Geometry Processing (601.458/658)

Misha Kazhdan

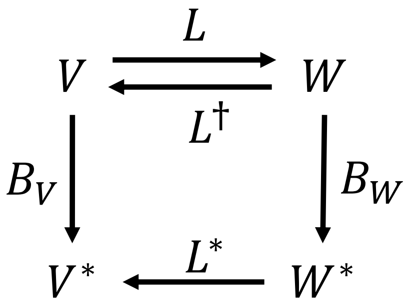
Outline

Recall

The Laplacian

Heat Diffusion

Recall: Adjoints



Given inner-product spaces $\{V, B_V: V \to V^*\}$ and $\{W, B_W: W \to W^*\}$, and a linear map $L \in \mathrm{Hom}(V, W)$, the adjoint of L is the map $L^\dagger \in \mathrm{Hom}(W, V)$:

$$\left\langle v, L^\dagger(w) \right\rangle_{B_V} = \langle L(v), w \rangle_{B_W}$$

$$\Updownarrow$$

$$\left[B_V\left(L^\dagger(w)\right)\right](v) = [B_W(w)]\big(L(v)\big)$$

$$\Updownarrow$$

$$\left[(B_V \circ L^\dagger)(w)\ \right](v) = [(L^* \circ B_W)(w)](v)$$

$$\Updownarrow$$

$$B_V \circ L^\dagger = L^* \circ B_W$$

$$\Updownarrow$$

$$L^\dagger = B_V^{-1} \circ L^* \circ B_W$$

Recall: Adjoints

Given inner-product spaces $\{V, B_V: V \to V^*\}$ and $\{W, B_W: W \to W^*\}$, and a linear map $L \in \mathrm{Hom}(V, W)$, the adjoint of L is the map $L^\dagger \in \mathrm{Hom}(W, V)$:

$$\left\langle v, L^\dagger(w)\right\rangle_{B_V} = \langle L(v), w\rangle_{B_W}$$

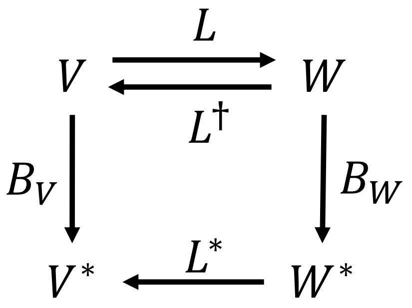
The adjoint of the adjoint is the original linear map:

$$L^{\dagger\dagger} = L.$$

Given linear maps $L \in \mathrm{Hom}(U, V)$ and $M \in \mathrm{Hom}(V, W)$, the adjoint of the composition is the composition of the adjoints, in the opposite order.

$$(M \circ L)^\dagger = L^\dagger \circ M^\dagger$$

Recall: Adjoints



Given inner-product spaces $\{V, B_V: V \to V^*\}$ and $\{W, B_W: W \to W^*\}$, and a linear map $L \in \mathrm{Hom}(V, W)$, the adjoint of L is the map $L^\dagger \in \mathrm{Hom}(W, V)$:

$$\left\langle v, L^\dagger(w) \right\rangle_{B_V} = \langle L(v), w \rangle_{B_W}$$
$$L^{\dagger\dagger} = L$$
$$(M \circ L)^\dagger = L^\dagger \circ M^\dagger$$

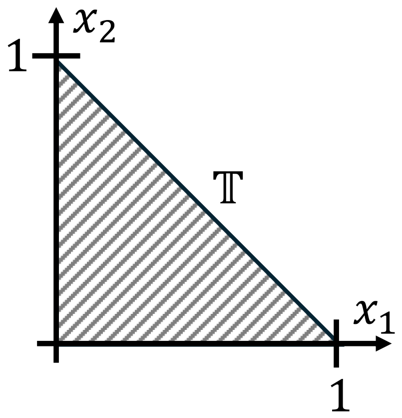
Corollary:

The composition of L with its adjoint is self-adjoint.

Proof:

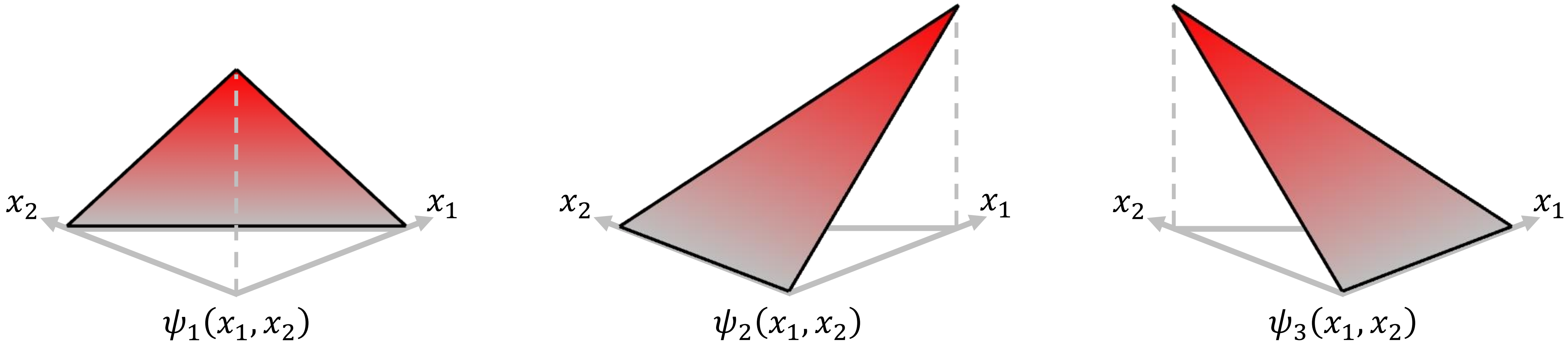
$$\begin{aligned} \left(L^\dagger \circ L\right)^\dagger &= L^\dagger \circ L^{\dagger\dagger} \\ &= L^\dagger \circ L \end{aligned}$$

Recall

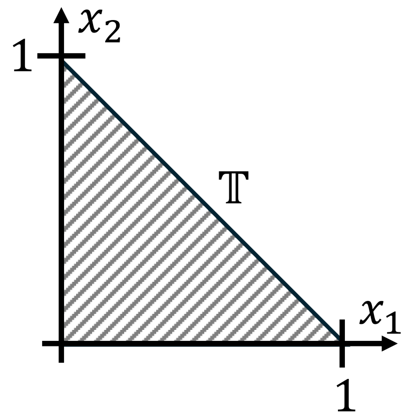


For the unit right triangle $\mathbb{T}$, we have a basis for linear functions:

$$\psi_1(x_1, x_2) = 1 - x_1 - x_2, \qquad \psi_2(x_1, x_2) = x_1, \qquad \psi_3(x_1, x_2) = x_2$$



Recall



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This defines a vector space $V = \mathrm{Span}(\psi_1, \psi_2, \psi_3)$.

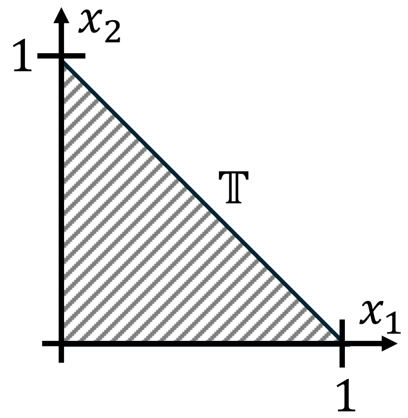
Which we turn into an inner-product space $\{V, m^\tau : V \to V^*\}$:

$$m^\tau(f, g) = \langle\langle f, g \rangle\rangle_\tau$$

We also have a symmetric, positive semi-definite, bilinear stiffness form:

$$s^\tau(f, g) = \langle\langle df, dg \rangle\rangle_\tau$$

Recall

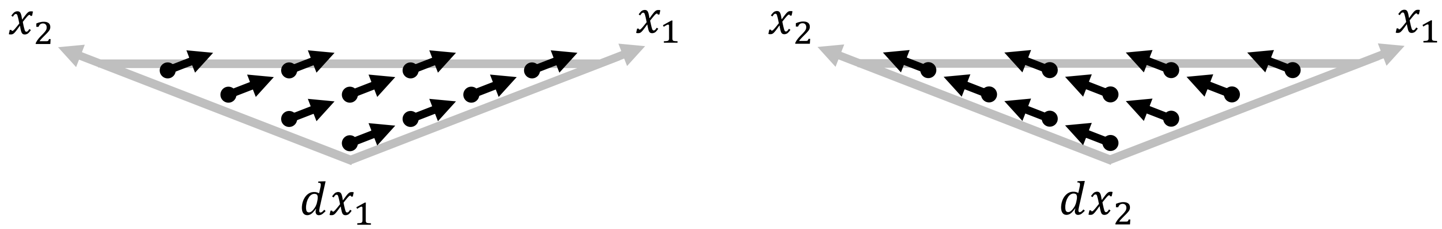


We also have a basis for constant cotangent vector fields $\{dx_1, dx_2\}$.

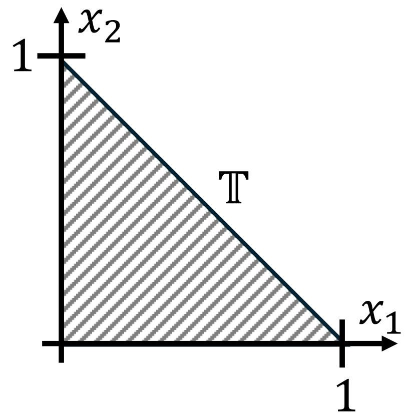
This defines a vector space $\overline{V} = \mathrm{Span}(dx_1, dx_2)$.

Which we turn into an inner-product space $\left\{\overline{V}, \overline{m}^{\tau}: \overline{V} \to \overline{V}^*\right\}$:

$$\overline{m}^{\tau}(\eta, \nu) = \langle\langle \eta, \nu \rangle\rangle_{\tau}$$



Recall



$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
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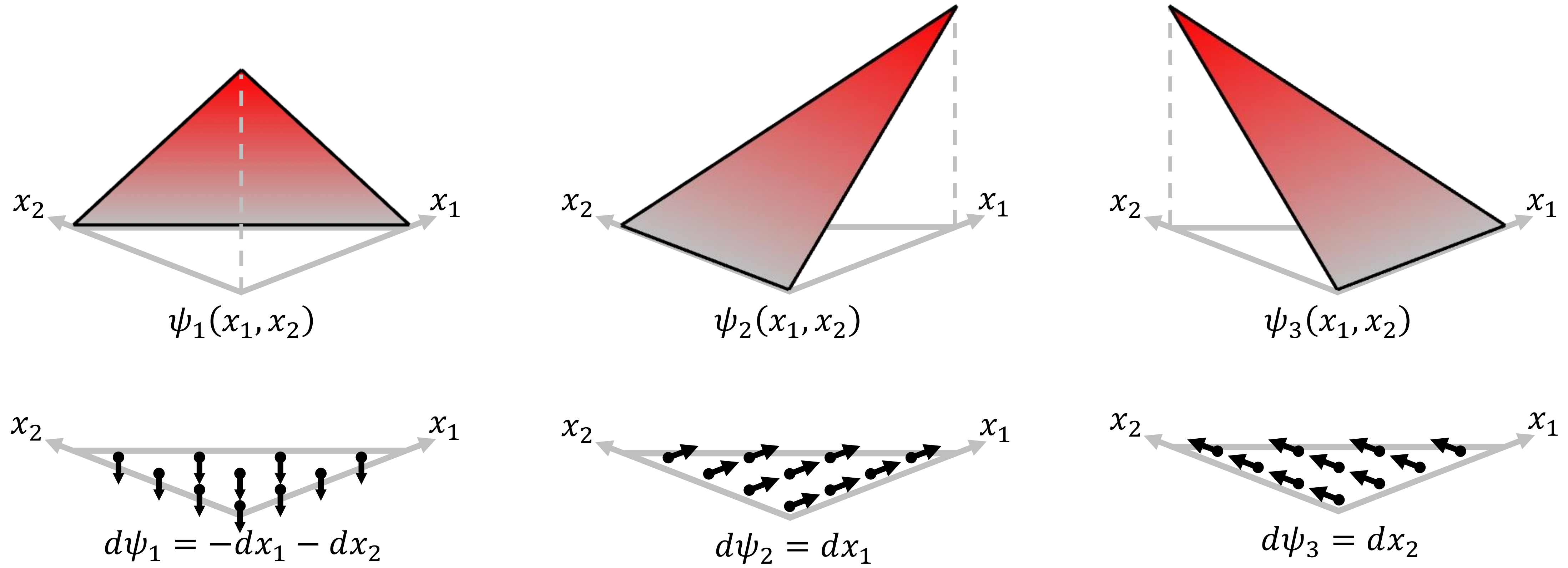
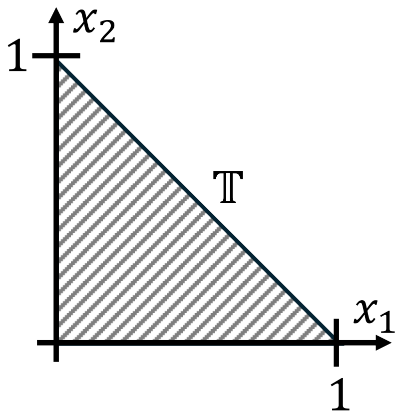
We can express the differentials of $\{\psi_1, \psi_2, \psi_3\}$ in terms of $\{dx_1, dx_2\}$.



Recall



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$d\psi_1 = -dx_1 - dx_2$
$d\psi_2 = dx_1$
$d\psi_3 = dx_2$

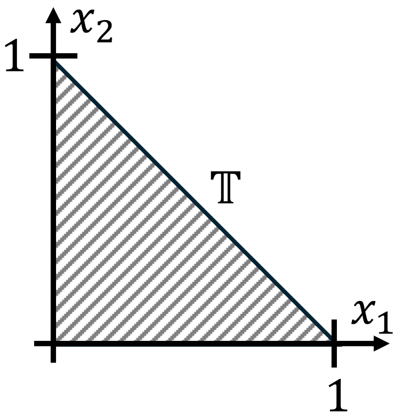
We can express the differentials of $\{\psi_1, \psi_2, \psi_3\}$ in terms of $\{dx_1, dx_2\}$.

This defines a linear map $d \in \mathrm{Hom}\left(V, \overline{V}\right)$.

Using the differential to pull-back the inner-product gives:

$$d^* \circ \overline{m}^\tau \circ d = s^\tau$$

Recall



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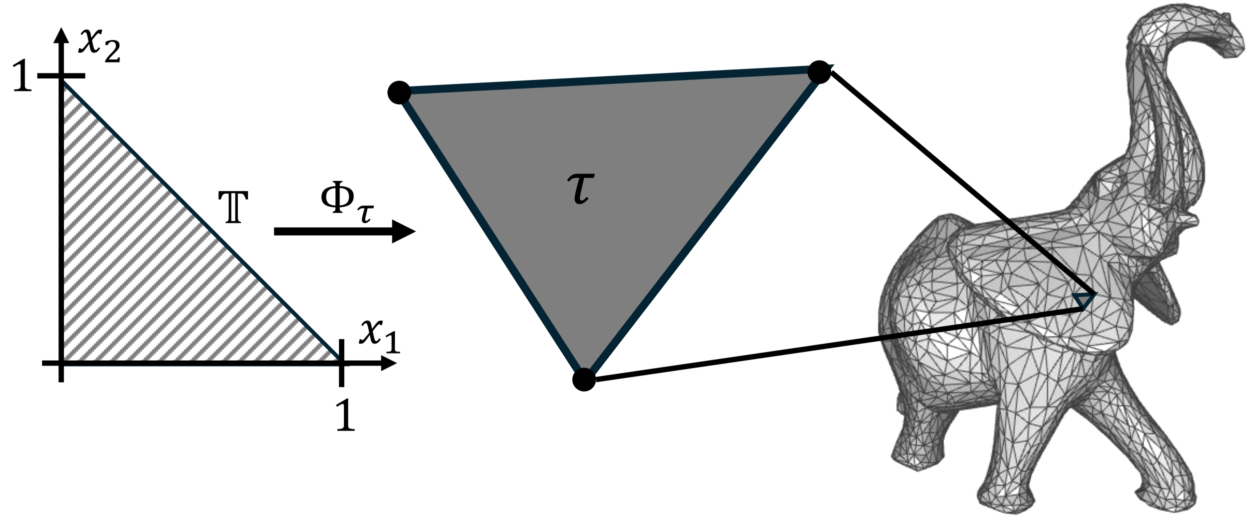
$d\psi_1 = -dx_1 - dx_2$
$d\psi_2 = dx_1$
$d\psi_3 = dx_2$

With respect to the bases $\{\psi_1, \psi_2, \psi_3\}$ and $\{dx_1, dx_2\}$ we have:

- The scalar field mass matrix: $\mathbf{m}^\tau \in \mathbb{R}^{3\times 3}$
- The scalar field stiffness matrix: $\mathbf{s}^\tau \in \mathbb{R}^{3\times 3}$
- The cotangent vector field mass matrix: $\overline{\mathbf{m}}^\tau \in \mathbb{R}^{2\times 2}$
- The differential matrix: $\mathbf{d} \in \mathbb{R}^{2\times 3}$
- The factorization of the scalar field stiffness matrix: $\mathbf{s}^\tau = \mathbf{d}^\top \cdot \overline{\mathbf{m}}^\tau \cdot \mathbf{d}$

Recall



Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we extend $\{\psi_1, \psi_2, \psi_3\}$ and $\{dx_1, dx_2\}$ to functions on the mesh.

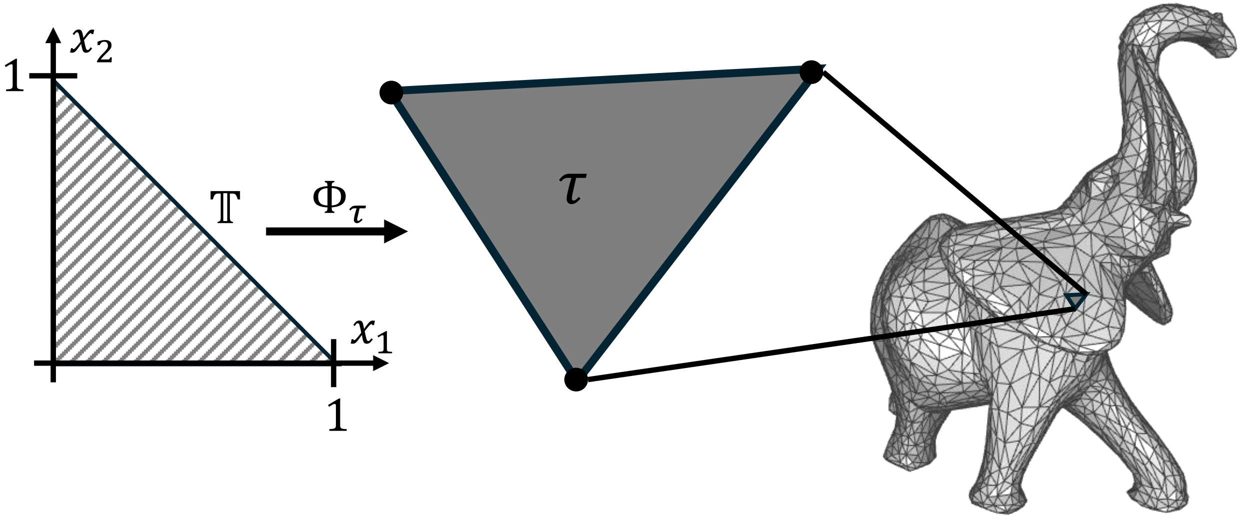
$\{\phi_v\}_{v \in \mathcal{V}}$:

Continuous scalar fields on the mesh , **indexed by vertices**, whose restriction to a triangle $\tau \in \mathcal{T}$ is a linear function (over $\mathbb{T}$) equal to 1 at v and 0 at the other vertices.

$\{\eta_\tau^1, \eta_\tau^2\}_{\tau \in \mathcal{T}}$:

Cotangent vector fields on the mesh, **indexed by triangles**, whose restriction to a triangle $\tau \in \mathcal{T}$ is a constant cotangent vector field (over $\mathbb{T}$)

Recall



These define vector spaces $V = \mathrm{Span}(\{\phi_v\}_{v\in\mathcal{V}})$ and $\overline{V} = \mathrm{Span}(\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}})$.

We turn these into inner-product spaces $\{V, M\colon V \to V^*\}$ and $\{\overline{V}, \overline{M}\colon \overline{V} \to \overline{V}^*\}$

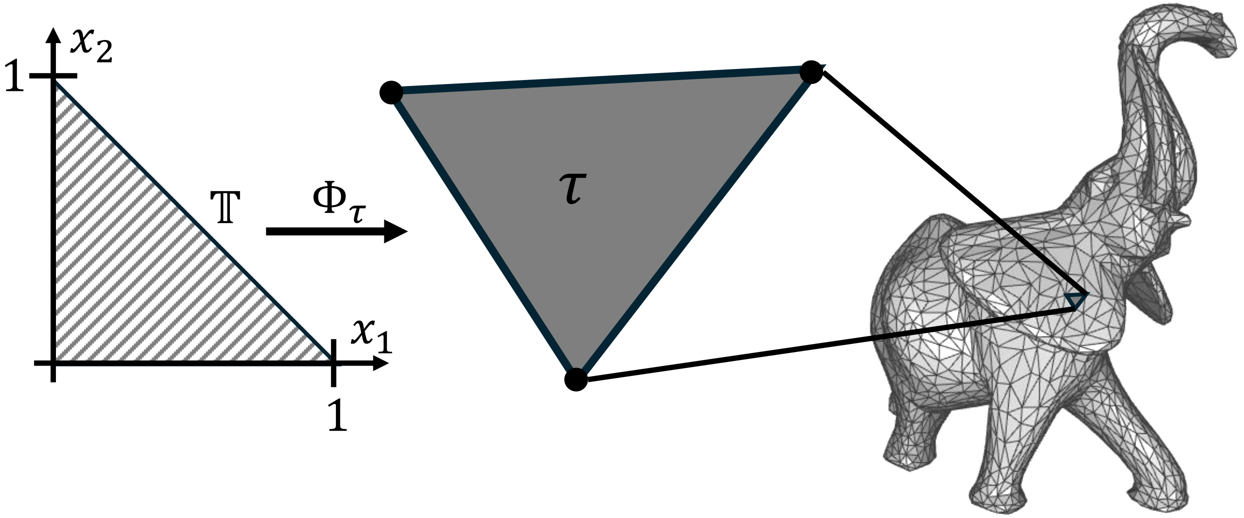
$$M(f,g) = \langle\langle f,g\rangle\rangle_{\mathcal{M}} \qquad \overline{M}(\eta,\nu) = \langle\langle \eta,\nu\rangle\rangle_{\mathcal{M}}$$

We also have a symmetric, positive semi-definite, bilinear stiffness form:

$$S(f,g) = \langle\langle df, dg\rangle\rangle_{\mathcal{M}}$$

And a differential $d \in \mathrm{Hom}(V, \overline{V})$.

Recall

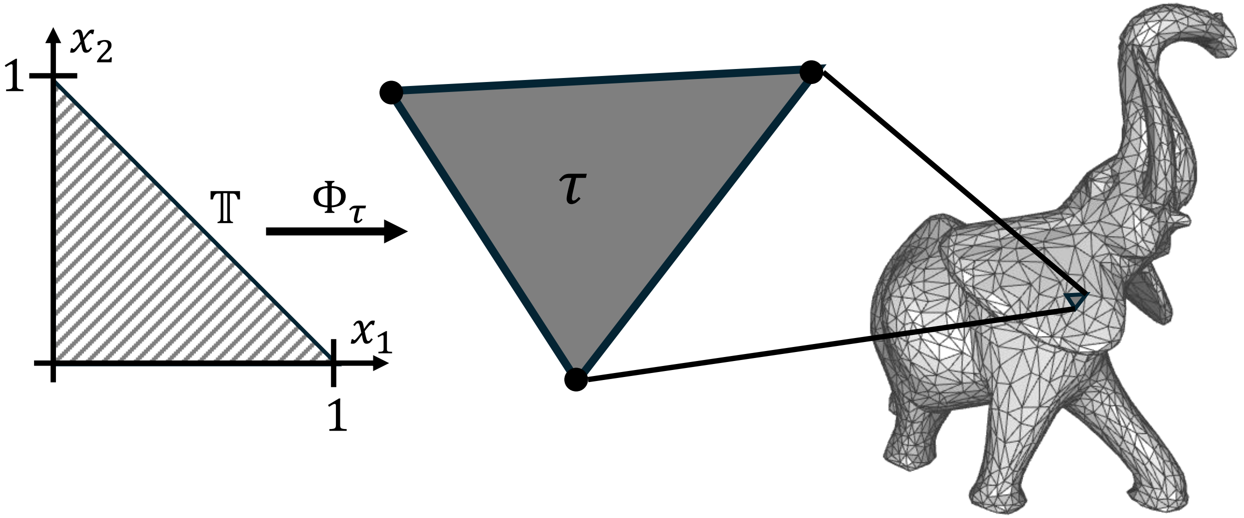


These define vector spaces $V = \text{Span}(\{\phi_v\}_{v\in\mathcal{V}})$ and $\overline{V} = \text{Span}(\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}})$.

The symmetric, positive semi-definite, bilinear stiffness form factors as:

$$S = d^* \circ \overline{M} \circ d$$

Recall



With respect to the bases $\{\phi_v\}_{v\in\mathcal{V}}$ and $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$ we have:

- The scalar field mass matrix: $\mathbf{M} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$
- The scalar field stiffness matrix: $\mathbf{S} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$
- The cotangent vector field mass matrix: $\overline{\mathbf{M}} \in \mathbb{R}^{2|\mathcal{T}|\times 2|\mathcal{T}|}$
- The differential matrix: $\mathbf{D} \in \mathbb{R}^{2|\mathcal{T}|\times|\mathcal{V}|}$
- The factorization of the scalar field stiffness matrix: $\mathbf{S} = \mathbf{D}^\top \cdot \overline{\mathbf{M}} \cdot \mathbf{D}$

Recall: Smoothing

Performing gradient-descent on the Dirichlet energy amounts to solving a PDE for the temporally evolving coefficients $\mathbf{a} \in \mathbb{R}^{|\mathcal{V}|}$:

Explicit:

$$\mathbf{a}^{t+\varepsilon} = \mathbf{M}^{-1} \cdot (\mathbf{M} - \varepsilon \cdot \boldsymbol{S}) \cdot \mathbf{a}^{t}$$

Implicit:

$$\mathbf{a}^{t+\varepsilon} = (\mathbf{M} + \varepsilon \cdot \mathbf{S})^{-1} \cdot \mathbf{M} \cdot \mathbf{a}^{t}$$

with $\mathbf{a}^t$ the coefficients at time t.

Recall: Multivariable Calculus (Euclidean)

Given a **Euclidean** domain $\Omega \subset \mathbb{R}^d$:

For a scalar field $f\colon \Omega \to \mathbb{R}$, the *gradient*, $\nabla f\colon \Omega \to \mathbb{R}^d$, gives the direction of steepest change of f:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_d}\right)^{\top}$$

For a vector field $\vec{v}\colon \Omega \to \mathbb{R}^d$, the divergence, $\mathrm{div}(\vec{v})\colon \Omega \to \mathbb{R}$, describes the extent to which the vector field converges/diverges at a point:

$$\mathrm{div}(\vec{v}) = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_d}{\partial x_d}$$

where $\vec{v} = (v_1, \cdots, v_d)^{\top}$.

For scalar fields $f, g\colon \Omega \to \mathbb{R}$ and a vector field $\vec{v}\colon \Omega \to \mathbb{R}^d$:

$$\mathrm{div}(f \cdot \vec{v}) = \langle \nabla f, \vec{v} \rangle + f \cdot \mathrm{div}(\vec{v})$$

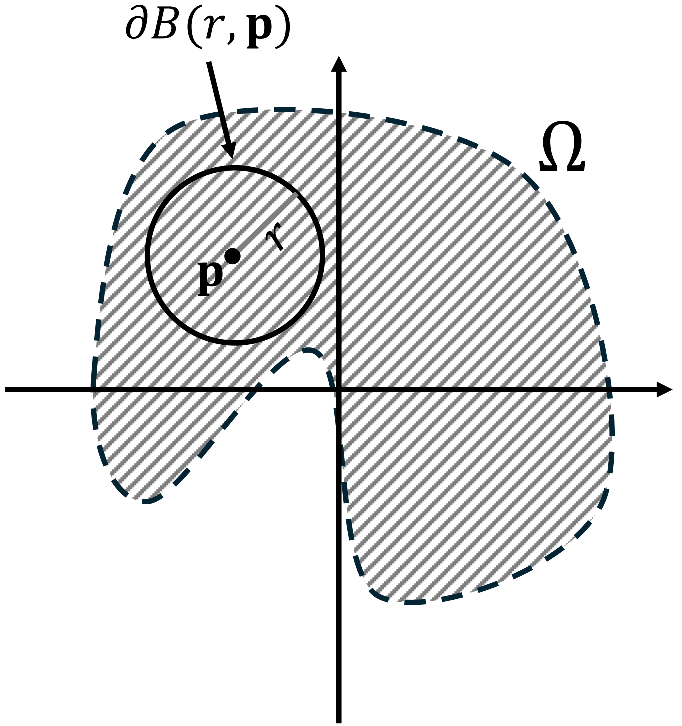
Recall: Multivariable Calculus (Euclidean)

Given a Euclidean domain $\Omega \subset \mathbb{R}^d$:

The *Laplacian* of a scalar function $f : \Omega \to \mathbb{R}$ is the divergence of its gradient:

$$\begin{aligned}\Delta f &= \mathrm{div}(\nabla f) \\ &= \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_d^2}\end{aligned}$$

The Laplacian measures the difference between the value of the function at a point and the average value nearby.



Formally:

$$\Delta f\Big|_{\mathbf{p}} = \lim_{r\to 0}\frac{2d}{r^2}\left(f(\mathbf{p}) - \frac{\int_{\partial B(r,\mathbf{p})} f(\mathbf{q})}{\int_{\partial B(r,\mathbf{p})} 1}\right)$$

with $\partial B(r, \mathbf{p})$ is the set of points at a distance of r from $\mathbf{p}$.

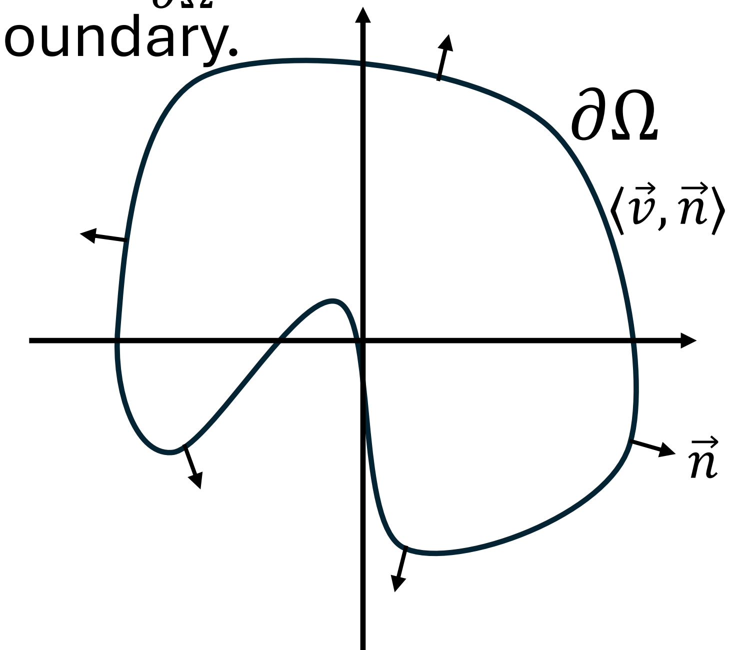
Recall: Multivariable Calculus (Euclidean)

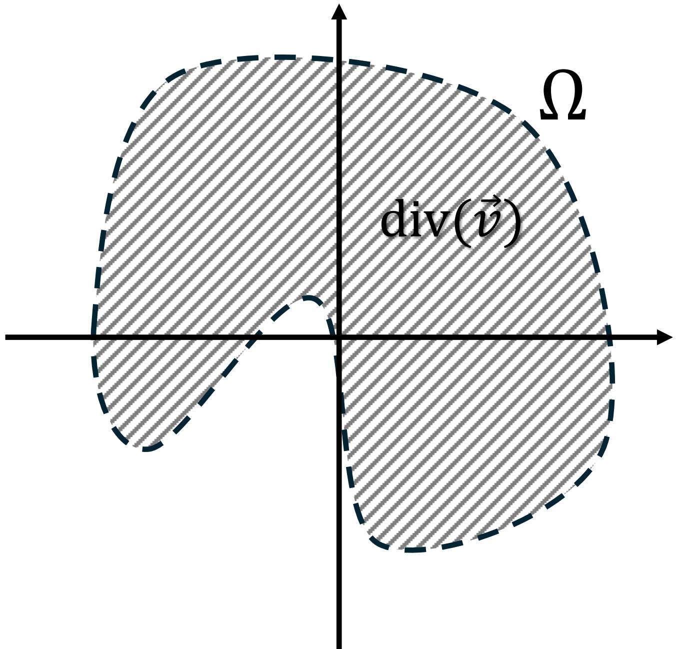
Given a Euclidean domain $\Omega \subset \mathbb{R}^d$:

For a vector field $\vec{v}: \Omega \to \mathbb{R}^d$, the integral of the divergence of $\vec{v}$ over the interior of the domain is the surface integral over the boundary:

$$\int_{\Omega} \nabla \cdot \vec{v} = \int_{\partial\Omega} \langle \vec{v}, \vec{n} \rangle$$

with $\vec{n}$ the unit normal on the boundary.





$\mathrm{div}(f \cdot \vec{v}) = \langle \nabla f, \vec{v} \rangle + f \cdot \mathrm{div}(\vec{v})$

Recall: Multivariable Calculus (Euclidean)

Given a Euclidean domain $\Omega \subset \mathbb{R}^d$:

For scalar fields $f, g: \Omega \to \mathbb{R}$, we have:

$$\begin{aligned}\int_\Omega \mathrm{div}(\nabla f) \cdot g &= \int_\Omega \mathrm{div}(\nabla f \cdot g) - \int_\Omega \langle \nabla f, \nabla g \rangle \\ &= \int_{\partial\Omega} \langle \nabla f \cdot g, \vec{n} \rangle - \int_\Omega \langle \nabla f, \nabla g \rangle\end{aligned}$$

In particular, if the boundary integral vanishes, we have:

$$\int_\Omega \mathrm{div}(\nabla f) \cdot g = -\int_\Omega \langle \nabla f, \nabla g \rangle$$

This occurs when:

- Ω does not have a boundary $\partial\Omega = \emptyset$, or
- g vanishes on the boundary (Dirichlet), or
- f is constant across the boundary (Neumann)

$\mathrm{div}(f \cdot \vec{v}) = \langle \nabla f, \vec{v} \rangle + f \cdot \mathrm{div}(\vec{v})$

Recall: Multivariable Calculus (Euclidean)

Given a Euclidean domain $\Omega \subset \mathbb{R}^d$:

For scalar fields $f, g: \Omega \to \mathbb{R}$, we have:

$$\int_\Omega \mathrm{div}(\nabla f) \cdot g = \int_\Omega \mathrm{div}(\nabla f \cdot g) - \int_\Omega \langle \nabla f, \nabla g \rangle$$
$$= \int_{\partial\Omega} \langle \nabla f \cdot g, \vec{n} \rangle - \int_\Omega \langle \nabla f, \nabla g \rangle$$

In partic

That is, under appropriate boundary conditions,
the divergence is the negative adjoint of the gradient.

This occurs when:

- Ω does not have a boundary $\partial\Omega = \emptyset$, or
- g vanishes on the boundary (Dirichlet), or
- f is constant across the boundary (Neumann)

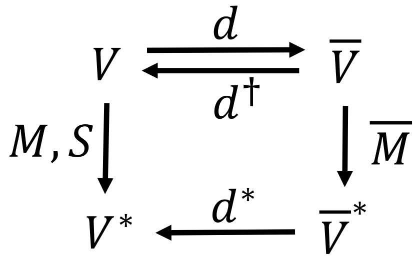
Outline

Recall

The Laplacian

Heat Diffusion

Adjoint of the Differential



Definition:

Given the inner-product spaces $\{V, M\colon V \to V^*\}$ and $\left\{\overline{V}, \overline{M}\colon \overline{V} \to \overline{V}^*\right\}$, and the differential map $d \in \mathrm{Hom}\left(V, \overline{V}\right)$ we can define its adjoint:

$$d^\dagger = M^{-1} \circ d^* \circ \overline{M}$$

Property:

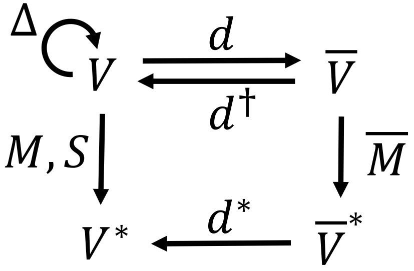
If $f \in V$ is a scalar field on $\mathcal{M}$ and $v \in \overline{V}$ is a cotangent vector field:

$$\langle\langle v, df \rangle\rangle_{\mathcal{M}} = \langle\langle d^\dagger v, f \rangle\rangle_{\mathcal{M}}$$

On the left we are using the inner-product on cotangent vector fields.

On the right we are using the inner-product on scalar fields.

The Laplacian



Definition:

Given the inner-product spaces $\{V, M\colon V \to V^*\}$ and $\left\{\overline{V}, \overline{M}\colon \overline{V} \to \overline{V}^*\right\}$, we define the *Laplacian*, $\Delta \in \mathrm{Hom}(V, V)$, to be the linear operator defined by:

$$\begin{aligned}\Delta &= -d^\dagger \circ d \\ &= -M^{-1} \circ d^* \circ \overline{M} \circ d \\ &= -M^{-1} \circ S\end{aligned}$$

Given the bases $\{\phi_v\}_{v\in\mathcal{V}}$ and $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$, the matrix expression for the Laplacian becomes:

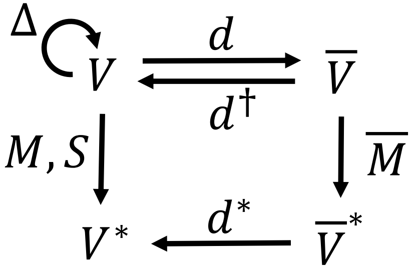
$$\boldsymbol{\Delta} = -\mathbf{M}^{-1} \cdot \mathbf{S}$$

Property:

Since the Laplacian is the (negative) of the composition of a linear map with its adjoint, it is self-adjoint:

$$\Delta^\dagger = \Delta$$

Adjoint of the Differential



Aside:

The adjoint of the differential is **almost** the negative divergence operator.

Formally:

The divergence operator is the adjoint of the gradient.

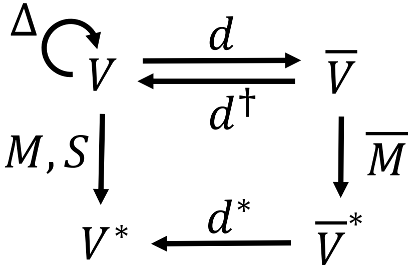
Instead of defining $\overline{V}$ to be a discretization of cotangent vector fields, we could have used the metric to define a discretization of tangent vector fields, $\{\xi_\tau^1, \xi_\tau^2\}_{\tau\in\mathcal{T}}$, with:

$$\xi_\tau^k \equiv g_\tau^{-1}\left(\eta_\tau^k\right)$$

✓ We could define the discrete gradient, whose adjoint is the divergence.

✗ We would not have a metric-dependent/independent factorization.

Adjoint of the Differential



Aside:

The adjoint of the differential is **almost** the negative divergence operator.

Formally:

The adjoint of the differential is the *co-differential* and is denoted δ:

$$\delta = d^\dagger \in \mathrm{Hom}(\overline{V}, V)$$

Outline

Recall

The Laplacian

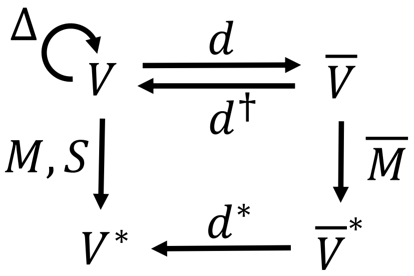
Heat Diffusion

Heat Diffusion

Newton's law of cooling:

> "The rate of change in the temperature of a body is directly proportional to the temperature difference between the body and the surroundings"

Heat Diffusion



Newton's law of cooling:

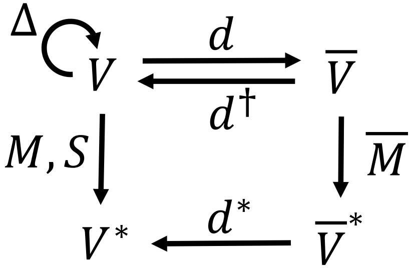
"The rate of change in the temperature of a body is directly proportional to the temperature difference between the body and the surroundings"

$$\Downarrow$$

The PDE describing the temporal evolution of a signal $f^t: \mathcal{M} \to \mathbb{R}$ (for $t \geq 0$) undergoing heat diffusion:

$$\frac{\partial f^t}{\partial t} = \Delta f^t$$

Heat Diffusion



$$\frac{\partial f^t}{\partial t} = \Delta f^t$$

Express f^t in terms of the basis $\{\phi_v\}_{v\in\mathcal{V}}$:

$$f^t(p) = \sum_{v\in\mathcal{V}} \mathbf{a}_v^t \cdot \phi_v(p)$$

with $\mathbf{a}^t \in \mathbb{R}^{|\mathcal{V}|}$ the temporally varying coefficients.

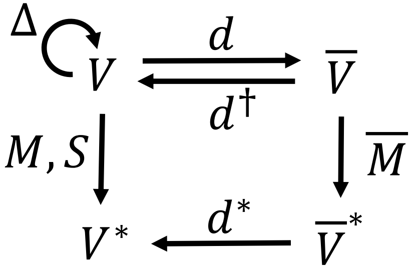
$$\Downarrow$$

$$\frac{\partial}{\partial t}\left(\sum_{v\in\mathcal{V}} \mathbf{a}_v^t \cdot \phi_v\right) = \Delta\left(\sum_{v\in\mathcal{V}} \mathbf{a}_v^t \cdot \phi_v\right)$$

$$\Downarrow$$

$$\sum_{v\in\mathcal{V}} \frac{\partial \mathbf{a}_v^t}{\partial t} \cdot \phi_v = \sum_{v\in\mathcal{V}} \mathbf{a}_v^t \cdot \Delta\phi_v$$

Heat Diffusion



$$\sum_{v\in\mathcal{V}} \frac{\partial \mathbf{a}_v^t}{\partial t} \cdot \phi_v = \sum_{v\in\mathcal{V}} \mathbf{a}_v^t \cdot \Delta\phi_v$$

For every vertex $w \in \mathcal{V}$, we would like to relate the w-th coefficient on the left with to the w-th coefficient on the right.

We can do this by evaluating with the dual basis vector ϕ_w^*, where $\{\phi_v^*\}_{v\in\mathcal{V}}$ is the canonical dual basis to $\{\phi_v\}_{v\in\mathcal{V}}$.

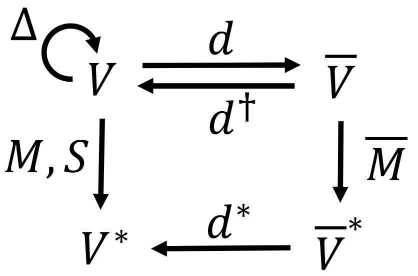
Recall:

$\phi_w^*(\phi_v)$ is one if $v = w$ and zero otherwise

For endomorphism $L \in \mathrm{Hom}(V, V)$ the (v, w)-th coefficient in its matrix expression is:

$$\mathbf{L}_{vw} = \phi_v^*\big(L(\phi_w)\big)$$

Heat Diffusion



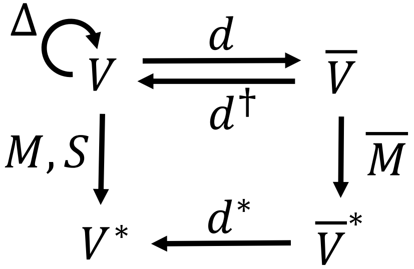
Computing the w-th coefficient:

$$\phi_w^*\left(\sum_{v\in\mathcal{V}} \frac{\partial \mathbf{a}_v^t}{\partial t} \cdot \phi_v\right) = \phi_w^*\left(\sum_{v\in\mathcal{V}} \mathbf{a}_v^t \cdot \Delta\phi_v\right)$$

$$\Updownarrow$$

$$\begin{aligned}
\frac{\partial \mathbf{a}_w^t}{\partial t} &= \sum_{v\in\mathcal{V}} \mathbf{a}_v^t \cdot \phi_w^*(\Delta\phi_v) \\
&= \sum_{v\in\mathcal{V}} \mathbf{a}_v^t \cdot \mathbf{\Delta}_{wv} \\
&= \sum_{v\in\mathcal{V}} \mathbf{a}_v^t \cdot (-\mathbf{M}^{-1} \cdot \mathbf{S})_{wv} \\
&= -\sum_{v\in\mathcal{V}} (\mathbf{M}^{-1} \cdot \mathbf{S})_{wv} \cdot \mathbf{a}_v^t \\
&= -(\mathbf{M}^{-1} \cdot \mathbf{S} \cdot \mathbf{a}^t)_w
\end{aligned}$$

Heat Diffusion



$$\frac{\partial \mathbf{a}_w^t}{\partial t} = -(\mathbf{M}^{-1} \cdot \mathbf{S} \cdot \mathbf{a}^t)_w$$

Since this is true for all vertices $w \in \mathcal{V}$, the PDE for the coefficients is:

$$\frac{\partial \mathbf{a}^t}{\partial t} = -\mathbf{M}^{-1} \cdot \mathbf{S} \cdot \mathbf{a}^t$$

Discretizing using (e.g.) implicit time-stepping:

$$\frac{\mathbf{a}^{t+\varepsilon} - \mathbf{a}^t}{\varepsilon} = -\mathbf{M}^{-1} \cdot \mathbf{S} \cdot \mathbf{a}^{t+\varepsilon}$$

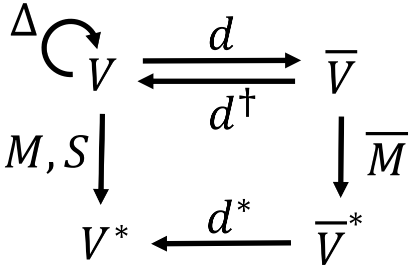
$$\Rightarrow \quad \mathbf{a}^{t+\varepsilon} - \mathbf{a}^t = -\varepsilon \cdot \mathbf{M}^{-1} \cdot \mathbf{S} \cdot \mathbf{a}^{t+\varepsilon}$$

$$\Rightarrow \quad \mathbf{a}^{t+\varepsilon} + \varepsilon \cdot \mathbf{M}^{-1} \cdot \mathbf{S} \cdot \mathbf{a}^{t+\varepsilon} = \mathbf{a}^t$$

$$\Rightarrow \quad \mathbf{M} \cdot \mathbf{a}^{t+\varepsilon} + \varepsilon \cdot \mathbf{S} \cdot \mathbf{a}^{t+\varepsilon} = \mathbf{M} \cdot \mathbf{a}^t$$

$$\Rightarrow \quad (\mathbf{M} + \varepsilon \cdot \mathbf{S}) \cdot \mathbf{a}^{t+\varepsilon} = \mathbf{M} \cdot \mathbf{a}^t$$

Heat Diffusion



$$\frac{\partial \mathbf{a}_w^t}{\partial t} = -(\mathbf{M}^{-1} \cdot \mathbf{S} \cdot \mathbf{a}^t)_w$$

Since this is true for all vertices $w \in \mathcal{V}$, the PDE for the coefficients is:

$\partial \mathbf{a}^t$

Performing gradient descent to reduce the Dirichlet energy

$\Updownarrow$

Solving the heat diffusion PDE

Discre

$$\frac{\mathbf{a}^{t+\varepsilon} - \mathbf{a}^t}{\varepsilon} = -\mathbf{M}^{-1} \cdot \mathbf{S} \cdot \mathbf{a}^{t+\varepsilon}$$

$$\Rightarrow \quad \mathbf{a}^{t+\varepsilon} - \mathbf{a}^t = -\varepsilon \cdot \mathbf{M}^{-1} \cdot \mathbf{S} \cdot \mathbf{a}^{t+\varepsilon}$$

$$\Rightarrow \quad \mathbf{a}^{t+\varepsilon} + \varepsilon \cdot \mathbf{M}^{-1} \cdot \mathbf{S} \cdot \mathbf{a}^{t+\varepsilon} = \mathbf{a}^t$$

$$\Rightarrow \quad \mathbf{M} \cdot \mathbf{a}^{t+\varepsilon} + \varepsilon \cdot \mathbf{S} \cdot \mathbf{a}^{t+\varepsilon} = \mathbf{M} \cdot \mathbf{a}^t$$

$$\Rightarrow \quad (\mathbf{M} + \varepsilon \cdot \mathbf{S}) \cdot \mathbf{a}^{t+\varepsilon} = \mathbf{M} \cdot \mathbf{a}^t$$