Geometry Processing (601.458/658)

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Outline

Recall

Inner-products on Linear Maps

Calculus on Surfaces

Trace

Given a square matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$, the *trace* of the matrix is the sum of diagonal elements:

$$\mathrm{tr}(\mathbf{M}) = \sum_{i=1}^{n} \mathbf{M}_{ii}$$

For $\mathbf{L}, \mathbf{M} \in \mathbb{R}^{n \times n}$ and $\mathbf{L}$ is invertible:

$$\mathrm{tr}(\mathbf{L}^{-1} \cdot \mathbf{M} \cdot \mathbf{L}) = \mathrm{tr}(\mathbf{L})$$

If $\mathbf{M} \in \mathbb{R}^{n \times m}$ and $\mathbf{L} \in \mathbb{R}^{m \times n}$ then:

$$\mathrm{tr}(\mathbf{L} \cdot \mathbf{M}) = \mathrm{tr}(\mathbf{M} \cdot \mathbf{L})$$

Frobenius Inner-Product

Given matrices $\mathbf{L}, \mathbf{M} \in \mathbb{R}^{m\times n}$, the *Frobenius inner-product* is defined as:

$$\begin{aligned}\langle \mathbf{M}, \mathbf{N} \rangle_F &= \operatorname{tr}(\mathbf{M}^\top \cdot \mathbf{N}) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{M}_{ij} \cdot \mathbf{N}_{ij}\end{aligned}$$

This is symmetric, and for $\mathbf{M} \in \mathbb{R}^{m\times n}$ we have:

$$\langle \mathbf{M}, \mathbf{M} \rangle_F = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{M}_{ij}^2$$

so it is positive definite as well.

Recall

Given vector spaces V with basis $\{v_1, \dots, v_n\}$, we can define the canonical dual basis $\{v_1^*, \dots, v_n^*\}$ for V^* satisfying:

$$v_i^*(v_j) = \delta_{ij}$$

Recall

Given vector spaces V and W, bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$, and a linear map $M \in \mathrm{Hom}(V, W)$, the matrix expression for L w.r.t. the bases is:

$$\mathbf{M}_{ij} = w_i^*\left(M(v_j)\right)$$

Given vectors spaces V and W, an invertible linear $L \in \mathrm{Hom}(V, W)$, and a basis $\{v_1, \ldots, v_n\}$ for V, the set $\{L(v_1), \ldots, L(v_n)\} \subset W$ is a basis for W.

Given vectors spaces V and W, an invertible linear $L \in \mathrm{Hom}(V, W)$, and a basis $\{v_1, \ldots, v_n\}$ for V, the set $\{L^{-*}(v_1^*), \ldots, L^{-*}(v_n^*)\} \subset W^*$ is the canonical dual basis for $\{L(v_1), \ldots, L(v_n)\}$.

Recall

The dual of a **symmetric** bilinear form $B: V \to V^*$ is itself:

$$B^* = B$$

Given an inner-product space $\{V, B: V \to V^*\}$, the pulled-back inner-product on V^* is $B^{-1}: V^* \to V^{**} \simeq V$.

Given an n-dimensional vector space V with inner-products, $B_1, B_2: V \to V^*$, if $\omega_1, \omega_2 \in \Lambda^n V^*$ are unit volume forms w.r.t. B_1 and B_2 respectively, the two inner products, then:

$$\omega_2 = \pm\sqrt{\det(B_1^{-1} \circ B_2)} \cdot \omega_1$$

Euclidean Space

Notation:

We denote by $\mathbb{E}^d$ the d-dimensional Euclidean space – this it the set of d-tuples of real values.

As a set, this is the same as $\mathbb{R}^d$, but it does not carry a vector space structure.

Points in $\mathbb{E}^d$ are represented by their *cartesian coordinates*, expressed with respect to the coordinate axes $\{x_1, \ldots, x_d\}$.

Differentiation

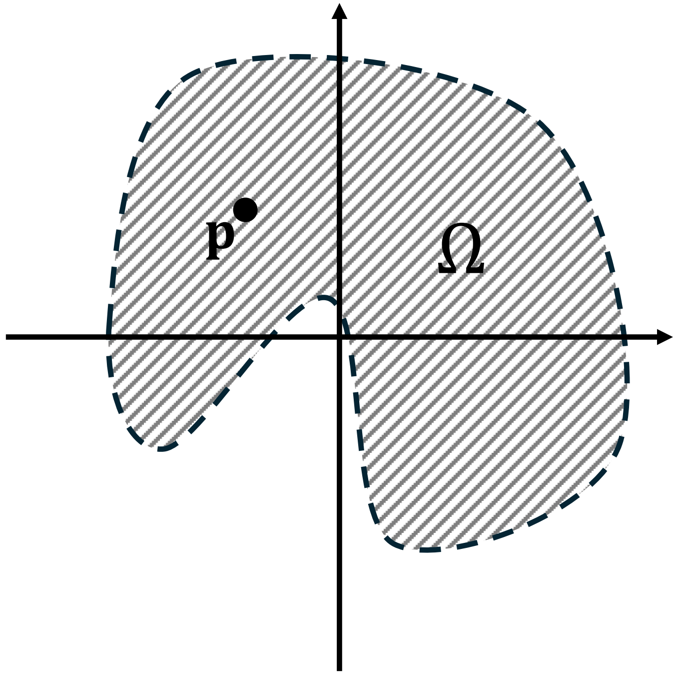
Definition:

Given an open set $\Omega \subset \mathbb{E}^d$, a function $F\colon \Omega \to \mathbb{E}^n$ is *differentiable* at $\mathbf{p} \in \Omega$ if can locally be approximated by a linear map.

We call the linear map the *differential* of F at $\mathbf{p}$, and denote it $dF|_{\mathbf{p}} \in \mathbb{R}^{n\times d}$:

$$F(\mathbf{q}) - F(\mathbf{p}) \approx dF\Big|_{\mathbf{p}} (\mathbf{q} - \mathbf{p})$$

for points $\mathbf{q}$ near $\mathbf{p}$.



Differentiation

We can write the function $F\colon \Omega \to \mathbb{E}^n$ in terms of its components:

$$F(\mathbf{p}) = \begin{pmatrix} F_1(\mathbf{p}) \\ \vdots \\ F_n(\mathbf{p}) \end{pmatrix}$$

$\Rightarrow$ The entries of $dF|_{\mathbf{p}}$ are given in terms of the partial derivatives:

$$\begin{aligned} \left(dF\Big|_{\mathbf{p}}\right)_{ij} &= \frac{\partial F_i}{\partial x_j}\Big|_{\mathbf{p}} \\ &= \lim_{t\to 0} \frac{F_i\left(\mathbf{p} + t\cdot \mathbf{e}_j\right) - F_i(\mathbf{p})}{t} \end{aligned}$$

Integration

Definition:

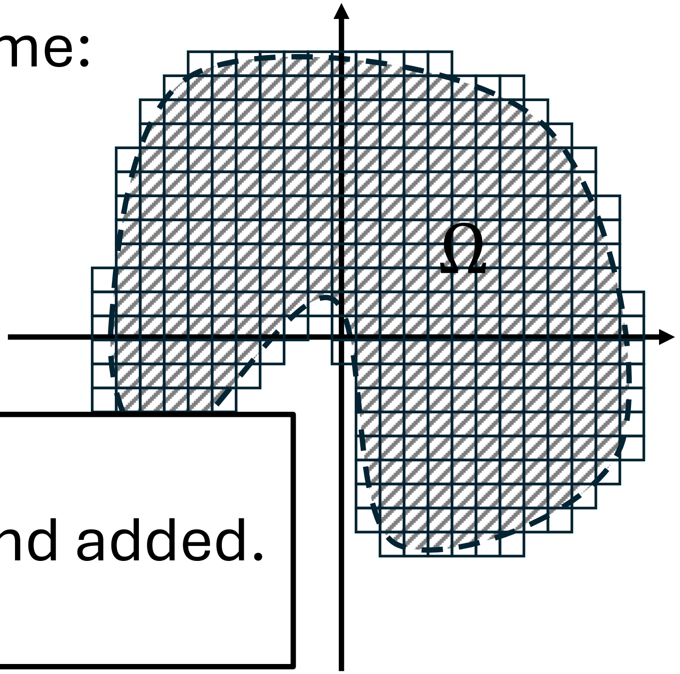
Given an open set $\Omega \subset \mathbb{E}^d$, we can partition the domain into the union of "tiny" cells:

$$\Omega \approx \cup_i C_i$$

The *integral* of $f : \Omega \to \mathbb{R}$ is computed as the sum of the values of the functions within the cells, weighted by the cell's volume:

$$\int_\Omega f \approx \sum_i |C_i| \cdot f(\mathbf{p}_i)$$

with $f(\mathbf{p}_i)$ the value of the function somewhere inside the cell and $|C_i|$ the cell's volume.



Note:
The function range only requires that values can be scaled and added.
$\Rightarrow$ Integration is defined for $f : \Omega \to V$, with V a vector space.

Outline

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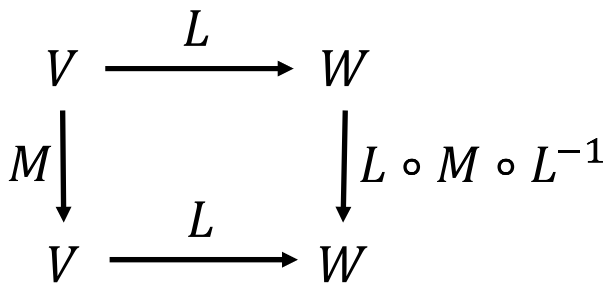
Calculus on Surfaces

Dual Basis



Recall:

Given vector spaces V and W, an invertible $L \in \mathrm{Hom}(V, W)$, and a basis $\{v_1, \ldots, v_n\}$ for V, then $\{L(v_1), \ldots, L(v_n)\}$ is a basis for W with dual basis $\{L^{-*}(v_1^*), \ldots, L^{-*}(v_n^*)\}$.

Given an endomorphism $M \in \mathrm{Hom}(V, V)$ we can associate an endomorphism $L \circ M \circ L^{-1} \in \mathrm{Hom}(W, W)$.

Dual Basis

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ M \big\downarrow & & \big\downarrow L \circ M \circ L^{-1} \\ V & \xrightarrow{L} & W \end{array}$$

Claim:

The matrix expression for M w.r.t. $\{v_1, \dots, v_n\}$ is the same as the matrix expression for $L \circ M \circ L^{-1}$ w.r.t. $\{L(v_1), \dots, L(v_n)\}$.

Proof:

The (i, j)-th coefficient of the matrix expression is:

$$\begin{aligned} [L^{-*}(v_i^*)]\left((L \circ M \circ L^{-1})\left(L(v_j)\right)\right) &= [L^{-*}(v_i^*)]\left(L\left(M(v_j)\right)\right) \\ &= v_i^*\left(L^{-1}\left(L\left(M(v_j)\right)\right)\right) \\ &= v_i^*\left(M(v_j)\right) \\ &= \mathbf{M}_{ij} \end{aligned}$$

Linear Maps and Primal/Dual Pairs

Dual/Primal Pairs as Linear Maps:

Given vector spaces V and W, we can think of a pair of dual and primal vectors, $(w, v^*) \in W \times V^*$, as a linear map in $\mathrm{Hom}(V, W)$:

$$\begin{aligned}(w, v^*)\colon V &\to V\\ v &\mapsto w \cdot v^*(v)\end{aligned}$$

Linear Maps as Dual/Primal Pairs:

Any linear map $L \in \mathrm{Hom}(V, W)$ can be expressed as the linear combination of dual and primal pairs.*

In particular:

If $W = V$, pairs in $V \times V^*$ are associated with endomorphisms in $\mathrm{Hom}(V, V)$.

*Will show an explicit construction later.

Linear Maps and Primal/Dual Pairs

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Linear Maps as Dual/Primal Pairs:

Any linear map $L \in \mathrm{Hom}(V, W)$ can be expressed as the linear combination of dual and primal pairs.*

Observation:

We can think of the flipped pair as $(v^*, w) \in V^* \times W^{**}$.
$\Rightarrow$ This should correspond to an endomorphism in $\mathrm{Hom}(V^*, W^*)$.

Linear Maps and Primal/Dual Pairs

$$(w, v^*)(v) \equiv w \cdot v^*(v)$$

Claim:

The flipped pair $(v^*, w) \in V^* \times W$ corresponds to the linear map dual of the linear map:

$$(w, v^*)^* = (v^*, w)$$

Proof:

For any $w^* \in W^*$, we have:

$$\begin{aligned}(w, v^*)^*(w^*) &= w^* \circ (w, v^*)\\ &= w^*(w) \cdot v^*\\ &= w(w^*) \cdot v^*\\ &= v^* \cdot w(w^*)\\ &= (v^*, w)(w^*)\end{aligned}$$

Trace of an Endomorphism

Definition:

For a vector space V, we define the *trace* of a pair of dual/primal vectors as:

$$\begin{aligned} \mathrm{tr}\colon V \times V^* &\to \mathbb{R} \\ (v, v^*) &\mapsto v^*(v) \end{aligned}$$

Observation:

The map is bilinear.

By the association of pairs in $V \times V^*$ with endomorphisms in $\mathrm{Hom}(V,V)$, this extends to a linear map:*

$$\mathrm{tr}\colon \mathrm{Hom}(V,V) \to \mathbb{R}$$

*E.g. using the universal property of tensor product spaces, from category theory.

Trace of an Endomorphism

Property:

Given a vector space V and pairs $\{(v_i, \alpha_i)\} \subset V \times V^*$, we can associate a linear map to the pairs:

$$\sum_i (v_i, \alpha_i) \equiv L \in \mathrm{Hom}(V, V)$$

The dual of this map is:

$$\begin{aligned} L^* &= \left(\sum_i (v_i, \alpha_i)\right)^* \\ &= \sum_i (v_i, \alpha_i)^* \\ &= \sum_i (\alpha_i, v_i) \end{aligned}$$

Trace of an Endomorphism

Property:

Given a vector space V and pairs $\{(v_i, \alpha_i)\} \subset V \times V^*$, we can associate a linear map to the pairs:

$$\sum_i (v_i, \alpha_i) \equiv L \in \mathrm{Hom}(V, V)$$

$$L^* = \sum_i (\alpha_i, v_i)$$

Taking the trace, this gives:

$$\begin{aligned} \mathrm{tr}(L^*) &= \sum_i v_i(\alpha_i) \\ &= \sum_i \alpha_i(v_i) \\ &= \mathrm{tr}(L) \end{aligned}$$

Trace of an Endomorphism

Since every endomorphism $L \in \mathrm{Hom}(V, V)$ can be expressed as the linear combination of the dual/primal pairs, it follows that for any $L \in \mathrm{Hom}(V, V)$:

$$\mathrm{tr}(L^*) = \mathrm{tr}(L)$$

Trace of an Endomorphism

Property:

Given vector spaces V and W, and given pairs $\{(w_i,\alpha_i)\}\subset W\times V^*$ and $\{(v_j,\beta_j)\}\subset V\times W^*$, we can associate linear maps to the two sets:

$$\sum_i (w_i,\alpha_i)\equiv L\in\mathrm{Hom}(V,W)\qquad\qquad \sum_j (v_j,\beta_j)\equiv M\in\mathrm{Hom}(W,V)$$

The composition of these maps is:

$$\begin{aligned}L\circ M &= \sum_i\sum_j (w_i,\alpha_i)\circ(v_j,\beta_j)\\ &= \sum_i\sum_j \alpha_i(v_j)\cdot(w_i,\beta_j)\end{aligned}$$

Similarly:

$$M\circ L=\sum_i\sum_j \beta_j(w_i)\cdot(v_j,\alpha_i)$$

Trace of an Endomorphism

$$L \circ M = \sum_i \sum_j \alpha_i(v_j) \cdot (w_i, \beta_j) \qquad M \circ L = \sum_j \sum_i \beta_j(w_i) \cdot (v_j, \alpha_i)$$

Taking the trace we get:

$$\begin{aligned}\operatorname{tr}(L \circ M) &= \operatorname{tr}\left(\sum_i \sum_j \alpha_i(v_j) \cdot (w_i, \beta_j)\right) \\ &= \sum_i \sum_j \alpha_i(v_j) \cdot \operatorname{tr}(w_i, \beta_j) \\ &= \sum_i \sum_j \alpha_i(v_j) \cdot \beta_j(w_i)\end{aligned}$$

Similarly, we get:

$$\operatorname{tr}(M \circ L) = \sum_j \sum_i \beta_j(w_i) \cdot \alpha_i(v_j)$$

$\Rightarrow \operatorname{tr}(L \circ M) = \operatorname{tr}(M \circ L)$

Trace of an Endomorphism

Since every linear map $L \in \mathrm{Hom}(V,W)$ can be expressed as the linear combination of the dual/primal pairs, it follows that for any $L \in \mathrm{Hom}(V,W)$ and $M \in \mathrm{Hom}(W,V)$ the trace of the composition is order-independent:

$$\mathrm{tr}(L \circ M) = \mathrm{tr}(M \circ L)$$

Corollary:

Given an endomorphism $L \in \mathrm{Hom}(V,V)$ and given an **invertible** linear map $M \in \mathrm{Hom}(V,W)$:

$$\begin{aligned} \mathrm{tr}(L) &= \mathrm{tr}(M^{-1} \circ M \circ L) \\ &= \mathrm{tr}(M \circ L \circ M^{-1}) \end{aligned}$$

Endomorphisms and Primal/Dual Pairs

Claim:

Given a vector space V and a basis $\{v_1,\ldots,v_n\}$, an endomorphism $L \in \mathrm{Hom}(V,V)$ can be expressed as the sum of pairs:

$$L = \sum_{k,l=1}^{n} (v_k, v_l^*) \cdot \mathbf{L}_{kl}$$

where $\mathbf{L} \in \mathbb{R}^{n\times n}$ is the matrix expression for L w.r.t. the basis.

Proof:

The (i,j)-the coefficient of the matrix expression w.r.t $\{v_1,\ldots,v_n\}$ is:

$$\begin{aligned} v_i^*\left(\left(\sum_{k,l=1}^{n} (v_k, v_l^*) \cdot \mathbf{L}_{kl}\right)(v_j)\right) &= v_i^*\left(\sum_{k,l=1}^{n} v_k \cdot v_l^*(v_j) \cdot \mathbf{L}_{kl}\right) \\ &= v_i^*\left(\sum_{k=1}^{n} v_k \cdot \mathbf{L}_{kj}\right) \\ &= \sum_{k=1}^{n} \mathbf{L}_{kj} \cdot v_i^*(v_k) \\ &= \mathbf{L}_{ij} \end{aligned}$$

Trace of an Endomorphism

In particular, given a basis $\{v_1, \ldots, v_n\}$ and the expansion:

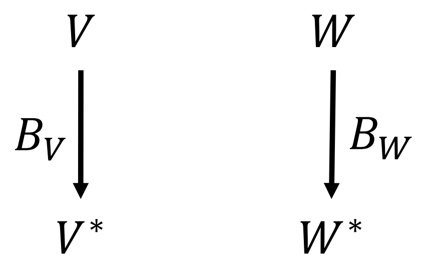
$$L = \sum_{i,j=1}^{n} \mathbf{L}_{ij} \cdot \left(v_i, v_j^*\right)$$

we have, by linearity:

$$\begin{aligned}
\operatorname{tr}(L) &= \operatorname{tr}\left(\sum_{i,j=1}^{n} \mathbf{L}_{ij} \cdot (v_i, v_j^*)\right) \\
&= \sum_{i,j=1}^{n} \mathbf{L}_{ij} \cdot \operatorname{tr}(v_i, v_j^*) \\
&= \sum_{i,j=1}^{n} \mathbf{L}_{ij} \cdot v_j^*(v_i) \\
&= \sum_{i,=1}^{n} \mathbf{L}_{ii}
\end{aligned}$$

… Which matches our definition of *trace* for matrices.

Inner-Products on Linear Maps



Given inner-product spaces $\{V, B_V: V \to V^*\}$ and $\{W, B_W: W \to W^*\}$, we have notions of "size" for both V and W.

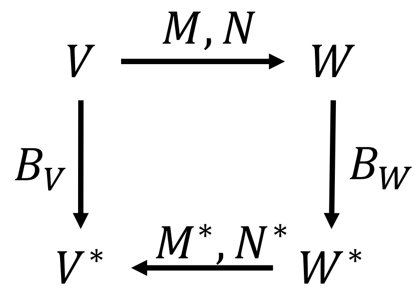
Question:

How do we use those to construct a notion of "size" (i.e. an inner-product) on the space of linear maps $\mathrm{Hom}(V, W)$.

Approach:

Transform the pair of linear maps into an endomorphism and take the trace.

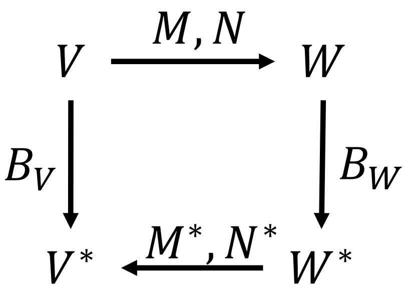
Inner-Products on Linear Maps



To define a bilinear form on the space $\mathrm{Hom}(V, W)$:

1. Given linear maps $M, N \in \mathrm{Hom}(V, W)$, define the endomorphism:
$$B_V^{-1} \circ M^* \circ B_W \circ N \in \mathrm{Hom}(V, V)$$
2. Define the bilinear form to be:
$$B_{\mathrm{Hom}(V,W)}(M, N) = \mathrm{tr}(B_V^{-1} \circ M^* \circ B_W \circ N)$$

Inner-Products on Linear Maps



Claim:

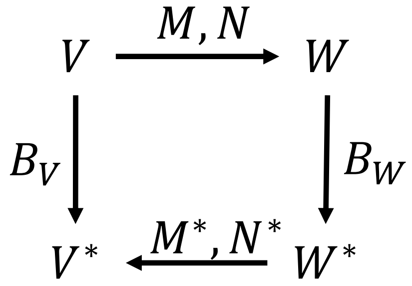
The bilinear form is symmetric:

$$\mathrm{tr}(M \circ N) = \mathrm{tr}(N \circ M)$$
$$\mathrm{tr}(N) = \mathrm{tr}(N^*)$$
$$(M \circ N)^* = N^* \circ M^*$$

Proof:

$$\begin{aligned} B_{\mathrm{Hom}(V,W)}(M,N) &= \mathrm{tr}(B_V^{-1} \circ M^* \circ B_W \circ N) \\ &= \mathrm{tr}(M^* \circ B_W \circ N \circ B_V^{-1}) \\ &= \mathrm{tr}\left((M^* \circ B_W \circ N \circ B_V^{-1})^*\right) \\ &= \mathrm{tr}\left((B_V^{-1})^* \circ N^* \circ B_W^* \circ (M^*)^*\right) \\ &= \mathrm{tr}(B_V^{-1} \circ N^* \circ B_W \circ M) \\ &= B_{\mathrm{Hom}(V,W)}(N,M) \end{aligned}$$

Inner-Products on Linear Maps



Claim:

The bilinear form is symmetric positive definite (i.e. an inner-product).

Harder to show this without choosing a basis.

Proof:

Given orthogonal bases $\{v_1, \ldots, v_n\}$ for V and $\{w_1, \ldots, w_m\}$ for W, the matrix expression for $B_V^{-1} \circ M^* \circ B_W \circ N$ is:

$$\mathbf{B}_V^{-1} \cdot \mathbf{M}^\top \cdot \mathbf{B}_W \cdot \mathbf{N} = \mathbf{M}^\top \cdot \mathbf{N}$$

In particular, for $M = N$, with $M \neq 0$, we get:

$$\begin{aligned} B_{\mathrm{Hom}(V,W)}(M, M) &= \mathrm{tr}(\mathbf{M}^\top \cdot \mathbf{M}) \\ &= \langle \mathbf{M}, \mathbf{M} \rangle_F \\ &> 0 \end{aligned}$$

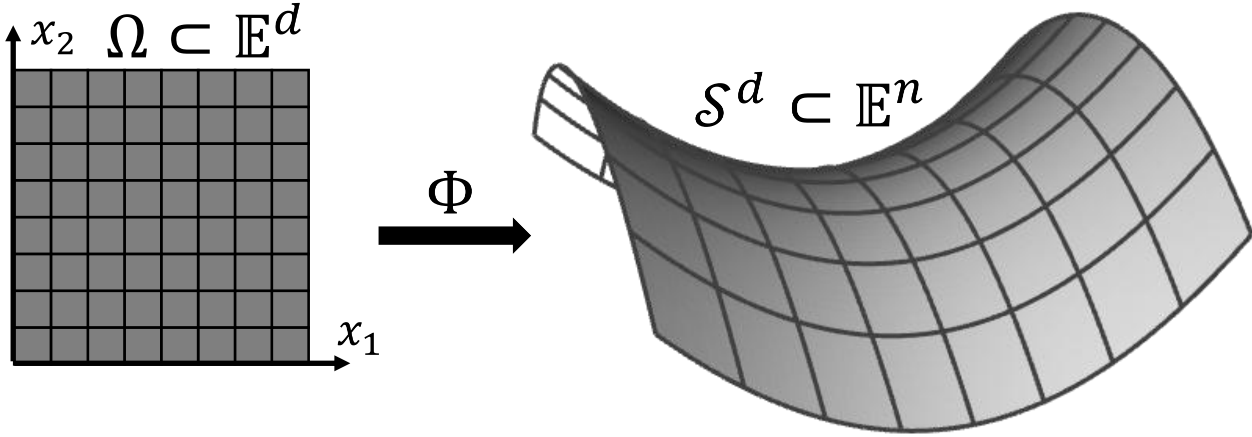
Outline

Recall

Inner-products on Linear Maps

Calculus on Surfaces

Surfaces



Goal:

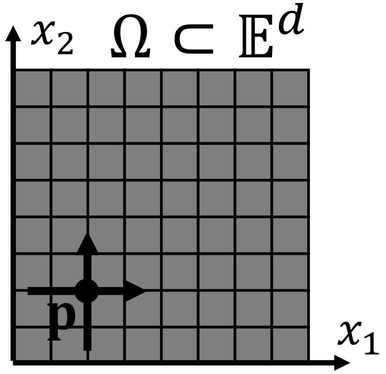
We want to differentiate and integrate functions defined on a d-dimensional surface $\mathcal{S}^d$ embedded in n-dimensional Euclidean space.

Approach:

Given a parameterization of $\mathcal{S}$ over a Euclidean domain Ω, we would like to:

- Perform the computation over the Euclidean domain,
- With respect to the geometry of the surface

Tangent Spaces



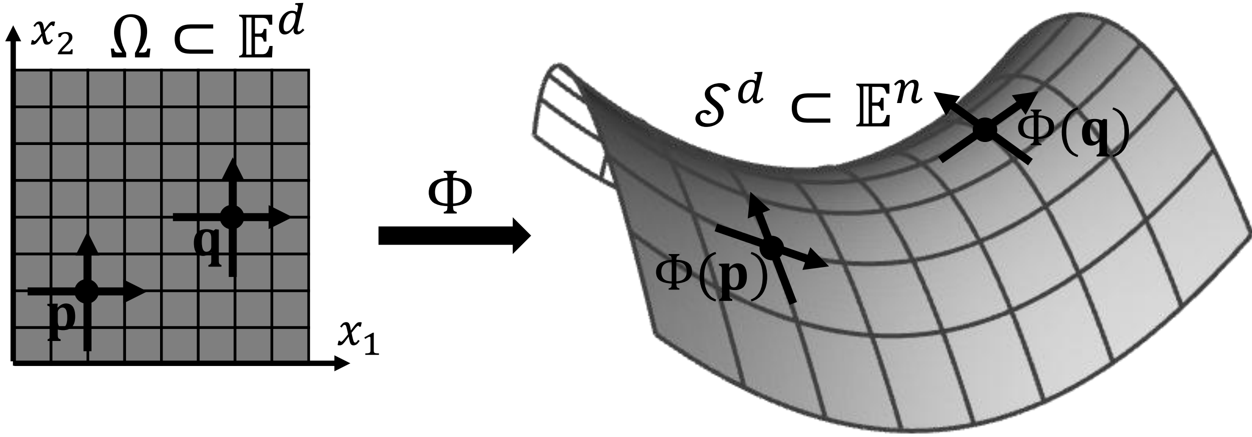
Definition:

For an open set $\Omega \subset \mathbb{E}^d$, we associate to each point $\mathbf{p} \in \Omega$ a *tangent space*, denoted $T_{\mathbf{p}}\Omega$, describing the "offsets from $\mathbf{p}$ to its neighbors".

Property:

The tangent space at $\mathbf{p}$ is a vector space and can be identified with $\mathbb{R}^d$.

Tangent Spaces



Definition:

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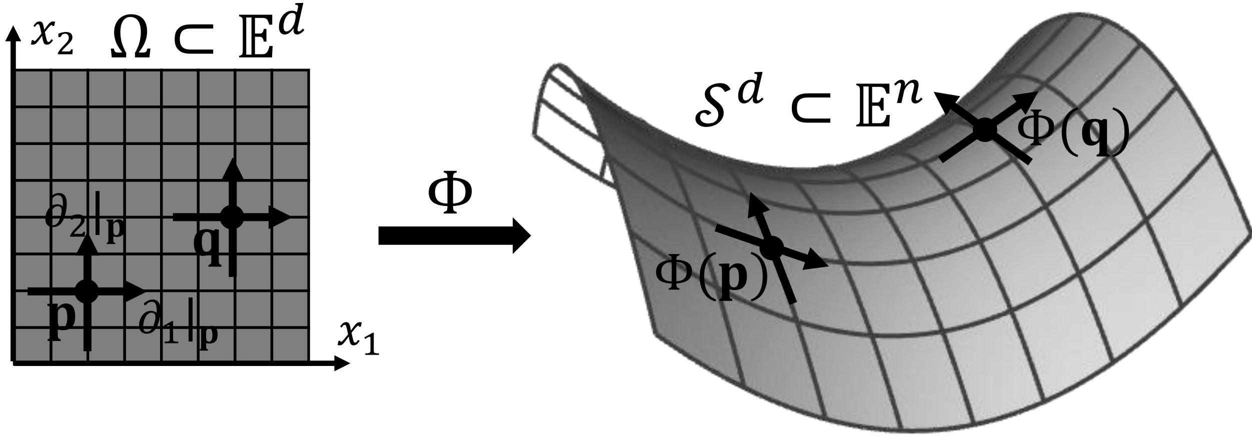
Picture:

In the context of an embedded surface, we think of the tangent space as the best fit (d-dimensional) plane to the surface at the point $\Phi(\mathbf{p})$.

Though the tangent spaces "look the same" on Ω, they are different when viewed on the embedding.

Tangent Spaces



Definition:

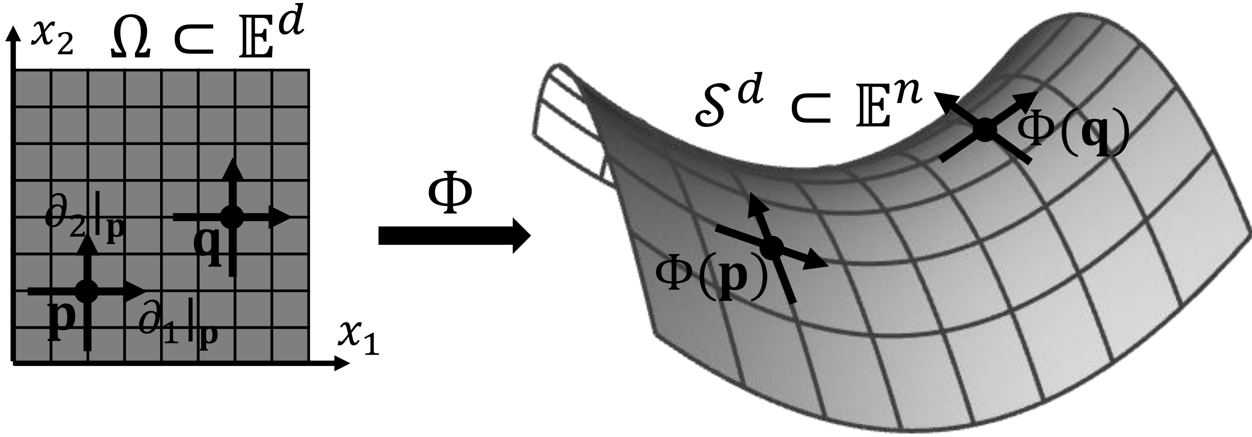
For an open set $\Omega \subset \mathbb{E}^d$, we associate to each point $\mathbf{p} \in \Omega$ a *tangent space*, denoted $T_\mathbf{p}\Omega$, describing the "offsets from $\mathbf{p}$ to its neighbors".

The basis for $T_\mathbf{p}\Omega$ associated to the cartesian axis is denoted $\{\partial_1|_\mathbf{p}, \ldots, \partial_d|_\mathbf{p}\}$.

The dual of $T_\mathbf{p}\Omega$, denoted $T_\mathbf{p}^*\Omega$, is called the *cotangent space*.

The canonical dual basis is denoted $\{dx_1|_\mathbf{p}, \ldots, dx_d|_\mathbf{p}\}$.

Tangent Spaces



Definition:

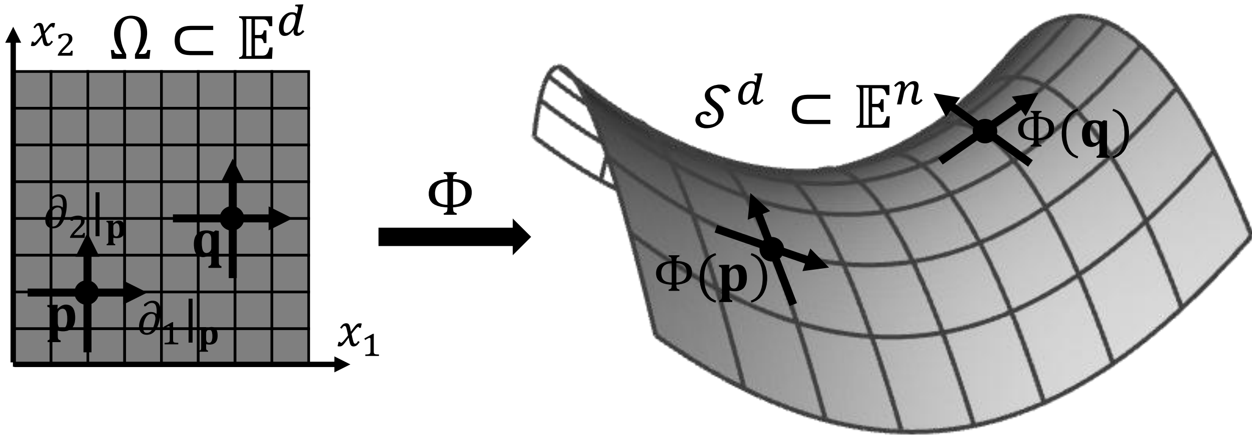
For an open set $\Omega \subset \mathbb{E}^d$, we associate to each point $\mathbf{p} \in \Omega$ a *tangent space*, denoted $T_{\mathbf{p}}\Omega$, describing the "offsets from $\mathbf{p}$ to its neighbors".

The *Euclidean inner-product* is a bilinear form at each point, $E_{\mathbf{p}}: T_p\Omega \to T_{\mathbf{p}}^*\Omega$, with respect to which the basis $\{\partial_1|_{\mathbf{p}}, \dots, \partial_d|_{\mathbf{p}}\}$ is orthogonal.

⇒ The matrix expression for $E_{\mathbf{p}}$ w.r.t. the basis $\{\partial_1|_{\mathbf{p}}, \dots, \partial_d|_{\mathbf{p}}\}$ is the identity:

$$\mathbf{E}_{\mathbf{p}} = \mathbf{Id} \in \mathbb{R}^{d\times d}$$

Tangent Spaces



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For an open set $\Omega \subset \mathbb{E}^d$, we associate to each point $\mathbf{p} \in \Omega$ a *tangent space*, denoted $T_{\mathbf{p}}\Omega$, describing the "offsets from $\mathbf{p}$ to its neighbors".

Definition:

The *Euclidean volume form* at a point $\mathbf{p} \in \Omega$, denoted $\omega_E|_{\mathbf{p}}$ is:

$$\omega_E\Big|_{\mathbf{p}} \equiv dx_1\Big|_{\mathbf{p}} \wedge \cdots \wedge dx_d\Big|_{\mathbf{p}} : \left(T_{\mathbf{p}}\Omega\right)^d \to \mathbb{R}$$

By construction:

$$\omega_E\Big|_{\mathbf{p}}\left(\partial_1\Big|_{\mathbf{p}}, \dots, \partial_d\Big|_{\mathbf{p}}\right) = 1$$

Since $\left\{\partial_1|_{\mathbf{p}}, \dots, \partial_d|_{\mathbf{p}}\right\}$ is $E_{\mathbf{p}}$-orthogonal, this is a unit volume form.

Differentials

Recall:

Given a map $F: \Omega \to \mathbb{E}^n$, we call the linear map the *differential* of F at $\mathbf{p}$, and denote it $dF|_{\mathbf{p}}$, with:

$$F(\mathbf{q}) - F(\mathbf{p}) \approx dF\Big|_{\mathbf{p}} (\mathbf{q} - \mathbf{p})$$

for points $\mathbf{q}$ near $\mathbf{p}$.

Thinking of:

$\mathbf{q} - \mathbf{p}$ as "the offset from $\mathbf{p}$ to $\mathbf{q}$", and

$F(\mathbf{q}) - F(\mathbf{p})$ as "the offset from $F(\mathbf{p})$ to $F(\mathbf{q})$"

$\Rightarrow$ The differential at $\mathbf{p}$ is a linear map from the tangent space $T_{\mathbf{p}}\Omega$ to the tangent space $T_{F(\mathbf{p})}\mathbb{E}^n$ describing how local directions are transformed by F.

Differentials

Real-valued Functions:

For the case $f: \Omega \to \mathbb{E}$, the differential is a linear map:

$$df\Big|_{\mathbf{p}}: T_{\mathbf{p}}\Omega \to T_{f(\mathbf{p})}\mathbb{E} \simeq \mathbb{R}$$

To get the change of the function in direction $v \in T_{\mathbf{p}}\Omega$, we evaluate the differential along direction v:

$$df\Big|_{\mathbf{p}}(v) = \frac{\partial f}{\partial v}\Big|_{\mathbf{p}}$$

For $v = \mathbf{a}_1 \cdot \partial_1|_{\mathbf{p}} + \cdots + \mathbf{a}_d \cdot \partial_d|_{\mathbf{p}}$, we have:

$$df\Big|_{\mathbf{p}}(v) = \mathbf{a}_1 \cdot \frac{\partial f}{\partial x_1} + \cdots + \mathbf{a}_d \cdot \frac{\partial f}{\partial x_d}$$

Differentials

Real-valued Functions:

For $v = \mathbf{a}_1 \cdot \partial_1|_{\mathbf{p}} + \cdots + \mathbf{a}_d \cdot \partial_d|_{\mathbf{p}}$, we have:

$$df\Big|_{\mathbf{p}}(v) = \mathbf{a}_1 \cdot \frac{\partial f}{\partial x_1} + \cdots + \mathbf{a}_d \cdot \frac{\partial f}{\partial x_d}$$

$$\Downarrow$$

$$df\Big|_{\mathbf{p}} = \frac{\partial f}{\partial x_1} \cdot dx_1 + \cdots + \frac{\partial f}{\partial x_d} \cdot dx_n \in T_{\mathbf{p}}^*\Omega$$

The differential describes how the function changes as we go in different directions.

It is **not** a tangent vector describing the “direction of steepest change”.

Gradients

Definition:

If we have an inner-product on the tangent space, $g_{\mathbf{p}}: T_{\mathbf{p}}\Omega \to T_{\mathbf{p}}^*\Omega$, we could use its inverse to identify the differential $df|_{\mathbf{p}}$ with a tangent vector*:

$$\nabla_{g_{\mathbf{p}}} f \Big|_{\mathbf{p}} \equiv g_{\mathbf{p}}^{-1}\left(df \Big|_{\mathbf{p}}\right).$$

We call the associated tangent vector the *gradient* of f at $\mathbf{p}$.

The subscript in "$\nabla_{g_{\mathbf{p}}}$" indicates that the associated tangent vector depends on the choice of inner-product.

The gradient **is** the tangent vector describing the "direction of steepest change".

*We denote inner-products on tangent spaces using lower case g.

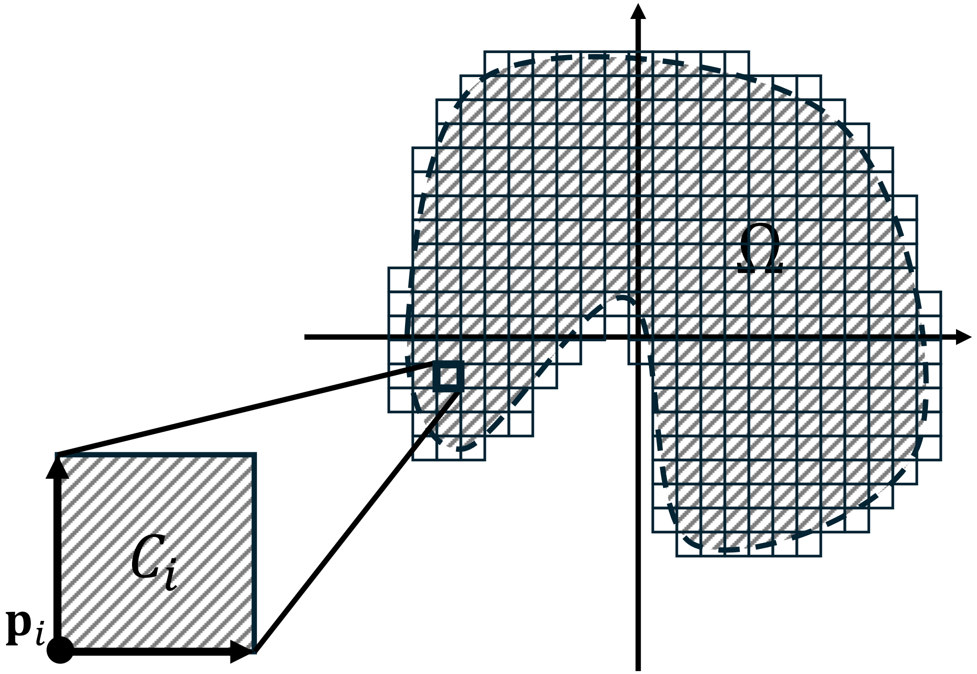
Integration (Redux)

Recall:

Given an open set $\Omega \subset \mathbb{E}^d$ and $f: \Omega \to \mathbb{R}$, we integrate f by partitioning the domain into cells and taking the volume-weighted sum of the values of f:

$$\int_\Omega f \approx \sum_i |C_i| \cdot f(\mathbf{p}_i)$$

Zooming in on a summand:



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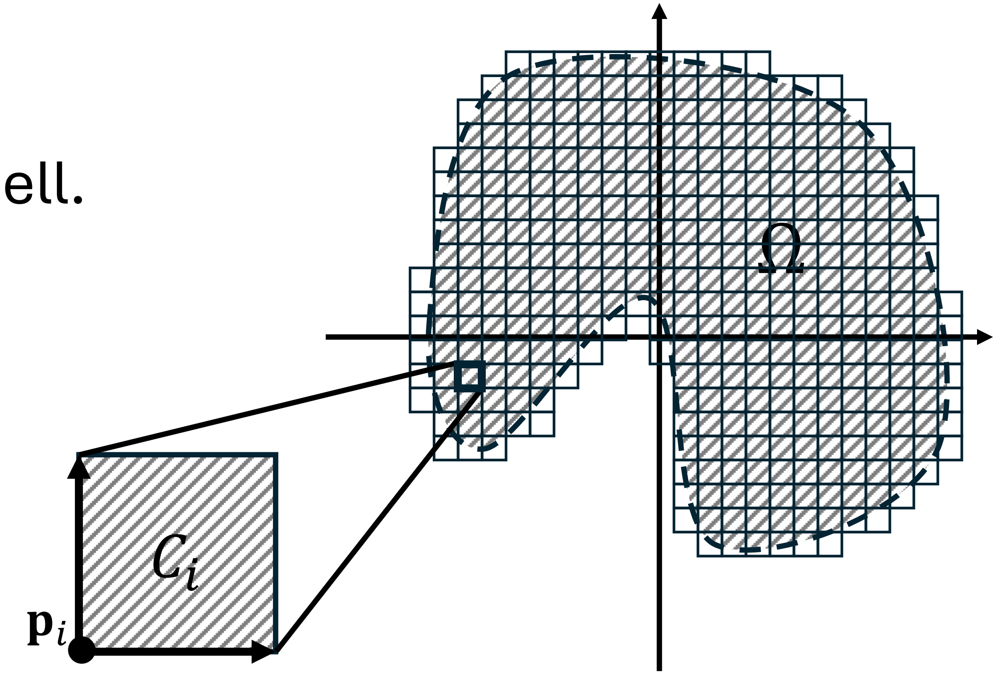

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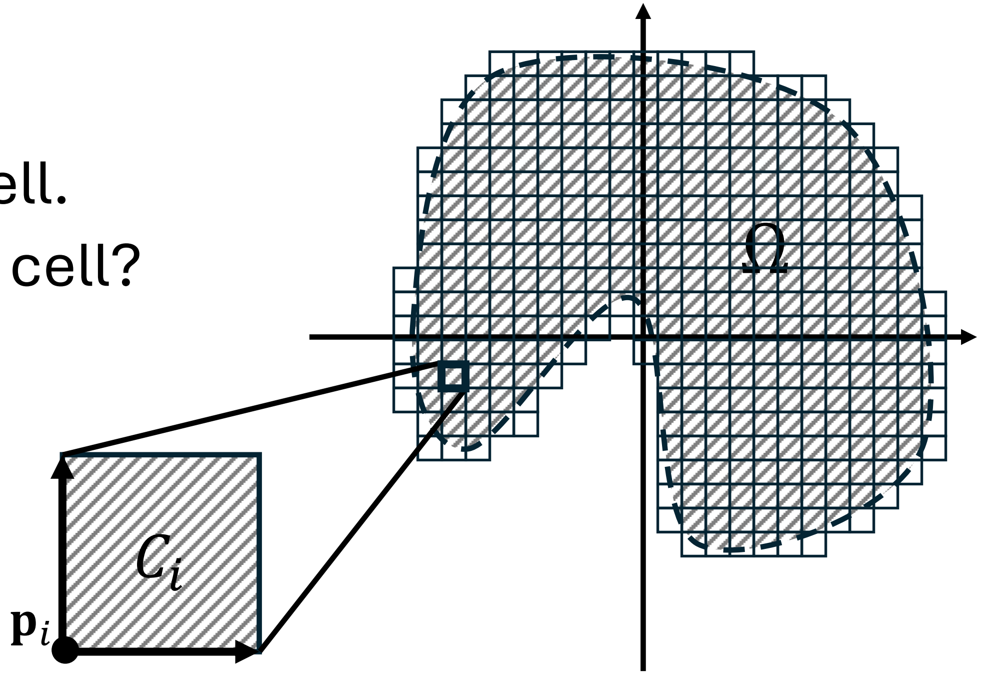
✓We can evaluate f at the corner of the cell.

✗ How should we define the volume of the cell?



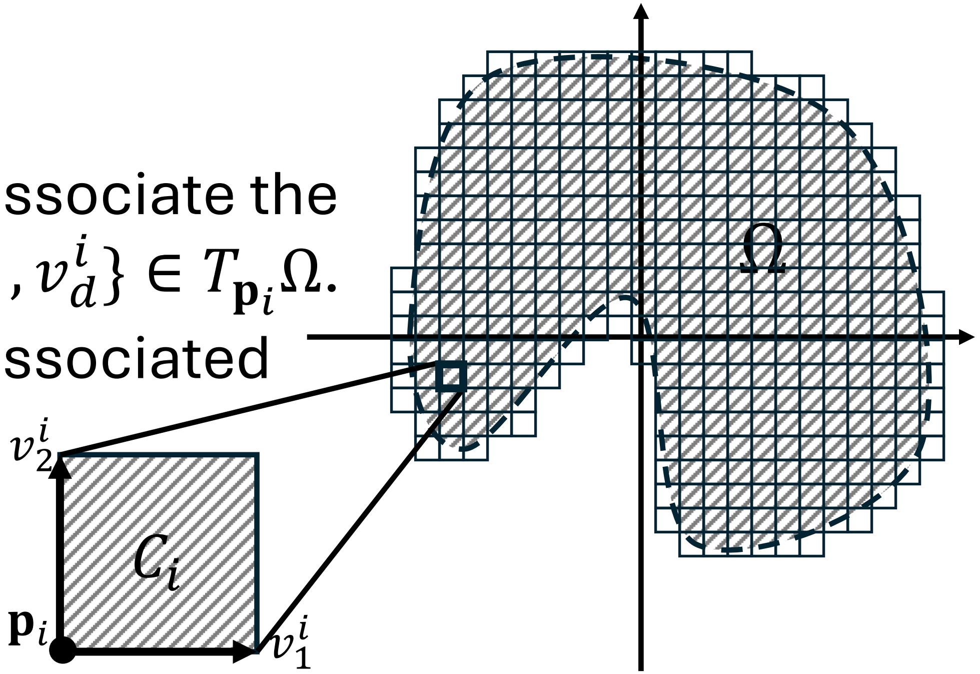
Integration (Redux)

$$\int_\Omega f \approx \sum_i \boxed{|C_i|} \cdot f(\mathbf{p}_i)$$

How should we define the volume of a cell?

Equivalently:

Given a parallelepiped cell in $\mathbb{E}^d$, we can associate the axes of the cell with tangent vectors $\{v_1^i, \ldots, v_d^i\} \in T_{\mathbf{p}_i}\Omega$. What is the volume of the parallelepiped associated with $\{v_1^i, \ldots, v_d^i\}$?



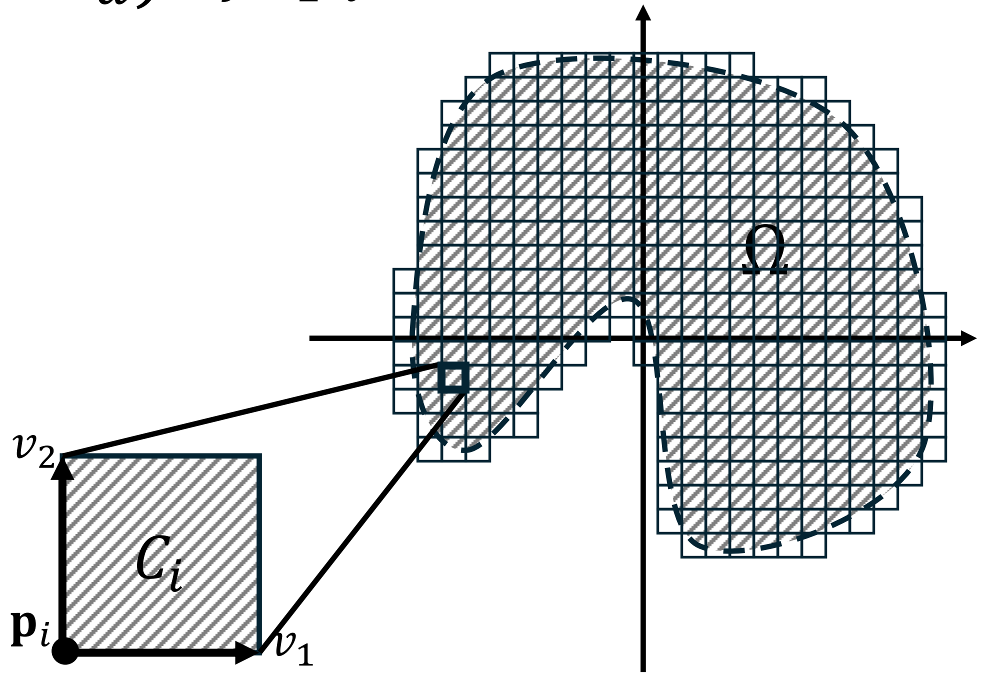
$\Rightarrow$ Need a volume form.

Integration (Redux)

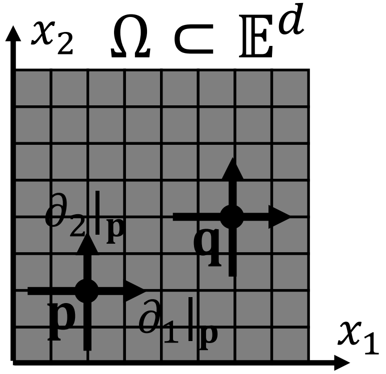
If we are given a function $f: \Omega \to \mathbb{R}$, and a function ω assigning a volume form to $T_{\mathbf{p}}\Omega$ for every point $\mathbf{p} \in \Omega$, we can define the integral of f with respect to ω as:

$$\int_\Omega f \cdot \omega \approx \sum_i \omega_{\mathbf{p}_i}\left(v_1^i, \dots, v_d^i\right) \cdot f(\mathbf{p}_i)$$

where $\{v_1^i, \dots, v_d^i\}$ are the "sides" of C_i.



Inner-Product



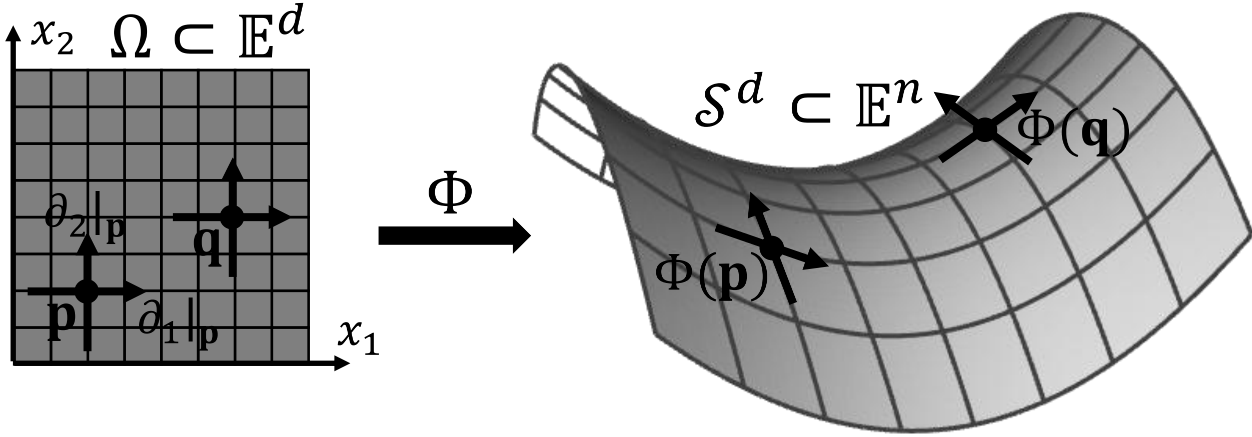
Defining the gradient and evaluating an integral are expressed in terms of measurements (lengths, angles, and volumes).

An inner-product, $g_{\mathbf{p}}: T_{\mathbf{p}}\Omega \to T_{\mathbf{p}}^*\Omega$, addresses both issues:

- Can apply the inverse $g_{\mathbf{p}}^{-1}: T_{\mathbf{p}}^*\Omega \to T_{\mathbf{p}}\Omega$ to turn differentials into gradients.
- Can use the inner-product to define a unit volume form

A simple choice is the Euclidean inner-product.

Inner-Product



Defining the gradient and evaluating an integral are expressed in terms of measurements (lengths, angles, and volumes).

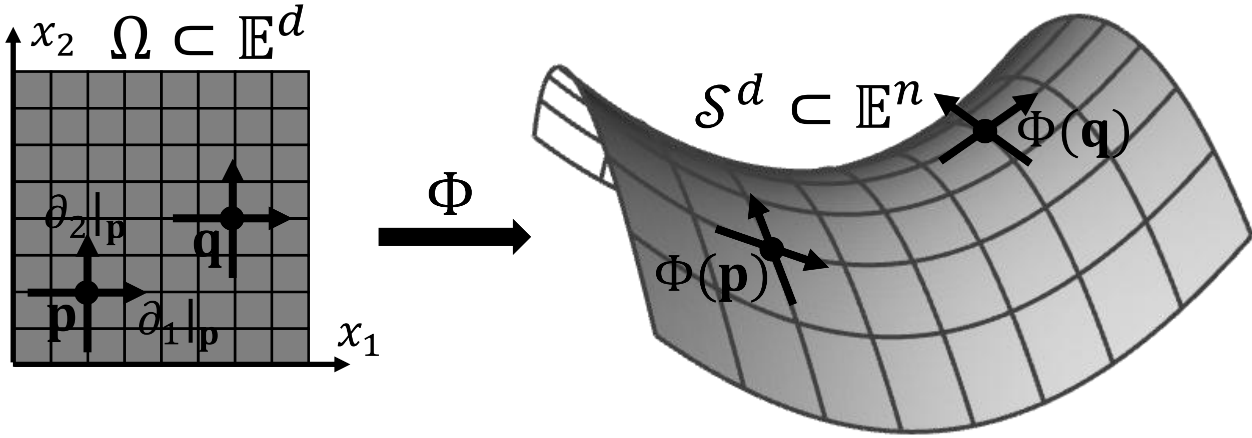
An inner-product, $g_{\mathbf{p}}: T_{\mathbf{p}}\Omega \to T^*_{\mathbf{p}}\Omega$, addresses both issues:

- Can apply the inverse $g_{\mathbf{p}}^{-1}: T^*_{\mathbf{p}}\Omega \to T_{\mathbf{p}}\Omega$ to turn differentials into gradients.
- Can use the inner-product to define a unit volume form

A simple choice is the Euclidean inner-product.

- ✗ This describes the geometry of the parameterization domain.
  We want the geometry of the embedded surface.

Inner-Product



Goal:

Define an inner-product on $T_{\mathbf{p}}\Omega$ that captures the geometry of the embedded surface.

Recall:

The differential $d\Phi|_{\mathbf{p}}: T_{\mathbf{p}}\Omega \to T_{\Phi(\mathbf{p})}\mathbb{E}^n$ maps tangent vectors in the parameterization domain to tangent vectors in the embedded space.

In the embedding space, we have a Euclidean inner-product.

$\Rightarrow$ Pull back the Euclidean inner-product from $T_{\Phi(\mathbf{p})}\mathbb{E}^n$:

$$g_{\mathbf{p}} \equiv d\Phi\Big|_{\mathbf{p}}^* \circ E_{\Phi(\mathbf{p})} \circ d\Phi\Big|_{\mathbf{p}}$$

Inner-Product: Matrix Representation

Pulling back the inner-product:

$$g_{\mathbf{p}} \equiv d\Phi\Big|_{\mathbf{p}}^{*} \circ E_{\Phi(\mathbf{p})} \circ d\Phi\Big|_{\mathbf{p}}$$

With respect to the basis $\{\partial_1|_{\mathbf{p}}, \ldots, \partial_d|_{\mathbf{p}}\}$ for $T_{\mathbf{p}}\Omega$ we have:

$$\begin{aligned}\mathbf{g}_{\mathbf{p}} &\equiv \mathbf{d}\Phi\Big|_{\mathbf{p}}^{\top} \cdot \mathbf{E}_{\Phi(\mathbf{p})} \cdot \mathbf{d}\Phi\Big|_{\mathbf{p}} \\ &= \mathbf{d}\Phi\Big|_{\mathbf{p}}^{\top} \cdot \mathbf{d}\Phi\Big|_{\mathbf{p}}\end{aligned}$$

Notationally we're distinguishing between:

$d\Phi|_{\mathbf{p}}$ -- the differential as a linear map from $T_{\mathbf{p}}\Omega$ to $T_{\Phi(\mathbf{p})}\mathbb{E}^n$

$\mathbf{d}\Phi|_{\mathbf{p}}$ -- the differential as a matrix in $\mathbb{R}^{n\times d}$, expressed with respect to the cartesian bases.

Inner-Product: Matrix Representation

Pulling back the inner-product:

$$g_{\mathbf{p}} \equiv d\Phi\Big|_{\mathbf{p}}^{*} \circ E_{\Phi(\mathbf{p})} \circ d\Phi\Big|_{\mathbf{p}}$$

Recall:

Given a vector space V with inner-products, $B_1, B_2 : V \to V^*$, the unit volume forms ω_1 and ω_2 are related by:

$$\omega_2 = \pm\sqrt{\det\left(B_1^{-1} \circ B_2\right)} \cdot \omega_1$$

$\Rightarrow$ Taking $V = T_{\mathbf{p}}\Omega$, setting $B_1 = E_{\mathbf{p}}$ (the Euclidean inner-product on $T_{\mathbf{p}}\Omega$) and $B_2 = g_{\mathbf{p}}$ (the pulled back inner-product on $T_{\mathbf{p}}\Omega$), the unit volume form defined by the pulled back inner-product is:

$$\omega_g\Big|_{\mathbf{p}} = \pm\sqrt{\det\left(E_{\mathbf{p}}^{-1} \circ \left(d\Phi\Big|_{\mathbf{p}}^{*} \circ E_{\Phi(\mathbf{p})} \circ d\Phi\Big|_{\mathbf{p}}\right)\right)} \cdot \omega_E\Big|_{\mathbf{p}}$$

Inner-Product: Matrix Representation

$$\omega_g\Big|_{\mathbf{p}} = \pm\sqrt{\det\left(E_{\mathbf{p}}^{-1}\circ\left(d\Phi\Big|_{\mathbf{p}}^{*}\circ E_{\Phi(\mathbf{p})}\circ d\Phi\Big|_{\mathbf{p}}\right)\right)}\cdot\omega_E\Big|_{\mathbf{p}}$$

Matrix representation:

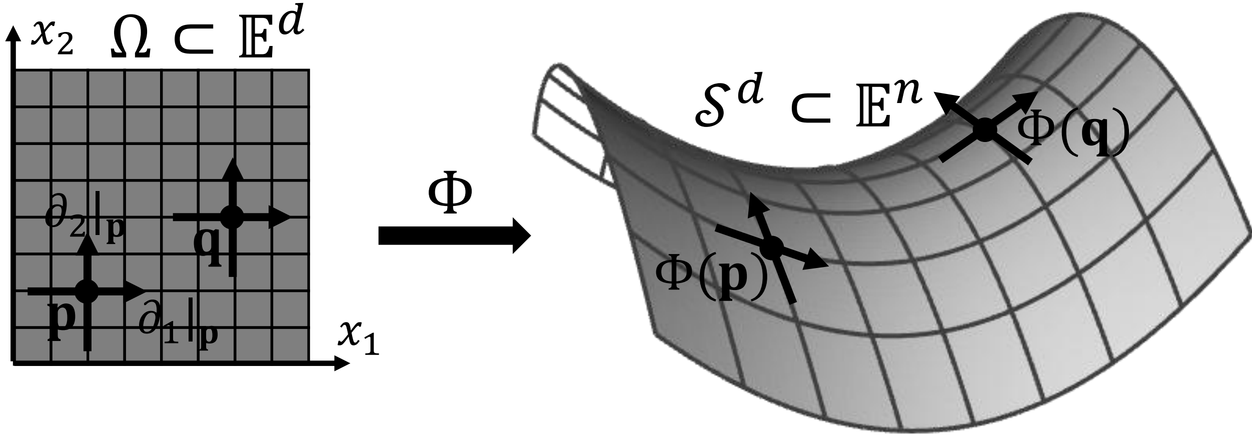
With respect to the cartesian bases $\{\partial_1|_{\mathbf{p}},\dots,\partial_d|_{\mathbf{p}}\}$ and $\{\partial_1|_{\Phi(\mathbf{p})},\dots,\partial_n|_{\Phi(\mathbf{p})}\}$, the pulled back volume-form is:*

$$\omega_g\Big|_{\mathbf{p}} = \pm\sqrt{\det\left(\mathbf{E}_{\mathbf{p}}^{-1}\cdot\left(\mathbf{d}\Phi\Big|_{\mathbf{p}}^{\top}\cdot\mathbf{E}_{\Phi(\mathbf{p})}\cdot\mathbf{d}\Phi\Big|_{\mathbf{p}}\right)\right)}\cdot\omega_E\Big|_{\mathbf{p}}$$

$$= \pm\sqrt{\det\left(\mathbf{d}\Phi\Big|_{\mathbf{p}}^{\top}\cdot\mathbf{d}\Phi\Big|_{\mathbf{p}}\right)}\cdot\omega_E\Big|_{\mathbf{p}}$$

*We chose the sign that makes the evaluation on $\{\partial_1|_{\mathbf{p}},\dots,\partial_d|_{\mathbf{p}}\}$ positive.

Inner-Product



Definition:

A function assigning an inner product $g_{\mathbf{p}}: T_{\mathbf{p}}\Omega \to T^*_{\mathbf{p}}\Omega$ to every point $\mathbf{p} \in \Omega$ is called a (*Riemannian*) *metric*.

The metric may be defined by an embedding $\Phi: \Omega \to \mathbb{E}^n$, but we do not require the embedding itself to compute gradients and integrals.*

Definition:

Surface properties that can be computed from the metric alone are referred to as *intrinsic*.

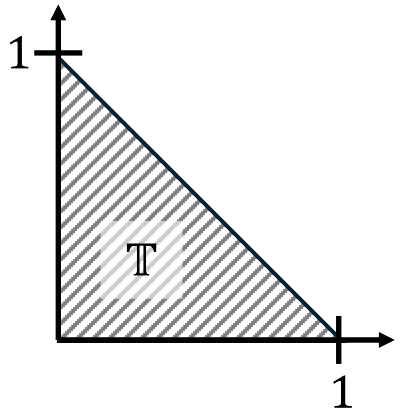
*Nash's embedding theorem states that every Riemannian manifold can be isometrically embedded in Euclidean space. That is, it can be embedded in such a way that the pull-back metric matches the Riemannian metric.

Triangle Meshes

Notation:

We denote by $\mathbb{T} \subset \mathbb{E}^2$ the unit right triangle:

$$\mathbb{T} = \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$$



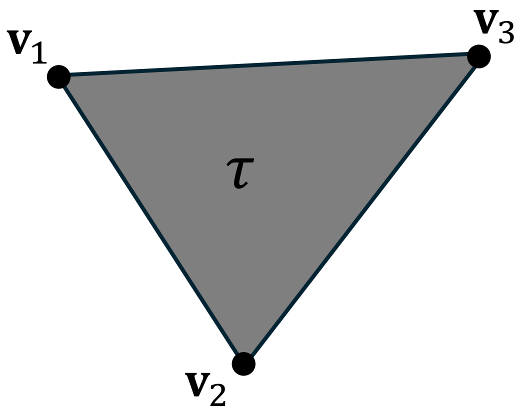
Triangle Meshes

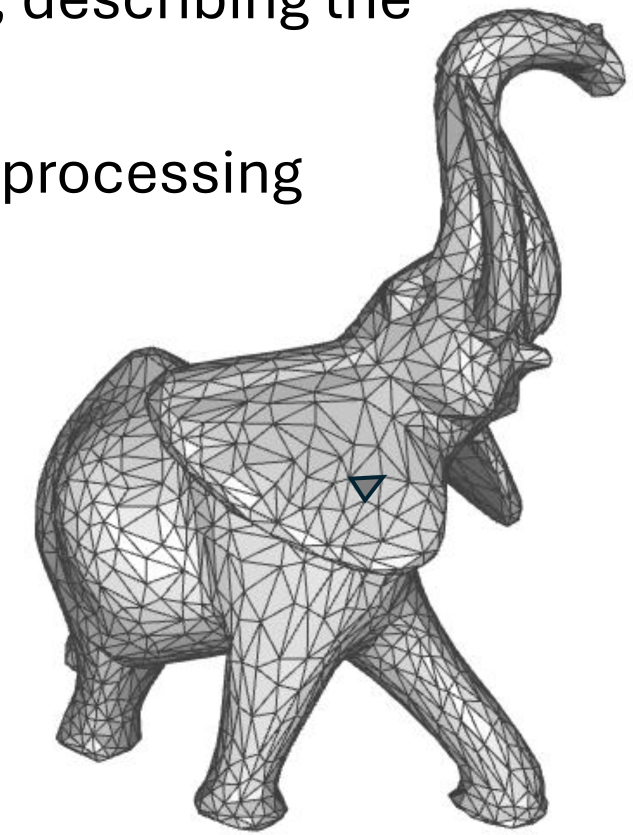
Motivation:

We will work with surfaces represented as triangle meshes in 3D.

We will use a "piece-wise" representation of functions, describing the function separately for each triangle.

$\Rightarrow$ To do geometry processing, it suffices to be able to the processing on a triangle.





Triangles

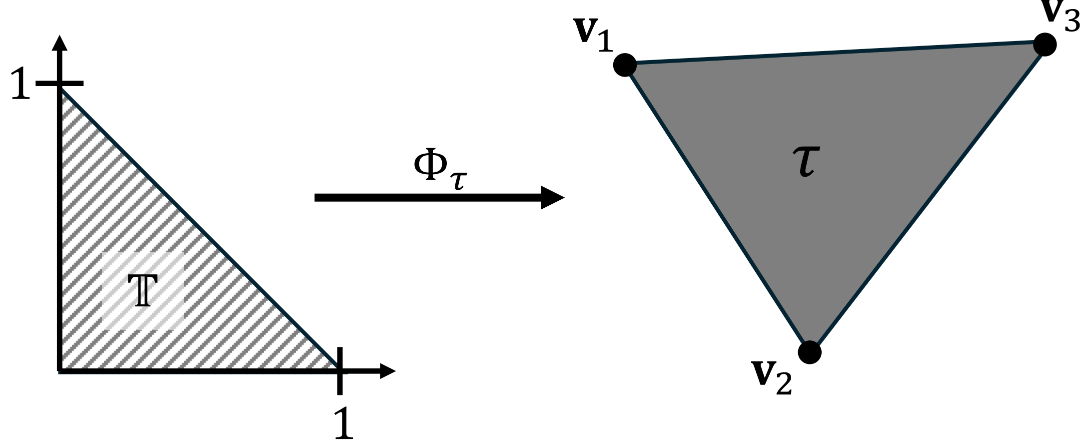
Parameterization:

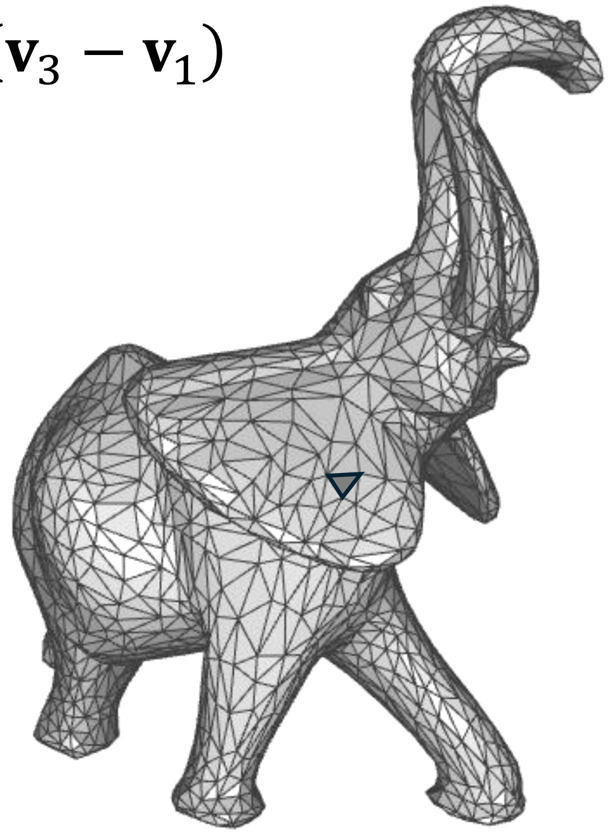
We parameterize the triangle over the unit right triangle:

$$\Phi : \mathbb{T} \to \tau$$
$$(x_1, x_2) \mapsto \mathbf{v}_1 + x_1 \cdot (\mathbf{v}_2 - \mathbf{v}_1) + x_2 \cdot (\mathbf{v}_3 - \mathbf{v}_1)$$

We describe a (e.g. real-valued) function f on the mesh triangle in terms of its values on the unit right triangle:

$$f : \mathbb{T} \to \mathbb{R}$$





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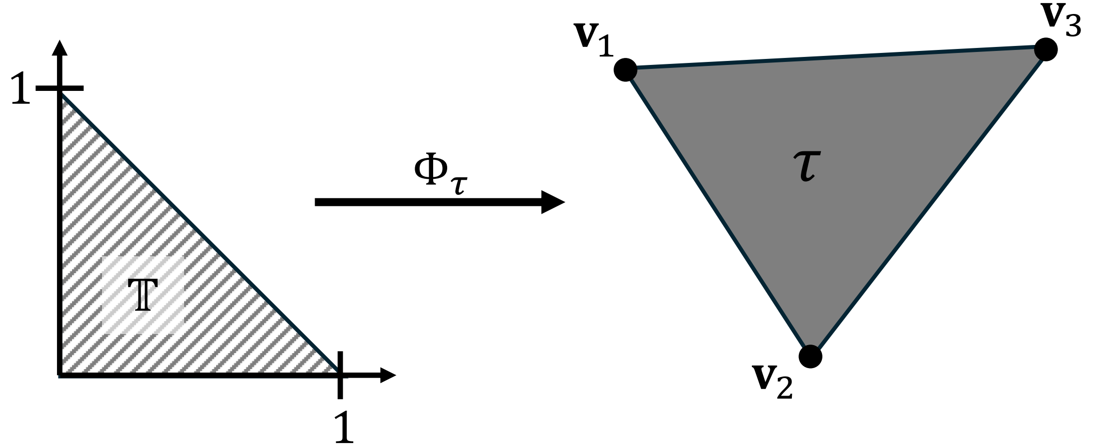
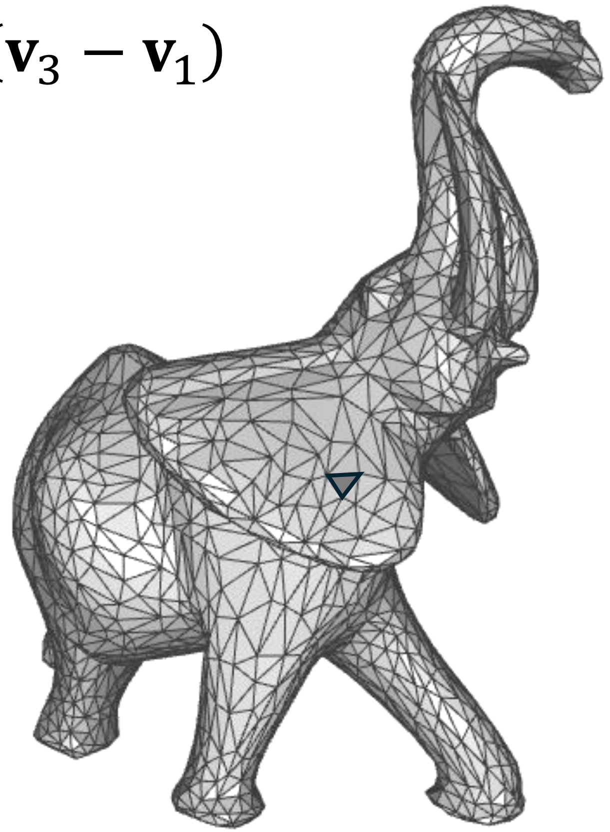
$$(x_1, x_2) \mapsto \mathbf{v}_1 + x_1 \cdot (\mathbf{v}_2 - \mathbf{v}_1) + x_2 \cdot (\mathbf{v}_3 - \mathbf{v}_1)$$

Computing the differential:

$$\mathbf{d}\Phi\Big|_{\mathbf{p}} = \begin{pmatrix} \mathbf{v}_2 - \mathbf{v}_1 & \mathbf{v}_3 - \mathbf{v}_1 \end{pmatrix}$$





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$$\mathbf{d}\Phi\Big|_{\mathbf{p}} = \begin{pmatrix} \mathbf{v}_2 - \mathbf{v}_1 & \mathbf{v}_3 - \mathbf{v}_1 \end{pmatrix}$$
$$\Downarrow$$
$$\mathbf{g}_{\mathbf{p}} = \mathbf{d}\Phi\Big|_{\mathbf{p}}^{\top} \cdot \mathbf{d}\Phi\Big|_{\mathbf{p}}$$
$$= \begin{pmatrix} \langle \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1 \rangle & \langle \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1 \rangle \\ \langle \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1 \rangle & \langle \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1 \rangle \end{pmatrix}$$

When it's clear from the context, we will drop the $\mathbf{p}$ subsc

Triangles

Parameterization:

We parameterize the triangle over the unit right triangle:

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$$(x_1, x_2) \mapsto \mathbf{v}_1 + x_1 \cdot (\mathbf{v}_2 - \mathbf{v}_1) + x_2 \cdot (\mathbf{v}_3 - \mathbf{v}_1)$$

The representation of the inner-product is constant within the triangle
⇒ We will refer to the matrix as $\mathbf{g}$.

More generally, when it's clear from the context, we will drop the $\mathbf{p}$ subscript.

$$= \begin{pmatrix} \langle \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1 \rangle & \langle \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1 \rangle \\ \langle \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1 \rangle & \langle \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1 \rangle \end{pmatrix}$$

Triangles

Recall:

We define an inner-product on the space of functions by integrating the product of two functions.

We define a smoothness energy on the space of functions by integrating the square norm of a function's differential.

For a function $f\colon \mathbb{T} \to \mathbb{R}$, the differential is a cotangent vector, $df|_{\mathbf{p}} \in T_{\mathbf{p}}^*\mathbb{T}$, and the inner-product on the space of cotangent vectors is $g_{\mathbf{p}}^{-1}\colon T_{\mathbf{p}}^*\mathbb{T} \to T_{\mathbf{p}}\mathbb{T}$.

Notation

For functions $f, h: \mathbb{T} \to \mathbb{R}$:

We will write the inner-product of f and h as:

$$\langle\langle f, h \rangle\rangle_g \equiv \int_{\mathbb{T}} f \cdot h \cdot \omega_g$$

computed by integrating the product of the functions w.r.t. the unit volume-form defined by the metric g.

We will write the inner-product of the differentials of f and h as:

$$\langle\langle df, dh \rangle\rangle_g \equiv \int_{\mathbb{T}} \langle df, dh \rangle_{g^{-1}} \cdot \omega_g$$

computed by integrating the inner-product of the differentials w.r.t. the unit volume-form defined by the metric g.

Triangles: Inner-Product

Given functions $f, h: \mathbb{T} \to \mathbb{R}$, their integral w.r.t. the pulled back inner-product is:

$$\langle\langle f, h \rangle\rangle_g \equiv \int_{\mathbb{T}} f \cdot h \cdot \sqrt{\det\left(E^{-1} \circ (d\Phi^* \circ E \circ d\Phi)\right)} \cdot \omega_E$$

With respect to the cartesian basis $\{\partial_1, \partial_2\}$, this gives:

$$\begin{aligned}\langle\langle f, h \rangle\rangle_g &= \sqrt{\det(\mathbf{g})} \cdot \int_{\mathbb{T}} f \cdot h \cdot \omega_E \\ &= \sqrt{\det(\mathbf{g})} \cdot \langle\langle f, h \rangle\rangle_E\end{aligned}$$

with $\langle\langle f, h \rangle\rangle_E$ the inner-product on real-valued functions on $\mathbb{T}$, w.r.t. the Euclidean inner-product.

Triangles: Dirichlet Energy

Given a function $f\colon \mathbb{T} \to \mathbb{R}$, its Dirichlet energy is the integral of the square norm of its differential:

$$
\begin{aligned}
E_D(f) &\equiv \langle\langle df, df \rangle\rangle_g \\
&= \int_{\mathbb{T}} \|df\|^2_{g^{-1}} \cdot \sqrt{\det\left(E^{-1} \circ (d\Phi^* \circ E \circ d\Phi)\right)} \cdot \omega_E
\end{aligned}
$$

With respect to the cartesian basis $\{\partial_1, \partial_2\}$, this gives:

$$
\begin{aligned}
E_D(f) &= \int_{\mathbb{T}} \mathbf{d}f^\top \cdot \mathbf{g}^{-1} \cdot \mathbf{d}f \cdot \sqrt{\det(\mathbf{g})} \cdot \omega_E \\
&= \sqrt{\det(\mathbf{g})} \cdot \int_{\mathbb{T}} \mathbf{d}f^\top \cdot \mathbf{g}^{-1} \cdot \mathbf{d}f \cdot \omega_E
\end{aligned}
$$

Triangles: Dirichlet Energy

Given a function $f: \mathbb{T} \to \mathbb{R}$, its Dirichlet energy is the integral of the square norm of its differential:

$$
\begin{aligned}
E_D(f) &= \sqrt{\det(\mathbf{g})} \cdot \int_{\mathbb{T}} \mathbf{d}f^\top \cdot \mathbf{g}^{-1} \cdot \mathbf{d}f \cdot \omega_E \\
&= \sqrt{\det(\mathbf{g})} \cdot \int_{\mathbb{T}} \mathrm{trace}(\mathbf{d}f^\top \cdot \mathbf{g}^{-1} \cdot \mathbf{d}f) \cdot \omega_E \\
&= \sqrt{\det(\mathbf{g})} \cdot \int_{\mathbb{T}} \mathrm{trace}(\mathbf{g}^{-1} \cdot \mathbf{d}f \cdot \mathbf{d}f^\top) \cdot \omega_E \\
&= \sqrt{\det(\mathbf{g})} \cdot \int_{\mathbb{T}} \langle \mathbf{g}^{-1}, \mathbf{d}f \cdot \mathbf{d}f^\top \rangle_F \cdot \omega_E \\
&= \left\langle \sqrt{\det(\mathbf{g})} \cdot \mathbf{g}^{-1}, \int_{\mathbb{T}} \mathbf{d}f \cdot \mathbf{d}f^\top \cdot \omega_E \right\rangle_F
\end{aligned}
$$

Triangles

Inner-Product:

$$\langle\langle f, h\rangle\rangle_g = \boxed{\sqrt{\det(\mathbf{g})}} \cdot \boxed{\langle\langle f, h\rangle\rangle_E}$$

Dirichlet Energy:

$$E_D(f) = \left\langle \boxed{\sqrt{\det(\mathbf{g})} \cdot \mathbf{g}^{-1}}, \boxed{\int_{\mathbb{T}} \mathbf{d}f \cdot \mathbf{d}f^\top \cdot \omega_E} \right\rangle_F$$

⇒ Both factor in terms of computation that:

1. Involves the function(s) but does not involve the inner-product
2. Involves the inner-product but does not involve function(s).