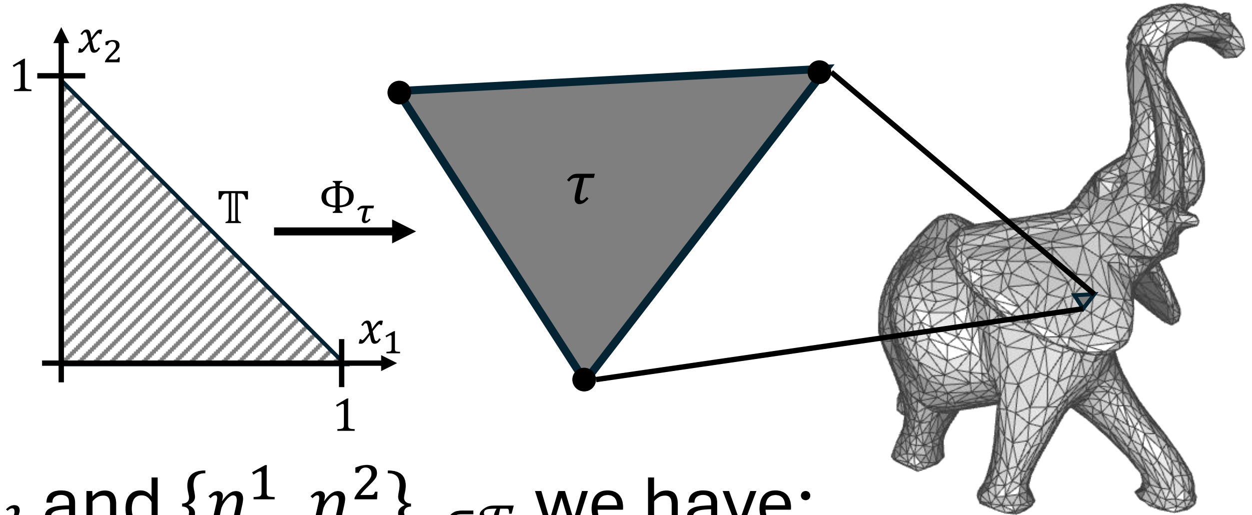
Recall



With respect to the bases $\{\phi_v\}_{v\in\mathcal{V}}$ and $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$ we have:

- The scalar field mass matrix: $\mathbf{M} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$
- The scalar field stiffness matrix: $\mathbf{S} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$
- The cotangent vector field mass matrix: $\overline{\mathbf{M}} \in \mathbb{R}^{2|\mathcal{T}|\times 2|\mathcal{T}|}$
- The differential matrix: $\mathbf{D} \in \mathbb{R}^{2|\mathcal{T}|\times|\mathcal{V}|}$
- The factorization of the scalar field stiffness matrix: $\mathbf{S} = \mathbf{D}^\top \cdot \overline{\mathbf{M}} \cdot \mathbf{D}$

Recall



With respect to the bases $\{\phi_v\}_{v\in\mathcal{V}}$ and $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$ we have:

Given functions $f, h \in V$, expressed w.r.t. the basis $\{\phi_v\}_{v\in\mathcal{V}}$ as $\mathbf{f}, \mathbf{h} \in \mathbb{R}^{|\mathcal{V}|}$:

$$f = \mathbf{f}_1 \cdot \phi_1 + \cdots + \mathbf{f}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$
$$h = \mathbf{h}_1 \cdot \phi_1 + \cdots + \mathbf{h}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$

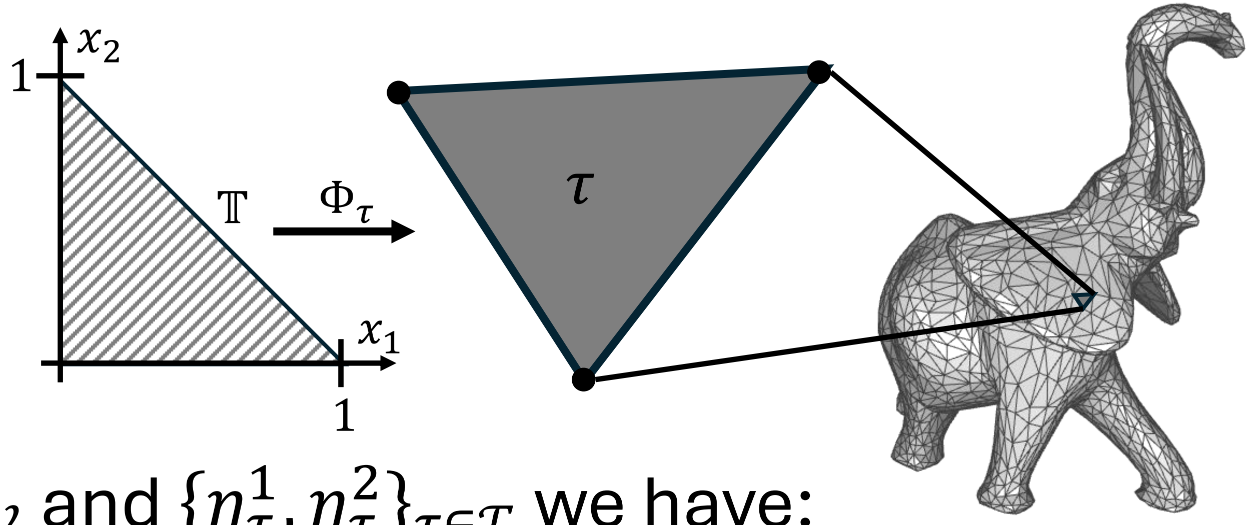
the inner-product of f with h is given by:

$$\langle\langle f, h\rangle\rangle_{\mathcal{M}} = \mathbf{f}^\top \cdot \mathbf{M} \cdot \mathbf{h}$$

The inner-product of the differential of f with the differential of h is given by:

$$\langle\langle df, dh\rangle\rangle_{\mathcal{M}} = \mathbf{f}^\top \cdot \mathbf{S} \cdot \mathbf{h}$$

Recall



With respect to the bases $\{\phi_v\}_{v\in\mathcal{V}}$ and $\{\eta_\tau^1,\eta_\tau^2\}_{\tau\in\mathcal{T}}$ we have:

Given a function $f \in V$, expressed w.r.t. the basis $\{\phi_v\}_{v\in\mathcal{V}}$ as $\mathbf{f} \in \mathbb{R}^{|\mathcal{V}|}$:

$$f = \mathbf{f}_1 \cdot \phi_1 + \cdots + \mathbf{f}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$

we can compute $\mathbf{w} = \mathbf{D} \cdot \mathbf{f} \in \mathbb{R}^{2|\mathcal{T}|}$.

This gives the expression of the differential of f w.r.t the basis $\{\eta_\tau^1,\eta_\tau^2\}_{\tau\in\mathcal{T}}$:

$$df = \mathbf{w}_1^1 \cdot \eta_1^1 + \mathbf{w}_1^2 \cdot \eta_1^2 + \cdots + \mathbf{w}_{|\mathcal{T}|}^1 \cdot \eta_{|\mathcal{T}|}^1 + \mathbf{w}_{|\mathcal{T}|}^2 \cdot \eta_{|\mathcal{T}|}^2$$

Outline

Recall

Quadratic Optimization

Gradient Domain Processing

Quadratic Optimization

Definition:

Given a vector space V and a bilinear form $B: V \to V^*$, the associated *quadratic form* is the real-valued function on V:

$$\begin{aligned} Q_B: V &\to \mathbb{R} \\ v &\mapsto \tfrac{1}{2} B(v, v) \end{aligned}$$

Recall:

For a bilinear form $B: V \to V^*$, we can define the dual $B^*: V \to V^*$.

For all $v, w \in V$, we have $B(v, w) = B^*(w, v)$

$\Rightarrow$ Replacing B with its symmetrization:

$$\begin{aligned} Q_{(B+B^*)/2}(v) &= \tfrac{1}{2}\big(B(v, v) + B^*(v, v)\big) \\ &= Q_B(v) \end{aligned}$$

Quadratic Optimization

Definition:

Given a vector space V and a bilinear form $B: V \to V^*$, the associated *quadratic form* is the real-valued function on V:

$$\begin{aligned} Q_B: V &\to \mathbb{R} \\ v &\mapsto \tfrac{1}{2} B(v, v) \end{aligned}$$

Recall:

For a bilinear form $B: V \to V^*$, we can define the dual $B^*: V \to V^*$.

For all $v, w \in V$

In working with quadratic forms, we will assume that B is symmetric.

$\Rightarrow$ Replacing B with its symmetrization:

$$\begin{aligned} Q_{(B+B^*)/2}(v) &= \tfrac{1}{2}\left(B(v,v) + B^*(v,v)\right) \\ &= Q_B(v) \end{aligned}$$

Quadratic Optimization

Definition:

Given a vector space V and a bilinear form $B: V \to V^*$, the associated *quadratic form* is the real-valued function on V:

$$Q_B: V \to \mathbb{R}$$
$$v \mapsto \frac{1}{2} B(v, v)$$

We'll show that this generalizes a quadratic polynomial in one variable:

If B is positive semi-definite, the form is bounded below.

If B is positive definite, it has a unique minimizer.

If the bilinear forms $B_1, \ldots, B_k: V \to V^*$ are positive semi-definite, their sum is as well.

If the bilinear forms $B_1, \ldots, B_k: V \to V^*$ are positive semi-definite, and one of them is strictly definite, their sum is positive definite.

Given a positive definite bilinear form and a dual vector, we can complete the square.

Quadratic Optimization

Recall:

Given a vector space V and a function $f: V \to \mathbb{R}$, the differential $df|_v$, along direction $w \in V$ gives the change of F as we move from v along direction w:

$$df\Big|_v(w) = \frac{\partial f}{\partial w}\Big|_v = \lim_{\varepsilon \to 0} \frac{f(v + \varepsilon \cdot w) - f(v)}{\varepsilon}$$

The differential of f is an element of the dual space, $df|_v \in V^*$

The differential of f vanishes at local extrema $v \in V$.

Quadratic Optimization

Given a vector space V, a dual vector $l \in V^*$ is a real-valued function on V:

$$\begin{aligned} l: V &\to \mathbb{R} \\ v &\mapsto l(v) \end{aligned}$$

By linearity, for any $w \in V$, the differential of l at $v \in V$ satisfies:

$$\begin{aligned} dl\Big|_v(w) &= \lim_{\varepsilon \to 0} \frac{l(v + \varepsilon \cdot w) - l(v)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{l(v) + \varepsilon \cdot l(w) \ - l(v)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\varepsilon \cdot l(w)}{\varepsilon} \\ &= l(w) \end{aligned}$$

$$\boxed{dl\Big|_v = l}$$

$$V \xrightarrow{L} W$$
$$V^* \xleftarrow{L^*} W^*$$

Quadratic Optimization

Given vector spaces V and W, and a linear map $L \in \mathrm{Hom}(V, W)$, the pull-back of a dual vector $l \in W^*$ is a real-valued function on V:

$$\begin{aligned} L^*(l)\colon V &\to \mathbb{R} \\ v &\mapsto l\big(L(v)\big) \end{aligned}$$

Corollary:

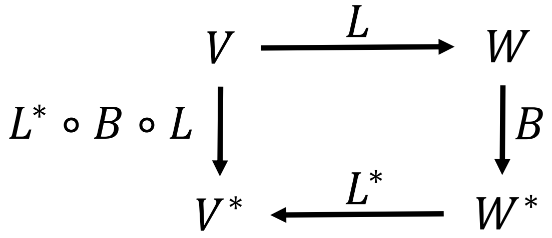
$$d\big(L^*(l)\big)\Big|_v = L^*(l)$$

Quadratic Optimization

Given a vector space V and a **symmetric** bilinear form $B: V \to V^*$, the differential of the associated quadratic form at $v \in V$ satisfies:

$$
\begin{aligned}
dQ_B\Big|_v(w) &= \lim_{\varepsilon\to 0} \frac{Q_B(v+\varepsilon\cdot w) - Q_B(v)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\frac{1}{2}[B(v+\varepsilon\cdot w)](v+\varepsilon\cdot w) - \frac{1}{2}[B(v)](v)}{\varepsilon} \\
&= \frac{1}{2}\lim_{\varepsilon\to 0} \frac{[B(v)](v) + \varepsilon\cdot[B(v)](w) + \varepsilon\cdot[B(w)](v) + \varepsilon^2\cdot[B(w)](w) - [B(v)](v)}{\varepsilon} \\
&= \frac{1}{2}\lim_{\varepsilon\to 0} \frac{\varepsilon\cdot[B(v)](w) + \varepsilon\cdot[B(v)](w) + \varepsilon^2\cdot[B(w)](w)}{\varepsilon} \\
&= [B(v)](w)
\end{aligned}
$$

$$
\boxed{dQ_B\Big|_v = B(v)}
$$



Quadratic Optimization

Given vector spaces V and W, and a linear map $L \in \mathrm{Hom}(V, W)$, the pull-back of symmetric bilinear form $B: W \to W^*$ is a symmetric bilinear form:

$$L^* \circ B \circ L: V \to V^*$$

Corollary:

$$dQ_{L^* \circ B \circ L}\Big|_v = (L^* \circ B \circ L)(v)$$

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ {\scriptstyle L^* \circ B \circ L}\downarrow & & \downarrow{\scriptstyle B} \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Quadratic Optimization

Given a vector space V, a symmetric bilinear form $B: V \to V^*$, and a linear functional $l \in V^*$, we can define the function:

$$f(v) = Q_B(v) - l(v)$$

Taking the differential at $v \in V$ gives:

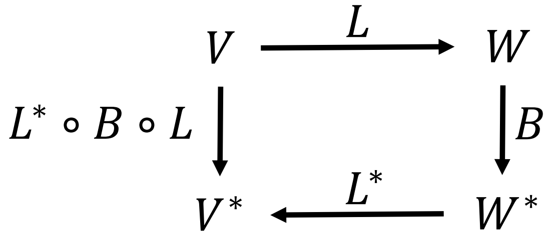
$$df\Big|_v = B(v) - l$$

$\Rightarrow$ The function is extremized at:

$$B(v) = l$$

$$dl\Big|_v = l$$
$$dQ_B\Big|_v = B(v)$$

Quadratic Optimization



Given vector spaces V and W, a linear map $L \in \mathrm{Hom}(V, W)$, a symmetric bilinear form $B: W \to W^*$, and a vector $w \in W$, we can define the function:

$$
\begin{aligned}
f(v) &= Q_B(L(v) - w) \\
&= \tfrac{1}{2}[B(L(v) - w)](L(v) - w) \\
&= \tfrac{1}{2}\Big(\big[B\big(L(v)\big)\big]\big(L(v)\big) - \big[B\big(L(v)\big)\big](w) - [B(w)]\big(L(v)\big) + [B(w)](w)\Big) \\
&= \tfrac{1}{2}[(L^* \circ B \circ L)(v)](v) - [B(w)]\big(L(v)\big) + \tfrac{1}{2}[B(w)](w) \\
&= \tfrac{1}{2}[(L^* \circ B \circ L)(v)](v) - [(L^* \circ B)(w)](v) + \tfrac{1}{2}[B(w)](w) \\
&= Q_{L^* \circ B \circ L}(v) - [(L^* \circ B)(w)](v) + Q_B(w)
\end{aligned}
$$

Taking the differential at $v \in V$ gives:

$$df\Big|_v = (L^* \circ B \circ L)(v) - (L^* \circ B)(w)$$

$\Rightarrow$ The function is extremized at:

$$(L^* \circ B \circ L)(v) = (L^* \circ B)(w)$$

$$d\big(L^*(l)\big)\Big|_v = L^*(l)$$

$$dQ_{L \circ B \circ L}\Big|_v = (L^* \circ B \circ L)(v)$$

Quadratic Optimization

Note:

Given a vector space V and symmetric bilinear forms $B_1, \ldots, B_k : V \to V^*$, the sum of the bilinear forms can be expanded as:

$$\begin{aligned}[(B_1 + \cdots + B_k)(v)](w) &= [B_1(v) + \cdots + B_k(v)](w) \\ &= [B_1(v)](w) + \cdots + [B_k(v)](w)\end{aligned}$$

⇒ The sum is symmetric.

⇒ If all the B_i are positive semi-definite, their sum is positive semi-definite.

⇒ If, additionally, one of the B_i is strictly positive definite, so is their sum.

⇒ If all the B_i are positive semi-definite and one of them is positive definite, their sum is invertible.

$$\frac{1}{2}\cdot q\cdot x^2 - l\cdot x = \frac{1}{2}\cdot q\cdot\left(x-\frac{l}{q}\right)^2 - \frac{l^2}{q}$$

Quadratic Optimization

Completing the Square:

Given a vector space V, a symmetric positive definite form $B: V \to V^*$, and a dual vector $l \in V^*$, we can define the energy:

$$\begin{aligned} f(v) &= Q_B(v) - l(v) \\ &= Q_B\left(v - B^{-1}(l)\right) + Q_{B^{-1}}(l) \end{aligned}$$

$\Rightarrow$ The function is bounded below by $Q_{B^{-1}}(l)$ and extremized at $v = B^{-1}(l)$.

$\Rightarrow$ The function has a global minimum at $v = B^{-1}(l)$.

$$dl\Big|_v = l$$
$$dQ_B\Big|_v = B(v)$$

Outline

Recall

Quadratic Optimization

Gradient Domain Processing

Gradient* Domain Processing

General Problem:

Given a scalar function $h \in V$ and a cotangent vector field $\zeta \in \overline{V}$, solve for the function $f \in V$ which minimizes the energy:

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \underbrace{\langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}}}_{\text{value fitting}} + \beta \cdot \frac{1}{2} \cdot \underbrace{\langle\langle df - \zeta, df - \zeta \rangle\rangle_{\mathcal{M}}}_{\text{differential fitting}}$$

The function f should have:

Values matching those of h,

Differentials matching those of ζ.

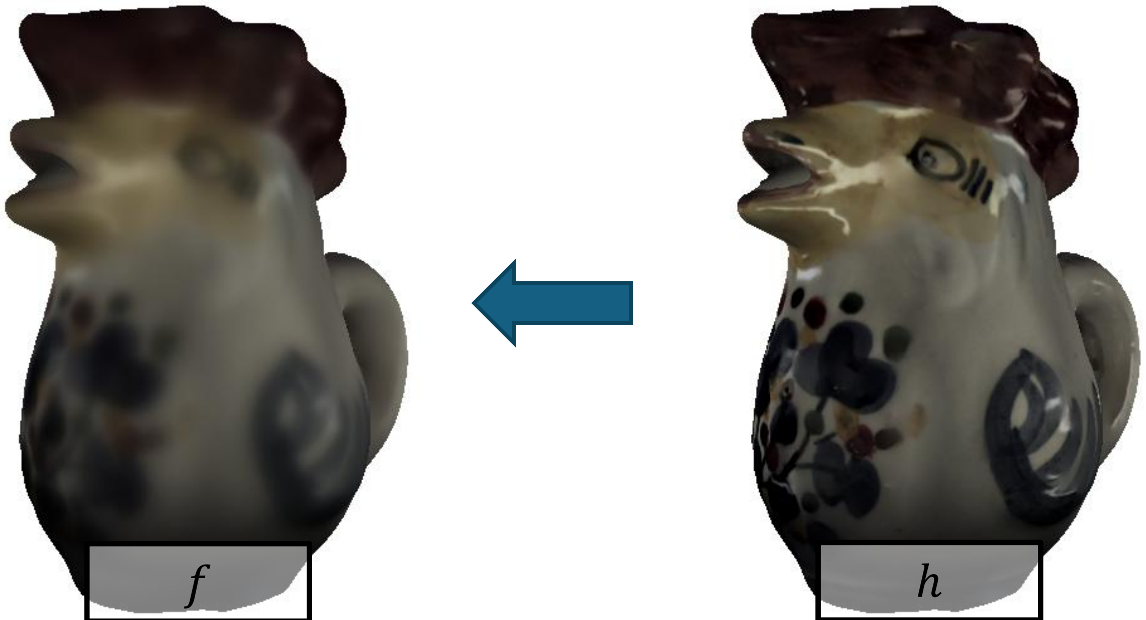
*Classically, the formulation of gradient-domain processing is in terms of the gradient, but it is equivalently formulated in terms of the differential.

Gradient Domain Processing: Smoothing

Given a signal $h: \mathcal{M} \to \mathbb{R}$, **smooth** the input:

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h\rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df, df\rangle\rangle_{\mathcal{M}}$$

⇒ Values should match those of the input, but differences should be small.

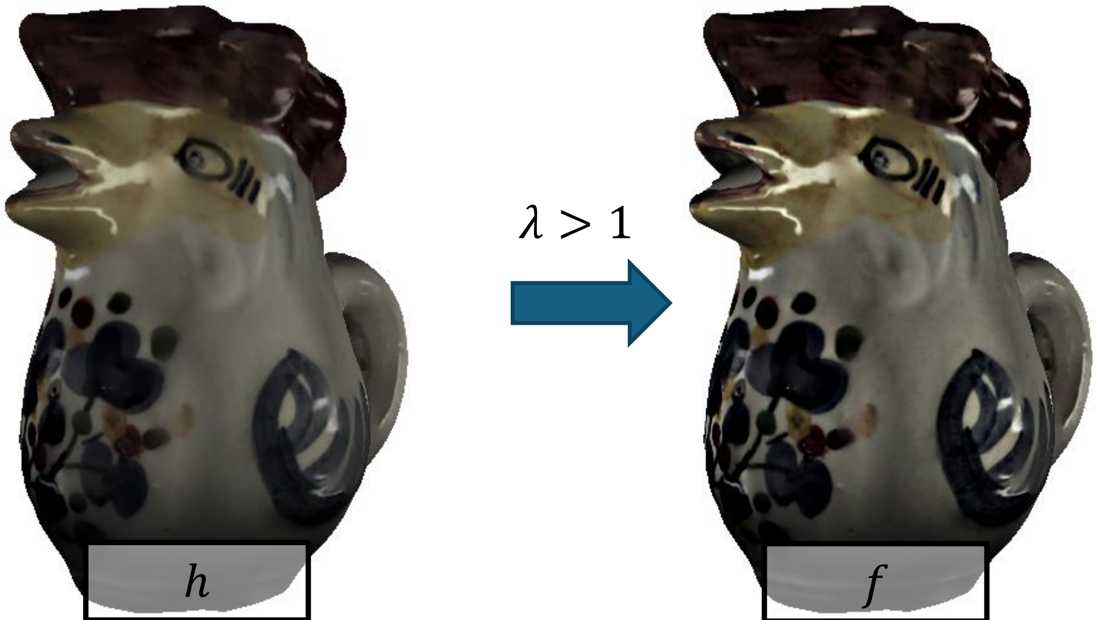


Gradient Domain Processing: Sharpening

Given a signal $h: \mathcal{M} \to \mathbb{R}$ and a scaling factor $\lambda \in \mathbb{R}$, **sharpen** the input:

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \lambda \cdot dh, df - \lambda \cdot dh \rangle\rangle_{\mathcal{M}}$$

⇒ Values should match those of the input but differences should be amplified (with extent of amplification parameterized by λ).

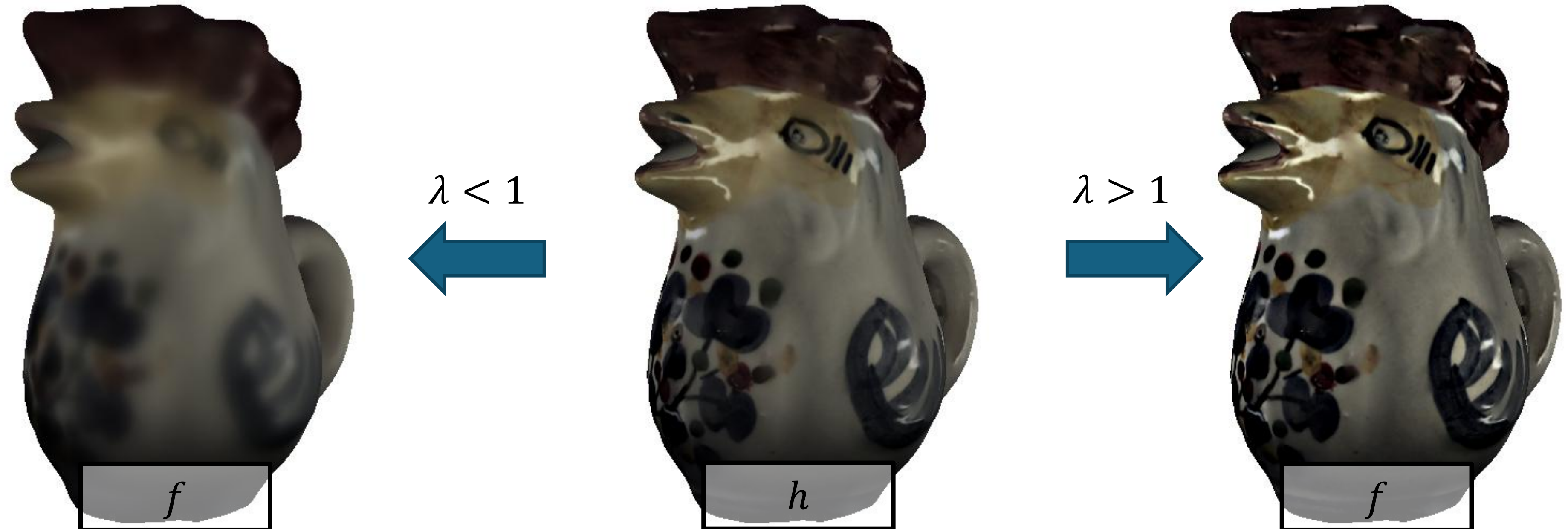


Gradient Domain Processing: Modulation

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \lambda \cdot df, df - \lambda \cdot df \rangle\rangle_{\mathcal{M}}$$

Gradient modulation:

- For $\lambda > 1$ differences are amplified, resulting in sharpening.
- For $\lambda < 1$ differences are dampened, resulting smoothing.



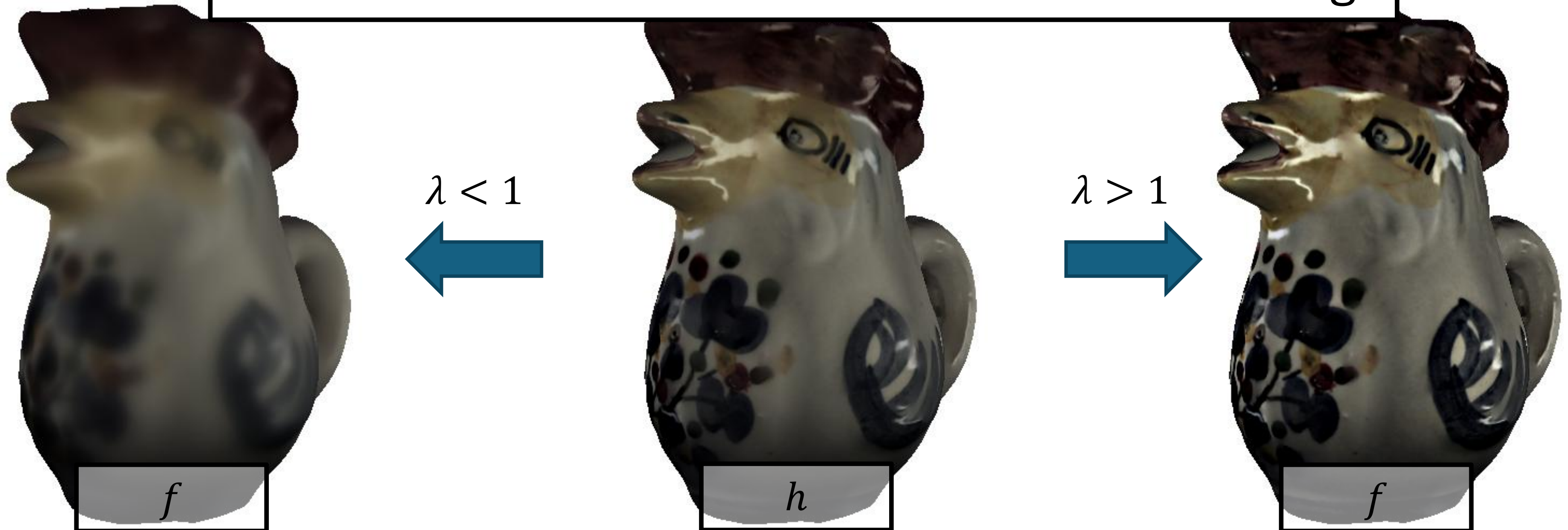
Gradient Domain Processing: Modulation

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \lambda \cdot df, df - \lambda \cdot df \rangle\rangle_{\mathcal{M}}$$

Gradient modulation:

- For $\lambda > 1$ differences are amplified, resulting in sharpening.
- For $\lambda < 1$ differences are dampened, resulting smoothing.

The case $\lambda = 0$ is an extreme case of smoothing.

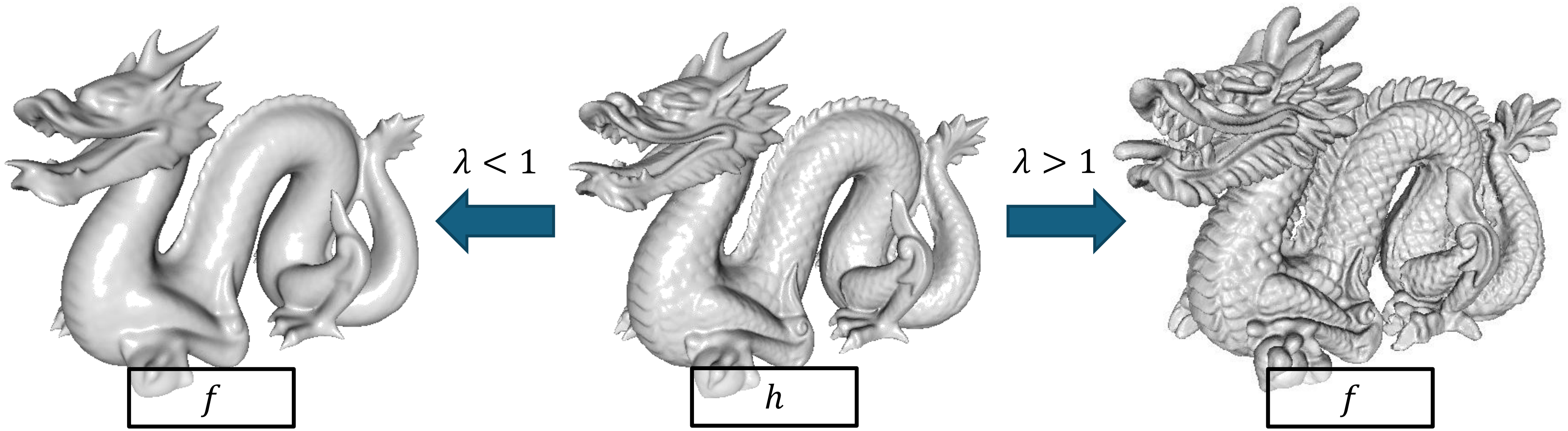


Gradient Domain Processing: Modulation

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \lambda \cdot df, df - \lambda \cdot df \rangle\rangle_{\mathcal{M}}$$

Gradient modulation:

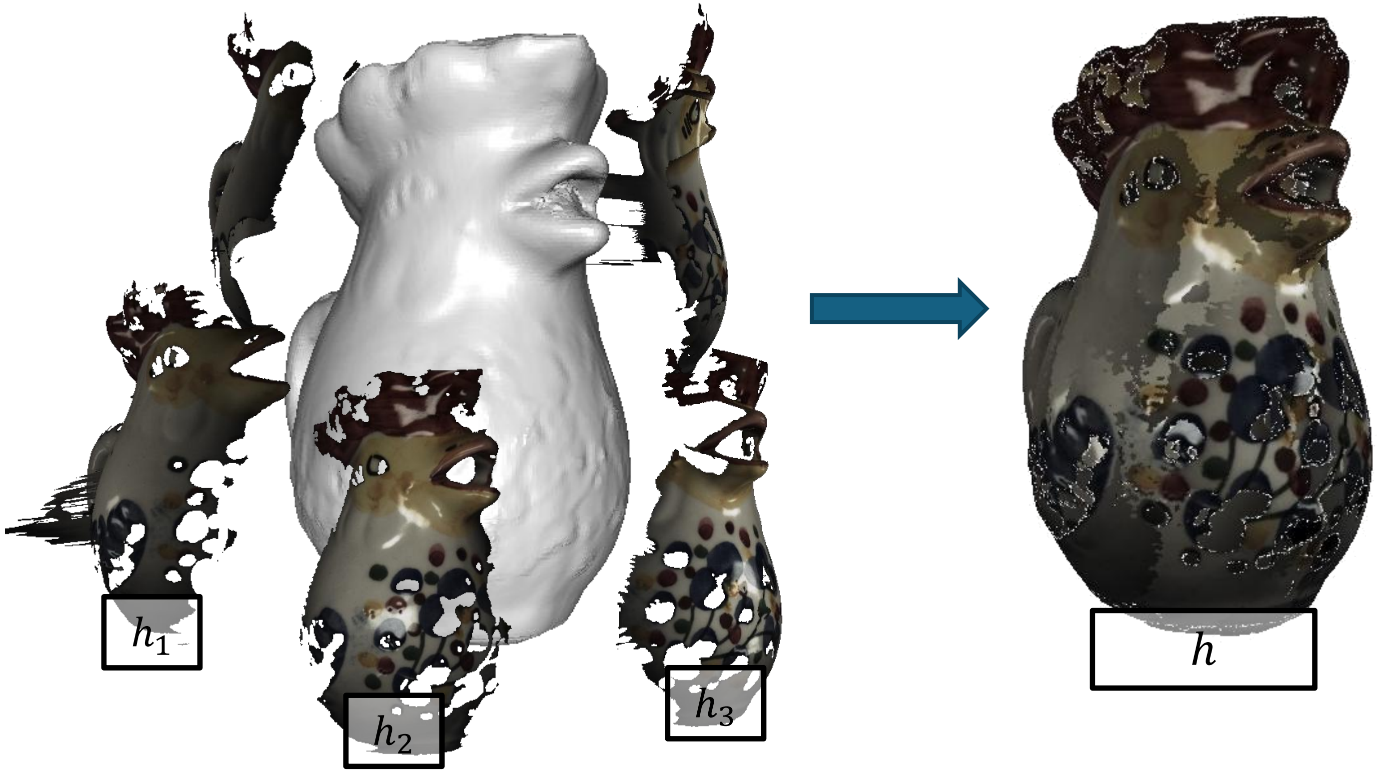
We can also apply this to the x-, y-, and z- coordinates of the geometry to sharpen smooth the surface.



Gradient Domain Processing: Stitching

Challenge:

Given a collection of partial signals $h_1, \ldots, h_k : \mathcal{M} \to \mathbb{R}$, the composite signal $h : \mathcal{M} \to \mathbb{R}$ may exhibit stitching discontinuities.

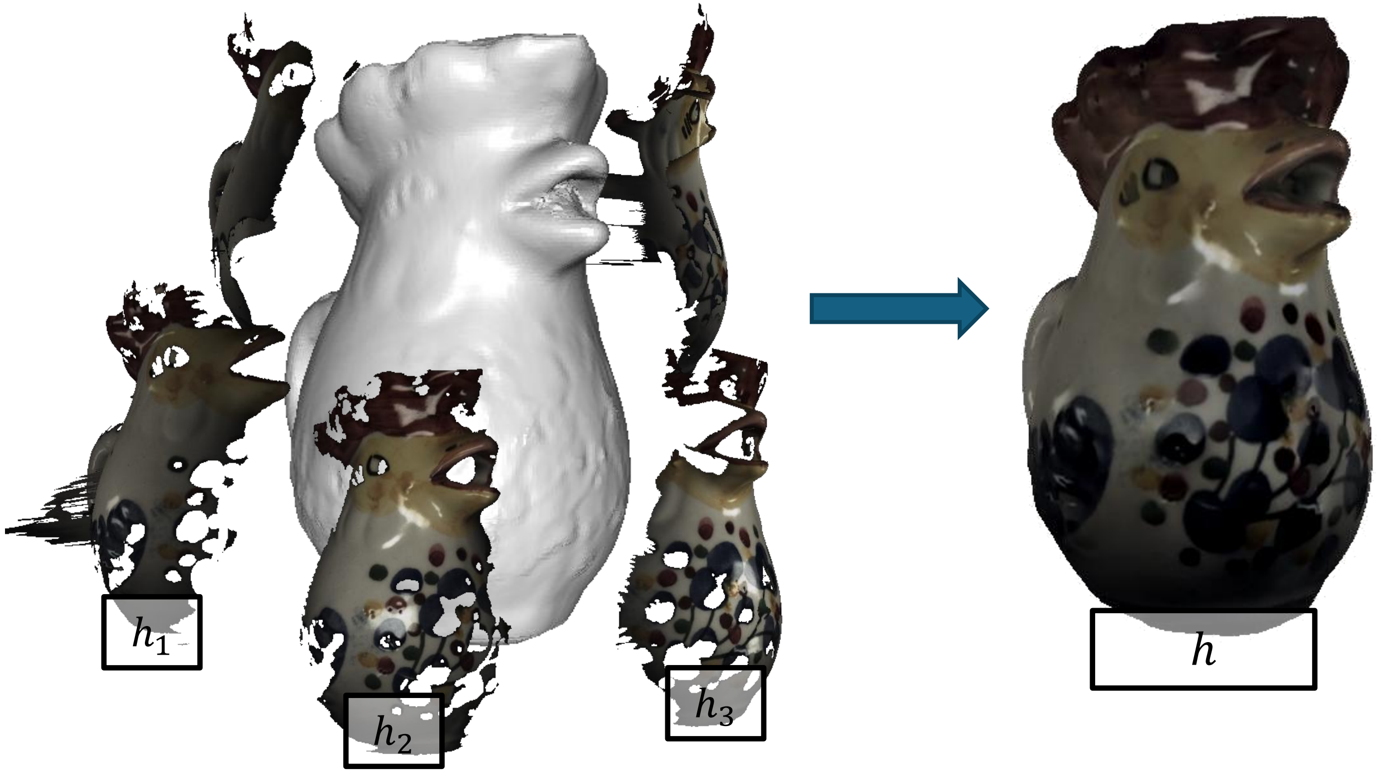


Gradient Domain Processing: Stitching

Approach:

Composite the differentials $\{dh_1, \dots, dh_k\} \to \zeta$ and solve:

$$\varepsilon \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \zeta, df - \zeta \rangle\rangle_{\mathcal{M}}$$



Gradient Domain Processing: Stitching

Approach:

Composite the differentials $\{dh_1, \ldots, dh_k\} \rightarrow \zeta$ and solve:

$$\varepsilon \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \zeta, df - \zeta \rangle\rangle_{\mathcal{M}}$$

The composite ζ assigns a constant cotangent vector field to each triangle.

For each triangle $\tau \in \mathcal{T}$:
  if one of the h_i is defined on all corners of τ:
    set ζ_τ to be the differential of h_i
  if multiple h_i are defined on all corners of τ:
    set ζ_t to be the average of the differentials
  if no h_i is defined on all corners of τ:
    set ζ_τ to zero

$$dQ_B(v-w)\Big|_v = B(v) - B(w)$$

$$dQ_B(L(v)-w)\Big|_v = (L^* \circ B \circ L)(v) - (L^* \circ B)(w)$$

Gradient Domain Processing

$$E(f) = \alpha \cdot \tfrac{1}{2} \cdot \langle\langle f-h, f-h \rangle\rangle_{\mathcal{M}} + \beta \cdot \tfrac{1}{2} \cdot \langle\langle df-\zeta, df-\zeta \rangle\rangle_{\mathcal{M}}$$

In terms of the bilinear forms $M: V \to V^*$ and $\overline{M}: \overline{V} \to \overline{V}^*$, this becomes:

$$\begin{aligned} E(f) &= \alpha \cdot \tfrac{1}{2} \cdot M(f-h, f-h) + \beta \cdot \tfrac{1}{2} \cdot \overline{M}(df-\zeta, df-\zeta) \\ &= \alpha \cdot Q_M(f-h) + \beta \cdot Q_{\overline{M}}(df-\zeta) \end{aligned}$$

Taking the differential, we get:

$$\begin{aligned} dE\Big|_f &= \alpha \cdot \big(M(f) - M(h)\big) + \beta \cdot \Big(\big(d^* \circ \overline{M} \circ d\big)(f) - \big(d^* \circ \overline{M}\big)(\zeta)\Big) \\ &= \big(\alpha \cdot M + \beta \cdot d^* \circ \overline{M} \circ d\big)(f) - \Big(\alpha \cdot M(h) + \beta \cdot \big(d^* \circ \overline{M}\big)(\zeta)\Big) \\ &= (\alpha \cdot M + \beta \cdot S)(f) - \Big(\alpha \cdot M(h) + \beta \cdot \big(d^* \circ \overline{M}\big)(\zeta)\Big) \end{aligned}$$

$$d\left(Q_B(L(v)-w)\right)\Big|_v = (L^* \circ B \circ L)(v) - (L^* \circ B)(w)$$

Gradient Domain Processing

$$dE\Big|_f = (\alpha \cdot M + \beta \cdot S)(f) - \left(\alpha \cdot M(h) + \beta \cdot \left(d^* \circ \overline{M}\right)(\zeta)\right)$$

If f is a minimizer of the energy, we must have:

$$(\alpha \cdot M + \beta \cdot S)(f) = \alpha \cdot M(h) + \beta \cdot \left(d^* \circ \overline{M}\right)(\zeta)$$

Since M is symmetric positive definite and S is symmetric positive semi-definite, the operator $(\alpha \cdot M + \beta \cdot S)$ is invertible whenever $\alpha > 0$ and $\beta \geq 0$:

$$f = (\alpha \cdot M + \beta \cdot S)^{-1}\left(\alpha \cdot M(h) + \beta \cdot \left(d^* \circ \overline{M}\right)(\zeta)\right)$$

Expressed in terms of the scalar field and cotangent vector field bases $\{\phi_v\}_{v\in\mathcal{V}}$ and $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$, this gives the linear system:

$$\mathbf{f} = (\alpha \cdot \mathbf{M} + \beta \cdot \mathbf{S})^{-1} \cdot \left(\alpha \cdot \mathbf{M} \cdot \mathbf{h} + \beta \cdot \mathbf{d}^\top \cdot \overline{\mathbf{M}} \cdot \boldsymbol{\zeta}\right)$$

Gradient Domain Modulation

Recall that for gradient domain modulation, the energy is:

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \lambda \cdot dh, df - \lambda \cdot dh \rangle\rangle_{\mathcal{M}}$$

In terms of the bilinear forms $M: V \to V^*$ and $\overline{M}: \overline{V} \to \overline{V}^*$, this becomes:

$$\begin{aligned} E(f) &= \alpha \cdot \frac{1}{2} \cdot M(f - h, f - h) + \beta \cdot \frac{1}{2} \cdot \overline{M}(df - \lambda \cdot dh, df - \lambda \cdot dh) \\ &= \alpha \cdot Q_M(f - h) + \beta \cdot Q_{\overline{M}}(df - \lambda \cdot dh) \\ &= \alpha \cdot Q_M(f - h) + \beta \cdot Q_{d^* \circ \overline{M} \circ d}(f - \lambda \cdot h) \\ &= \alpha \cdot Q_M(f - h) + \beta \cdot Q_S(f - \lambda \cdot h) \end{aligned}$$

So the differential of the energy, at $f \in V$, is:

$$\begin{aligned} dE\Big|_f &= \alpha \cdot M(f - h) + \beta \cdot S(f - \lambda \cdot h) \\ &= (\alpha \cdot M + \beta \cdot S)(f) - (\alpha \cdot M + \beta \cdot \lambda \cdot h) \end{aligned}$$

Gradient Domain Modulation

For gradient domain modulation, the differential of the energy is:

$$dE\Big|_f = (\alpha\cdot M + \beta\cdot S)(f) - (\alpha\cdot M + \beta\cdot\lambda\cdot h)$$

Setting the differential to zero, the minimizer is:

$$f = (\alpha\cdot M + \beta\cdot S)^{-1}\big((\alpha\cdot M + \beta\cdot\lambda\cdot S)(h)\big)$$

Expressed in terms of the scalar field and cotangent vector field bases $\{\phi_v\}_{v\in\mathcal{V}}$, this gives the linear system:

$$\mathbf{f} = (\alpha\cdot\mathbf{M} + \beta\cdot\mathbf{S})^{-1}\cdot(\alpha\cdot\mathbf{M} + \beta\cdot\lambda\cdot\mathbf{S})\cdot\mathbf{h}$$

$\Rightarrow$ Do not need to differentials (or cotangent vector fields).

Gradient Domain Modulation

For gradient domain modulation, the minimizer is:

$$\mathbf{f} = (\alpha \cdot \mathbf{M} + \beta \cdot \mathbf{S})^{-1} \cdot (\alpha \cdot \mathbf{M} + \beta \cdot \lambda \cdot \mathbf{S}) \cdot \mathbf{h}$$

⇒ In the case of extreme smoothing, $\lambda = 0$, this reduces to:

$$\begin{aligned}\mathbf{f} &= (\alpha \cdot \mathbf{M} + \beta \cdot \mathbf{S})^{-1} \cdot (\alpha \cdot \mathbf{M}) \cdot \mathbf{h} \\ &= \left(\mathbf{M} + \frac{\beta}{\alpha} \cdot \mathbf{S}\right)^{-1} \cdot \mathbf{M} \cdot \mathbf{h}\end{aligned}$$

This is an implicit time-step of the gradient descent PDE with $\mathbf{h} = \mathbf{a}^t$ the current solution, $\mathbf{f} = \mathbf{a}^{t+\boldsymbol{\varepsilon}}$ the next solution, and $\varepsilon = \frac{\beta}{\alpha}$ the time-step.

Gradient Domain Processing

$$E(f) = \alpha \cdot \frac{1}{2} \cdot \langle\langle f - h, f - h \rangle\rangle_{\mathcal{M}} + \beta \cdot \frac{1}{2} \cdot \langle\langle df - \zeta, df - \zeta \rangle\rangle_{\mathcal{M}}$$

Note:

If $\beta = 0$, setting $f = h$ gives a solution with zero energy.

If $\alpha = 0$, even ignoring issues of positive-definiteness, we are not guaranteed to get a solution with zero energy.

This is because the differential of a function is always **curl-free**, while the target differential field may not be:

$$\int_0^1 df\Big|_{c(t)} \big(c'(t)\big) dt = f\big(c(1)\big) - f\big(c(0)\big)$$

for all curves $c: [0,1] \to \mathcal{M}$.