Geometry Processing (601.458/658)

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Outline

Recall

Bases and Endomorphisms

Volume Forms and Determinants

$$A \xrightarrow{\Phi} B \xrightarrow{\Psi} C$$

Pull-Backs

Given sets A, B, and C, if we have functions $\Phi: A \to B$ and $\Psi: B \to C$, we can pull back functions from B to A using Φ^* and we can pull-back functions from C to B using Ψ^*.

Claim:

The pull-back of the composition is the composition of the pull-backs, in reversed order:

$$(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$$

$$A \xrightarrow{\Phi} B \xrightarrow{\Psi} C$$

Pull-Backs

Given sets A, B, and C, if we have functions $\Phi: A \to B$ and $\Psi: B \to C$, we can pull back functions from B to A using Φ^* and we can pull-back functions from C to B using Ψ^*.

Proof:

Given a function f on C and given $a \in A$:

$$\begin{aligned}
[(\Psi \circ \Phi)^* f](a) &= f\big((\Psi \circ \Phi)(a)\big) \\
&= f\big(\Psi\big(\Phi(a)\big)\big) \\
&= [\Psi^*(f)]\big(\Phi(a)\big) \\
&= \big[\Phi^*\big(\Psi^*(f)\big)\big](a) \\
&= [(\Phi^* \circ \Psi^*)(f)](a)
\end{aligned}$$

Permutations

We denote by S_n the group of *permutations* on n elements – invertible maps:

$$\sigma: \{1, \dots n\} \to \{1, \dots n\}.$$

We denote a permutation $\sigma \in S_n$ as a reordering of the set $\{1, \dots n\}$:

$$\sigma = (i_1, \dots, i_n)$$

with $1 \leq i_j \leq n$ and $i_j \neq i_k$.

This describes the order of the elements after permutation, so that:

$$\sigma: \{1, \dots, n\} \to \{1, \dots, n\}$$
$$j \mapsto i_j$$

Permutations

We denote by S_n the group of *permutations* on n elements – invertible maps:

$$\sigma: \{1, \ldots n\} \to \{1, \ldots n\}.$$

An *inversion* in $\sigma \in S_n$ is a pair of indices (j, k), with $1 \leq j < k \leq n$, for which the associated values of the permutation are out of order:

$$\sigma(j) > \sigma(k)$$

The *sign* of a permutation $\sigma \in S_n$, denoted $\mathrm{sign}(\sigma)$, is $+1$ if the number of inversions is even and -1 if it is odd.

The sign of the identity permutation is $+1$.

Permutations

We denote by S_n the group of *permutations* on n elements – invertible maps:

$$\sigma: \{1, \dots n\} \to \{1, \dots n\}.$$

We denote by $\tau_{jk} \in S_n$ (with $1 \leq j \neq k \leq n$) the *transposition* that swaps the j-th and k-th elements.

Permutations

Claim:

Given any permutation $\sigma \in S_n$ and a transposition τ_{jk}:

$$\text{sign}(\tau_{jk} \circ \sigma) = -\text{sign}(\sigma)$$

Proof:

We can partition the elements as:

$$(i_1, \dots, i_{j-1}, \boldsymbol{i_j}, i_{j+1}, \dots, i_{k-1}, \boldsymbol{i_k}, i_{k+1}, \dots, i_n)$$

Note that transposing i_j and i_k can only introduce/remove inversions between pairs of indices if (at least) one of the indices is either i_j or i_k.

Permutations

Claim:

Given any permutation $\sigma \in S_n$ and a transposition τ_{jk}:

$$\operatorname{sign}(\tau_{jk} \circ \sigma) = -\operatorname{sign}(\sigma)$$

Proof:

We can partition the elements as:

$$(i_1, \ldots, i_{j-1}, \boldsymbol{i_j}, i_{j+1}, \ldots, i_{k-1}, \boldsymbol{i_k}, i_{k+1}, \ldots, i_n)$$

$\{i_1, \ldots, i_{j-1}\}$:

An element in this range is inverted w.r.t. i_j (resp. i_k) before the transposition if and only if it will be inverted w.r.t. i_j (resp. i_k) after the transposition.

⇒ Transposing i_j and i_k won't change the number of inversions contributed from this range.

⇒ This will not affect the parity.

Permutations

Claim:

Given any permutation $\sigma \in S_n$ and a transposition τ_{jk}:

$$\text{sign}(\tau_{jk} \circ \sigma) = -\text{sign}(\sigma)$$

Proof:

We can partition the elements as:

$$(i_1, \dots, i_{j-1}, \boldsymbol{i_j}, i_{j+1}, \dots, i_{k-1}, \boldsymbol{i_k}, \boxed{i_{k+1}, \dots, i_n})$$

$\{i_{k+1}, \dots, i_n\}$:

An element in this range is inverted w.r.t. i_j (resp. i_k) before the transposition if and only if it will be inverted w.r.t. i_j (resp. i_k) after the transposition.

⇒ Transposing i_j and i_k won't change the number of inversions contributed from this range.

⇒ This will not affect the parity.

Permutations

Claim:

Given any permutation $\sigma \in S_n$ and a transposition τ_{jk}:

$$\text{sign}(\tau_{jk} \circ \sigma) = -\text{sign}(\sigma)$$

Proof:

We can partition the elements as:

$$(i_1, \dots, i_{j-1}, \boldsymbol{i_j}, \boxed{i_{j+1}, \dots, i_{k-1}}, \boldsymbol{i_k}, i_{k+1}, \dots, i_n)$$

$\{i_{j+1}, \dots, i_{k-1}\}$:

An element in this range is inverted w.r.t. i_j (resp. i_k) before the transposition if and only if it will be not be inverted w.r.t. i_j (resp. i_k) after the transposition.

⇒ The change in the number of inversions contributed from this range will be even.

⇒ This will not affect the parity.

Permutations

Claim:

Given any permutation $\sigma \in S_n$ and a transposition τ_{jk}:

$$\mathrm{sign}\left(\tau_{jk} \circ \sigma\right) = -\mathrm{sign}(\sigma)$$

Proof:

We can partition the elements as:

$$\left(i_1, \dots, i_{j-1}, \boldsymbol{i_j}, i_{j+1}, \dots, i_{k-1}, \boldsymbol{i_k}, i_{k+1}, \dots, i_n\right)$$

$\{i_j, i_k\}$:

These elements are inverted before the transposition if and only if they will be not be inverted after the transposition.

$\Rightarrow$ The transposition changes the number of inversions by one.

$\Rightarrow$ This will change the parity.

Permutations

Claim:

Given any permutation $\sigma \in S_n$ and a transposition τ_{jk}:

$$\mathrm{sign}(\tau_{jk} \circ \sigma) = -\mathrm{sign}(\sigma)$$

Corollary:

Since the sign of the identity is $+1$, and since we can decompose any permutation as the composition of transpositions:

$$\sigma = \tau_{i_1 j_1} \circ \cdots \circ \tau_{i_k j_k}$$

⇒ The sign of a permutation is defined by the parity of the count of constituent transpositions*:

$$\mathrm{sign}(\sigma) = (-1)^k$$

*Though the decomposition into transpositions is not unique, the parity is.

Permutations

Claim:

The sign of a permutation is defined by the parity of the count of constituent transpositions:

$$\sigma = \tau_{i_1 j_1} \circ \cdots \circ \tau_{i_k j_k}$$
$$\Downarrow$$
$$\operatorname{sign}(\sigma) = (-1)^k$$

Corollary:

For $\sigma_1, \sigma_2 \in S_n$ with decompositions:

$$\sigma_1 = \tau_{i_1 j_1} \circ \cdots \circ \tau_{i_k j_k} \quad \text{and} \quad \sigma_2 = \tau_{m_1 n_1} \circ \cdots \circ \tau_{m_l n_l}$$

we have:

$$\sigma_1 \circ \sigma_2 = \tau_{i_1 j_1} \circ \cdots \circ \tau_{i_k j_k} \circ \tau_{m_1 n_1} \circ \cdots \circ \tau_{m_l n_l}$$
$$\Downarrow$$
$$\operatorname{sign}(\sigma_1 \circ \sigma_2) = (-1)^{k+l} = (-1)^k \cdot (-1)^l = \operatorname{sign}(\sigma_1) \cdot \operatorname{sign}(\sigma_2)$$

Permutations

Claim:

The sign of the composition of transpositions is the product of their signs:

$$\text{sign}(\sigma_1 \circ \sigma_2) = \text{sign}(\sigma_1) \cdot \text{sign}(\sigma_2)$$

Corollary:

Since the sign of the identity is $+1$, for $\sigma \in S_n$ we have:

$$\begin{aligned} 1 &= \text{sign}(\sigma \circ \sigma^{-1}) \\ &= \text{sign}(\sigma) \cdot \text{sign}(\sigma^{-1}) \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \text{sign}(\sigma^{-1}) &= \frac{1}{\text{sign}(\sigma)} \\ &= \text{sign}(\sigma) \end{aligned}$$

Determinant

Given a square matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$, the *determinant* of the matrix is the signed sum of permuted products:

$$\det(\mathbf{M}) = \sum_{\sigma\in S_n} \text{sign}(\sigma) \cdot \prod_{i=1}^{n} \mathbf{M}_{i,\sigma(i)}$$

```
Determinant:
det ← 0
for i ∈ [1,n]
    det ← det + (−1)^{i+1} · M_{0i} ·Determinant( Sub( M , 0 , i ) )
return det
```

With Sub($\mathbf{L}$, i, j) returning the sub-matrix obtained by removing the i-th row and j-th column.

Determinant: Example

$$\det(\mathbf{M}) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \cdot \prod_{i=1}^{n} \mathbf{M}_{i,\sigma(i)}$$

Example:

In the case of a 2×2 matrix:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix}$$

the group of permutations is:

$$S_2 = \{(1,2), (2,1)\}$$

$$\Downarrow$$

$$\det(\mathbf{M}) = \mathbf{M}_{11} \cdot \mathbf{M}_{22} - \mathbf{M}_{12} \cdot \mathbf{M}_{21}$$

Outline

Recall

Bases and Endomorphisms

Volume Forms and Determinants

Bases and Endomorphisms

$$V \underset{L^{-1}}{\overset{L}{\rightleftarrows}} W$$

Claim:

Given vectors spaces V and W, an invertible linear $L \in \mathrm{Hom}(V, W)$, and a basis $\{v_1, \ldots, v_n\}$ for V, the set $\{L(v_1), \ldots, L(v_n)\} \subset W$ is a basis for W.

Proof:

If $\{L(v_1), \ldots, L(v_n)\}$ are linearly dependent, there exist $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}$ with at least one $\mathbf{a}_i \neq 0$ such that:

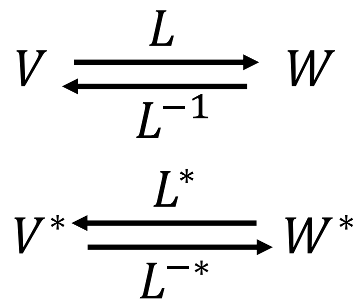
$$\mathbf{a}_1 \cdot L(v_1) + \cdots + \mathbf{a}_n \cdot L(v_n) = 0$$

Applying L^{-1} to both sides gives:

$$\mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_n \cdot v_n = 0$$

contradicting the linear independence of $\{v_1, \ldots, v_n\}$.

Bases and Endomorphisms



Claim:

Given vectors spaces V and W, an invertible linear $L \in \mathrm{Hom}(V, W)$, and a basis $\{v_1, \dots, v_n\}$ for V, the set $\{L(v_1), \dots, L(v_n)\} \subset W$ is a basis for W.

Claim:

The set $\{L^{-*}(v_1^*), \dots, L^{-*}(v_n^*)\} \subset W^*$ is the dual basis for $\{L(v_1), \dots, L(v_n)\}$.

Proof:

Consider how the i-th dual basis vector acts on the j-th primal vector:

$$\begin{aligned}[L^{-*}(v_i^*)]\left(L(v_j)\right) &= v_i^*\left(L^{-1}\left(L(v_j)\right)\right) \\ &= v_i^*(v_j) \\ &= \delta_{ij}\end{aligned}$$

Outline

Recall

Bases and Endomorphisms

Volume Forms and Determinants

Volume Forms

Recall:

Given a vector space V, we say that a map $B: V \times V \to \mathbb{R}$ is *bilinear* if it is linear in each entry:

$$B(\alpha \cdot u_1 + \beta \cdot u_2, v) = \alpha \cdot B(u_1, v) + \beta \cdot B(u_2, v)$$
$$B(u, \alpha \cdot v_1 + \beta \cdot v_2) = \alpha \cdot B(u, v_1) + \beta \cdot B(u, v_2)$$

The space of bilinear maps is a vector space.

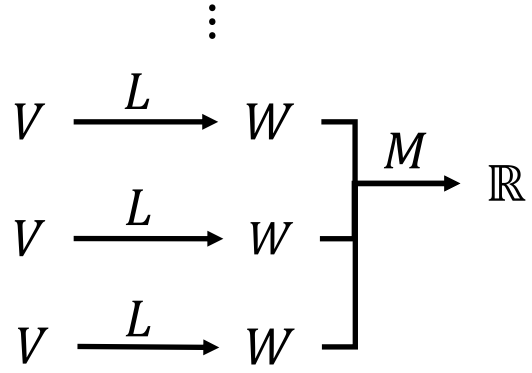
A map $M: V^n \to \mathbb{R}$ is *multi-linear* if it is linear in each entry.

The space of multi-linear maps is a vector space denoted $\otimes^n V^*$.

$M \in \otimes^n V^*$ is *alternating* if swapping two arguments negates the output.

The space of alternating multi-linear maps is a vector space denoted $\Lambda^n V^*$.

Volume Forms

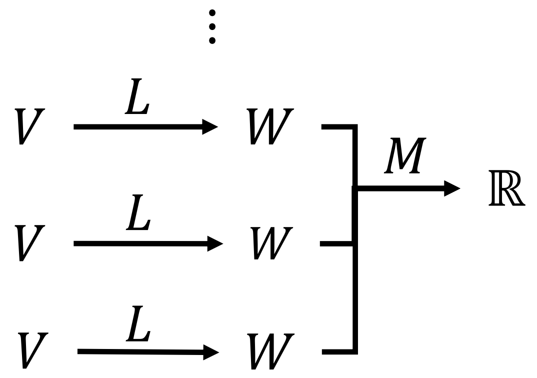


Recall:

Given vector spaces V and W, a multi-linear map $M \in \bigotimes^n W^*$, and a linear map $L \in \mathrm{Hom}(V, W)$, we can pull-back M to a map:

$$L^*(M): V^n \to \mathbb{R}$$
$$(v_1, \dots, v_n) \mapsto M\big(L(v_1), \dots, L(v_n)\big)$$

Volume Forms



Claim:

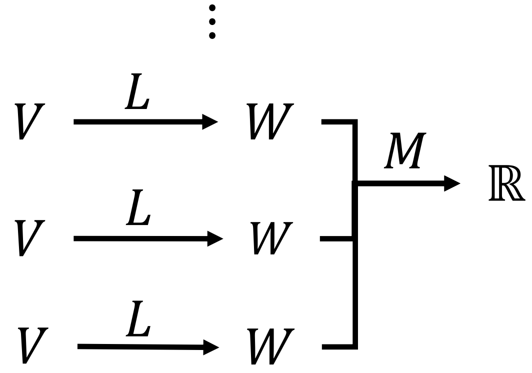
The pull-back of a multi-linear map, $M \in \bigotimes^n W^*$ is also multi-linear, $L^*(M) \in \bigotimes^n V^*$.

Proof:

$$
\begin{aligned}
[L^*(M)](\alpha \cdot v_1 + \beta \cdot w, v_2, \dots, v_n) &= M\big(L(\alpha \cdot v_1 + \beta \cdot w), L(v_2), \dots, L(v_n)\big) \\
&= M\big(\alpha \cdot L(v_1) + \beta \cdot L(w), L(v_2), \dots, L(v_n)\big) \\
&= \alpha \cdot M\big(L(v_1), L(v_2), \dots, L(v_n)\big) + \beta \cdot M\big(L(w), L(v_2), \dots, L(v_n)\big) \\
&= \alpha \cdot [L^*(M)](v_1, v_2, \dots, v_n) + \beta \cdot [L^*(M)](w, v_2, \dots, v_n)
\end{aligned}
$$

A similar argument shows that this is true for *any* index.

Volume Forms



Claim:

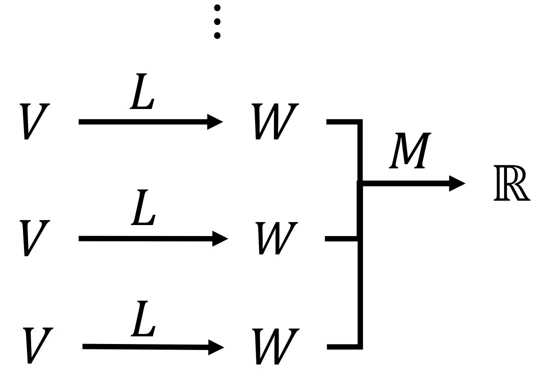
The pull-back of an alternating multi-linear map, $M \in \Lambda^n W^*$, is also an alternating multi-linear map, $L^*(M) \in \Lambda^n V^*$.

Proof:

$$\begin{aligned}[L^*(M)](v_1, v_2, v_3, \ldots, v_n) &= M\big(L(v_1), L(v_2), L(v_3), \ldots, L(v_n)\big) \\ &= -M\big(L(v_2), L(v_1), L(v_3), \ldots, L(v_n)\big) \\ &= -[L^*(M)](v_2, v_1, v_3, \ldots, v_n)\end{aligned}$$

A similar argument shows that this is true for *any* pair of indices.

Volume Forms



Recall:

The pull-back is a linear map between functions spaces.

⇒ As a map from (alternating) multi-linear maps on W to (alternating) multi-linear maps on V the pull-back is a linear map:

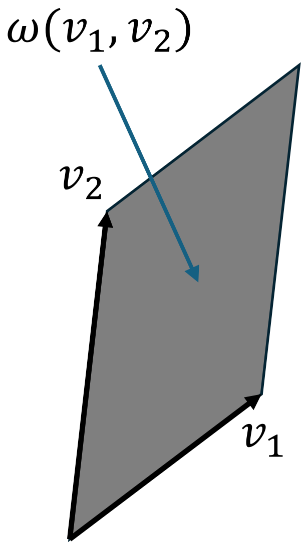
$$L^* \in \mathrm{Hom}\left(\bigotimes^n W^*, \bigotimes^n V^*\right)$$
$$L^* \in \mathrm{Hom}\left(\wedge^n W^*, \wedge^n V^*\right)$$

Volume Forms

Goal:

Given an n-dimensional vector space V, we would like to consider maps that takes n vectors in V and returns the **signed** volume of the associated parallelepiped:

$$\omega : V^n \to \mathbb{R}$$



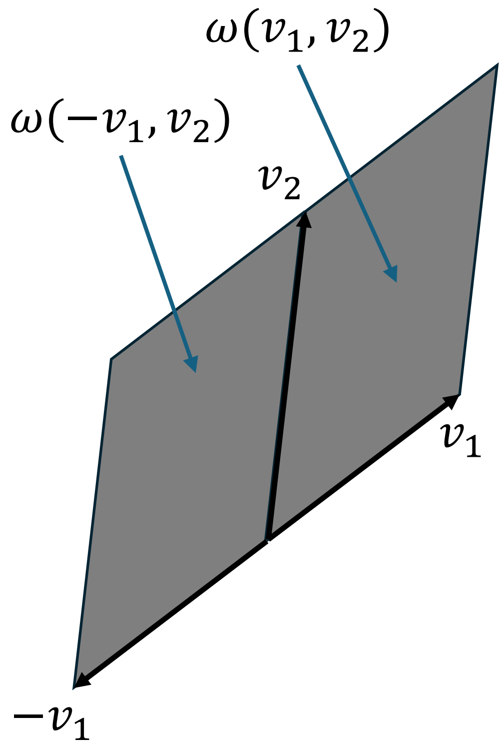
We begin by identifying the properties such a map should satisfy.

Volume Forms

Property 1:

Negating an argument should negate the signed volume:

$$\omega(-v_1, v_2, \dots, v_n) = -\omega(v_1, v_2, \dots, v_n)$$

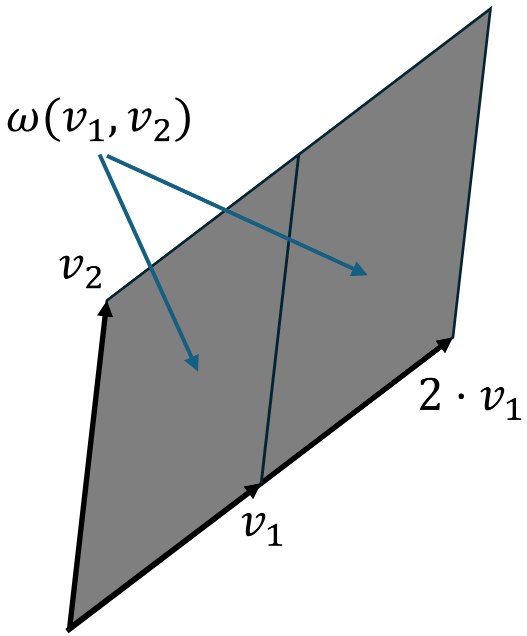


Volume Forms

Property 2:

For any positive integer $k \in \mathbb{N}$, we have:

$$\omega(k \cdot v_1, v_2, \dots, v_n) = k \cdot \omega(v_1, v_2, \dots, v_n)$$



Volume Forms

Property 2:

For any positive integer $k \in \mathbb{N}$, we have:

$$\omega(k \cdot v_1, v_2, \dots, v_n) = k \cdot \omega(v_1, v_2, \dots, v_n)$$

⇒ For any positive integer $k \in \mathbb{N}$, we have:

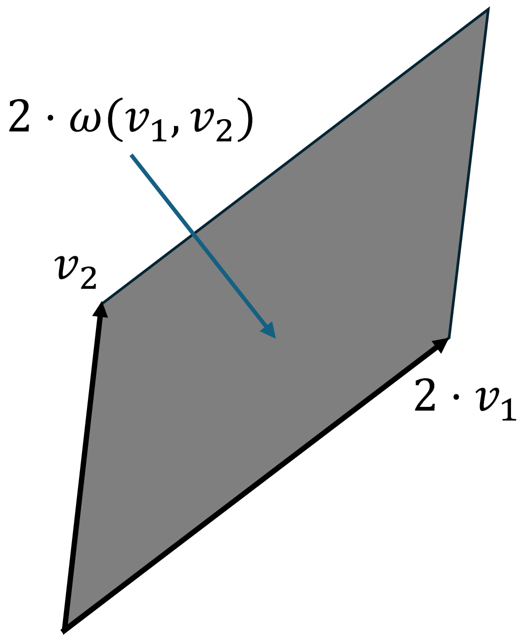
$$\omega\left(\frac{v_1}{k}, v_2, \dots, v_n\right) = \frac{1}{k} \cdot \omega(v_1, v_2, \dots, v_n)$$

⇒ For any rational number $q \in \mathbb{Q}$:

$$\omega(q \cdot v_1, v_2, \dots, v_n) = q \cdot \omega(v_1, v_2, \dots, v_n)$$

⇒ By continuity, for any $\alpha \in \mathbb{R}$:

$$\alpha \cdot \omega(q \cdot v_1, v_2, \dots, v_n) = \alpha \cdot \omega(v_1, v_2, \dots, v_n)$$

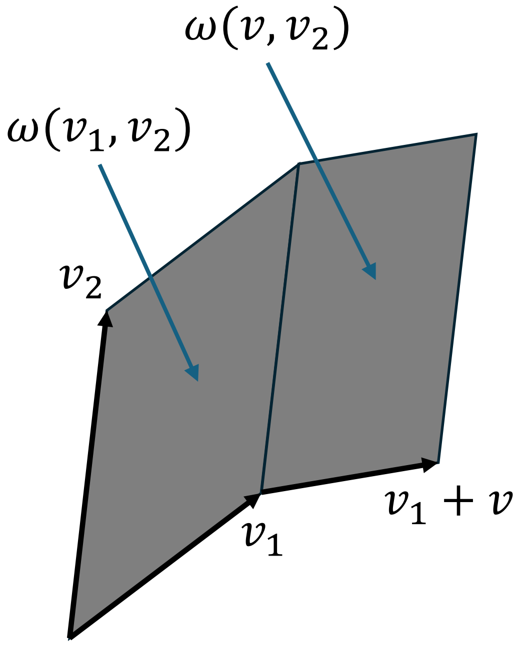


Volume Forms

Property 3:

For any vector $v \in V$ we have:

$$\omega(v_1, v_2, \dots, v_n) + \omega(v, v_2, \dots, v_n)$$



Volume Forms

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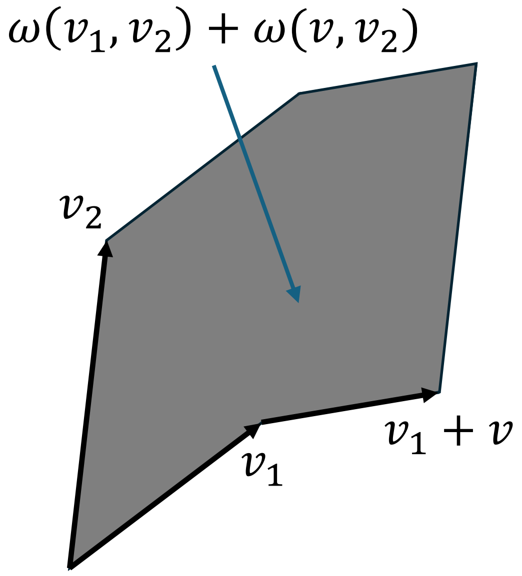

Volume Forms

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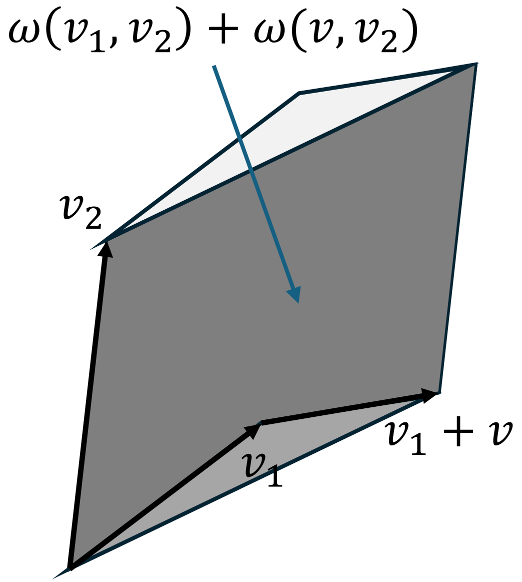
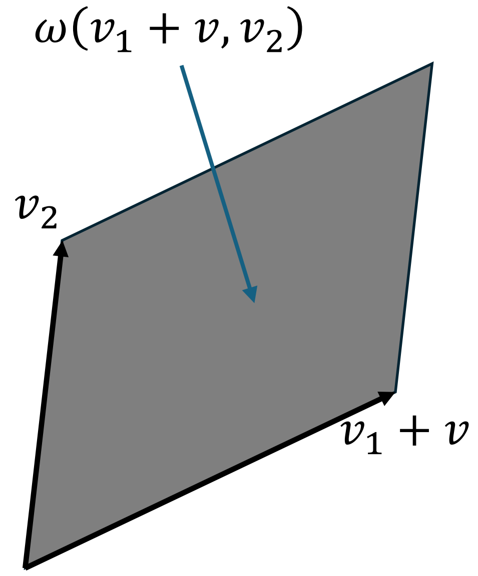


Volume Forms

Property 3:

For any vector $v \in V$ we have:

$$\omega(v_1, v_2, \ldots, v_n) + \omega(v, v_2, \ldots, v_n) = \omega(v_1 + v, v_2, \ldots, v_n)$$



Volume Forms

Property 1-3:

For any $\alpha \in \mathbb{R}$ and any vector $v \in V$, we should have:

$$\omega(\alpha \cdot v_1, v_2, \dots, v_n) = \alpha \cdot \omega(v_1, v_2, \dots, v_n)$$

$$\omega(v_1 + v, v_2, \dots, v_n) = \omega(v_1, v_2, \dots, v_n) + \omega(v, v_2, \dots, v_n)$$

$\Rightarrow$ Since this is true for all arguments, the map ω should be multi-linear:

$$\omega \in \bigotimes^n V^*$$

Volume Forms

Additionally:

If for any $1 \leq i \neq j \leq n$ the i-th and j-th arguments are the same, we get a degenerate parallelepiped and the volume should be zero.

Using multi-linearity, this gives:

$$
\begin{aligned}
0 &= \omega(v_1 + v_2, v_1 + v_2, v_3, \ldots, v_n) \\
&= \cancel{\omega(v_1, v_1, v_3, \ldots, v_n)} \\
&\quad +\omega(v_1, v_2, v_3, \ldots, v_n) \\
&\quad +\omega(v_2, v_1, v_3, \ldots, v_n) \\
&\quad +\cancel{\omega(v_2, v_2, v_3, \ldots, v_n)} \\
&= \omega(v_1, v_2, v_3, \ldots, v_n) + \omega(v_2, v_1, v_3, \ldots, v_n)
\end{aligned}
$$

$$\Downarrow$$

$$\omega(v_1, v_2, v_3, \ldots, v_n) = -\omega(v_2, v_1, v_3, \ldots, v_n)$$

Volume Forms

Additionally:

If for any $1 \leq i \neq j \leq n$ the i-th and j-th arguments are the same, we get a degenerate parallelepiped and the volume should be zero.

⇒ Swapping the first two arguments should negate the volume:

$$\omega(v_1, v_2, v_3, \dots, v_n) = -\omega(v_2, v_1, v_3, \dots, v_n)$$

⇒ Since this is true for any pair of indices, the map ω should be alternating:

$$\omega \in \Lambda^n V^*$$

Volume Forms

Definition:

For an n-dimensional vector space V, we say that an alternating, multi-linear map on n arguments, $\omega \in \Lambda^n V^*$, is a *volume form*.

Claim:

The vector space of volume forms is one-dimensional.

Equivalently:

If we know the value of the volume form on one n-tuple of (linearly independent) vectors, we know its value on all n-tuples.

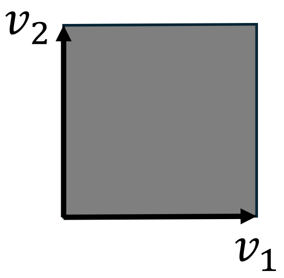
Volume Forms

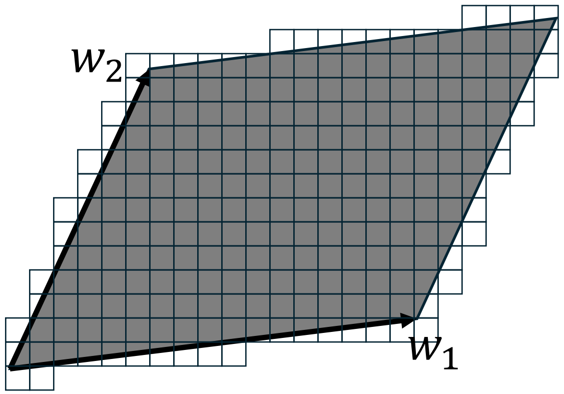
Claim:

If we know the value of the volume form on one n-tuple of (linearly independent) vectors, we know its value on all n-tuples.

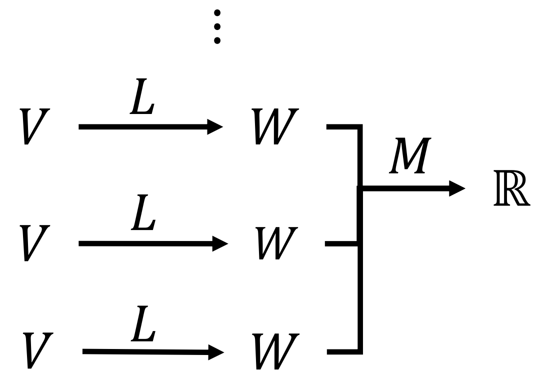
"Proof":

If we know the volume of some parallelepiped $\{v_1, \ldots, v_n\}$, we can define the value for any other parallelepiped $\{w_1, \ldots, w_n\}$ by scaling and tiling.





Determinants



Recall:

For any endomorphism $L \in \mathrm{Hom}(V, V)$, the pull-back L^* is an endomorphism, $L^* \in \mathrm{Hom}(\Lambda^n V^*, \Lambda^n V^*)$.

Since the space of volume forms is one-dimensional, for any endomorphism $L \in \mathrm{Hom}(V, V)$ there exists $\lambda_L \in \mathbb{R}$ such that:

$$L^*(\omega) = \lambda_L \cdot \omega$$

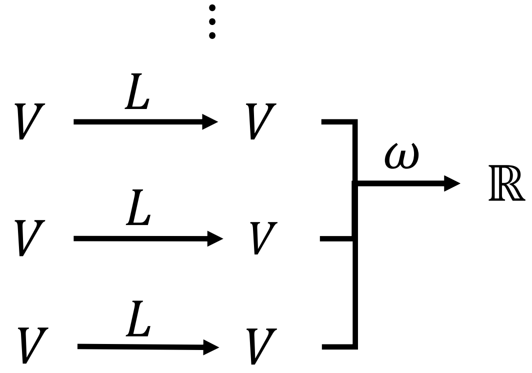
for all volume forms ω.

Definition:

For an endomorphism $L \in \mathrm{Hom}(V, V)$, the *determinant* of L, denoted $\det(L)$, is the scalar satisfying:

$$L^*(\omega) = \det(L) \cdot \omega$$

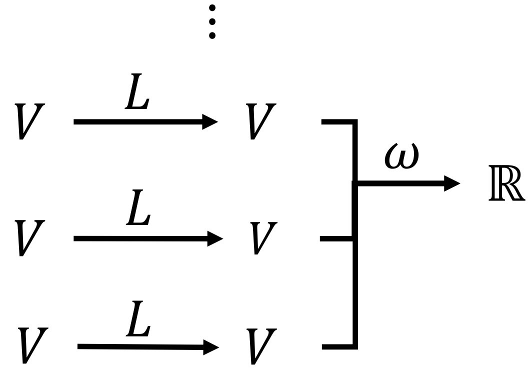
Determinants



Given a vector space V, a volume form ω, and an endomorphism $L \in \mathrm{Hom}(V, V)$, we have:

$$\det(L) \cdot \omega(v_1, \ldots, v_n) = [L^*(\omega)](v_1, \ldots, v_n)$$
$$= \omega\big(L(v_1), \ldots, L(v_n)\big)$$
$$\Downarrow$$
$$\det(L) = \frac{\omega\big(L(v_1), \ldots, L(v_n)\big)}{\omega(v_1, \ldots, v_n)}$$

⇒ The determinant of an endomorphism L is the *change in volume* that a parallelepiped goes through after being transformed by L.



Determinants

Given a vector space V, a volume form ω, and an endomorphism $L \in \mathrm{Hom}(V, V)$, we have:

The determinant does **not** define the volume of a parallelepiped.
It only defines the change relative to an existing measure of volume.

$$\det(L) = \frac{\omega\big(L(v_1), \dots, L(v_n)\big)}{\omega(v_1, \dots, v_n)}$$

⇒ The determinant of an endomorphism L is the *change in volume* that a parallelepiped goes through after being transformed by L.

Determinants: Properties

Linear dependence:

Given vectors $\{v_1, \dots, v_n\}$ that are linearly **dependent**, any volume form will evaluate to zero on $\{v_1, \dots, v_n\}$.

Proof:

Without loss of generality, assume:

$$v_n = \mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_{n-1} \cdot v_{n-1}$$

Then for any volume form $\omega: V^n \to \mathbb{R}$ we have:

$$\begin{aligned}\omega(v_1, \dots, v_{n-1}, v_n) &= \omega(v_1, \dots, v_{n-1}, \mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_{n-1} \cdot v_{n-1}) \\ &= \mathbf{a}_1 \cdot \omega(\boxed{v_1}, \dots, v_{n-1}, \boxed{v_1}) + \cdots + \mathbf{a}_{n-1} \cdot \omega(v_1, \dots, \boxed{v_{n-1}}, \boxed{v_{n-1}}) \\ &= 0\end{aligned}$$

Determinants: Properties

Linear independence:

Given vectors $\{v_1, \dots, v_n\}$ that are linearly **independent**, if a volume form evaluates to zero on $\{v_1, \dots, v_n\}$ then it evaluates to zero on every n-tuple.

Proof:

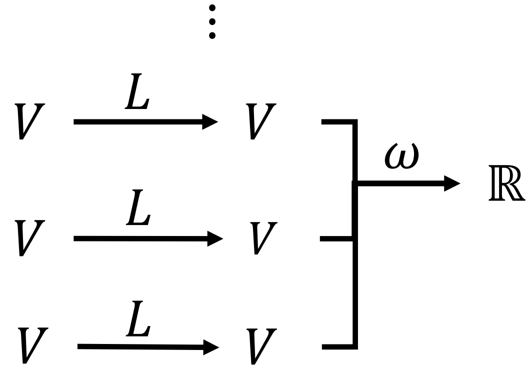
Assume that $\omega \in \Lambda^n V^*$ evaluates to zero on $\{v_1, \dots, v_n\}$ but there is some other set of vectors $\{w_1, \dots, w_n\}$ with:

$$\omega(w_1, \dots, w_n) \neq 0$$

Let $L \in \mathrm{Hom}(V, V)$ be an endomorphism taking v_i to w_i:

$$\begin{aligned} 0 \neq\ & \omega(w_1, \dots, w_n) \\ =\ & \omega\big(L(v_1), \dots, L(v_n)\big) \\ =\ & [L^*(\omega)](v_1, \dots, v_n) \\ =\ & \det(L) \cdot \omega(v_1, \dots, v_n) \end{aligned}$$

Determinants: Properties



Singular endomorphisms:

If $L \in \mathrm{Hom}(V, V)$ is not invertible, the determinant is zero.

Proof:

Let $\{v_1, \ldots, v_n\}$ be a basis and $\omega \in \Lambda^n V^*$ be some non-zero volume form.

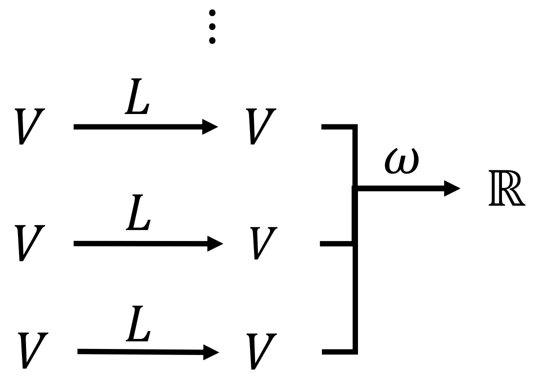
Since L is not invertible, the set $\{L(v_1), \ldots, L(v_n)\}$ is linearly dependent.

$$\begin{aligned} 0 &= \omega\big(L(v_1), \ldots, L(v_n)\big) \\ &= [L^*(\omega)](v_1, \ldots, v_n) \\ &= \det(L) \cdot \omega(v_1, \ldots, v_n) \end{aligned}$$

Since $\{v_1, \ldots, v_n\}$ is linearly independent, $\omega(v_1, \ldots, v_n) \neq 0$

$\Rightarrow \det(L) = 0$

Determinants: Properties

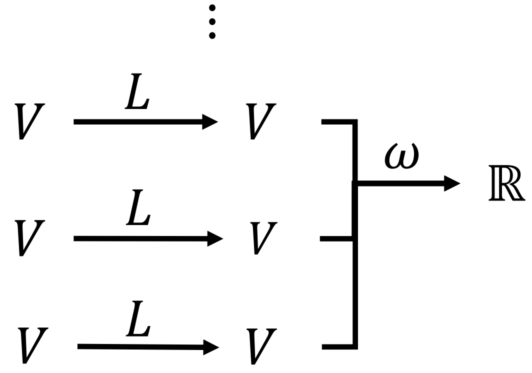


Properties:

The determinant of the identity is one:

$$\det(\mathrm{Id}_V) = \frac{\omega\big(\mathrm{Id}_V(v_1), \ldots, \mathrm{Id}_V(v_n)\big)}{\omega(v_1, \ldots, v_n)}$$
$$= \frac{\omega(v_1, \ldots, v_n)}{\omega(v_1, \ldots, v_n)}$$
$$= 1$$

Determinants: Properties



Properties:

For $L, M \in \mathrm{Hom}(V, V)$, the determinant of the composition is the product of the determinants:

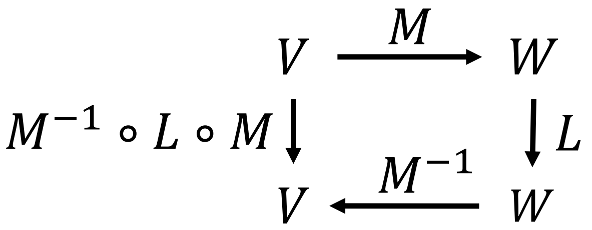
$$\det(L \circ M) = \det(L) \cdot \det(M)$$

Proof:

For any $\omega \in \Lambda^n V^*$ we have:

$$\begin{aligned} \det(L \circ M) \cdot \omega &= (L \circ M)^*(\omega) \\ &= (M^* \circ L^*)(\omega) \\ &= M^*\big(L^*(\omega)\big) \\ &= \det(M) \cdot L^*(\omega) \\ &= \det(M) \cdot \det(L) \cdot \omega \end{aligned}$$

Determinants: Properties



Properties:

Given vector spaces V and W, an invertible linear map $M \in \mathrm{Hom}(V, W)$, and an endomorphism $L \in \mathrm{Hom}(V, V)$, we can define an endomorphism:

$$M^{-1} \circ L \circ M : V \to V$$

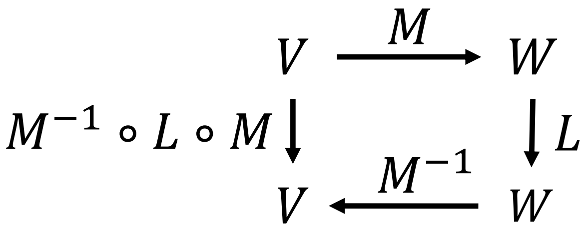
The determinant of this endomorphism satisfies:

$$\det(M^{-1} \circ L \circ M) = \det(L)$$

Note:

We cannot simply use the fact the determinant of the composition is the composition of determinants because M is not an endomorphism.

Determinants: Properties

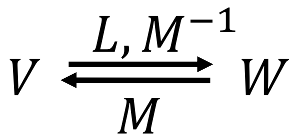


Proof:

Let ω_W be a volume form on W, let $\omega_V \equiv M^*(\omega_W)$ be its pull-back to V, and let $\{v_1, \ldots, v_n\}$ be a basis for V.

Expanding, we get:

$$
\begin{aligned}
\det(M^{-1} \circ L \circ M) \cdot \omega_V(v_1, \ldots, v_n) &= [(M^{-1} \circ L \circ M)^*(\omega_V)](v_1, \ldots, v_n) \\
&= \left[M^*\left(L^*\left(M^{-*}(\omega_V)\right)\right)\right](v_1, \ldots, v_n) \\
&= \left[M^*\left(L^*\left(M^{-*}\left(M^*(\omega_W)\right)\right)\right)\right](v_1, \ldots, v_n) \\
&= \left[M^*\left(L^*(\omega_W)\right)\right](v_1, \ldots, v_n) \\
&= [L^*(\omega_W)](M(v_1), \ldots, M(v_n)) \\
&= \det(L) \cdot \omega_W(M(v_1), \ldots, M(v_n)) \\
&= \det(L) \cdot [M^*(\omega_W)](v_1, \ldots, v_n) \\
&= \det(L) \cdot \omega_V(v_1, \ldots, v_n)
\end{aligned}
$$



Determinants: Properties

Properties:

Given vector spaces V and W, an invertible linear map $M \in \mathrm{Hom}(V, W)$, and an endomorphism $L \in \mathrm{Hom}(V, V)$, we can define an endomorphism:

$$M^{-1} \circ L \circ M: V \to V$$

The determinant of this endomorphism satisfies:

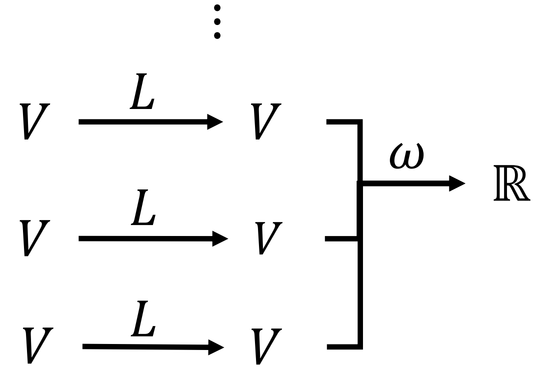
$$\det(M^{-1} \circ L \circ M) = \det(L)$$

Corollary:

Given vector spaces V and W, an **invertible** linear map $M \in \mathrm{Hom}(W, V)$, and a linear map $L \in \mathrm{Hom}(V, W)$:

$$\begin{aligned}\det(M \circ L) &= \det(M^{-1} \circ M \circ L \circ M)\\ &= \det(L \circ M)\end{aligned}$$

Determinants: Properties



Properties:

If $L \in \mathrm{Hom}(V,V)$ is invertible, the determinant of the inverse is the reciprocal of the determinant:

$$\det(L^{-1}) = \frac{1}{\det(L)}$$

Proof:

$$\begin{aligned} 1 &= \det(\mathrm{Id}_V) \\ &= \det(L \circ L^{-1}) \\ &= \det(L) \cdot \det(L^{-1}) \end{aligned}$$

$$\Downarrow$$

$$\det(L^{-1}) = \frac{1}{\det(L)}$$

Volume Forms

Claim:

Given a vector space V and vectors $\{w_1^*, \dots, w_n^*\} \subset V^*$, the map:

$$w_1^* \wedge \cdots \wedge w_n^* \colon V^n \to \mathbb{R}$$
$$(v_1, \dots, v_n) \mapsto \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \cdot \prod_{i=1}^{n} w_i^*(v_{\sigma(i)})$$

is a volume form on V.

Proof:

Cumbersome, but straightforward.

Volume Forms

Note:

$$\sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \cdot \prod_{i=1}^{n} w_i^*(v_{\sigma(i)}) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma^{-1}) \cdot \prod_{i=1}^{n} w_i^*(v_{\sigma^{-1}(i)})$$

$$= \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \cdot \prod_{i=1}^{n} w_i^*(v_{\sigma^{-1}(i)})$$

$$= \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \cdot \prod_{i=1}^{n} w_{\sigma(i)}^*(v_i)$$

It doesn't matter if we apply σ to the index of w_i^* or v_i.

Volume Forms

Note:

Given a vector space V with basis $\{v_1, \ldots, v_n\}$, let $\{v_1^*, \ldots, v_n^*\}$ be the dual basis, we have:

$$[v_1^* \wedge \cdots \wedge v_n^*](v_1, \ldots, v_n) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot \prod_{i=1}^{n} v_i^*\left(v_{\sigma(i)}\right)$$

For any permutation $\sigma \in S_n$ that is not the identity, there must be some index $i \in \{1, \ldots, n\}$ such that:

$$\sigma(i) \neq i$$

$\Rightarrow$ The product vanishes for all σ that are not the identity.

$$[w_1^* \wedge \cdots \wedge w_n^*](v_1, \ldots, v_n) = \text{sign}(\text{Id}) \cdot \prod_{i=1}^{n} v_i^*(v_i)$$

$$= 1$$

Volume Forms

Note:

Given a vector space V with basis $\{v_1, \dots, v_n\}$ and an endomorphism $L \in \mathrm{Hom}(V, V)$, let $\{v_1^*, \dots, v_n^*\}$ be the dual basis, we have:

$$\begin{aligned}
\det(L) \cdot [w_1^* \wedge \cdots \wedge w_n^*](v_1, \dots, v_n) &= [L^*(w_1^* \wedge \cdots \wedge w_n^*)](v_1, \dots, v_n) \\
&= [w_1^* \wedge \cdots \wedge w_n^*]\big(L(v_1), \dots, L(v_n)\big) \\
&= \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \cdot \prod_{i=1}^{n} v_i^*\big(L(v_{\sigma(i)})\big) \\
&= \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \cdot \prod_{i=1}^{n} \mathbf{L}_{i,\sigma(i)} \\
&= \det(\mathbf{L}) \cdot [w_1^* \wedge \cdots \wedge w_n^*](v_1, \dots, v_n)
\end{aligned}$$

Volume Forms

Claim:

If $L \in \mathrm{Hom}(V,V)$ is an endomorphism, we have:

$$[w_1^* \wedge \cdots \wedge w_n^*](L(v_1),\ldots,L(v_n)) = [L^*(w_1^*) \wedge \cdots \wedge L^*(w_n^*)](v_1,\ldots,v_n)$$

Proof:

$$\begin{aligned}[w_1^* \wedge \cdots \wedge w_n^*](L(v_1),\ldots,L(v_n)) &= \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) \cdot \prod_{i=1}^{n} w_i^*\left(L\left(v_{\sigma(i)}\right)\right) \\ &= \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) \cdot \prod_{i=1}^{n} [L^*(w_i^*)](v_{\sigma(i)}) \\ &= [L^*(w_1^*) \wedge \cdots \wedge L^*(w_n^*)](v_1,\ldots,v_n)\end{aligned}$$

Volume Forms

Claim:

If $L \in \mathrm{Hom}(V,V)$ is an endomorphism, we have:

$$[w_1^* \wedge \cdots \wedge w_n^*](L(v_1),\ldots,L(v_n)) = [L^*(w_1^*) \wedge \cdots \wedge L^*(w_n^*)](v_1,\ldots,v_n)$$

Corollary:

If $L \in \mathrm{Hom}(V,V)$ is an endomorphism, we have:

$$L^*(w_1^* \wedge \cdots \wedge w_n^*) = L^*(w_1^*) \wedge \cdots \wedge L^*(w_n^*)$$

Proof:

For any $\{v_1,\ldots,v_n\} \subset V$ we have:

$$\begin{aligned}[L^*(w_1^* \wedge \cdots \wedge w_n^*)](v_1,\ldots,v_n) &= [w_1^* \wedge \cdots \wedge w_n^*]\big(L(v_1),\ldots,L(v_n)\big) \\ &= [L^*(w_1^*) \wedge \cdots \wedge L^*(w_n^*)](v_1,\ldots,v_n)\end{aligned}$$

Volume Forms

Claim:

Under the association $V \simeq V^{**}$, for any basis $\{v_1, \dots, v_n\}$ for V and any (not necessarily dual) basis $\{w_1^*, \dots, w_n^*\}$ for V^* we have:

$$[w_1^* \wedge \cdots \wedge w_n^*](v_1, \dots, v_n) = [v_1 \wedge \cdots \wedge v_n](w_1^*, \dots, w_n^*)$$

Proof:

$$\begin{aligned}
[w_1^* \wedge \cdots \wedge w_n^*](v_1, \dots, v_n) &= \sum_{\sigma \in S_k} \text{sign}(\sigma) \cdot \prod_{i=1}^{n} w_i^*\left(v_{\sigma(i)}\right) \\
&= \sum_{\sigma \in S_k} \text{sign}(\sigma) \cdot \prod_{i=1}^{n} v_{\sigma(i)}(w_i^*) \\
&= \sum_{\sigma \in S_k} \text{sign}(\sigma) \cdot \prod_{i=1}^{n} v_i\left(w_{\sigma(i)}^*\right) \\
&= [v_1 \wedge \cdots \wedge v_n](w_1^*, \dots, w_n^*)
\end{aligned}$$

Volume Forms

Properties:

For $L \in \mathrm{Hom}(V, V)$:

$$\det(L) = \det(L^*)$$

Volume Forms

Proof:

Let $\{v_1, \dots, v_n\}$ be a basis for V, let $\{w_1^*, \dots, w_n^*\}$ and let be a basis for V^*.

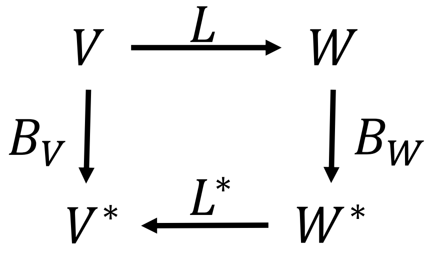
Under the association $V \simeq V^{**}$ we can define the volume form on V^*:

$$[v_1 \wedge \cdots \wedge v_n] \in \Lambda^n V^{**}$$

Pulling back via the dual L^* and evaluating on $\{w_1^*, \dots, w_n^*\}$ gives:

$$\begin{aligned}
\det(L^*) \cdot [v_1 \wedge \cdots \wedge v_n](w_1^*, \dots, w_n^*) &= [L^{**}(v_1 \wedge \cdots \wedge v_n)](w_1^*, \dots, w_n^*) \\
&= [L^{**}(v_1) \wedge \cdots \wedge L^{**}(v_n)](w_1^*, \dots, w_n^*) \\
&= [L(v_1) \wedge \cdots \wedge L(v_n)](w_1^*, \dots, w_n^*) \\
&= [w_1^* \wedge \cdots \wedge w_n^*](L(v_1), \dots, L(v_n)) \\
&= [L^*(w_1^* \wedge \cdots \wedge w_n^*)](v_1, \dots, v_n) \\
&= \det(L) \cdot [w_1^* \wedge \cdots \wedge w_n^*](v_1, \dots, v_n) \\
&= \det(L) \cdot [v_1 \wedge \cdots \wedge v_n](w_1^*, \dots, w_n^*)
\end{aligned}$$

Determinants



Recall:

Given inner-product spaces $\{V, B_V : V \to V^*\}$ and $\{W, B_W : W \to W^*\}$, a linear map $L \in \mathrm{Hom}(V, W)$ is *orthogonal* if the inner-product on V is the pull-back of the inner-product on W (as a bilinear form):

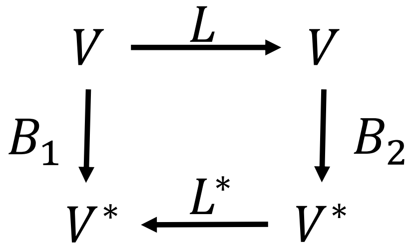
$$B_V = L^* \circ B_W \circ L$$

$\Rightarrow$ Taking $V = W$, if we have two inner-products $B_1, B_2 : V \to V^*$, an endomorphism $L \in \mathrm{Hom}(V, V)$ will be orthogonal if:

$$B_1 = L^* \circ B_2 \circ L$$

Q: What can we say about the determinant of L?

Determinants

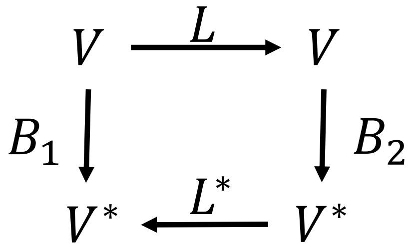


Claim:

Given a vector space V with two inner-product, $B_1, B_2 : V \to V$, if an endomorphism $L : \{V, B_1\} \to \{V, B_2\}$ is orthogonal, then:

$$\det(L) = \pm\sqrt{\det(B_2^{-1} \circ B_1)}$$

Determinants



Proof:

Since L is orthogonal, we have:

$$B_1 = L^* \circ B_2 \circ L$$

This gives:

$$\begin{aligned} 1 &= \det(B_1 \circ B_1^{-1}) \\ &= \det(L^* \circ B_2 \circ L \circ B_1^{-1}) \\ &= \det(L^*) \cdot \det(B_2 \circ L \circ B_1^{-1}) \\ &= \det(L) \cdot \det(L \circ B_1^{-1} \circ B_2) \\ &= \det(L) \cdot \det(L) \cdot \det(B_1^{-1} \circ B_2) \\ &= \frac{[\mathrm{det(L)}]^2}{\det(B_2^{-1} \circ B_1)} \end{aligned}$$

$$\Downarrow$$

$$\det(L) = \pm\sqrt{\det(B_2^{-1} \circ B_1)}$$

Determinants

Definition:

Given an inner-product space, $\{V, B: V \to V^*\}$, let $\{v_1, \ldots, v_n\}$ be a B-orthogonal basis. We say that a volume form $\omega \in \Lambda^n V^*$ is a *unit volume form* if:

$$\omega(v_1, \ldots, v_n) = \pm 1$$

Claim:

If ω is a unit volume form w.r.t. one B-orthogonal basis, it is a unit volume form for all B-orthogonal bases.

Determinants

Proof:

Let $\{w_1, \dots, w_n\}$ be some other B-orthogonal basis.

$\Rightarrow$ There exists an orthogonal transformation $L \in \mathrm{Hom}(V, V)$ such that:

$$L(v_i) = w_i$$

Applying ω to this basis gives:

$$\begin{aligned}
\omega(w_1, \dots, w_n) &= \omega\big(L(v_1), \dots, L(v_n)\big) \\
&= [L^*(\omega)](v_1, \dots, v_n) \\
&= \det(L) \cdot \omega(v_1, \dots, v_n) \\
&= \pm\sqrt{\det(B^{-1} \circ B)} \\
&= \pm 1
\end{aligned}$$

Determinants

Claim:

Given a vector space V with two inner-products, $B_1, B_2: V \to V^*$, two **unit** volume forms $\omega_1, \omega_2 \in \Lambda^n V^*$ are related by:

$$\omega_2 = \pm\sqrt{\det(B_1^{-1} \circ B_2)} \cdot \omega_1$$

Determinants

Proof:

Choose $L: \{V, B_2\} \rightarrow \{V, B_1\}$ to be an orthogonal transformation.

Since ω_2 is a unit-volume form, for any B_2-orthogonal basis $\{v_1, \dots, v_n\}$:

$$\pm 1 = \omega_2(v_1, \dots, v_n)$$

Since $\{L(v_1), \dots, L(v_n)\}$ is B_1-orthogonal:

$$\begin{aligned} \pm 1 &= \omega_1\big(L(v_1), \dots, L(v_n)\big) \\ &= \det(L) \cdot \omega_1(v_1, \dots, v_n) \end{aligned}$$

$$\Downarrow$$

$$\omega_2(v_1, \dots, v_n) = \pm \det(L) \cdot \omega_1(v_1, \dots, v_n)$$

$$\Downarrow$$

$$\omega_2 = \pm \det(L) \cdot \omega_1$$

$$\Downarrow$$

$$\omega_2 = \pm \sqrt{\det(B_1^{-1} \circ B_2)} \cdot \omega_1$$