Geometry Processing (601.458/658)

Misha Kazhdan

Outline

Recall

The Hat Basis on a Triangle

Recall

Integration by parts:

Given functions $f, g: \mathbb{R} \to \mathbb{R}$, the derivative of their product is:
$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Taking the integral of the derivative:
$$f \cdot g = \int (f \cdot g)' = \int (f' \cdot g + f \cdot g')$$

This gives:
$$\int f' \cdot g = f \cdot g - \int f \cdot g'$$

Recall

Consider the polynomial $P_{m,n}(x) = (1-x)^m \cdot x^n$:

Setting $f(x) = (1-x)^m$ and $g(x) = x^n$, integration by parts gives:

$$\int_0^1 P_{m,n}(x) \cdot dx = \int_0^1 (1-x)^m \cdot x^n \cdot dx$$

$$= \left[\frac{-1}{m+1}(1-x)^{m+1} \cdot x^n \right]_{x=0}^{x=1} - \int_0^1 \frac{-n}{m+1}(1-x)^{m+1} \cdot x^{n-1} \cdot dx$$

$$= \frac{n}{m+1} \int_0^1 P_{m+1,n-1}(x) \cdot dx$$

$$= \frac{n \cdot (n-1)}{(m+1) \cdot (m+2)} \int_0^1 P_{m+2,n-2}(x) \cdot dx$$

$$\vdots$$

$$= \frac{n! \cdot m!}{(m+n)!} \int_0^1 P_{m+n,0}(x) \cdot dx$$

Recall

Consider the polynomial $P_{m,n}(x) = (1-x)^m \cdot x^n$:

Setting $f(x) = (1-x)^m$ and $g(x) = x^n$, integration by parts gives:

$$\int_0^1 P_{m,n}(x) \cdot dx = \frac{n! \cdot m!}{(m+n)!} \int_0^1 P_{m+n,0}(x) \cdot dx$$

$$= \frac{n! \cdot m!}{(m+n)!} \int_0^1 (1-x)^{m+n} \cdot dx$$

$$= \frac{n! \cdot m!}{(m+n)!} \cdot \left[\frac{-1}{m+n+1}(1-x)^{m+n+1} \right]_{x=0}^{x=1}$$
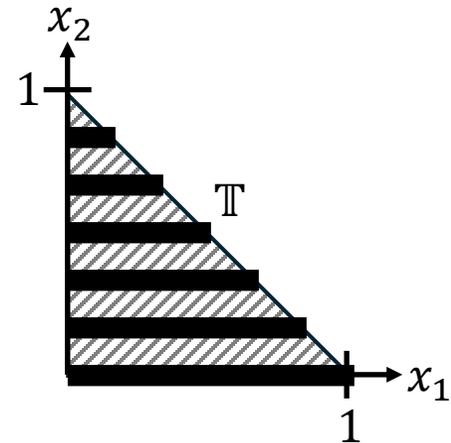
$$= \frac{n! \cdot m!}{(m+n+1)!}$$

Recall

$$\int_0^1 (1-x)^m \cdot x^n = \frac{n! \cdot m!}{(m+n+1)!}$$

Consider the monomial in two variables $P_{m,n}(x_1, x_2) = x_1^m \cdot x_2^n$:

The integral of $P_{m,n}$ over the unit right triangle (w.r.t. the Euclidean metric) is:

$$\int_{\mathbb{T}} x_1^m \cdot x_2^n \cdot dx_1 \wedge dx_2 = \int_0^1 \int_0^{1-x_2} x_1^m \cdot x_2^n \cdot dx_1 \wedge dx_2$$

$$= \int_0^1 \left[\int_0^{1-x_2} x_1^m \cdot dx_1 \right] \cdot x_2^n \wedge dx_2$$

$$= \int_0^1 \left[\frac{1}{m+1} x_1^{m+1} \right]_{x_1=0}^{x_1=1-x_2} \cdot x_2^n \cdot dx_2$$

$$= \frac{1}{m+1} \int_0^1 (1-x_2)^{m+1} \cdot x_2^n \cdot dx_2$$

$$= \frac{1}{m+1} \cdot \frac{n! \cdot (m+1)!}{(m+n+2)!}$$

$$= \frac{n! \cdot m!}{(m+n+2)!}$$

Recall

Given a triangle with angles a, b, and c, and opposite sides A, B, and C, the ratios of the sines of the angles and the lengths of the opposite edges are equal:

$$\frac{\sin(a)}{\sin(b)} = \frac{A}{B} \qquad \frac{\sin(b)}{\sin(c)} = \frac{B}{C} \qquad \frac{\sin(c)}{\sin(a)} = \frac{C}{A}$$
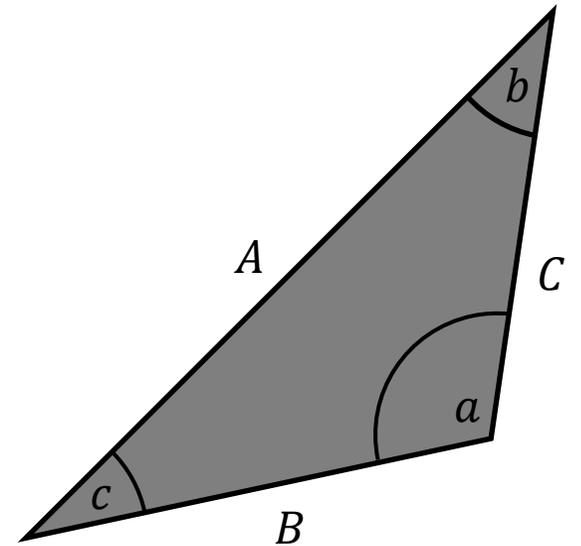
Recall

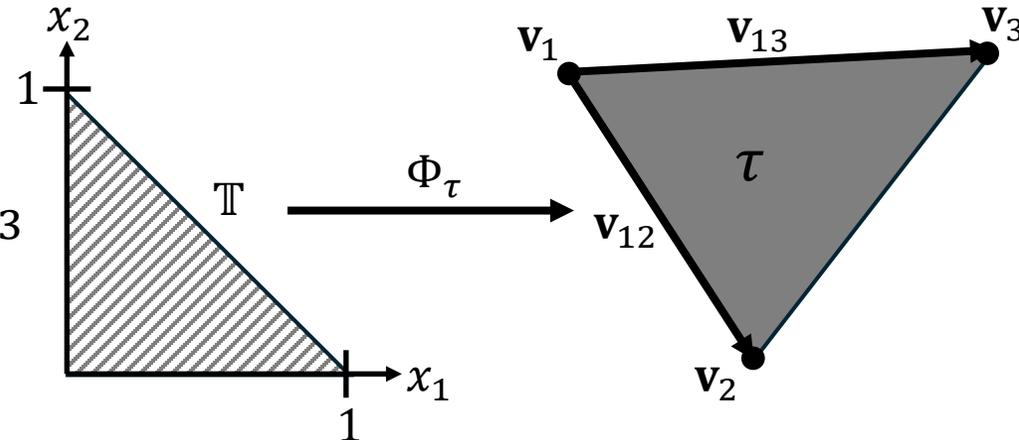We denote by $\mathbb{T} \subset \mathbb{E}^2$ the unit right triangle:
$$\mathbb{T} = \{(x_1, x_2) \in [0,1]^2 | x_1 + x_2 \leq 1\}$$

Given a triangle $\tau \subset \mathbb{E}^3$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{E}^3$, we parameterize the triangle over the unit-right triangle as:
$$\Phi_\tau(x_1, x_2) = \mathbf{v}_1 + x_1 \cdot (\mathbf{v}_2 - \mathbf{v}_1) + x_2 \cdot (\mathbf{v}_3 - \mathbf{v}_1)$$

Denoting by $\mathbf{v}_{ij} = \mathbf{v}_j - \mathbf{v}_i$ the direction from vertex $\mathbf{v}_i$ to vertex $\mathbf{v}_j$:
$$\Phi_\tau(x_1, x_2) = \mathbf{v}_1 + x_1 \cdot \mathbf{v}_{12} + x_2 \cdot \mathbf{v}_{13}$$

Recall

$$\Phi_\tau(x_1, x_2) = \mathbf{v}_1 + x_1 \cdot \mathbf{v}_{12} + x_2 \cdot \mathbf{v}_{13}$$

For $\mathbf{p} \in \mathbb{T}$, the differential $\mathbf{d}\Phi_\tau|_\mathbf{p} \in \mathbb{R}^{3 \times 2}$ is the matrix whose columns are the differences:

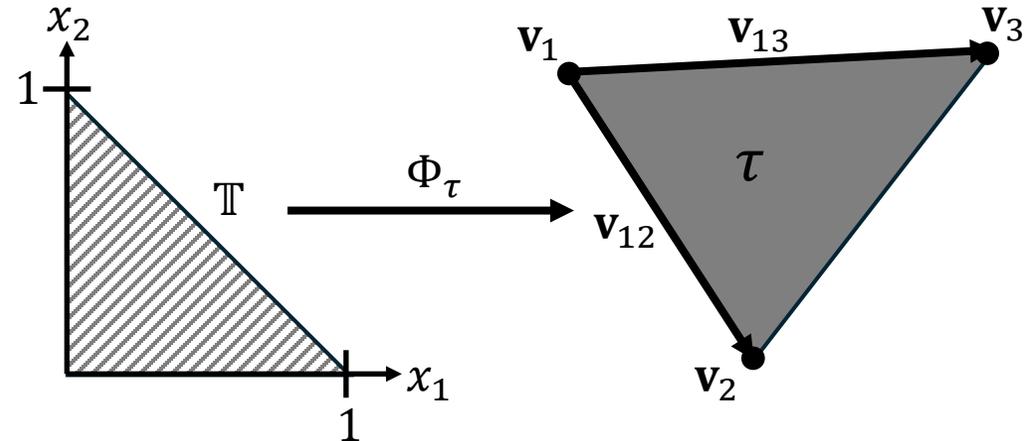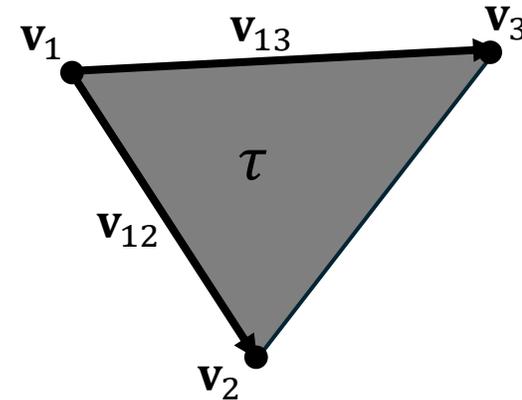$$\mathbf{d}\Phi_\tau\Big|_\mathbf{p} = (\mathbf{v}_{12} \quad \mathbf{v}_{13})$$

Recall

$$\Phi_\tau(x_1, x_2) = \mathbf{v}_1 + x_1 \cdot \mathbf{v}_{12} + x_2 \cdot \mathbf{v}_{13}$$

For $\mathbf{p} \in \mathbb{T}$, the differential $\mathbf{d}\Phi_\tau|_{\mathbf{p}} \in \mathbb{R}^{3 \times 2}$ is the matrix whose columns are the differences:

$$\mathbf{d}\Phi_\tau\Big|_{\mathbf{p}} = \begin{pmatrix} \mathbf{v}_{12} & \mathbf{v}_{13} \end{pmatrix}$$

The pulled-back inner-product, expressed in matrix form w.r.t. the cartesian basis $\{\partial_1|_{\mathbf{p}}, \partial_2|_{\mathbf{p}}\} \subset T_{\mathbf{p}}\mathbb{T}$ is:

$$\mathbf{g}_\tau = \mathbf{d}\Phi_\tau\Big|_{\mathbf{p}}^{\mathsf{T}} \cdot \mathbf{d}\Phi_\tau\Big|_{\mathbf{p}} = \begin{pmatrix} \langle \mathbf{v}_{12}, \mathbf{v}_{12} \rangle & \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ \langle \mathbf{v}_{13}, \mathbf{v}_{12} \rangle & \langle \mathbf{v}_{13}, \mathbf{v}_{13} \rangle \end{pmatrix}$$
$$= \begin{pmatrix} \|\mathbf{v}_{12}\|^2 & \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ \langle \mathbf{v}_{13}, \mathbf{v}_{12} \rangle & \|\mathbf{v}_{13}\|^2 \end{pmatrix}$$

Recall

Given a triangle $\tau \subset \mathbb{R}^n$, we define the inner-product on the space of real-valued functions on τ is:

$$\langle\langle f, h \rangle\rangle_{g_\tau} = \sqrt{\det(\mathbf{g}_\tau)} \cdot \langle\langle f, h \rangle\rangle_E$$

And the Dirichlet energy is defined as:

$$E_D(f) = \langle\langle df, df \rangle\rangle_{g_\tau}$$

$$= \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \int_{\mathbb{T}} \mathbf{d}f \cdot \mathbf{d}f^{\top} \cdot \omega_E \right\rangle_F$$

with ω_E the unit-volume form with respect to the Euclidean inner-product.

Outline

Recall

The Hat Basis on a Triangle

Notation

Often, we define the inner-product on the tangent space by pulling back the inner-product from an embedded surfaces.

We will subscript the inner-product on the space of functions and differentials with the geometry for brevity:

$$\langle\langle f, h \rangle\rangle_{g_\tau} \leftrightarrow \langle\langle f, h \rangle\rangle_\tau$$
$$\langle\langle df, dh \rangle\rangle_{g_\tau} \leftrightarrow \langle\langle df, dh \rangle\rangle_\tau$$

The Hat Basis

Definition:

We say that a function f on $\mathbb{T}$ is a *scalar field* if for all $\mathbf{p} \in \mathbb{T}$:
$$f(\mathbf{p}) \in \mathbb{R}$$

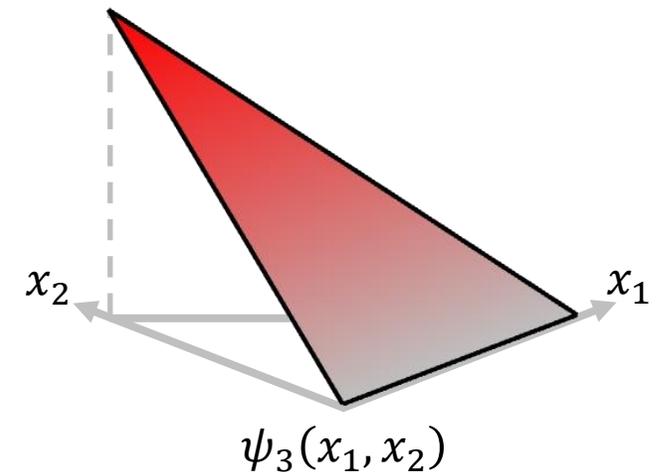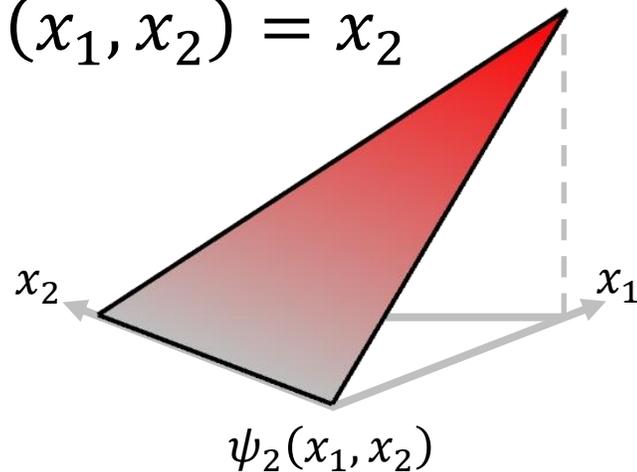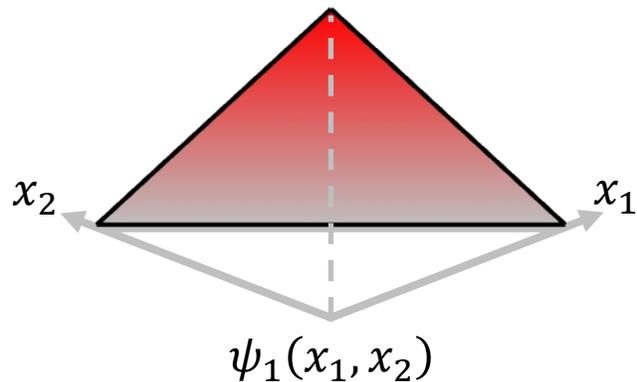# The Hat Basis

As we cannot work with all functions on the triangle, we restrict ourselves to those we can represent using a finite basis.

A standard basis is the "hat" basis, consisting of:

$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1, x_2) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

x_2 x_1

$\psi_1(x_1, x_2)$

x_2 x_1

$\psi_2(x_1, x_2)$

x_2 x_1

$\psi_3(x_1, x_2)$

The Hat Basis

$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1, x_2) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

Properties:

Partition of unity: The functions sum to one:
$$\psi_1(x_1, x_2) + \psi_2(x_1, x_2) + \psi_3(x_1, x_2) = 1$$

Lagrange property: The functions interpolate the vertex values:
$$\psi_1(0,0) = \psi_2(1,0) = \psi_3(0,1) = 1$$
$$\psi_2(0,0) = \psi_3(0,0) = \psi_1(1,0) = \psi_3(1,0) = \psi_1(0,1) = \psi_2(0,1) = 0$$
$$\Downarrow$$

Linear reproduction: Any linear polynomial $f: \mathbb{T} \to \mathbb{R}$ is expressed as:
$$f = f(0,0) \cdot \psi_0 + f(1,0) \cdot \psi_1 + f(0,1) \cdot \psi_2$$

The Hat Basis

$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1, x_2) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

Questions:

Given functions:
$$f(x_1, x_2) = \mathbf{f}_1 \cdot \psi_1(x_1, x_2) + \mathbf{f}_2 \cdot \psi_2(x_1, x_2) + \mathbf{f}_3 \cdot \psi_3(x_1, x_2)$$
$$h(x_1, x_2) = \mathbf{h}_1 \cdot \psi_1(x_1, x_2) + \mathbf{h}_2 \cdot \psi_2(x_1, x_2) + \mathbf{h}_3 \cdot \psi_3(x_1, x_2)$$
what is the inner-product of f and h?

Given a function:
$$f(x_1, x_2) = \mathbf{f}_1 \cdot \psi_1(x_1, x_2) + \mathbf{f}_2 \cdot \psi_2(x_1, x_2) + \mathbf{f}_3 \cdot \psi_3(x_1, x_2)$$
what is the Dirichlet energy of f?

$$f(x_1, x_2) = \mathbf{f}_1 \cdot \psi_1(x_1, x_2) + \mathbf{f}_2 \cdot \psi_2(x_1, x_2) + \mathbf{f}_3 \cdot \psi_3(x_1, x_2)$$
$$h(x_1, x_2) = \mathbf{h}_1 \cdot \psi_1(x_1, x_2) + \mathbf{h}_2 \cdot \psi_2(x_1, x_2) + \mathbf{h}_3 \cdot \psi_3(x_1, x_2)$$

The Hat Basis: Inner Product

The inner-product is:

$$\langle\langle f, h \rangle\rangle_\tau = \sqrt{\det(\mathbf{g}_\tau)} \cdot \boxed{\langle\langle f, h \rangle\rangle_{\mathbb{E}^2}}$$

Expanding the second term gives:

$$\langle\langle f, h \rangle\rangle_{\mathbb{E}^2} = \left\langle\!\!\left\langle \sum_{i=1}^{3} \mathbf{f}_i \cdot \psi_i, \sum_{j=1}^{3} \mathbf{h}_j \cdot \psi_j \right\rangle\!\!\right\rangle_{\mathbb{E}^2}$$

$$= \sum_{i,j=1}^{3} \mathbf{f}_i \cdot \mathbf{h}_j \cdot \langle\langle \psi_i, \psi_j \rangle\rangle_{\mathbb{E}^2}$$

$$f(x_1, x_2) = \mathbf{f}_1 \cdot \psi_1(x_1, x_2) + \mathbf{f}_2 \cdot \psi_2(x_1, x_2) + \mathbf{f}_3 \cdot \psi_3(x_1, x_2)$$
$$h(x_1, x_2) = \mathbf{h}_1 \cdot \psi_1(x_1, x_2) + \mathbf{h}_2 \cdot \psi_2(x_1, x_2) + \mathbf{h}_3 \cdot \psi_3(x_1, x_2)$$

The Hat Basis: Inner Product

The inner-product is:

$$\langle\langle f, h \rangle\rangle_\tau = \sqrt{\det(\mathbf{g}_\tau)} \cdot \boxed{\langle\langle f, h \rangle\rangle_\mathbb{T}}$$

Expanding the second term gives:

$$\langle\langle f, h \rangle\rangle_{\mathbb{E}^2} \sum_{i,j=1}^{3} \mathbf{f}_i \cdot \mathbf{h}_j \cdot \langle\langle \psi_i, \psi_j \rangle\rangle_\mathbb{T}$$

Setting $\mathbf{m}^\tau \in \mathbb{R}^{3 \times 3}$ to be the triangle's *mass matrix*:

$$\mathbf{m}_{ij}^\tau = \sqrt{\det(\mathbf{g}_\tau)} \cdot \langle\langle \psi_i, \psi_j \rangle\rangle_\mathbb{T}$$
$$\Downarrow$$
$$\langle\langle f, h \rangle\rangle_\tau = \mathbf{f}^\top \cdot \mathbf{m}^\tau \cdot \mathbf{h}$$

The Hat Basis: Inner Product

$$\mathbf{m}_{ij}^\tau = \sqrt{\det(\mathbf{g}_\tau)} \cdot \boxed{\langle\langle \psi_i, \psi_j \rangle\rangle_{\mathbb{T}}}$$
$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1, x_2) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

$$\langle\langle f, h \rangle\rangle_\tau = \mathbf{f}^\top \cdot \mathbf{m}^\tau \cdot \mathbf{h}$$

$\Rightarrow$ If we can integrate the product of the hat functions **over the unit right triangle**, computing the inner-product of functions expressed in the hat basis w.r.t. to the pull-back metric g_τ is not hard.

The value of $\langle\langle \psi_i, \psi_j \rangle\rangle_{\mathbb{T}}$ is independent of the inner-product g_τ.

The Hat Basis: Inner Product

$$\mathbf{m}_{ij}^{\tau} = \sqrt{\det(\mathbf{g}_{\tau})} \cdot \boxed{\langle\langle \psi_i, \psi_j \rangle\rangle_{\mathbb{T}}}$$
$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1, x_2) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

$$\langle\langle f, h \rangle\rangle_{\tau} = \mathbf{f}^{\top} \cdot \mathbf{m}^{\tau} \cdot \mathbf{h}$$

$\Rightarrow$ If we can integrate the product of the hat functions **over the unit right triangle**, computing the inner-product of functions expressed in the hat basis w.r.t. to the pull-back metric g_{τ} is not hard.

The value of $\langle\langle \psi_i, \psi_j \rangle\rangle_{\mathbb{T}}$ is independent of the inner-product g_{τ}.

$\Downarrow$

It can be computed once in a pre-processing stop.

The Hat Basis: Inner Product

$$\mathbf{m}_{ij}^\tau = \sqrt{\det(\mathbf{g}_\tau)} \cdot \boxed{\langle\langle \psi_i, \psi_j \rangle\rangle_{\mathbb{T}}}$$

$$\psi_1(x_1, x_2) = 1 - x_1 - x_2$$
$$\psi_2(x_1, x_2) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

$$\int_{\mathbb{T}} x_1^m \cdot x_2^n \cdot dx_1 \wedge dx_2 = \frac{n! \cdot m!}{(m + n + 2)!}$$

Example:

$$\langle\langle \psi_2, \psi_3 \rangle\rangle_{\mathbb{T}} = \int_{\mathbb{T}} x_1 \cdot x_2 \cdot dx_1 \wedge dx_2$$
$$= \frac{1}{4!}$$
$$= \frac{1}{24}$$

You'll need to work this out for the other pairs.

$$f(x_1, x_2) = \mathbf{f}_1 \cdot \psi_1(x_1, x_2) + \mathbf{f}_2 \cdot \psi_2(x_1, x_2) + \mathbf{f}_3 \cdot \psi_3(x_1, x_2)$$

The Hat Basis: Dirichlet Energy

The Dirichlet energy is defined as:

$$E_D(f) = \langle\langle df, df \rangle\rangle_\tau = \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \boxed{\int_{\mathbb{T}} \mathbf{d}f \cdot \mathbf{d}f^\top \cdot \omega_E} \right\rangle_F$$

Expanding the second term gives:

$$\int_{\mathbb{T}} \mathbf{d}f \cdot \mathbf{d}f^\top \cdot \omega_E = \int_{\mathbb{T}} \left(\mathbf{d}\left(\sum_{i=1}^{3} \mathbf{f}_i \cdot \psi_i \right) \right) \cdot \left(\mathbf{d}\left(\sum_{j=1}^{3} \mathbf{f}_j \cdot \psi_j \right) \right)^\top \cdot \omega_E$$

$$= \int_{\mathbb{T}} \sum_{i,j=1}^{3} \mathbf{f}_i \cdot \mathbf{f}_j \cdot \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^\top \cdot \omega_E$$

$$= \sum_{i,j=1}^{3} \mathbf{f}_i \cdot \mathbf{f}_j \cdot \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^\top \cdot \omega_E$$

$$f(x_1, x_2) = \mathbf{f}_1 \cdot \psi_1(x_1, x_2) + \mathbf{f}_2 \cdot \psi_2(x_1, x_2) + \mathbf{f}_3 \cdot \psi_3(x_1, x_2)$$

The Hat Basis: Dirichlet Energy

The Dirichlet energy is defined as:

$$E_D(f) = \langle\langle df, df \rangle\rangle_\tau = \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \boxed{\int_\mathbb{T} \mathbf{d}f \cdot \mathbf{d}f^\top \cdot \omega_E} \right\rangle_F$$

Expanding the second term gives:

$$\int_\mathbb{T} \mathbf{d}f \cdot \mathbf{d}f^\top \cdot \omega_E = \sum_{i,j=1}^{3} \mathbf{f}_i \cdot \mathbf{f}_j \cdot \boxed{\int_\mathbb{T} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^\top \cdot \omega_E}$$

Setting $\mathbf{d}^{ij} \in \mathbb{R}^{2\times2}$ to be the integrated outer-product of derivatives:

$$\mathbf{d}^{ij} = \int_\mathbb{T} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^\top \cdot \omega_E$$

$$\Downarrow$$

$$E_D(f) = \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \sum_{i,j=1}^{3} \mathbf{f}_i \cdot \mathbf{f}_j \cdot \mathbf{d}^{ij} \right\rangle_F$$

$$f(x_1, x_2) = \mathbf{f}_1 \cdot \psi_1(x_1, x_2) + \mathbf{f}_2 \cdot \psi_2(x_1, x_2) + \mathbf{f}_3 \cdot \psi_3(x_1, x_2)$$

The Hat Basis: Dirichlet Energy

$$\mathbf{d}^{ij} = \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j{}^{\top} \cdot \omega_E$$

With respect to $\mathbf{d}^{ij} \in \mathbb{R}^{2 \times 2}$, the Dirichlet energy becomes:

$$
\begin{aligned}
E_D(f) &= \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \sum_{i,j=1}^{3} \mathbf{f}_i \cdot \mathbf{f}_j \cdot \mathbf{d}^{ij} \right\rangle_F \\
&= \sum_{i,j=1}^{3} \mathbf{f}_i \cdot \mathbf{f}_j \cdot \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \mathbf{d}^{ij} \right\rangle_F
\end{aligned}
$$

The Hat Basis: Dirichlet Energy

$$\mathbf{d}^{ij} = \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^{\top} \cdot \omega_E$$

With respect to $\mathbf{d}^{ij} \in \mathbb{R}^{2 \times 2}$, the Dirichlet energy becomes:

$$E_D(f) = \sum_{i,j=1}^{3} \mathbf{f}_i \cdot \mathbf{f}_j \cdot \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \mathbf{d}^{ij} \right\rangle_F$$

Setting $\mathbf{s}^{\tau} \in \mathbb{R}^{3 \times 3}$ to be the *stiffness matrix*:

$$\mathbf{s}_{ij}^{\tau} = \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \mathbf{d}^{ij} \right\rangle_F$$

$$\Downarrow$$

$$E_D(f) = \mathbf{f}^{\top} \cdot \mathbf{s}^{\tau} \cdot \mathbf{f}$$

The Hat Basis: Dirichlet Product

$$\mathbf{s}_{ij}^{\tau} = \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1} \boxed{\mathbf{d}^{ij}} \right\rangle_F$$

$$\boxed{\mathbf{d}^{ij} = \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^{\top} \cdot \omega_E}$$

$$E_D(f) = \mathbf{f}^{\top} \cdot \mathbf{s}^{\tau} \cdot \mathbf{f}$$

$\Rightarrow$ If we can integrate the product of the partial derivatives of the hat functions **over the unit right triangle**, computing the Dirichlet energy of functions expressed in the hat basis w.r.t. to the pull-back metric g_τ is not hard.

The matrices $\mathbf{d}^{ij}$ are independent of the inner-product g_τ.

The Hat Basis: Dirichlet Product

$$\mathbf{s}_{ij}^{\tau} = \left\langle \sqrt{\det(\mathbf{g}_{\tau})} \cdot \mathbf{g}_{\tau}^{-1} \boxed{\mathbf{d}^{ij}} \right\rangle_F$$

$$\boxed{\mathbf{d}^{ij} = \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^{\top} \cdot \omega_E}$$

$$E_D(f) = \mathbf{f}^{\top} \cdot \mathbf{s}^{\tau} \cdot \mathbf{f}$$

$\Rightarrow$ If we can integrate the product of the partial derivatives of the hat functions **over the unit right triangle**, computing the Dirichlet energy of functions expressed in the hat basis w.r.t. to the pull-back metric g_{τ} is not hard.

The matrices $\mathbf{d}^{ij}$ are independent of the inner-product g_{τ}.

$\Downarrow$

Can be computed once in a pre-processing stop.

The Hat Basis: Dirichlet Product

$$\mathbf{d}^{ij} = \int_{\mathbb{T}} \mathbf{d}\psi_i \cdot \mathbf{d}\psi_j^\top \cdot \omega_E$$

$$\psi_1(x_1, x_2) = 1 - x_1 - x_1$$
$$\psi_2(x_1, x_1) = x_1$$
$$\psi_3(x_1, x_2) = x_2$$

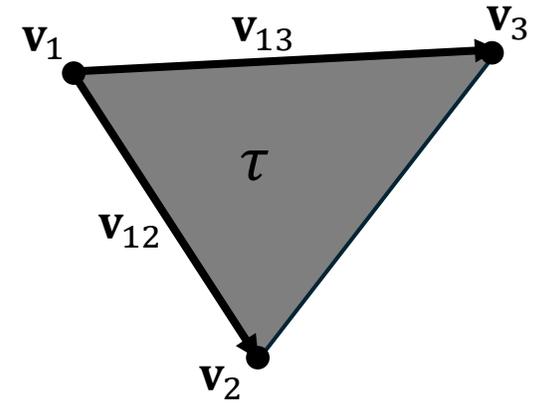$$\int_{\mathbb{T}} x_1^m \cdot x_2^n \cdot dx_1 \wedge dx_2 = \frac{n! \cdot m!}{(m + n + 2)!}$$

Example:

$$\mathbf{d}^{23} = \int_{\mathbb{T}} \mathbf{d}\psi_2 \cdot \mathbf{d}\psi_3^\top \cdot dx_1 \wedge dx_2$$

$$= \int_{\mathbb{T}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\top \cdot dx_1 \wedge dx_2$$

$$= \int_{\mathbb{T}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (0 \quad 1) \cdot dx_1 \wedge dx_2$$

$$= \int_{\mathbb{T}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot dx_1 \wedge dx_2$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}$$

You'll need to work this out for the other pairs.

The Hat Basis

$$\mathbf{m}_{ij}^{\tau} = \sqrt{\det(\mathbf{g}_\tau)} \cdot \langle\langle \psi_i, \psi_j \rangle\rangle_E$$

$$\mathbf{s}_{ij}^{\tau} = \left\langle \sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}, \mathbf{d}^{ij} \right\rangle_F$$

$$\mathbf{g}_\tau = \begin{pmatrix} \|\mathbf{v}_{12}\|^2 & \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ \langle \mathbf{v}_{13}, \mathbf{v}_{12} \rangle & \|\mathbf{v}_{13}\|^2 \end{pmatrix}$$
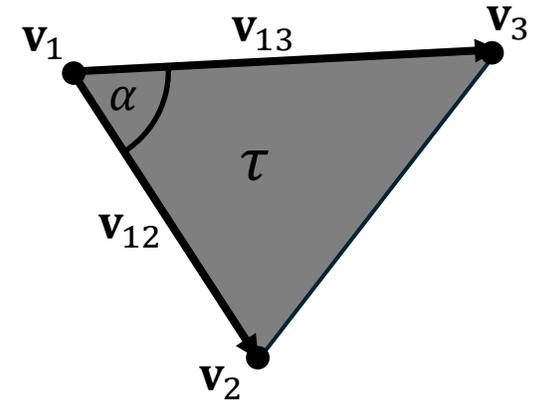

What components of the inner-product affect the mass and stiffness matrices?

The Hat Basis

$$\mathbf{m}_{ij}^{\tau} = \boxed{\sqrt{\det(\mathbf{g}_{\tau})}} \; \langle\langle \psi_i, \psi_j \rangle\rangle_E$$

$$\mathbf{s}_{ij}^{\tau} = \left\langle \sqrt{\det(\mathbf{g}_{\tau})} \cdot \mathbf{g}_{\tau}^{-1}, \mathbf{d}^{ij} \right\rangle_F$$

$$\mathbf{g}_{\tau} = \begin{pmatrix} \|\mathbf{v}_{12}\|^2 & \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ \langle \mathbf{v}_{13}, \mathbf{v}_{12} \rangle & \|\mathbf{v}_{13}\|^2 \end{pmatrix}$$
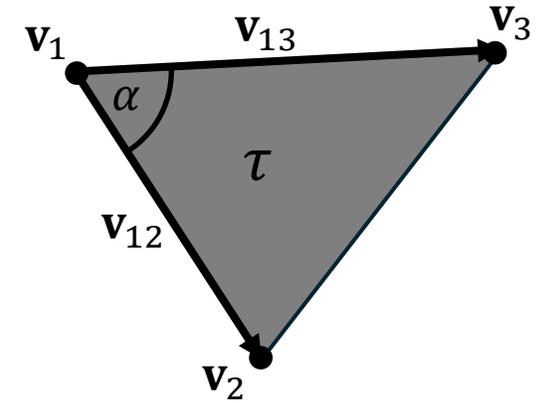
For the mass matrix:

$$\sqrt{\det(\mathbf{g}_{\tau})} = \sqrt{\|\mathbf{v}_{12}\|^2 \cdot \|\mathbf{v}_{13}\|^2 - \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle^2}$$

$$= \|\mathbf{v}_{12}\| \cdot \|\mathbf{v}_{13}\| \cdot \sqrt{1 - \frac{\langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle^2}{\|\mathbf{v}_{12}\|^2 \cdot \|\mathbf{v}_{13}\|^2}}$$

$$= \|\mathbf{v}_{12}\| \cdot \|\mathbf{v}_{13}\| \cdot \sqrt{1 - \left(\frac{\langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle}{\|\mathbf{v}_{12}\| \cdot \|\mathbf{v}_{13}\|} \right)^2}$$

$$= \|\mathbf{v}_{12}\| \cdot \|\mathbf{v}\| \cdot \sqrt{1 - \cos^2(\alpha)}$$

$$= \|\mathbf{v}_{12}\| \cdot \|\mathbf{v}_{13}\| \cdot \sqrt{\sin^2(\alpha)}$$

$$= \|\mathbf{v}_{12}\| \cdot \|\mathbf{v}_{13}\| \cdot \sin(\alpha)$$

$$= 2 \cdot |\tau|$$

The Hat Basis

$$\mathbf{m}_{ij}^{\tau} = \boxed{\sqrt{\det(\mathbf{g}_{\tau})}} \langle\langle \psi_i, \psi_j \rangle\rangle_E$$

$$\mathbf{s}_{ij}^{\tau} = \left\langle \sqrt{\det(\mathbf{g}_{\tau})} \cdot \mathbf{g}_{\tau}^{-1}, \mathbf{d}^{ij} \right\rangle_F$$

$$\mathbf{g}_{\tau} = \begin{pmatrix} \|\mathbf{v}_{12}\|^2 & \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ \langle \mathbf{v}_{13}, \mathbf{v}_{12} \rangle & \|\mathbf{v}_{13}\|^2 \end{pmatrix}$$

For the mass matrix:

$$\sqrt{\det(\mathbf{g}_{\tau})} = 2 \cdot |\tau|$$

⇒ The mass matrix only "sees" area, not angles.

⇒ It is invariant under transformations of the triangle that preserve the area.
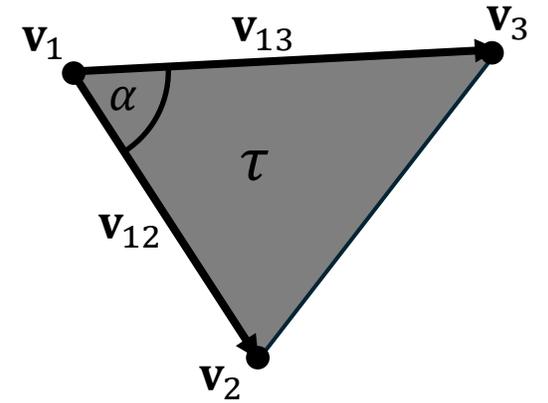
⇒ It is defined by the *authalic* structure of the geometry.

The Hat Basis

$$\mathbf{m}_{ij}^{\tau} = \sqrt{\det(\mathbf{g}_{\tau})} \cdot \langle\langle \psi_i, \psi_j \rangle\rangle_E$$

$$\mathbf{s}_{ij}^{\tau} = \left\langle \boxed{\sqrt{\det(\mathbf{g}_{\tau})} \cdot \mathbf{g}_{\tau}^{-1}}, \mathbf{d}^{ij} \right\rangle_F$$

$$\mathbf{g}_{\tau} = \begin{pmatrix} \|\mathbf{v}_{12}\|^2 & \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ \langle \mathbf{v}_{13}, \mathbf{v}_{12} \rangle & \|\mathbf{v}_{13}\|^2 \end{pmatrix}$$

For the stiffness matrix:

$$\sqrt{\det(\mathbf{g}_{\tau})} \cdot \mathbf{g}_{\tau}^{-1} = \sqrt{\det(\mathbf{g}_{\tau})} \cdot \begin{pmatrix} \|\mathbf{v}_{12}\|^2 & \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle & \|\mathbf{v}_{13}\|^2 \end{pmatrix}^{-1}$$

$$= \sqrt{\det(\mathbf{g}_{\tau})} \cdot \frac{1}{\det(\mathbf{g}_{\tau})} \begin{pmatrix} \|\mathbf{v}_{13}\|^2 & -\langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ -\langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle & \|\mathbf{v}_{12}\| \end{pmatrix}$$

$$= \frac{1}{\sqrt{\det(\mathbf{g}_{\tau})}} \begin{pmatrix} \|\mathbf{v}_{13}\|^2 & -\langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ -\langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle & \|\mathbf{v}_{12}\|^2 \end{pmatrix}$$

$$= \frac{1}{\|\mathbf{v}_{12}\| \cdot \|\mathbf{v}_{13}\| \cdot \sin(\alpha)} \begin{pmatrix} \|\mathbf{v}_{13}\|^2 & -\langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ -\langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle & \|\mathbf{v}_{12}\|^2 \end{pmatrix}$$

$$= \frac{1}{\sin(\alpha)} \begin{pmatrix} \dfrac{\|\mathbf{v}_{13}\|}{\|\mathbf{v}_{12}\|} & -\dfrac{\langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle}{\|\mathbf{v}_{12}\| \cdot \|\mathbf{v}_{13}\|} \\ -\dfrac{\langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle}{\|\mathbf{v}_{12}\| \cdot \|\mathbf{v}_{13}\|} & \dfrac{\|\mathbf{v}_{12}\|}{\|\mathbf{v}_{13}\|} \end{pmatrix}$$

$$= \frac{1}{\sin(\alpha)} \begin{pmatrix} \|\mathbf{v}_{13}\|/\|\mathbf{v}_{12}\| & -\cos(\alpha) \\ -\cos(\alpha) & \|\mathbf{v}_{12}\|/\|\mathbf{v}_{13}\| \end{pmatrix}$$
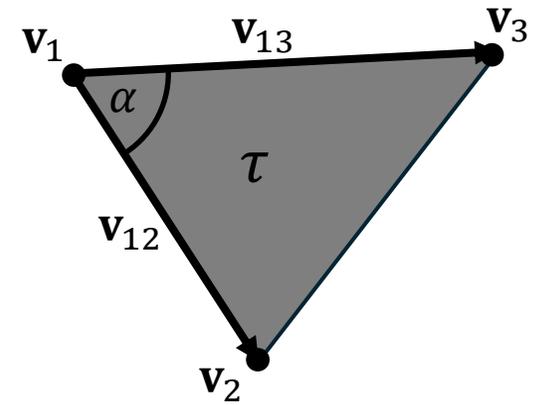
The Hat Basis

$$\mathbf{m}_{ij}^\tau = \sqrt{\det(\mathbf{g}_\tau)} \cdot \langle\langle \psi_i, \psi_j \rangle\rangle_E$$

$$\mathbf{s}_{ij}^\tau = \left\langle \boxed{\sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1}}, \mathbf{d}^{ij} \right\rangle_F$$

$$\mathbf{g}_\tau = \begin{pmatrix} \|\mathbf{v}_{12}\|^2 & \langle \mathbf{v}_{12}, \mathbf{v}_{13} \rangle \\ \langle \mathbf{v}_{13}, \mathbf{v}_{12} \rangle & \|\mathbf{v}_{13}\|^2 \end{pmatrix}$$

For the stiffness matrix:

$$\sqrt{\det(\mathbf{g}_\tau)} \cdot \mathbf{g}_\tau^{-1} = \frac{1}{\sin(\alpha)} \begin{pmatrix} \|\mathbf{v}_{13}\|/\|\mathbf{v}_{12}\| & -\cos(\alpha) \\ -\cos(\alpha) & \|\mathbf{v}_{12}\|/\|\mathbf{v}_{13}\| \end{pmatrix}$$

$\Rightarrow$ The stiffness matrix only "sees" angles, not areas.

$\Rightarrow$ It is invariant under similarity transformations.

$\Rightarrow$ It is defined by the *conformal* structure of the geometry.

Algebraic Interpretation

The linear scalar field basis defines a 3-dimensional space of functions:
$$V = \text{Span}(\psi_1, \psi_2, \psi_3)$$

Mass matrix:

Letting m^τ be the symmetric, positive-definite bilinear map:
$$m^\tau : V \times V \rightarrow \mathbb{R}$$
$$(f, h) \mapsto \langle\langle f, h \rangle\rangle_\tau$$

this makes V into an inner-product space $\{V, m^\tau : V \rightarrow V^*\}$.

$\Rightarrow$ The mass matrix $\mathbf{m}^\tau \in \mathbb{R}^{3 \times 3}$ gives the expression for the map $m^\tau : V \rightarrow V^*$
w.r.t. the bases $\{\psi_1, \psi_2, \psi_3\}$ and $\{\psi_1^*, \psi_2^*, \psi_3^*\}$.

Algebraic Interpretation

The linear scalar field basis defines a 3-dimensional space of functions:
$$V = \text{Span}(\psi_1, \psi_2, \psi_3)$$

Stiffness matrix:

Similarly, we have a symmetric, positive semi-definite bilinear map:
$$s^\tau: V \times V \to \mathbb{R}$$
$$(f, h) \mapsto \langle\langle df, dh \rangle\rangle_\tau$$

$\Rightarrow$ The stiffness matrix $\mathbf{s}^\tau \in \mathbb{R}^{3\times3}$ gives the expression for the map $s^\tau: V \to V^*$ w.r.t. the bases $\{\psi_1, \psi_2, \psi_3\}$ and $\{\psi_1^*, \psi_2^*, \psi_3^*\}$.

Caution

We are now working with two types of inner-product spaces.

$\{T_{\mathbf{p}}\mathbb{T}, g_{\mathbf{p}}: T_{\mathbf{p}}\mathbb{T} \to T_{\mathbf{p}}^*\mathbb{T}\}$:

The space of tangent vectors at the point $\mathbf{p} \in \mathbb{T}$, with inner-product $g_{\mathbf{p}}$ defined by pulling back the Euclidean inner-product using the differential of the parametrization $\Phi_\tau: \mathbb{T} \to \tau$.

$\{V, M: V \to V^*\}$:

The space of piecewise-linear functions on the triangle, with inner-product M defined by integrating functions over the triangle.