Geometry Processing (601.458/658)

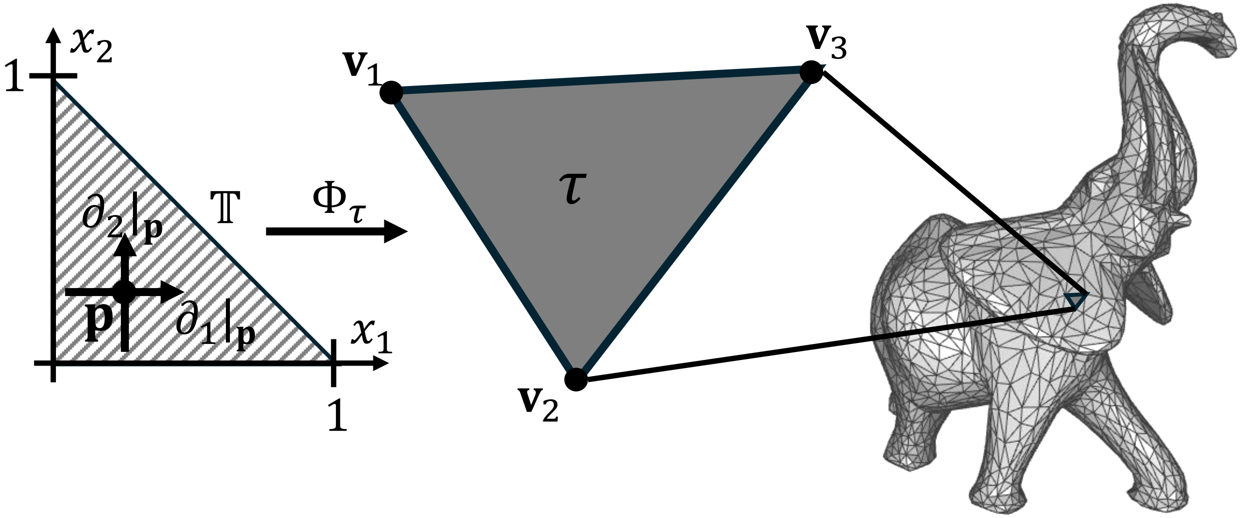
Misha Kazhdan

Outline

Recall

Constant Vector Fields

Recall



Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we perform integration/differentiation over the unit right triangle, $\mathbb{T}$.

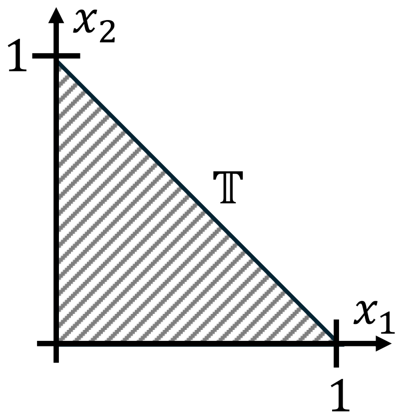
At each point $\mathbf{p} \in \mathbb{T}$ we have a tangent space $T_{\mathbf{p}}\mathbb{T}$ spanned by $\{\partial_1|_{\mathbf{p}}, \partial_2|_{\mathbf{p}}\}$.

The dual space $T_{\mathbf{p}}^*\mathbb{T}$ is spanned by the canonical dual basis $\{dx_1|_{\mathbf{p}}, dx_2|_{\mathbf{p}}\}$.

We denote the pulled back inner-product as $g_\tau|_{\mathbf{p}}: T_{\mathbf{p}}\mathbb{T} \to T_{\mathbf{p}}^*\mathbb{T}$.

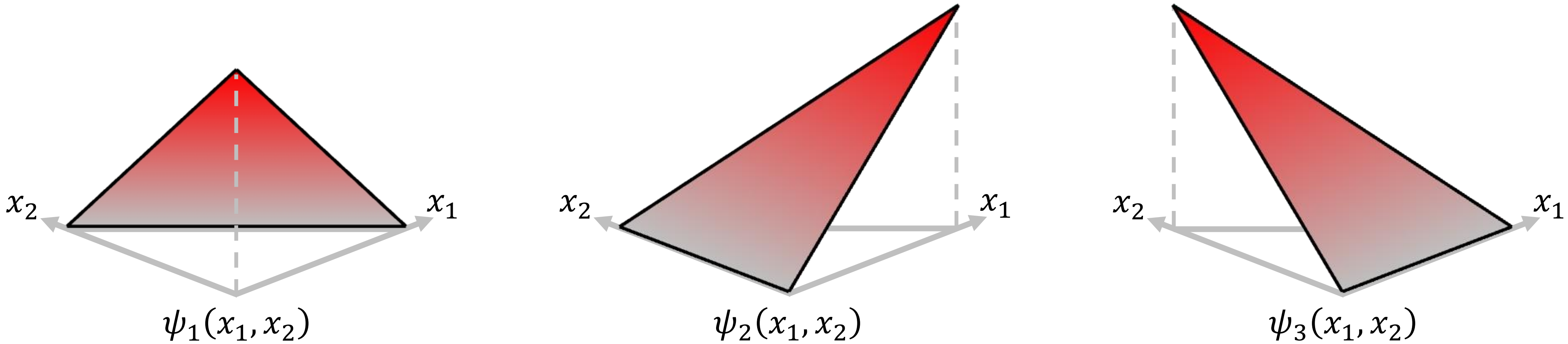
Since the inner-product is constant, we will often drop the subscript $\mathbf{p}$.

Recall

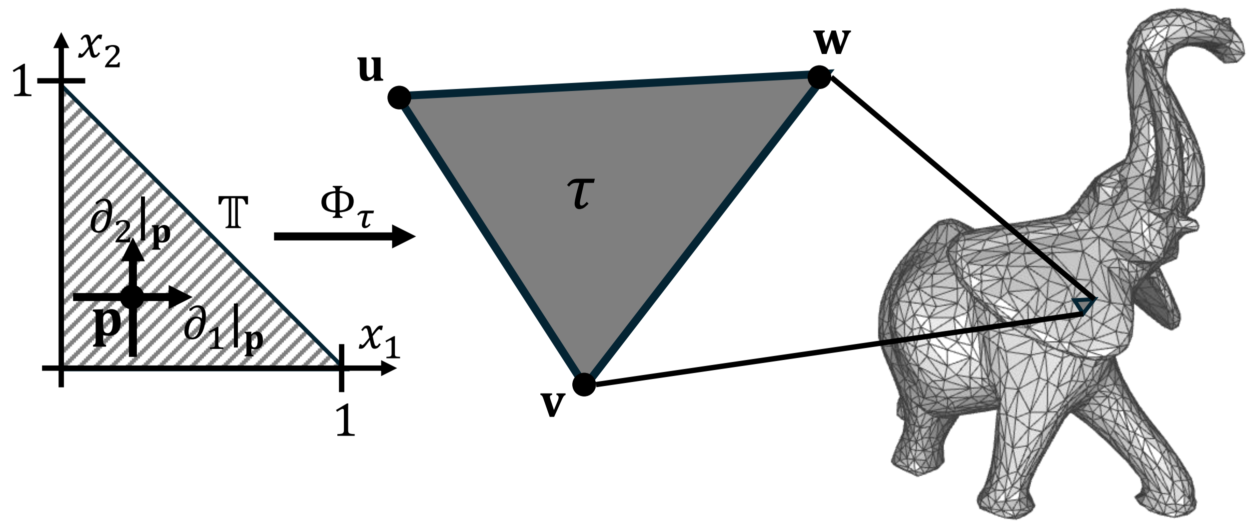


For the unit right triangle $\mathbb{T}$, we have the basis for linear functions:

$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1, x_2) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

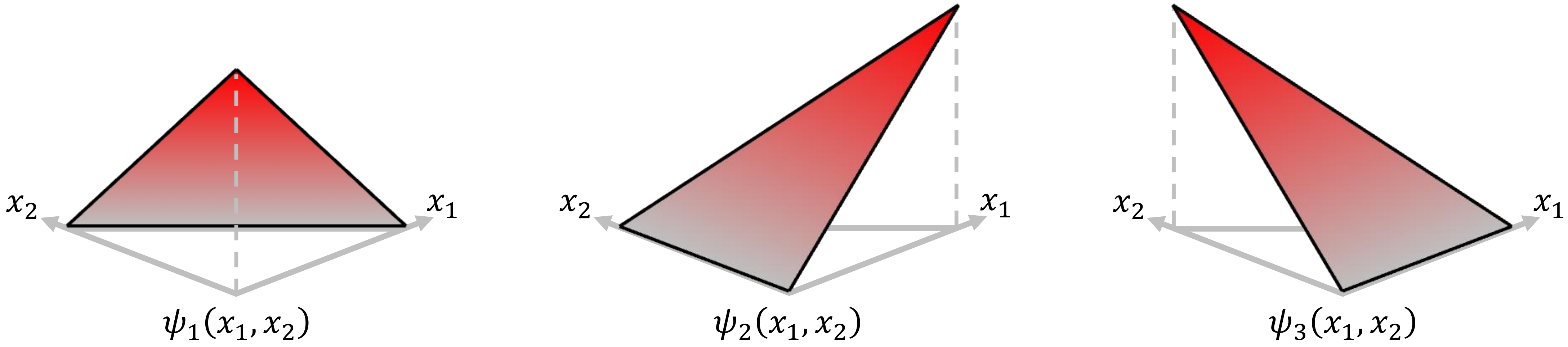


Recall



Given a triangle mesh $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we define a vector space V of piecewise-linear functions, spanned by the “hat” functions $\{\phi_v\}_{v\in\mathcal{V}}$.

Given a function $f \in V$, we denote by f_τ the restriction of the function to triangle $\tau \subset \mathcal{T}$, realized as a function on $\mathbb{T}$.

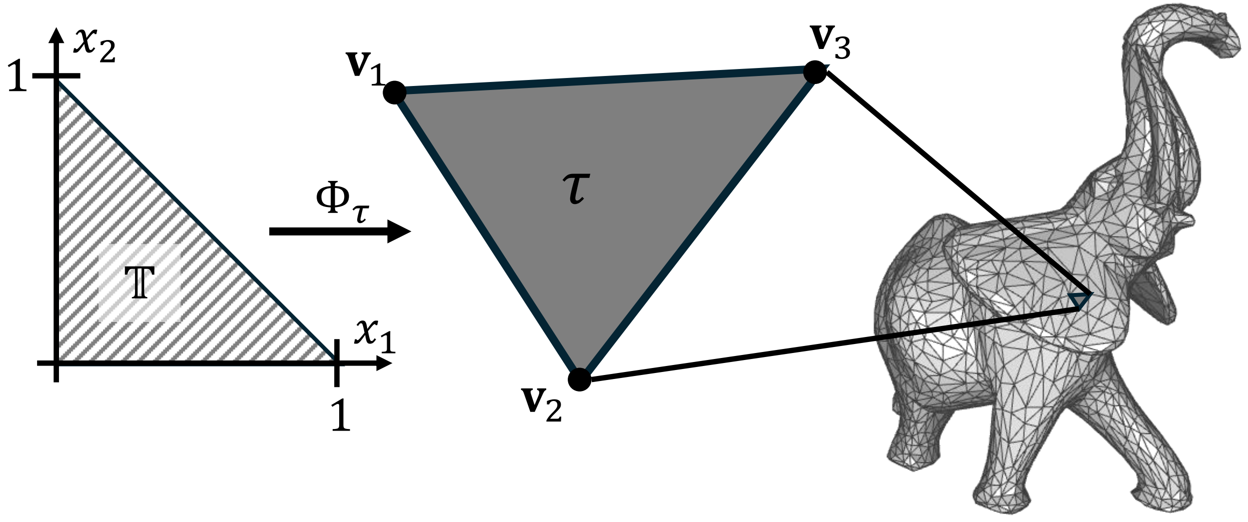


Outline

Recall

Constant Vector Fields

Vector Fields



Definition:

We say that a function ν on $\mathbb{T}$ is a *cotangent vector field* if for all $\mathbf{p} \in \mathbb{T}$:

$$\nu(\mathbf{p}) \in T^*_{\mathbf{p}}\mathbb{T}$$

We say that a function v on $\mathbb{T}$ is a *tangent vector field* if for all $\mathbf{p} \in \mathbb{T}$:

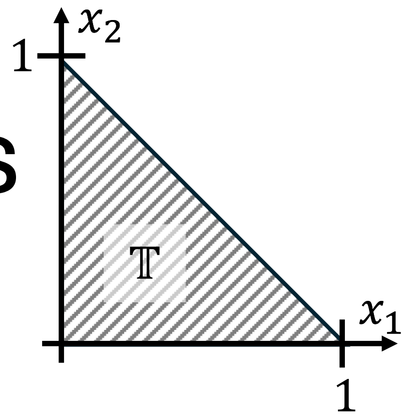
$$v(\mathbf{p}) \in T_{\mathbf{p}}\mathbb{T}$$

If we have a metric g_τ, we can turn cotangent vector fields into tangent vector fields:

$$v \equiv g_\tau(\nu)$$

and vice versa.

Constant Vector Fields



Scalar Functions:

Real-valued linear functions on a triangle are spanned by the "hat" basis:

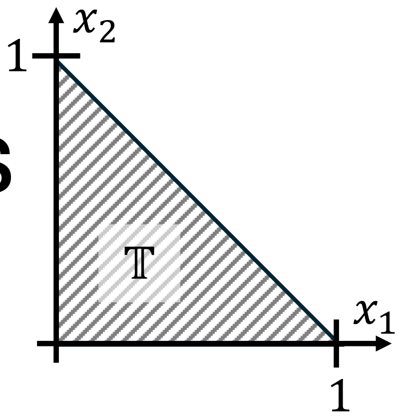
$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1, x_2) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

These are defined independent of a metric.

Using these, we define the mass and stiffness matrices, $\mathbf{m}^\tau, \mathbf{s}^\tau \in \mathbb{R}^{3\times 3}$.

These incorporate the inner-product, g_τ, pulled back from the embedded triangle $\tau \subset \mathbb{R}^3$.

Constant Vector Fields



$\psi_1(x_1, x_2) = 1 - x_1 - x_2$
$\psi_2(x_1, x_2) = x_1$
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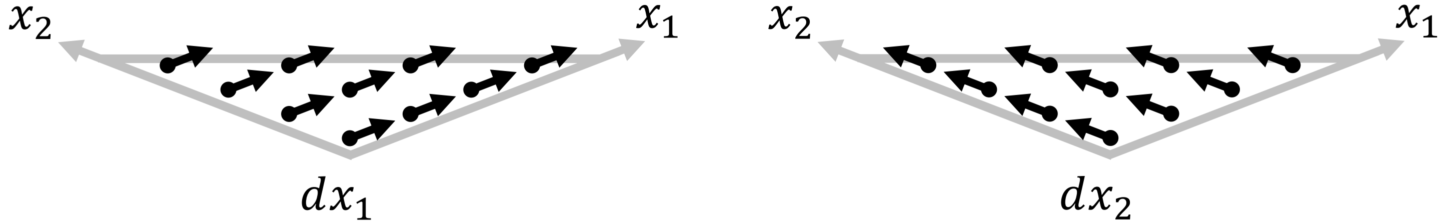
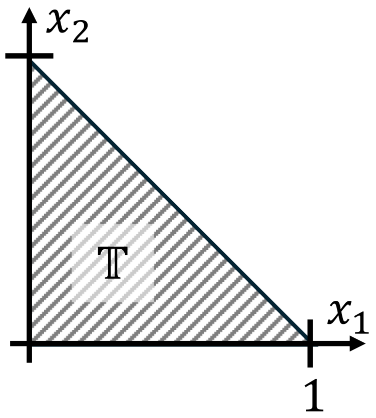
Cotangent Vector Fields:

We can also consider the space of **cotangent** vector fields that are constant w.r.t. the basis $\{dx_1, dx_2\}$.



Constant Vector Fields



$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1, x_2) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

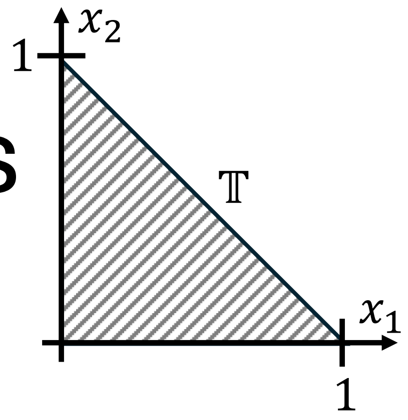
Properties:

Because the hat basis is linear within a triangle, its differentials are constant w.r.t. $\{dx_1, dx_2\}$.

$\Rightarrow$ The differentials can be expressed in the basis $\{dx_1, dx_2\}$:

$$d\psi_1 = \frac{\partial \psi_1}{\partial x_1} \cdot dx_1 + \frac{\partial \psi_1}{\partial x_2} \cdot dx_2 = -dx_1 - dx_2$$
$$d\psi_2 = \frac{\partial \psi_2}{\partial x_1} \cdot dx_1 + \frac{\partial \psi_2}{\partial x_2} \cdot dx_2 = dx_1$$
$$d\psi_3 = \frac{\partial \psi_3}{\partial x_1} \cdot dx_1 + \frac{\partial \psi_3}{\partial x_2} \cdot dx_2 = dx_2$$

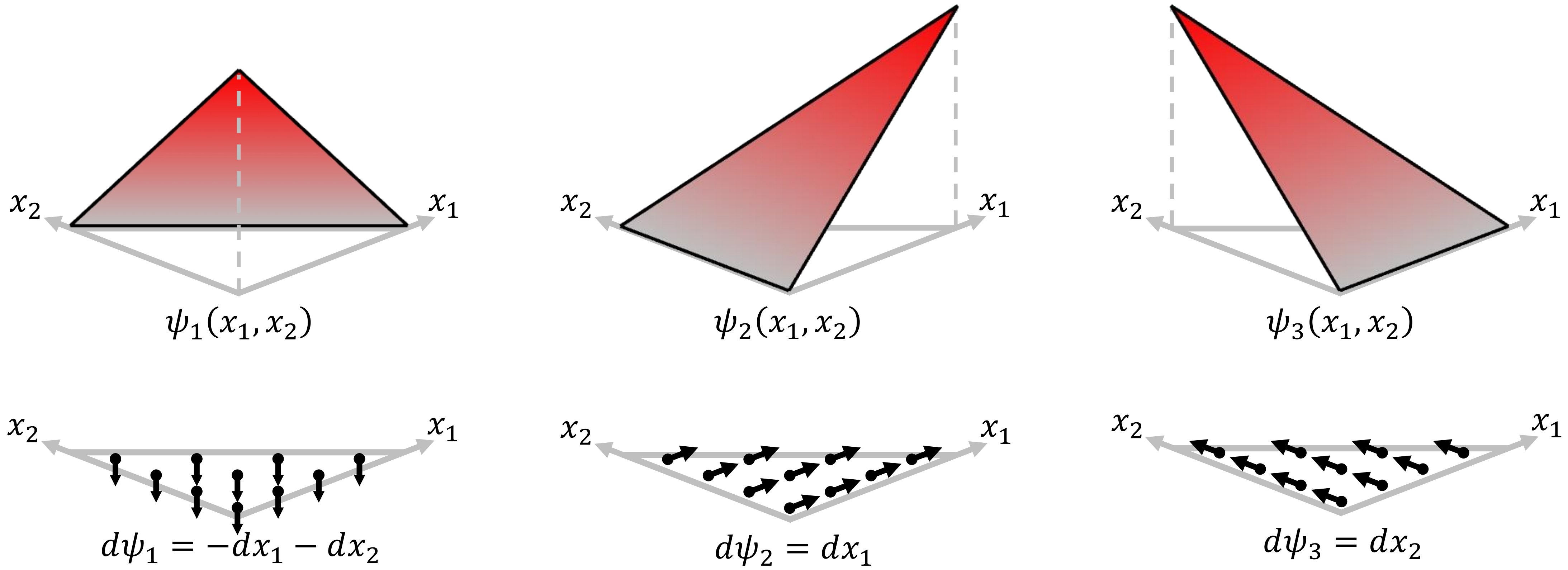
Constant Vector Fields



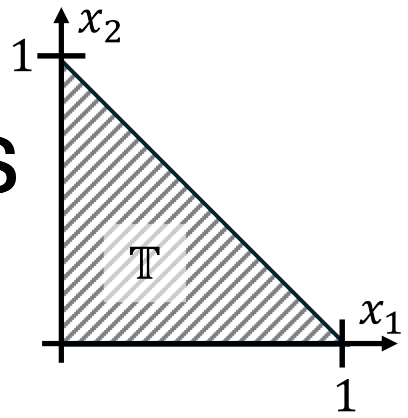
$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1, x_2) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

$$d\psi_1 = -dx_1 - dx_2$$
$$d\psi_2 = dx_1$$
$$d\psi_3 = dx_2$$

We can express the differentials of $\{\psi_1, \psi_2, \psi_3\}$ in terms of $\{dx_1, dx_2\}$.



Constant Vector Fields



$$\psi_1(x_1,x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1,x_2) = x_1$$
$$\psi_3(x_1,x_2) = x_2$$

$$d\psi_1 = -dx_1 - dx_2$$
$$d\psi_2 = dx_1$$
$$d\psi_3 = dx_2$$

Matrix Representation:

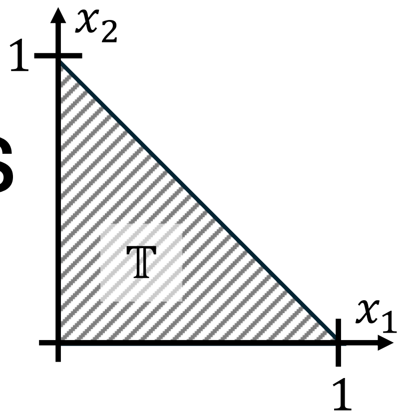
Given a scalar function $\psi = \mathbf{a}_1 \cdot \psi_1 + \mathbf{a}_2 \cdot \psi_2 + \mathbf{a}_3 \cdot \psi_3$, its differential is:

$$\begin{aligned} d\psi &= d(\mathbf{a}_1 \cdot \psi_1 + \mathbf{a}_2 \cdot \psi_2 + \mathbf{a}_3 \cdot \psi_3) \\ &= \mathbf{a}_1 \cdot d\psi_1 + \mathbf{a}_2 \cdot d\psi_2 + \mathbf{a}_3 \cdot d\psi_3 \\ &= \mathbf{a}_1 \cdot (-dx_1 - dx_2) + \mathbf{a}_2 \cdot dx_1 + \mathbf{a}_3 \cdot dx_2 \\ &= (-\mathbf{a}_1 + \mathbf{a}_2) \cdot dx_1 + (-\mathbf{a}_1 + \mathbf{a}_3) \cdot dx_2 \end{aligned}$$

$\Rightarrow$ With respect to the bases $\{\psi_1, \psi_2, \psi_3\}$ and $\{dx_1, dx_2\}$, the differential operator is expressed by the matrix $\mathbf{d} \in \mathbb{R}^{2\times 3}$:

$$\mathbf{d} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Constant Vector Fields



$\psi_1(x_1, x_2) = 1 - x_1 - x_2$
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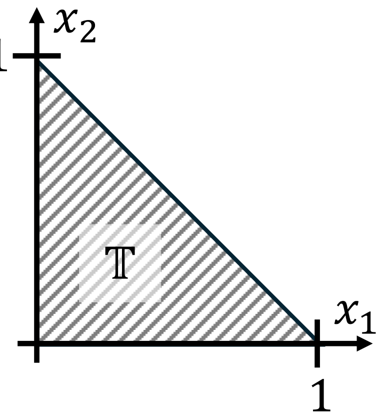
Note:
The matrix representation of the differential is metric-independent.

The coefficients of the differential are the differences of the function’s coefficients/values along the edges.

⇒ With resp… …ntial operator is expressed by the matrix $\mathbf{d} \in \mathbb{R}$…:

$$\mathbf{d} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Constant Vector Fields



$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
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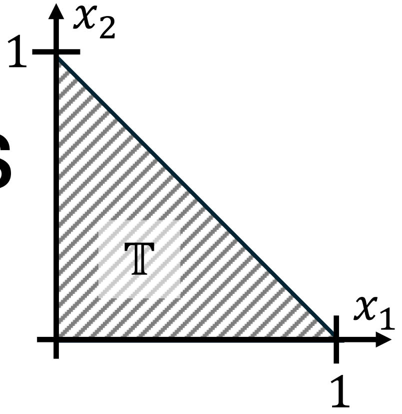
$$d\psi_1 = -dx_1 - dx_2$$
$$d\psi_2 = dx_1$$
$$d\psi_3 = dx_2$$

Matrix Representation:

Given two cotangent vector fields, $\nu = \mathbf{a}_1 \cdot dx_1 + \mathbf{a}_2 \cdot dx_2$ and $\eta = \mathbf{b}_1 \cdot dx_1 + \mathbf{b}_2 \cdot dx_2$, the inner product of the vector fields is:

$$\begin{aligned}
\langle\langle \nu, \eta \rangle\rangle_\tau &= \int_{\mathbb{T}} \langle \nu, \eta \rangle_{g_\tau^{-1}} \cdot \sqrt{\det(E^{-1} \circ g_\tau)} \cdot \omega_E \\
&= \sqrt{\det(\mathbf{g}_\tau)} \cdot \int_{\mathbb{T}} \left\langle \sum_{i=1}^{2} \mathbf{a}_i \cdot dx_i \, , \sum_{j=1}^{2} \mathbf{b}_j \cdot dx_j \right\rangle_{g_\tau^{-1}} \cdot \omega_E \\
&= \sqrt{\det(\mathbf{g}_\tau)} \cdot \sum_{i,j=1}^{2} \mathbf{a}_i \cdot \mathbf{b}_j \int_{\mathbb{T}} \langle dx_i, dx_j \rangle_{g_\tau^{-1}} \cdot \omega_E \\
&= \frac{\sqrt{\det(\mathbf{g}_\tau)}}{2} \cdot \sum_{i,j=1}^{2} \mathbf{a}_i \cdot \mathbf{b}_j \cdot (\mathbf{g}_\tau^{-1})_{ij}
\end{aligned}$$

Constant Vector Fields



$$\psi_1(x_1,x_2) = 1 - x_1 - x_2$$
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Given two cotangent vector fields, $\nu = \mathbf{a}_1 \cdot dx_1 + \mathbf{a}_2 \cdot dx_2$ and $\eta = \mathbf{b}_1 \cdot dx_1 + \mathbf{b}_2 \cdot dx_2$, the inner product of the vector fields is:

$$\langle\langle \nu, \eta \rangle\rangle_\tau = \frac{\sqrt{\det(\mathbf{g}_\tau)}}{2} \cdot \sum_{i,j=1}^{2} \mathbf{a}_i \cdot \mathbf{b}_j \cdot (\mathbf{g}_\tau^{-1})_{ij}$$

Setting $\overline{\mathbf{m}}^\tau \in \mathbb{R}^{2\times 2}$ to be the differential mass matrix:

$$\overline{\mathbf{m}}^\tau = \frac{\sqrt{\det(\mathbf{g}_\tau)}}{2} \cdot \mathbf{g}_\tau^{-1}$$

$$\Downarrow$$

$$\langle\langle \nu, \eta \rangle\rangle_\tau = \mathbf{a}^\top \cdot \overline{\mathbf{m}}^\tau \cdot \mathbf{b}$$

Algebraic Interpretation

The constant cotangent vector field basis defines a 2-dimensional space:

$$\overline{V} = \text{Span}(dx_1, dx_2)$$

Mass matrix:

Letting $\overline{m}^\tau$ be the symmetric, positive-definite bilinear map:

$$\overline{m}^\tau : \overline{V} \times \overline{V} \to \mathbb{R}$$
$$(\nu, \eta) \mapsto \langle\langle \nu, \eta \rangle\rangle_\tau$$

this makes $\overline{V}$ into an inner-product space $\left\{\overline{V}, \overline{m}^\tau : \overline{V} \to \overline{V}^*\right\}$.

$\Rightarrow$ The mass matrix $\overline{\mathbf{m}}^\tau \in \mathbb{R}^{2\times 2}$ gives us an expression for the map $\overline{m}^\tau : \overline{V} \to \overline{V}^*$ w.r.t. the bases $\{dx_1, dx_2\}$.

Algebraic Interpretation

The constant cotangent vector field basis defines a 2-dimensional space:

$$\overline{V} = \mathrm{Span}(dx_1, dx_2)$$

The linear scalar field basis defines a 3-dimensional space:

$$V = \mathrm{Span}(\psi_1, \psi_2, \psi_3)$$

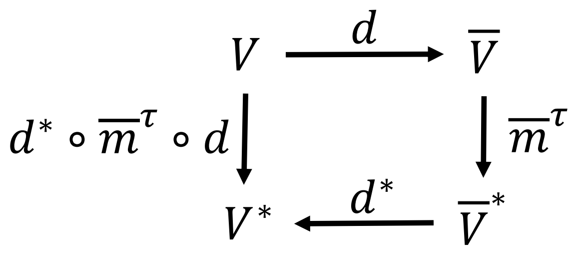
Differential matrix:

Let $d \in \mathrm{Hom}\left(V, \overline{V}\right)$ be the differential:

$$\begin{aligned} d: V &\to \overline{V} \\ f &\mapsto df \end{aligned}$$

$\Rightarrow$ The matrix $\mathbf{d} \in \mathbb{R}^{2\times 3}$ gives an expression for the map $d: V \to \overline{V}$ w.r.t. the bases $\{\psi_1, \psi_2, \psi_3\}$ and $\{dx_1, dx_2\}$.

Algebraic Interpretation



We have a vector space V, an inner-product space $\left\{\overline{V}, \overline{m}^\tau : \overline{V} \to \overline{V}^*\right\}$, and a linear map $d \in \mathrm{Hom}\left(V, \overline{V}\right)$.

$\Rightarrow$ We can pull back the bilinear form $\overline{m}^\tau$ to obtain a bilinear form on V:

$$d^* \circ \overline{m}^\tau \circ d$$

Properties:

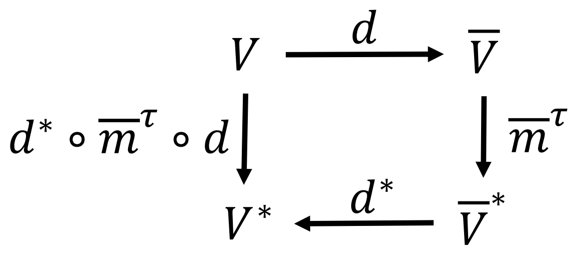
Since $\overline{m}^\tau$ is symmetric, so is the pull-back $d^* \circ \overline{m}^\tau \circ d$.

Since $\overline{m}^\tau$ is positive semi-definite, so is the pull-back $d^* \circ \overline{m}^\tau \circ d$.

Note:

Though $\overline{m}^\tau$ is positive definite, the pull-back is not since d is not invertible.

Algebraic Interpretation



We have a vector space V, an inner-product space $\left\{\overline{V}, \overline{m}^\tau : \overline{V} \to \overline{V}^*\right\}$, and a linear map $d \in \mathrm{Hom}\left(V, \overline{V}\right)$.

$\overline{m}^\tau$ is a bilinear map taking two constant cotangent vector-fields and returning their integrated inner-product:

$$\overline{m}^\tau(\nu, \eta) = \langle\langle \nu, \eta \rangle\rangle_\tau$$

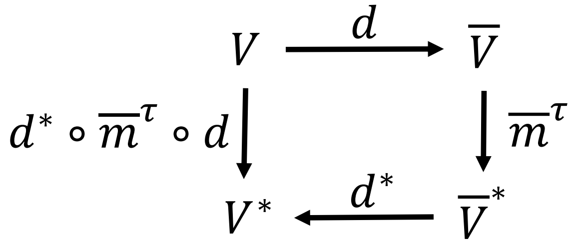
⇒ The pull-back to scalar fields is the bilinear map $\overline{m}^\tau$ applied to the differentials of the scalar fields:

$$\left(d^* \circ \overline{m}^\tau \circ d\right)(f, g) = \langle\langle df, dg \rangle\rangle_\tau$$

⇒ Applied to the same function:

$$\left(d^* \circ \overline{m}^\tau \circ d\right)(f, f) = E_D(f)$$

Algebraic Interpretation



We have a vector space V, an inner-product space $\left\{\overline{V}, \overline{m}^\tau : \overline{V} \to \overline{V}^*\right\}$, and a linear map $d \in \mathrm{Hom}\left(V, \overline{V}\right)$.

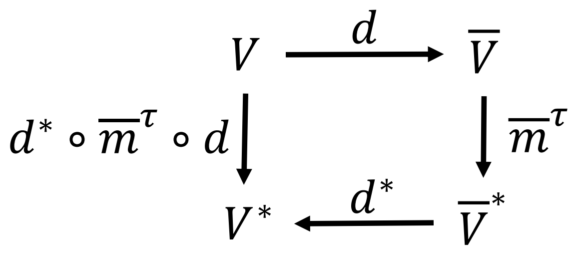
Applied to the same function, the pull-back of $\overline{m}^\tau$ gives the Dirichlet energy:

$$\left(d^* \circ \overline{m}^\tau \circ d\right)(f, f) = E_D(f)$$

This is how we defined the stiffness operator:

$$s^\tau : V \to V^*$$
$$\Downarrow$$
$$s^\tau = d^* \circ \overline{m}^\tau \circ d$$

Algebraic Interpretation



We have a vector space V, an inner-product space $\left\{\overline{V}, \overline{m}^\tau : \overline{V} \to \overline{V}^*\right\}$, and a linear map $d \in \mathrm{Hom}\left(V, \overline{V}\right)$.

$$s^\tau = d^* \circ \overline{m}^\tau \circ d$$

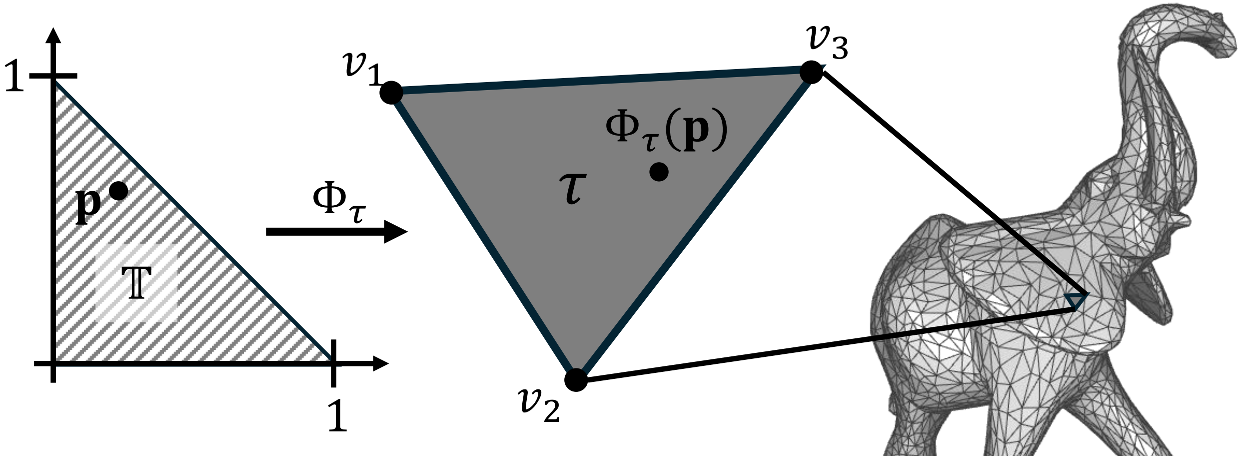
The stiffness operator factors as a composition of:

- Metric-independent components, and
- Metric-dependent components

Or, in terms of the bases $\{\psi_1, \psi_2, \psi_3\}$ and $\{\nu_1, \nu_2\}$:

$$\mathbf{s}^\tau = \mathbf{d}^\top \circ \overline{\mathbf{m}}^\tau \circ \mathbf{d}$$

Triangle Meshes



Recall:

Given a triangle mesh, $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we defined a space of scalar fields, assigning a piecewise-linear scalar field to each **vertex** $\{\phi_v\}_{v\in\mathcal{V}}$:

$$\phi_v\big(\Phi_\tau(\mathbf{p})\big) = \psi_{\tau(v)}(\mathbf{p})$$

Using finite-element assembly, we constructed mass and stiffness matrices, $\mathbf{M}, \boldsymbol{S} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$.

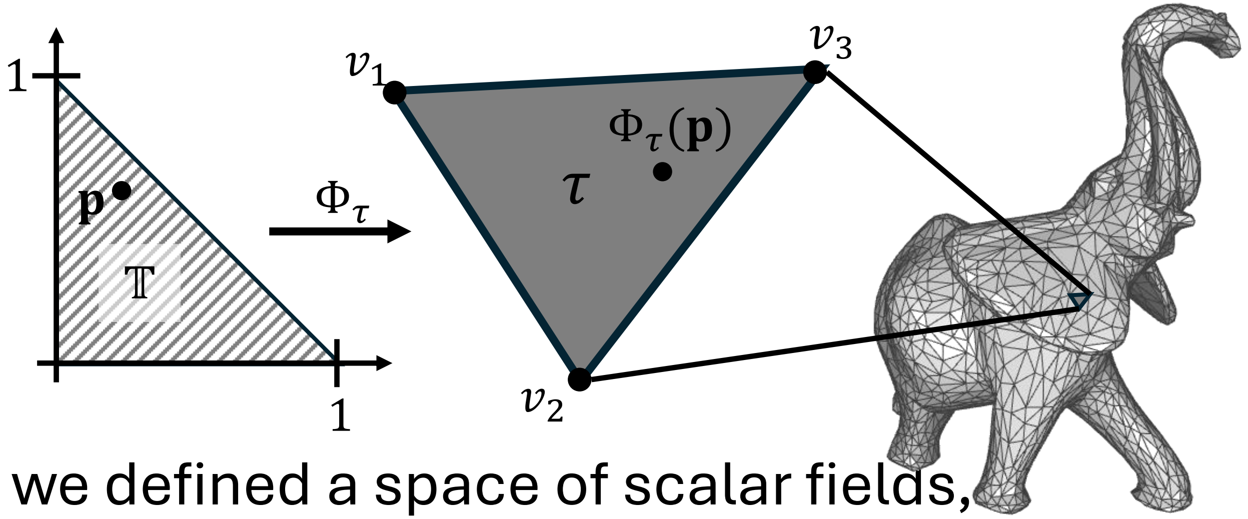
For piecewise-linear functions $f, h \in \mathrm{Span}\big(\phi_1, \ldots, \phi_{|\mathcal{V}|}\big)$:

$$f = \mathbf{f}_1 \cdot \phi_1 + \cdots + \mathbf{f}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$
$$h = \mathbf{h}_1 \cdot \phi_1 + \cdots + \mathbf{h}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$

Their inner-product is:

$$\langle\langle f, h\rangle\rangle_{\mathcal{M}} = \mathbf{f}^\top \cdot \mathbf{M} \cdot \mathbf{h}$$

Triangle Meshes



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$$\phi_v\big(\Phi_\tau(\mathbf{p})\big) = \psi_{\tau(v)}(\mathbf{p})$$

Using finite-element assembly, we constructed mass and stiffness matrices, $\mathbf{M}, \boldsymbol{S} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$.

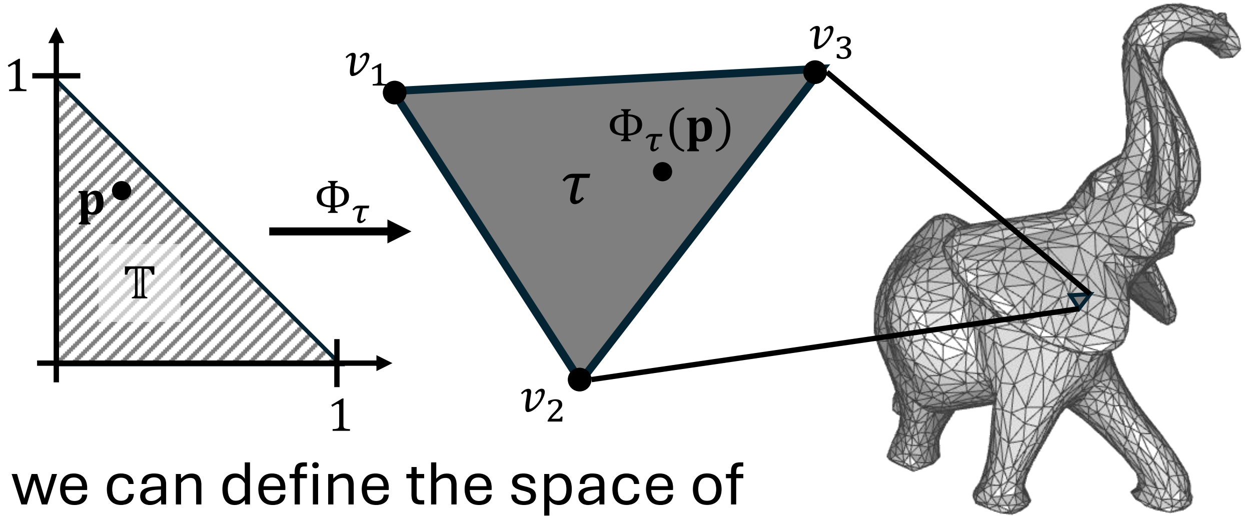
For piecewise-linear functions $f, h \in \mathrm{Span}\big(\phi_1, \ldots, \phi_{|\mathcal{V}|}\big)$:

$$f = \mathbf{f}_1 \cdot \phi_1 + \cdots + \mathbf{f}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$
$$h = \mathbf{h}_1 \cdot \phi_1 + \cdots + \mathbf{h}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$

The inner-product of their differentials is:

$$\langle\langle df, dh\rangle\rangle_{\mathcal{M}} = \mathbf{f}^\top \cdot \mathbf{S} \cdot \mathbf{h}$$

Triangle Meshes



Similarly:

Given a triangle mesh, $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we can define the space of cotangent vector fields, assigning two piecewise-constant cotangent vector fields to each **triangle** $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$:

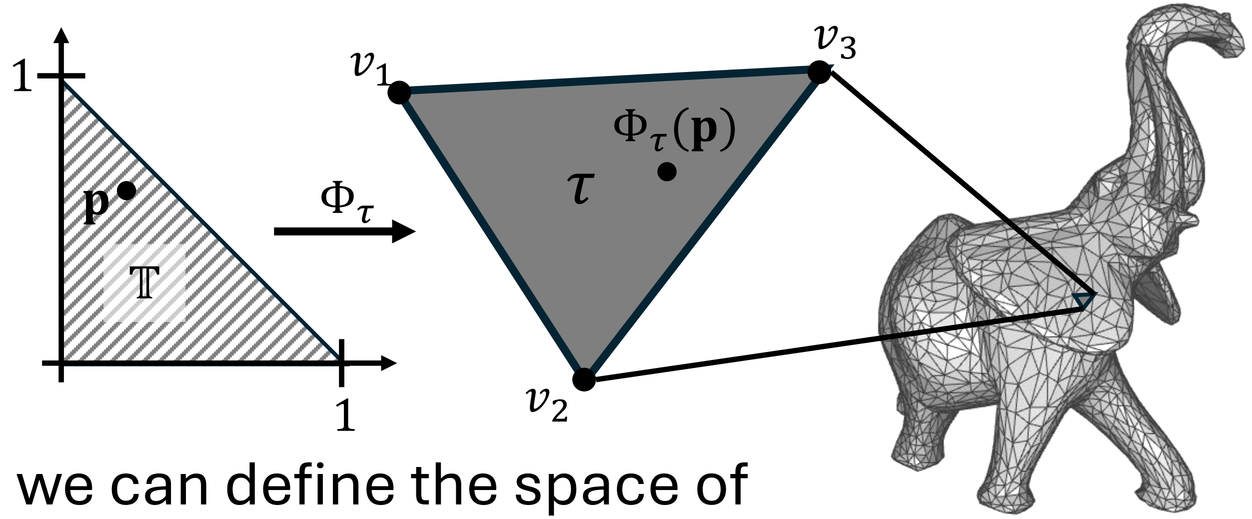
$$\eta_\tau^k\big(\Phi_\tau(\mathbf{p})\big) = dx_k$$

Note:

The basis functions η_τ^k are defined on the mesh, but return values in the cotangent space of the parameterization domain:

$$\eta_\tau^k\big(\Phi_\tau(\mathbf{p})\big) \in T_\mathbf{p}^*\mathbb{T}$$

Triangle Meshes



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$$\eta_\tau^k\big(\Phi_\tau(\mathbf{p})\big) = dx_k$$

Again, we can construct the mass matrix, $\overline{\mathbf{M}} \in \mathbb{R}^{2|\mathcal{T}|\times 2|\mathcal{T}|}$.

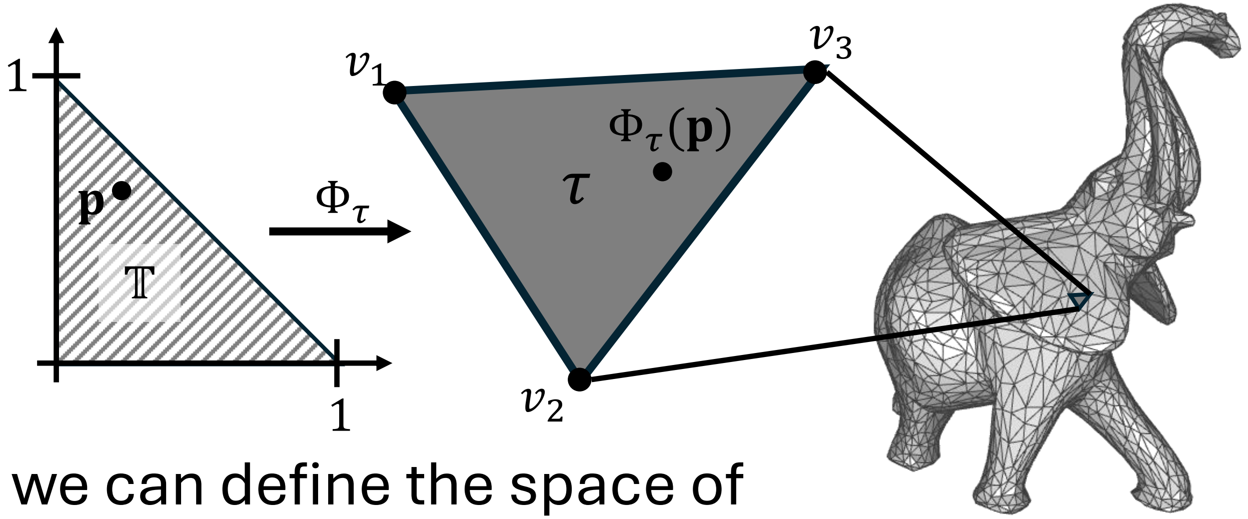
For piecewise-constant vector fields $v, w \in \mathrm{Span}\big(\eta_1^1, \eta_1^2, \ldots, \eta_{|\mathcal{T}|}^1, \eta_{|\mathcal{T}|}^2\big)$:

$$v = \mathbf{v}_1^1 \cdot \eta_1^1 + \mathbf{v}_1^2 \cdot \eta_1^2 + \cdots + \mathbf{v}_{|\mathcal{T}|}^1 \cdot \eta_{|\mathcal{T}|}^1 + \mathbf{v}_{|\mathcal{T}|}^2 \cdot \eta_{|\mathcal{T}|}^2$$
$$w = \mathbf{w}_1^1 \cdot \eta_1^1 + \mathbf{w}_1^2 \cdot \eta_1^2 + \cdots + \mathbf{w}_{|\mathcal{T}|}^1 \cdot \eta_{|\mathcal{T}|}^1 + \mathbf{w}_{|\mathcal{T}|}^2 \cdot \eta_{|\mathcal{T}|}^2$$

Their inner-product is:

$$\langle\langle v, w\rangle\rangle_{\mathcal{M}} = \mathbf{v}^\top \cdot \overline{\mathbf{M}} \cdot \mathbf{w}$$

Triangle Meshes



Similarly:

Given a triangle mesh, $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we can define the space of cotangent vector fields, assigning two piecewise-constant cotangent vector fields to each **triangle** $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$:

$$\eta_\tau^k\big(\Phi_\tau(\mathbf{p})\big) = dx_k$$

And, we can construct the differential matrix, $\mathbf{D} \in \mathbb{R}^{2|\mathcal{T}|\times|\mathcal{V}|}$.

For a piecewise-linear function $f \in \mathrm{Span}\big(\phi_1, \dots, \phi_{|\mathcal{V}|}\big)$:

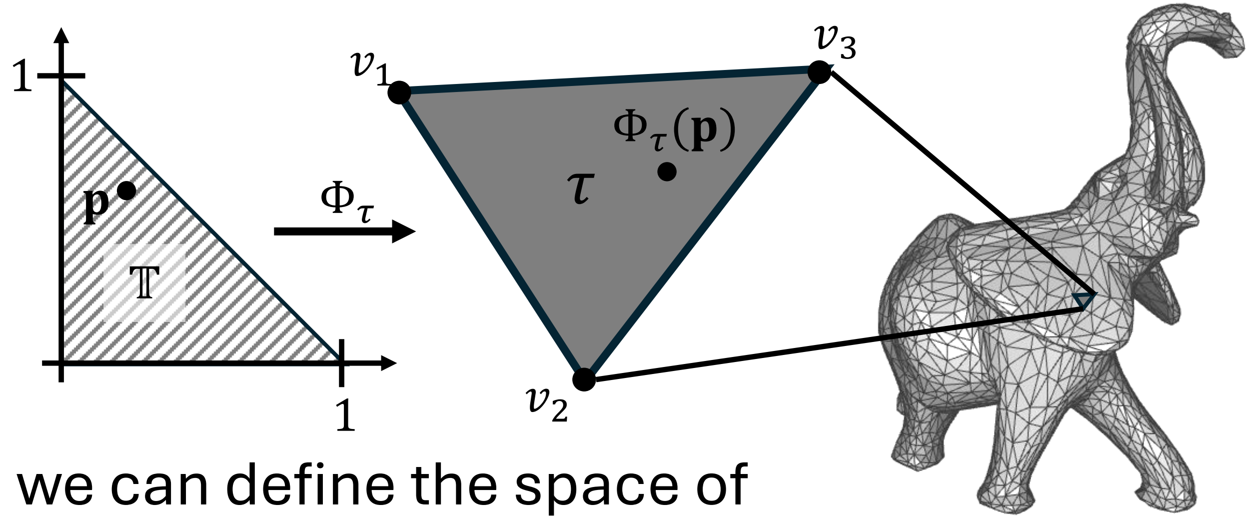
$$f = \mathbf{f}_1 \cdot \phi_1 + \cdots + \mathbf{f}_{|\mathcal{V}|} \cdot \phi_{|\mathcal{V}|}$$

We can set $\mathbf{v} = \mathbf{D} \cdot \mathbf{f} \in \mathbb{R}^{2|\mathcal{T}|}$.

This gives the coefficients of the differential of $\mathbf{f}$ in the basis $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$:

$$df = \mathbf{v}_1^1 \cdot \eta_1^1 + \mathbf{v}_1^2 \cdot \eta_1^2 + \cdots + \mathbf{v}_{|\mathcal{T}|}^1 \cdot \eta_{|\mathcal{T}|}^1 + \mathbf{v}_{|\mathcal{T}|}^2 \cdot \eta_{|\mathcal{T}|}^2$$

Triangle Meshes



Similarly:

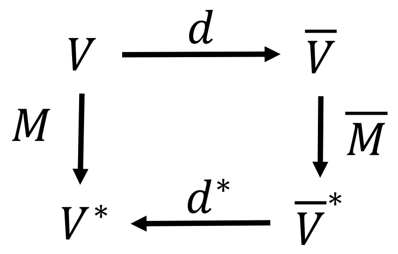
Given a triangle mesh, $\mathcal{M} = \{\mathcal{V}, \mathcal{T}\}$, we can define the space of cotangent vector fields, assigning two piecewise-constant cotangent vector fields to each **triangle** $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$:

$$\eta_\tau^k\big(\Phi_\tau(\mathbf{p})\big) = dx_k$$

As for a single triangle, this gives a factorization of the stiffness matrix:

$$\mathbf{S} = \mathbf{D}^\top \cdot \overline{\mathbf{M}} \cdot \mathbf{D}$$

Algebraic Interpretation



Given per-vertex scalar field basis $\{\phi_v\}_{v\in\mathcal{V}}$, we have a $|\mathcal{V}|$-dimensional inner-product space $\{V, M: V \to V^*\}$.

Given the per-triangle cotangent vector field basis $\{\eta_\tau^1, \eta_\tau^2\}_{\tau\in\mathcal{T}}$, we have a $2|\mathcal{T}|$-dimensional inner-product space $\{\overline{V}, \overline{M}: \overline{V} \to \overline{V}^*\}$.

The differential is a linear map from scalar fields to cotangent vector fields:

$$d \in \mathrm{Hom}\left(V, \overline{V}\right)$$

The (symmetric positive semi-definite) stiffness operator is the pull-back of the inner-product $\overline{M}: \overline{V} \to \overline{V}^*$ to a bilinear form on V, via the differential:

$$S = d^* \circ \overline{M} \circ d$$

Caution

We are now working with three types of inner-product spaces.

$\{T_{\mathbf{p}}\mathbb{T}, g_{\mathbf{p}}: T_{\mathbf{p}}\mathbb{T} \rightarrow T_{\mathbf{p}}^{*}\mathbb{T}\}$:

Tangent vectors at the point $\mathbf{p} \in \mathbb{T}$, with inner-product $g_{\mathbf{p}}$ defined by pulling back the Euclidean inner-product using the differential of the parametrization $\Phi_{\tau}: \mathbb{T} \rightarrow \tau$.

$\{V, M: V \rightarrow V^{*}\}$:

Piecewise-linear functions on the mesh, with inner-product M defined by integrating over the mesh.

$\{\overline{V}, \overline{M}: \overline{V} \rightarrow \overline{V}^{*}\}$:

Piecewise-constant cotangent vector fields on the mesh, with inner-product $\overline{M}$ defined by integrating over the mesh.