

Geometry Processing (601.458/658)

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Outline

Recall

Duals and Duals of Duals

Bilinear Forms

Recall

Given vector spaces V and W , the space of linear maps from V to W is denoted $\text{Hom}(V, W)$.

Given a vector space V , the dual space is denoted $V^* \equiv \text{Hom}(V, \mathbb{R})$.

Given vector spaces V and W , and a $L \in \text{Hom}(V, W)$, the pull-back/dual $L^* \in \text{Hom}(W^*, V^*)$ is defined by composition:

$$[L^*(v^*)](v) = v^*(L(v)) \quad \forall v \in V, v^* \in V^*$$

Given a vector space V with basis $\{v_1, \dots, v_n\}$, the canonical dual basis for V^* is $\{v_1^*, \dots, v_n^*\}$ with:

$$v_i^*(v_j) = \delta_{ij}$$

Recall

Given a vector spaces V and W with bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$, and $L \in \text{Hom}(V, W)$, the coefficients of the matrix expression of L are given by:

$$L_{ij} = w_i^* (L(v_j))$$

Given a vector space V , we have a canonical association $V \simeq V^{**}$ with $v \in V$ treated as a linear map acting on $v^* \in V^*$ through evaluation:

$$v(v^*) \equiv v^*(v)$$

Given vector spaces U, V , and W , and linear maps $L \in \text{Hom}(U, V)$ and $M \in \text{Hom}(V, W)$, we have:

$$(L \circ M)^* = M^* \circ L^*$$

Outline

Recall

Duals and Duals of Duals

Bilinear Forms

Duals

Claim:

Given a vector space V , the dual of the identity map $\text{Id}_V \in \text{Hom}(V, V)$ is the identity map $\text{Id}_{V^*} \in \text{Hom}(V^*, V^*)$ on the dual space:

$$\text{Id}_V^* = \text{Id}_{V^*}$$

Proof:

Given vectors $v \in V$ and $v^* \in V^*$ we have:

$$\begin{aligned} [\text{Id}_V^*(v^*)](v) &= v^*(\text{Id}_V(v)) \\ &= v^*(v) \end{aligned}$$

Since this is true for all $v \in V$:

$$\text{Id}_V^*(v^*) = v^*$$

Since this is true for all $v^* \in V^*$:

$$\text{Id}_V^* = \text{Id}_{V^*}$$

Duals and Inverses

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ V^* & \xleftarrow{L^*} & W^* \\ V^* & \xrightarrow{(L^*)^{-1}} & W^* \end{array} \quad \begin{array}{ccc} V & \xleftarrow{L^{-1}} & W \\ V^* & \xrightarrow{(L^{-1})^*} & W^* \end{array}$$

Given two vector spaces V and W , and an invertible linear map $L \in \text{Hom}(V, W)$, we can define the dual of the inverse.

Q: How does the dual of the inverse relate to the inverse of the dual?

Duals and Inverses

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ V^* & \xleftarrow{L^*} & W^* \\ V^* & \xrightarrow{(L^*)^{-1}} & W^* \end{array} \quad \begin{array}{ccc} V & \xleftarrow{L^{-1}} & W \\ V^* & \xrightarrow{(L^{-1})^*} & W^* \end{array}$$

Claim:

The dual of the inverse is the inverse of the dual:

$$(L^{-1})^* = (L^*)^{-1}$$

Proof:

$$\begin{aligned} \text{Id}_{V^*} &= \text{Id}_V^* \\ &= (L \circ L^{-1})^* \\ &= (L^{-1})^* \circ L^* \\ &\Downarrow \\ (L^{-1})^* &= (L^*)^{-1} \end{aligned}$$

Since order doesn't matter, we will write the inverse of the transpose as L^{-*}

Duals of Duals

$$V \xrightarrow{L} W$$

$$V^* \xleftarrow{L^*} W^*$$

Given vector spaces V and W , and a linear map $L \in \text{Hom}(V, W)$ we can define the dual map $L^*: W^* \rightarrow V^*$.

We can also define the dual dual map, $L^{**}: V^{**} \rightarrow W^{**}$.

Q: How does the dual dual map L^{**} relate to the map L ?

Duals of Duals

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ \Phi_W \downarrow & \curvearrowright & \downarrow \Phi_V \\ V^{**} & \xrightarrow{L^{**}} & W^{**} \end{array}$$

Claim:

Under the associations $V \simeq V^{**}$ and $W \simeq W^{**}$ given by evaluation, the maps L and L^{**} are the same.

Equivalently:

Denoting by $\Phi_V: V \rightarrow V^{**}$ and $\Phi_W: W \rightarrow W^{**}$ the associations, we have:
$$\Phi_W \circ L = L^{**} \circ \Phi_V$$

Duals of Duals

$$\begin{array}{ccc} U & \xrightarrow{L} & V \\ \Phi_W \downarrow & \curvearrowright & \downarrow \Phi_V \\ U^{**} & \xrightarrow{L^{**}} & V^{**} \end{array}$$

Claim:

Denoting by $\Phi_V: V \rightarrow V^{**}$ and $\Phi_W: W \rightarrow W^{**}$ the associations, we have:

$$\Phi_W \circ L = L^{**} \circ \Phi_V$$

Proof:

Consider how the image of $v \in V$ acts on $w^* \in W^*$.

Left-hand-side:

$$[\Phi_W(L(v))](w^*) = w^*(L(v))$$

Right-hand-side:

$$\begin{aligned} [L^{**}(\Phi_V(v))](w^*) &= [\Phi_V(v)](L^*(w^*)) \\ &= [L^*(w^*)](v) \\ &= w^*(L(v)) \end{aligned}$$

Outline

Recall

Duals and Duals of Duals

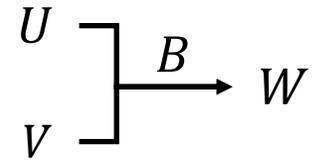
Bilinear Forms

Inner-product spaces

Adjoint

Orthonormal basis

Bilinear Maps



Given vector spaces U, V , and W , a map $B: U \times V \rightarrow W$ is *bilinear* if it is linear in both its inputs.

That is, for all $u, u_1, u_2 \in U$, all $v, v_1, v_2 \in V$, and all $\alpha, \beta \in \mathbb{R}$:

$$B(\alpha_1 \cdot u_1 + \beta \cdot u_2, v) = \alpha \cdot B(u_1, v) + \beta \cdot B(u_2, v)$$

$$B(u, \alpha \cdot v_1 + \beta \cdot v_2) = \alpha \cdot B(u, v_1) + \beta \cdot B(u, v_2)$$

Bilinear Maps: Pull-backs

$$\begin{array}{ccc} U & \xrightarrow{L} & V \\ U & \xrightarrow{L} & V \end{array} \left. \vphantom{\begin{array}{ccc} U & \xrightarrow{L} & V \\ U & \xrightarrow{L} & V \end{array}} \right\} \xrightarrow{B} W$$

Given vector spaces U, V , and W , a bilinear map $B: V \times V \rightarrow W$, and a linear map $L \in \text{Hom}(U, V)$:

The *pull-back* of the bilinear map $B: V \times V \rightarrow W$ through L is the bilinear map defined by composition on both arguments:

$$\begin{aligned} L^*(B): U \times U &\rightarrow W \\ (u, v) &\mapsto B(L(u), L(v)) \end{aligned}$$

Bilinear Forms

$$\begin{array}{c} V \\ \left. \vphantom{V} \right\} \\ V \end{array} \xrightarrow{B} \mathbb{R}$$

When $U = V$ and $W = \mathbb{R}$, we call the bilinear maps *bilinear forms*:

$$B: V \times V \rightarrow \mathbb{R}$$

Bilinear Forms: As linear maps

$$\begin{array}{c} V \\ \\ V \end{array} \left. \vphantom{\begin{array}{c} V \\ \\ V \end{array}} \right\} \xrightarrow{B} \mathbb{R}$$

Given a vector space V , and a bilinear form $B: V \times V \rightarrow \mathbb{R}$, fixing the second argument to a vector $v \in V$, we can define the map:

$$\begin{aligned} B^v: V &\rightarrow \mathbb{R} \\ w &\mapsto B(w, v) \end{aligned}$$

Since B is linear in the 1st argument, $B^v: V \rightarrow \mathbb{R}$ is a linear map.

$\Rightarrow B^v$ is an element of the dual space, $B^v \in V^*$.

$\Rightarrow B$ is equivalent to a map $B: V \rightarrow V^*$.

Since B is linear in the 2nd argument, the map $B: V \rightarrow V^*$ is linear.

$\Rightarrow B$ is equivalent to a linear map $B \in \text{Hom}(V, V^*)$:

$$[B(v)](u) \equiv B(u, v)$$

Bilinear Forms: As linear maps

$$\begin{array}{c} V \\ \\ V \end{array} \Big] \xrightarrow{B} \mathbb{R}$$

Given a vector space V , and a bilinear form $B: V \times V \rightarrow \mathbb{R}$, fixing the second argument to a vector $v \in V$, we can define the map:

$$\begin{aligned} B^v: V &\rightarrow \mathbb{R} \\ w &\mapsto B(w, v) \end{aligned}$$

Will think of a bilinear form as a linear map from the primal space to the dual.



This allows us to talk about the inverse of a bilinear form.
Which doesn't make sense for bilinear maps.

Sometimes the interpretation as a bilinear map will be clearer.

Bilinear Forms: As linear maps

$$\begin{array}{ccc} V & \xrightarrow{B} & V^* \\ V^* & \xleftarrow{B^*} & V^{**} \simeq V \end{array}$$

Given a vector space V , and a bilinear form $B: V \times V \rightarrow \mathbb{F}$, we have the dual linear map:

$$B^*: V^{**} \rightarrow V^*$$

Under the association $V \simeq V^{**}$, the dual of the bilinear map is also a map from the primal space to its dual.

$$B^*: V \rightarrow V^*$$

Q: How does B^* relate to B ?

Bilinear Forms: As linear maps

$$V \xrightarrow{B} V^*$$

$$V^* \xleftarrow{B^*} V^{**} \simeq V$$

Claim:

Under the association $V \simeq V^{**}$, the map B^* is the map B with inputs reversed.

Proof:

For vectors $u, v \in V$ we have:

$$\begin{aligned} [B^*(u)](v) &= u(B(v)) \\ &= [B(v)](u) \end{aligned}$$

Bilinear Maps: Matrix representation

Given a vector space V , a bilinear map $B: V \times V \rightarrow \mathbb{R}$, and a basis $\{v_1, \dots, v_n\}$, for all $u, v \in V$ we can associate u and v with column vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$:

$$\begin{aligned}u &= \mathbf{a}_1 \cdot v_1 + \dots + \mathbf{a}_n \cdot v_n \\v &= \mathbf{b}_1 \cdot v_1 + \dots + \mathbf{b}_n \cdot v_n\end{aligned}$$

This gives:

$$\begin{aligned}B(u, v) &= B\left(\sum_i \mathbf{a}_i \cdot v_i, \sum_i \mathbf{b}_i \cdot v_i\right) \\&= \sum_{i,j} \mathbf{a}_i \cdot \mathbf{b}_j \cdot B(v_i, v_j)\end{aligned}$$

Bilinear Maps: Matrix representation

$$B(u, v) = \sum_{i,j} \mathbf{a}_i \cdot \mathbf{b}_j \cdot B(v_i, v_j)$$

Setting $\mathbf{B} \in \mathbb{R}^{n \times n}$ to be the matrix:

$$\mathbf{B}_{ij} = B(v_i, v_j)$$

we get:

$$\begin{aligned} B(u, v) &= \sum_{i,j} \mathbf{a}_i \cdot \mathbf{b}_j \cdot \mathbf{B}_{ij} \\ &= \sum_i \mathbf{a}_i \cdot \sum_j \mathbf{B}_{ij} \cdot \mathbf{b}_j \\ &= \sum_i \mathbf{a}_i \cdot (\mathbf{B} \cdot \mathbf{b})_i \\ &= \mathbf{a}^\top \cdot \mathbf{B} \cdot \mathbf{b} \end{aligned}$$

Bilinear Forms: Matrix representation

Given a vector space V and a basis $\{v_1, \dots, v_n\}$, we have the dual dual basis $\{v_1^{**}, \dots, v_n^{**}\}$.

As a linear map $B: V \rightarrow V^*$, the matrix expression for B is given by:

$$\begin{aligned} \mathbf{B}_{ij} &= v_i^{**} \left(B(v_j) \right) \\ &= [B(v_j)] v_i \\ &= B^{vj}(v_i) \\ &= B(v_i, v_j) \end{aligned}$$

This is the same expression we got representing B as a bilinear map.

Bilinear Forms

Given a vector space V , and a bilinear form $B: V \rightarrow V^*$:

The form is *symmetric* if for all $u, v \in V$ we have:

$$\begin{aligned} & [B(u)](v) = [B(v)](u) \\ \Leftrightarrow & B(v, u) = B(u, v) \end{aligned}$$

The form is symmetric *positive semi-definite* if for all $v \in V$ we have:

$$\begin{aligned} & [B(v)](v) \geq 0 \\ \Leftrightarrow & B(v, v) \geq 0 \end{aligned}$$

The form is symmetric *positive definite* if for all $v \in V$ with $v \neq 0$ we have:

$$\begin{aligned} & [B(v)](v) > 0 \\ \Leftrightarrow & B(v, v) > 0 \end{aligned}$$

Bilinear Forms

Given a vector space V , and a bilinear form $B: V \times V \rightarrow \mathbb{R}$:

The form is *symmetric* if for all $u, v \in V$ we have:

$$\begin{aligned} & [B(u)](v) = [B(v)](u) \\ \Leftrightarrow & B(v, u) = B(u, v) \end{aligned}$$

The form is *symmetric positive semi-definite* if for all $v \in V$ we have:

$$\begin{aligned} & [B(v)](v) \geq 0 \\ \Leftrightarrow & B(v, v) \geq 0 \end{aligned}$$

Note:

It makes sense to talk about a bilinear map $B: V \times V \rightarrow \mathbb{R}$ being symmetric.

It makes sense to talk about a linear map $B: V \rightarrow V^*$ being symmetric.

It **does not** make sense to talk about an endomorphism being symmetric.

Bilinear Forms: Pull-Backs (redux)

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ V & \xrightarrow{L} & W \end{array} \left. \vphantom{\begin{array}{ccc} V & \xrightarrow{L} & W \\ V & \xrightarrow{L} & W \end{array}} \right\} \xrightarrow{B} \mathbb{R}$$

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ L^* \circ B \circ L \downarrow & & \downarrow B \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Recall:

Given vector spaces V and W , a linear map $L \in \text{Hom}(V, W)$, and a bilinear map $B: W \times W \rightarrow \mathbb{R}$, we defined the pull-back as:

$$[L^*(B)](u, v) \equiv B(L(u), L(v))$$

Q: What is the algebraic expression for the pull-back when viewed as a linear map from the primal to the dual?

Bilinear Forms: Pull-Backs (redux)

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ V & \xrightarrow{L} & W \end{array} \left. \vphantom{\begin{array}{ccc} V & \xrightarrow{L} & W \\ V & \xrightarrow{L} & W \end{array}} \right\} \xrightarrow{B} \mathbb{R}$$

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ L^* \circ B \circ L \downarrow & & \downarrow B \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Claim:

The pull-back of the bilinear form, expressed as a linear map is:

$$L^* \circ B \circ L: V \rightarrow V^*$$

Proof:

Given $v, w \in V$, we have:

$$\begin{aligned} [(L^* \circ B \circ L)(v)](w) &= \left[L^* \left(B(L(v)) \right) \right] (w) \\ &= [(B \circ L)(v)](L(w)) \\ &= [B(L(v))](L(w)) \\ &= B(L(v), L(w)) \end{aligned}$$

Bilinear Forms: Pull-Backs (redux)

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ V & \xrightarrow{L} & W \end{array} \left. \vphantom{\begin{array}{ccc} V & \xrightarrow{L} & W \\ V & \xrightarrow{L} & W \end{array}} \right\} \xrightarrow{B} \mathbb{R}$$

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ L^* \circ B \circ L \downarrow & & \downarrow B \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Claim:

The pull-back of the bilinear form, expressed as a linear map is:

$$L^* \circ B \circ L: V \rightarrow V^*$$

Note:

- \Rightarrow If B is symmetric, so is its pull-back.
- \Rightarrow If B is symmetric positive semi-definite, so is its pull-back
- \Rightarrow If L is non-singular (doesn't map a vector to zero) and B is symmetric positive definite, so is its pull-back

Bilinear Forms: Pull-Backs (redux)

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ V & \xrightarrow{L} & W \end{array} \left. \vphantom{\begin{array}{ccc} V & \xrightarrow{L} & W \\ V & \xrightarrow{L} & W \end{array}} \right\} \xrightarrow{B} \mathbb{R}$$

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ L^* \circ B \circ L \downarrow & \searrow^{B \circ L} & \downarrow B \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Claim:

The pull-back of the bilinear form, expressed as a linear map is:

$$L^* \circ B \circ L: V \rightarrow V^*$$

[DANGER (Overloaded Notation)]

As a linear map $B: W \rightarrow W^*$, the pull-back would be $B \circ L: V \rightarrow W^*$, which is not a bilinear form.

\Rightarrow To avoid confusion, will avoid writing $L^*(B)$.

positive definite, so is its pull-back

Inner-Product Spaces

$$V \xrightarrow{B} V^*$$

$$V^* \xleftarrow{B^*} V^{**}$$

Given a vector space V , and a symmetric positive definite bilinear form $B: V \rightarrow V^*$, we refer to B as an *inner-product* and to the pair $\{V, B: V \rightarrow V^*\}$ as an *inner-product space*.

Recall:

Under the association $V \simeq V^{**}$, the map B^* is the map B with inputs reversed.

\Rightarrow When B is symmetric (e.g. an inner-product), the two are equal:

$$B^* = B$$

$$V \xrightarrow{B} V^*$$

Inner-Product Spaces

Given a vector space V , and a symmetric positive definite bilinear form $B: V \rightarrow V^*$, we refer to B as an *inner-product* and to the pair $\{V, B: V \rightarrow V^*\}$ as an *inner-product space*.

Claim:

If B is an inner-product, then B is invertible.

Proof:

Otherwise there would exist $v \in V$ with $v \neq 0$ such that $B(v) = 0$.
 $\Rightarrow [B(v)](v) = 0$, contradicting positive definiteness.

Inner-Product Spaces

$$\begin{array}{c} V \simeq V^{**} \\ \downarrow B \\ V^* \end{array}$$

Given a vector space V , and a symmetric positive definite bilinear form $B: V \rightarrow V^*$, we refer to B as an *inner-product* and to the pair $\{V, B: V \rightarrow V^*\}$ as an *inner-product space*.

Given that B is invertible, we can also define a bilinear form:

$$B^{-1}: V^* \rightarrow V \simeq V^{**}$$

Q: Is this an inner-product on V^* ?

Inner-Product Spaces

$$\begin{array}{ccc} V^* & \xrightarrow{B^{-1}} & V \simeq V^{**} \\ & & \downarrow B \\ V^* & \xleftarrow{B^{-*}} & V^* \end{array}$$

Claim:

The bilinear form $B^{-1}: V^* \rightarrow V \simeq V^{**}$ is an inner-product.

Proof:

$$\begin{aligned} B^{-1} &= B^{-1} \circ B \circ B^{-1} \\ &= B^{-*} \circ B \circ B^{-1} \end{aligned}$$

$\Rightarrow B^{-1}$ is the pull-back of the bilinear form B from V to V^* .

Since B is symmetric positive definite and B^{-1} is invertible:

$\Rightarrow B^{-1}$ is also symmetric positive definite.

$\Rightarrow B^{-1}$ is an inner-product.

Inner-Product Spaces

$$\begin{array}{ccc}
 V^* & \xrightarrow{B^{-1}} & V \simeq V^{**} \\
 & & \downarrow B \\
 V^* & \xleftarrow{B^{-*}} & V^*
 \end{array}$$

Claim:

The bilinear form $B^{-1}: V^* \rightarrow V \simeq V^{**}$ is an inner-product.

Proof:

$$B^{-1} = \underbrace{B^{-1}}_{B^{-*}} \circ \underbrace{B}_{B} \circ \underbrace{B^{-1}}_{B^{-1}}$$

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 \end{array}
 \boxed{
 \begin{array}{c}
 \{V: B: V \rightarrow V^*\} \text{ is an inner-product space} \\
 \Updownarrow \\
 \{V^*, B^{-1}: V^* \rightarrow V \simeq V^{**}\} \text{ is an inner-product space}
 \end{array}
 }$$

$\Rightarrow B^{-1}$ is also symmetric positive definite.

$\Rightarrow B^{-1}$ is an inner-product.

Inner-Product Spaces

Given an inner-product space $\{V, B: V \rightarrow V^*\}$ we will denote the realization of B as a bilinear map using angle-bracket notation:

$$\begin{aligned}\langle \cdot, \cdot \rangle_B: V \times V &\rightarrow \mathbb{R} \\ (u, v) &\mapsto [B(v)](u)\end{aligned}$$

For all $v, w \in V$, an inner-product allows us to measure *length*:

$$\|v\|_B = \sqrt{\langle v, v \rangle_B}$$

and unsigned* *angle*:

$$\angle_B(v, w) = \cos^{-1} \left(\frac{\langle v, w \rangle_B}{\|v\|_B \cdot \|w\|_B} \right)$$

*Signing requires a notion of orientation, which is not encoded in an inner-product

Inner-Product Spaces: Examples

For the space of n -dimensional column vectors, \mathbb{R}^n , the standard inner-product is the *Euclidean inner-product*:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^n \mathbf{a}_i \cdot \mathbf{b}_i = \mathbf{a}^\top \cdot \mathbf{b}$$

Inner-Product Spaces: Examples

For the space of functions (e.g. bounded degree polynomials) defined on the interval $[0,1]$, we can define an inner-product through integration:

$$\langle f, g \rangle_B = \int_0^1 f(s) \cdot g(s) ds$$

If we have a (probability) density function $\rho: [0,1] \rightarrow \mathbb{R}^{>0}$, we can also define a density-modulated inner-product:

$$\langle f, g \rangle_B = \int_0^1 (f(s) \cdot g(s)) \cdot \rho(s) ds$$

Inner-Product Spaces: Examples

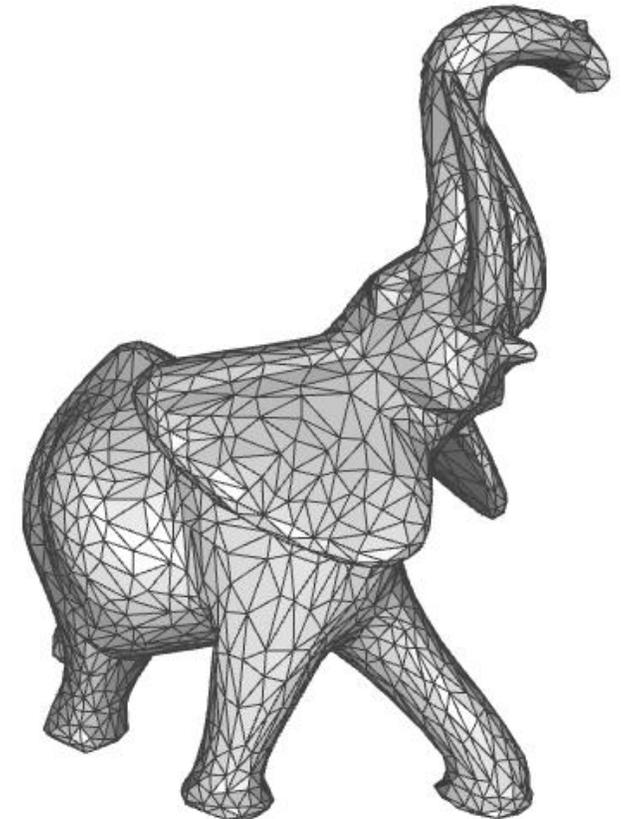
Give a triangle mesh $(\mathcal{V}, \mathcal{T})$, with \mathcal{V} the set of vertices and \mathcal{T} the set of triangles, we can define a discrete function space by assigning a value to each triangle:

$$V = \{\phi: \mathcal{T} \rightarrow \mathbb{R}\}.$$

We can define an inner product on the space of discrete functions using the area-weighted sum:

$$\langle \phi, \psi \rangle_B = \sum_{\tau \in \mathcal{T}} |\tau| \cdot \phi(\tau) \cdot \psi(\tau)$$

with $|\tau|$ the area of triangle τ .



Orthogonal Transformations

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ B_V \downarrow & & \downarrow B_W \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Given inner-product spaces $\{V, B_V: V \rightarrow V^*\}$ and $\{W, B_W: W \rightarrow W^*\}$, a linear map $L \in \text{Hom}(V, W)$ is *orthogonal* if the inner-product on V is the pull-back of the inner-product on W (as a bilinear form):

$$B_V = L^* \circ B_W \circ L$$

Note:

Thinking of inner-products as bilinear maps, this is equivalent to:

$$\langle v, w \rangle_{B_V} = \langle L(v), L(w) \rangle_{B_W}$$

for all $v, w \in V$.

Orthogonal Transformations

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ B_V \downarrow & & \downarrow B_W \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Given inner-product spaces $\{V, B_V: V \rightarrow V^*\}$ and $\{W, B_W: W \rightarrow W^*\}$, a linear map $L \in \text{Hom}(V, W)$ is *orthogonal* if the inner-product on V is the pull-back of the inner-product on W (as a bilinear form):

$$B_V = L^* \circ B_W \circ L$$

Note:

An orthogonal transformation must be injective. Otherwise, if there is a non-zero $v \in V$ such that $L(v) = 0$, we would have:

$$B_V(v) = (L^* \circ B_W \circ L)(v) = 0$$

contradicting the positive-definiteness of the inner-product.

An orthogonal transformation need not be surjective (unless the dimensions of the two spaces are the same).

Adjoint

$$\begin{array}{ccc} V & \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{L^\dagger} \end{array} & W \\ B_V \downarrow & \text{\(\circlearrowleft\}} & \downarrow B_W \\ V^* & \begin{array}{c} \xleftarrow{L^*} \end{array} & W^* \end{array}$$

Given inner-product spaces $\{V, B_V: V \rightarrow V^*\}$ and $\{W, B_W: W \rightarrow W^*\}$, and given a linear map $L \in \text{Hom}(V, W)$, the *adjoint* of L is the linear map $L^\dagger \in \text{Hom}(W, V)$ such that:

$$B_V \circ L^\dagger = L^* \circ B_W$$

Equivalently, for all $v \in V$ and $w \in W$:

$$\langle v, L^\dagger(w) \rangle_{B_V} = \langle L(v), w \rangle_{B_W}$$

\Rightarrow The adjoint is a linear map that keeps the inner-products equal when moving the linear map from one argument to the other.

Adjoint

$$\begin{array}{ccc} V & \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{L^\dagger} \end{array} & W \\ B_V \downarrow & & \downarrow B_W \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Given inner-product spaces $\{V, B_V: V \rightarrow V^*\}$ and $\{W, B_W: W \rightarrow W^*\}$, and given a linear map $L \in \text{Hom}(V, W)$, the *adjoint* of L is the linear map $L^\dagger \in \text{Hom}(W, V)$ such that:

$$B_V \circ L^\dagger = L^* \circ B_W$$

Since $B_V: V \rightarrow V^*$ is invertible, we can express the adjoint as:

$$L^\dagger = B_V^{-1} \circ L^* \circ B_W$$

Adjoint

$$\begin{array}{ccc} V & \xrightleftharpoons{L, L^{\dagger\dagger}} & W \\ B_V \downarrow & \xleftarrow{L^\dagger} & \downarrow B_W \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Given inner-product spaces $\{V, B_V: V \rightarrow V^*\}$ and $\{W, B_W: W \rightarrow W^*\}$, and given a linear map $L \in \text{Hom}(V, W)$, the *adjoint* of L is the linear map $L^\dagger \in \text{Hom}(W, V)$ such that:

$$\begin{aligned} B_V \circ L^\dagger &= L^* \circ B_W \\ &\Updownarrow \\ L^\dagger &= B_V^{-1} \circ L^* \circ B_W \end{aligned}$$

The adjoint of the adjoint is a map $L^{\dagger\dagger}: V \rightarrow W$.

Q: How does it relate to L ?

Adjoint

$$\begin{array}{ccc} V & \begin{array}{c} \xrightarrow{L, L^{\dagger\dagger}} \\ \xleftarrow{L^{\dagger}} \end{array} & W \\ B_V \downarrow & & \downarrow B_W \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Claim:

The adjoint of the adjoint is the original linear map:

$$L^{\dagger\dagger} = L.$$

Recall:

The dual of a composition is the composition of the duals, in reverse order:

$$(L \circ M)^* = M^* \circ L^*$$

If B is a symmetric bilinear form then it is its own dual: $B = B^*$.

If B is an inner-product then the inverse B^{-1} is also an inner-product.

The dual of the dual of a linear map is itself, $L^{**} = L$

Adjoint

$$\begin{array}{ccc} V & \xrightleftharpoons[L^\dagger]{L, L^{\dagger\dagger}} & W \\ B_V \downarrow & & \downarrow B_W \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Claim:

The adjoint of the adjoint is the original linear map:

$$L^{\dagger\dagger} = L.$$

Proof:

The adjoint can be expressed as:

$$L^\dagger = B_V^{-1} \circ L^* \circ B_W$$

\Rightarrow The adjoint of the adjoint is:

$$\begin{aligned} L^{\dagger\dagger} &= B_W^{-1} \circ (B_V^{-1} \circ L^* \circ B_W)^* \circ B_V \\ &= B_W^{-1} \circ B_W^* \circ L^{**} \circ B_V^{-*} \circ B_V \\ &= B_W^{-1} \circ B_W \circ L \circ B_V^{-1} \circ B_V \\ &= L \end{aligned}$$

Adjoint

Given an inner-product space $\{V, B: V \rightarrow V^*\}$ and an endomorphism $L \in \text{Hom}(V, V)$, we say that L is *self-adjoint* if $L^\dagger = L$.

Thinking of inner-products as bilinear maps:

Given $u, v \in V$:

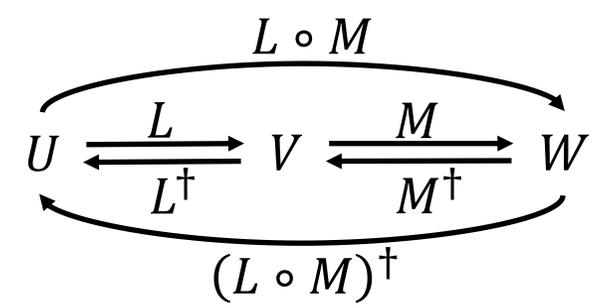
$$\langle L(u), v \rangle_B = \langle u, L(v) \rangle_B$$

\Rightarrow The linear map is self-adjoint if the inner-product is invariant to the choice of argument to which it is applied.

Note:

It only makes sense to talk about endomorphisms being self-adjoint, and only within the context of inner-product spaces.

Adjoint



Given inner-product spaces $\{U, B_U: U \rightarrow U^*\}$, $\{V, B_V: V \rightarrow V^*\}$ and $\{W, B_W: W \rightarrow W^*\}$, and invertible linear maps $L \in \text{Hom}(U, V)$ and $M \in \text{Hom}(V, W)$, we can define:

- The composition $L \circ M$

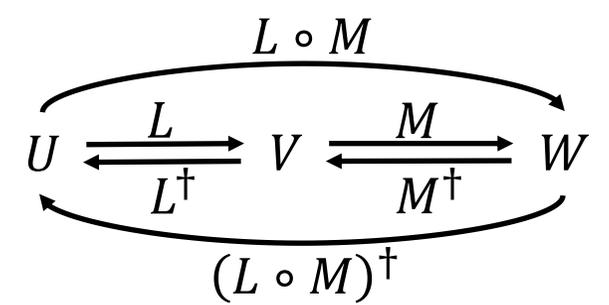
- The adjoint L^\dagger

- The adjoint M^\dagger

- The adjoint of the composition $(L \circ M)^\dagger$

Q: How do these relate?

Adjoint



Claim:

The adjoint of the composition is the composition of the adjoints, in the opposite order.

$$(M \circ L)^\dagger = L^\dagger \circ M^\dagger$$

Proof:

$$\begin{aligned} L^\dagger \circ M^\dagger &= B_U^{-1} \circ L^* \circ B_V \circ B_V^{-1} \circ M^* \circ B_W \\ &= B_U^{-1} \circ L^* \circ M^* \circ B_W \\ &= B_U^{-1} \circ (M \circ L)^* \circ B_W \\ &= (M \circ L)^\dagger \end{aligned}$$

Adjoint

$$\text{Id}_V \circlearrowleft V \begin{matrix} \xrightarrow{B} \\ \xleftarrow{B^{-1}} \end{matrix} V^* \circlearrowright \text{Id}_{V^*}$$

Claim:

Given an inner-product space $\{V, B: V \rightarrow V^*\}$, the identity, $\text{Id}_V \in \text{Hom}(V, V)$, is self-adjoint.

Proof:

$$\begin{aligned} \text{Id}_V^\dagger &= B^{-1} \circ \text{Id}_{V^*} \circ B \\ &= B^{-1} \circ \text{Id}_{V^*} \circ B \\ &= B^{-1} \circ B \\ &= \text{Id} \end{aligned}$$

Adjoint

$$\begin{array}{ccc} V & \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{L^\dagger} \end{array} & W \\ B_V \downarrow & & \downarrow B_W \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Claim:

Given inner-product spaces $\{V, B_V: V \rightarrow V^*\}$ and $\{W, B_W: W \rightarrow W^*\}$, and an invertible linear map $L \in \text{Hom}(V, W)$ The adjoint of the inverse is the inverse of the adjoint:

$$(L^\dagger)^{-1} = (L^{-1})^\dagger$$

Proof:

As with the inverse of the dual:

$$\begin{aligned} \text{Id}_W &= \text{Id}_W^\dagger \\ &= (L \circ L^{-1})^\dagger \\ &= (L^{-1})^\dagger \circ L^\dagger \\ &\Downarrow \\ (L^\dagger)^{-1} &= (L^{-1})^\dagger \end{aligned}$$

Adjoint

$$\begin{array}{ccc} V & \xrightleftharpoons[L^\dagger]{L} & W \\ B_V \downarrow & & \downarrow B_W \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Claim:

Given inner-product spaces $\{V, B_V: V \rightarrow V^*\}$ and $\{W, B_W: W \rightarrow W^*\}$, and an invertible linear map $L \in \text{Hom}(V, W)$ The adjoint of the inverse is the inverse of the adjoint:

$$(L^\dagger)^{-1} = (L^{-1})^\dagger$$

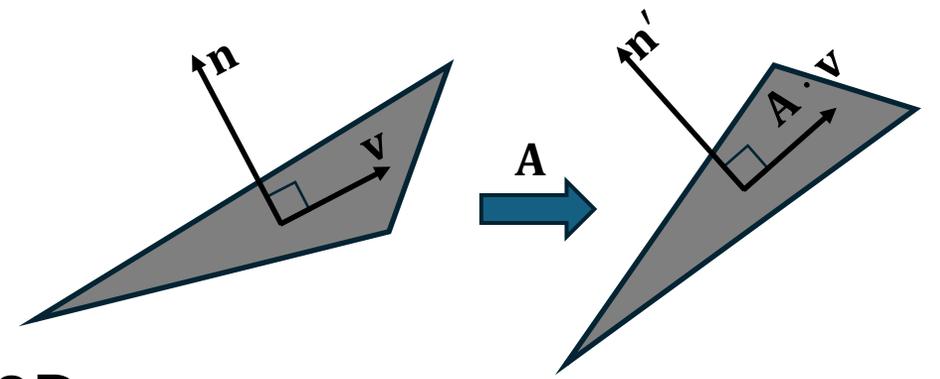
Proof:

Since order doesn't matter, we will write the inverse of the adjoint as $L^{-\dagger}$

As with the inverse of the adjoint.

$$\begin{aligned} \text{Id}_W &= \text{Id}_W^\dagger \\ &= (L \circ L^{-1})^\dagger \\ &= (L^{-1})^\dagger \circ L^\dagger \\ &\Downarrow \\ (L^\dagger)^{-1} &= (L^{-1})^\dagger \end{aligned}$$

Adjoint: Example



In graphics we often work with triangles in 3D.

We track a triangle's normal $\mathbf{n} \in \mathbb{R}^3$ as the vector perpendicular to all directions $\mathbf{v} \in \mathbb{R}^3$ that are tangent to the triangle:

$$\langle \mathbf{n}, \mathbf{v} \rangle = 0$$

We apply transformations $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ to the 3D space

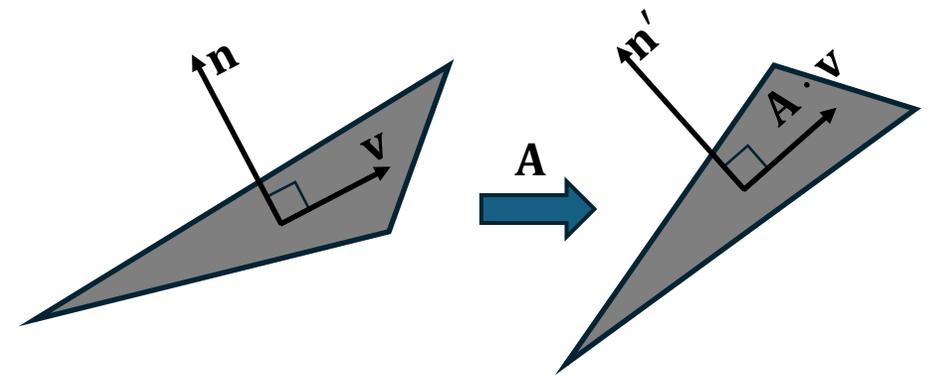
Q: What is the normal of the transformed triangle?

More generally:

Q: Since \mathbf{A} takes direction \mathbf{v} to $\mathbf{A} \cdot \mathbf{v}$, what is the $\mathbf{n}' \in \mathbb{R}^3$ such that:

$$\langle \mathbf{n}, \mathbf{v} \rangle = \langle \mathbf{n}', \mathbf{A} \cdot \mathbf{v} \rangle$$

Adjoint: Example



Claim:

The transformed normal, $\mathbf{n}' \in \mathbb{R}^3$ satisfying $\langle \mathbf{n}, \mathbf{v} \rangle = \langle \mathbf{n}', \mathbf{A} \cdot \mathbf{v} \rangle$ for all $\mathbf{v} \in \mathbb{R}^3$, is the inverse transpose of \mathbf{A} applied to the original normal:

$$\mathbf{n}' = \mathbf{A}^{-\top} \cdot \mathbf{n}$$

Recall:

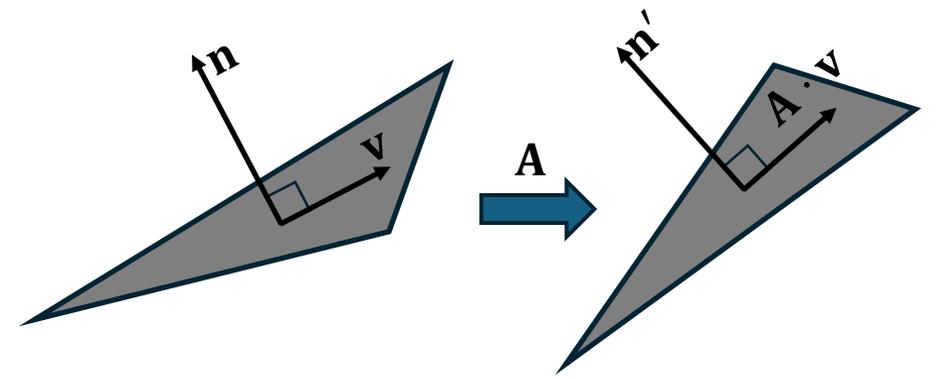
Given an inner-product space $\{V, B: V \rightarrow V^*\}$ and $L \in \text{Hom}(V, V)$, the adjoint L^\dagger is defined by:

$$L^\dagger = B^{-1} \circ L^* \circ B$$

and satisfies:

$$\langle L(u), v \rangle_B = \langle u, L^\dagger(v) \rangle_B$$

Adjoint: Example



Claim:

The transformed normal, $\mathbf{n}' \in \mathbb{R}^3$ satisfying $\langle \mathbf{n}, \mathbf{v} \rangle = \langle \mathbf{n}', \mathbf{A} \cdot \mathbf{v} \rangle$ for all $\mathbf{v} \in \mathbb{R}^3$, is the inverse transpose of \mathbf{A} applied to the original normal:

$$\mathbf{n}' = \mathbf{A}^{-\top} \cdot \mathbf{n}$$

Proof:

Assume there is a linear transformation $\mathbf{A}' \in \mathbb{R}^{3 \times 3}$ such that:

$$\mathbf{n}' = \mathbf{A}' \cdot \mathbf{n}$$

$$\Rightarrow \langle \mathbf{n}, \mathbf{v} \rangle = \langle \mathbf{A}' \cdot \mathbf{n}, \mathbf{A} \cdot \mathbf{v} \rangle$$

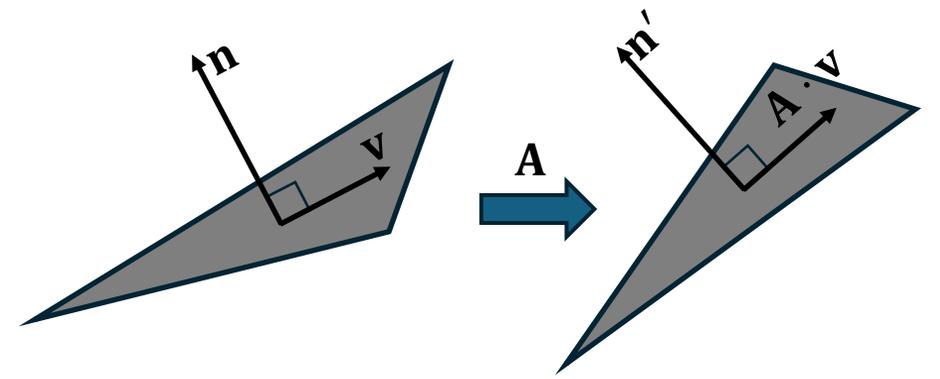
$$\Rightarrow \langle \mathbf{n}, \mathbf{v} \rangle = \langle \mathbf{n}, (\mathbf{A}')^\dagger \cdot \mathbf{A} \cdot \mathbf{v} \rangle$$

$$\Rightarrow (\mathbf{A}')^\dagger \cdot \mathbf{A} = \mathbf{Id}$$

$$\Rightarrow (\mathbf{A}')^\dagger = \mathbf{A}^{-1}$$

$$\Rightarrow \mathbf{A}' = \mathbf{A}^{-\dagger}$$

Adjoint: Example



Claim:

The transformed normal, $\mathbf{n}' \in \mathbb{R}^3$ satisfying $\langle \mathbf{n}, \mathbf{v} \rangle = \langle \mathbf{n}', \mathbf{A} \cdot \mathbf{v} \rangle$ for all $\mathbf{v} \in \mathbb{R}^3$, is the inverse transpose of \mathbf{A} applied to the original normal:

$$\mathbf{n}' = \mathbf{A}^{-\top} \cdot \mathbf{n}$$

Proof:

We've shown that:

$$\mathbf{n}' = \mathbf{A}^{-\dagger} \cdot \mathbf{n}$$

With respect to the standard Euclidean basis, the matrix representation for the inner-product is the identity:

$$\begin{aligned} \mathbf{A}^{\dagger} &= \mathbf{B}^{-1} \circ \mathbf{A}^{\top} \circ \mathbf{B} \\ &= \mathbf{A}^{\top} \end{aligned}$$

$$L \circlearrowleft V \begin{array}{c} \xrightarrow{B, D} \\ \xleftarrow{B^{-1}} \end{array} V^*$$

Adjoint

Given an inner-product space $\{V, B: V \rightarrow V^*\}$ and given a symmetric bilinear form $D: V \rightarrow V^*$, we can define the endomorphism:

$$B^{-1} \circ D: V \rightarrow V$$

Claim:

The endomorphism $L = B^{-1} \circ D$ is self-adjoint.

Proof:

$$\begin{aligned} L^\dagger &= B^{-1} \circ (B^{-1} \circ D)^* \circ B \\ &= B^{-1} \circ D^* \circ (B^{-1})^* \circ B \\ &= B^{-1} \circ D \circ B^{-1} \circ B \\ &= B^{-1} \circ D \\ &= L \end{aligned}$$

$$L \circlearrowleft V \begin{array}{c} \xrightarrow{B, D} \\ \xleftarrow{B^{-1}} \end{array} V^*$$

Adjoint

Given an inner-product space $\{V, B: V \rightarrow V^*\}$ and given a symmetric bilinear form $D: V \rightarrow V^*$, we can define the endomorphism:

$$B^{-1} \circ D: V \rightarrow V$$

Claim:

The endomorphism $L = B^{-1} \circ D$ is self-adjoint.

Similarly for a self-adjoint linear map $L: V \rightarrow V$, we can define:

$$B \circ L: V \rightarrow V^*$$

Claim:

The bilinear form $D = B \circ L$ is self-adjoint.

$$L \circlearrowleft V \begin{matrix} \xrightarrow{B, D} \\ \xleftarrow{B^{-1}} \end{matrix} V^*$$

Adjoint

Given an inner-product space $\{V, B: V \rightarrow V^*\}$ and given a symmetric bilinear form $D: V \rightarrow V^*$, we can define the endomorphism:

$$B^{-1} \circ D: V \rightarrow V$$

Claim:

The endomorphism $L = B^{-1} \circ D$ is self-adjoint.

Sim For inner-product spaces, there is an equivalence between symmetric bilinear forms and self-adjoint endomorphisms.

Claim: However, we can talk about symmetric bilinear forms without an inner-product!

The bilinear form $D = B \circ L$ is self-adjoint.

Orthogonal Basis

Given an inner-product space $\{V, B: V \rightarrow V^*\}$ we say that a basis $\{v_1, \dots, v_n\}$ is *orthogonal* if the basis vectors are unit-length and mutually perpendicular:

$$\langle v_i, v_j \rangle_B = [B(v_i)](v_j) = \delta_{ij}$$

\Rightarrow If $\{v_1, \dots, v_n\}$ is an orthogonal basis, then the expression for B w.r.t. the bases $\{v_1, \dots, v_n\}$ and $\{v_1^*, \dots, v_n^*\}$ is the identity:

$$\mathbf{B} = \text{Id} \in \mathbb{R}^{n \times n}$$

Orthogonal Basis

Given an inner-product space $\{V, B: V \rightarrow V^*\}$ and a basis $\{v_1, \dots, v_n\}$ we can turn it into an orthogonal basis using Gram-Schmidt:

$$\begin{aligned}v_1 &\leftarrow \frac{v_1}{\|v_1\|_B} \\v_2 &\leftarrow \frac{v_2 - v_1 \cdot \langle v_1, v_2 \rangle_B}{\|v_2 - v_1 \cdot \langle v_1, v_2 \rangle_B\|} \\v_3 &\leftarrow \frac{v_3 - v_1 \cdot \langle v_1, v_3 \rangle_B - v_2 \cdot \langle v_2, v_3 \rangle_B}{\|v_3 - v_1 \cdot \langle v_1, v_3 \rangle_B - v_2 \cdot \langle v_2, v_3 \rangle_B\|} \\&\vdots\end{aligned}$$

Gram-Schmidt:
for $i \in [1, n]$
 for $j \in [1, i)$
 $v_i \leftarrow v_i - v_j \cdot \langle v_j, v_i \rangle_B$
 $v_i \leftarrow \frac{v_i}{\|v_i\|_B}$

Orthogonal Basis

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ B_V \downarrow & & \downarrow B_W \\ V^* & \xleftarrow{L^*} & W^* \end{array}$$

Recall:

Given inner-product spaces $\{V, B_V: V \rightarrow V^*\}$ and $\{W, B_W: W \rightarrow W^*\}$, a linear map $L \in \text{Hom}(V, W)$ is orthogonal w.r.t. the two inner-products if:

$$B_V = L^* \circ B_W \circ L$$

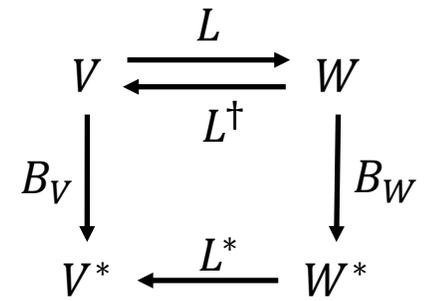
\Rightarrow Given orthogonal bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$, if $\mathbf{L} \in \mathbb{R}^{m \times n}$ is the matrix expression for L , we have:

$$\begin{aligned} \mathbf{B}_V &= \mathbf{L}^\top \cdot \mathbf{B}_W \cdot \mathbf{L} \\ \Leftrightarrow \mathbf{Id} &= \mathbf{L}^\top \cdot \mathbf{Id} \cdot \mathbf{L} \\ &= \mathbf{L}^\top \cdot \mathbf{L} \end{aligned}$$

If L is invertible, this gives:

$$\mathbf{L}^{-1} = \mathbf{L}^\top$$

Orthogonal Basis



Recall:

Given inner-product spaces $\{V, B_V: V \rightarrow V^*\}$ and $\{W, B_W: W \rightarrow W^*\}$, given a linear map $L \in \text{Hom}(V, W)$, the *adjoint* of L is:

$$L^\dagger = B_V^{-1} \circ L^* \circ B_W$$

\Rightarrow Given orthogonal bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$, if $\mathbf{L} \in \mathbb{R}^{m \times n}$ is the matrix expression for L , we have:

$$\begin{aligned} \mathbf{L}^\dagger &= \mathbf{B}_V^{-1} \cdot \mathbf{L}^\top \cdot \mathbf{B}_W \\ &= \mathbf{L}^\top \end{aligned}$$

\Rightarrow If the bases are orthogonal then, as for the dual, the matrix expression for the adjoint is the transpose:

$$\mathbf{L}^* = \mathbf{L}^\top$$