Geometry Processing (601.458/658)

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Outline

Recall

Vector spaces

Dual spaces

Recall

We denote by $\mathbb{R}$ the set of *real numbers*.

We denote by $\mathbb{R}^n$ the n-dim. *column vectors* (of real numbers):

$$\mathbb{R}^n \equiv \left\{ \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix} \middle| \mathbf{a}_i \in \mathbb{R} \right\}$$

We denote by $\mathbb{R}^{m \times n}$ the $(m \times n)$-dim. *matrices* (of real numbers):*

$$\mathbb{R}^{m \times n} \equiv \left\{ \begin{pmatrix} \mathbf{M}_{11} & \cdots & \mathbf{M}_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{m1} & \cdots & \mathbf{M}_{mn} \end{pmatrix} \middle| \mathbf{M}_{ij} \in \mathbb{R} \right\}$$

*First index is row, second is column

Recall

For $\mathbf{a} \in \mathbb{R}^n$ we denote the i*-th coefficient* as $\mathbf{a}_i$.

For $\mathbf{M} \in \mathbb{R}^{m\times n}$ we denote the (i,j)*-th coefficient* as $\mathbf{M}_{ij}$.

For matrix $\mathbf{M} \in \mathbb{R}^{m\times n}$ we denote by $\mathbf{M}^\top \in \mathbb{R}^{n\times m}$ the *transpose* of $\mathbf{M}$:

$$\mathbf{M}^\top_{ij} = \mathbf{M}_{ji}$$

Recall

For $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{M} \in \mathbb{R}^{m \times n}$ the *product* $\mathbf{b} = \mathbf{M} \cdot \mathbf{a}$ is the m-dimensional column vector with:

$$\mathbf{b}_i = \sum_{j=1}^{n} \mathbf{M}_{ij} \cdot \mathbf{a}_j$$

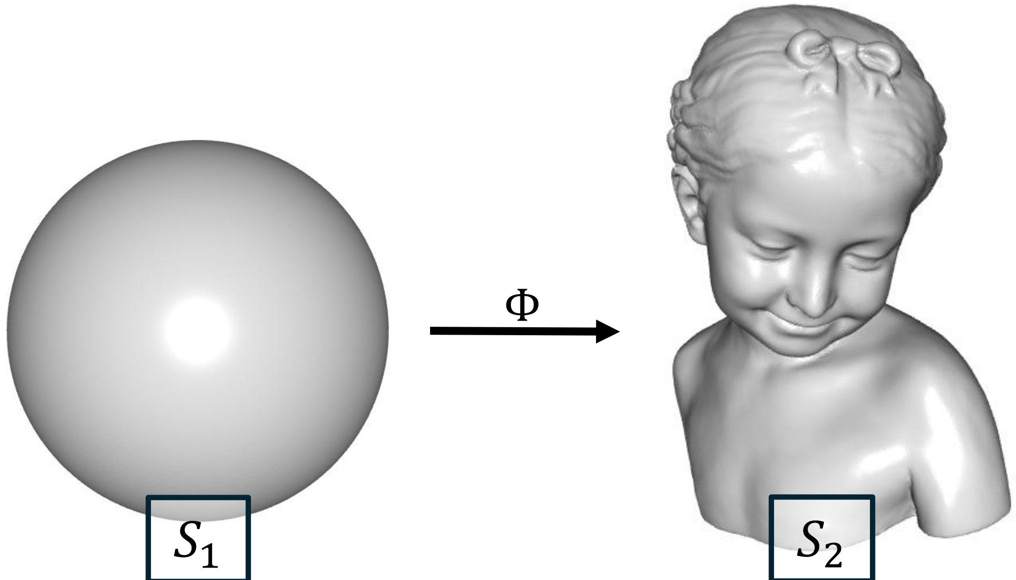
For $\mathbf{M} \in \mathbb{R}^{l \times m}$ and $\mathbf{N} \in \mathbb{R}^{m \times n}$ the *product* $\mathbf{L} = \mathbf{M} \cdot \mathbf{N}$ is the $(l \times n)$-dimensional matrix with:

$$\mathbf{L}_{ij} = \sum_{k=1}^{m} \mathbf{M}_{ik} \cdot \mathbf{N}_{kj}$$

Recall

Given sets S and D, we denote by S^D the *set of* D*-valued functions on* S.

Given sets S_1 and S_2, let $\Phi: S_1 \rightarrow S_2$ be some function from S_1 to S_2.



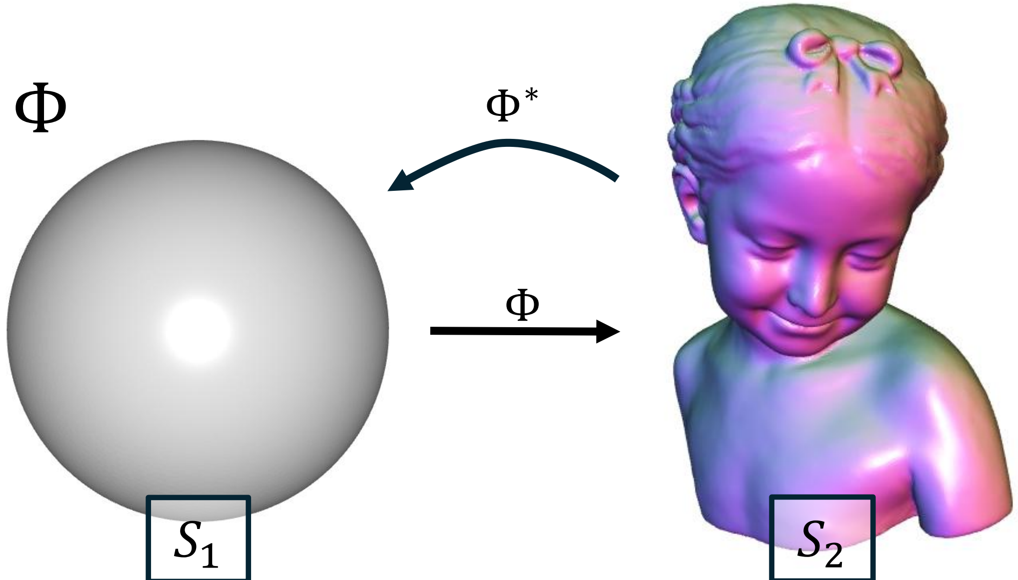
Recall

Given sets S and D, we denote by S^D the set of D-valued functions on S.

Given sets S_1 and S_2, let $\Phi: S_1 \to S_2$ be some function from S_1 to S_2.

The *pull-back*, Φ^*, is a map taking functions $S_2 \to D$ to functions $S_1 \to D$, defined by composition:

$$\begin{aligned} \Phi^*: S_2^D &\to S_1^D \\ f &\mapsto f \circ \Phi \end{aligned}$$



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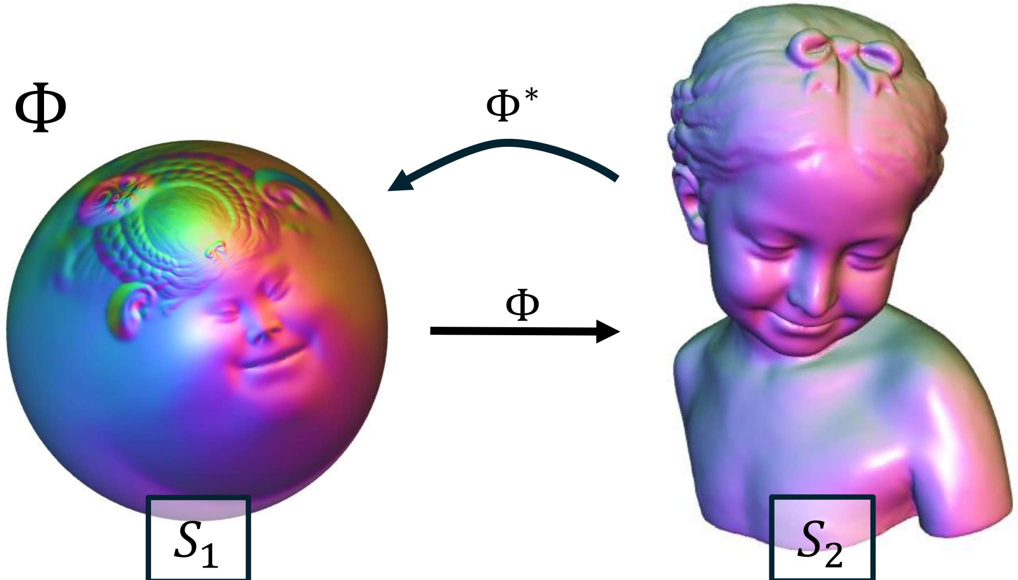
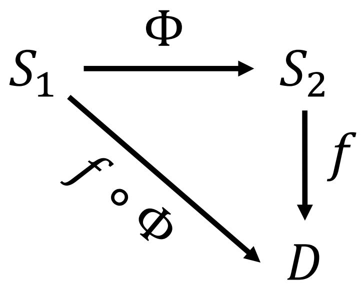


Recall



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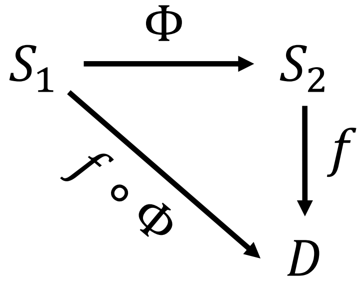
$$\begin{aligned} \Phi^*: S_2^D &\to S_1^D \\ f &\mapsto f \circ \Phi \end{aligned}$$

This is standard notation for functions.

Recall



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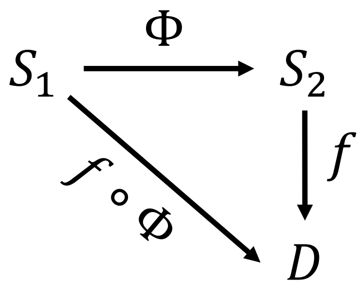
$$\Phi^*: S_2^D \to S_1^D$$
$$f \mapsto f \circ \Phi$$

The first line describes the domain/range of the function.

Recall



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Given sets S_1 and S_2, let $\Phi: S_1 \to S_2$ be some function from S_1 to S_2.

The *pull-back*, Φ^*, is a map taking functions $S_2 \to D$ to functions $S_1 \to D$, defined by composition:

$$\Phi^*: S_2^D \to S_1^D$$
$$\boxed{f \mapsto f \circ \Phi}$$

The second line describes what the function does to its input.

Outline

Recall

Vector spaces

- Linear maps
- Bases
- Matrices

Dual spaces

Vector Spaces

A (real) *vector space* V is a set with:

An addition operator "+",
⇒ Adding two vectors gives a vector

A scaling operator "$\cdot$"
⇒ Scaling a vector by a real number gives a vector

Vector Spaces

A *subspace* $W \subset V$ is a subset of V which is also a vector space.

Vector Spaces (Examples)

Vector Spaces:

- The set $V = \{0\}$
- The real numbers
- The space of n-dimensional column vectors
- The space of $(m \times n)$-dimensional matrices
- The space of bounded-degree polynomials
- The space of real-valued functions on a set

$$(\alpha \cdot f + \beta \cdot g)(x) \equiv \alpha \cdot f(x) + \beta \cdot g(x)$$

- The space of vector-valued functions on a set

$$V \xrightarrow{L} W$$

Linear Maps

Given vector spaces V and W:

We say that a map $L: V \to W$ is *linear* if for all vectors $v_1, v_2 \in \mathrm{V}$ and all $\alpha, \beta \in \mathbb{R}$:

$$L(\alpha \cdot v_1 + \beta \cdot v_2) = \alpha \cdot L(v_1) + \beta \cdot L(v_2)$$

We call a linear map from a vector space into itself, $L: V \to V$, an *endomorphism*.

$$V \xrightarrow{L} W$$

Linear Maps

Given vector spaces V and W:

We denote the set of linear maps by $\mathrm{Hom}(V,W)$.*

If $L, M \in \mathrm{Hom}(V,W)$ and $\alpha, \beta \in \mathbb{R}$ then the map:

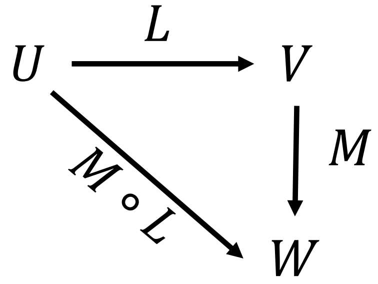
$$\begin{aligned} \alpha \cdot L + \beta \cdot M : V &\to W \\ v &\mapsto \alpha \cdot L(v) + \beta \cdot M(v) \end{aligned}$$

is also a linear map.

$\Rightarrow \mathrm{Hom}(V,W)$ is a vector space.

*Short for *homomorphism*

Linear Maps

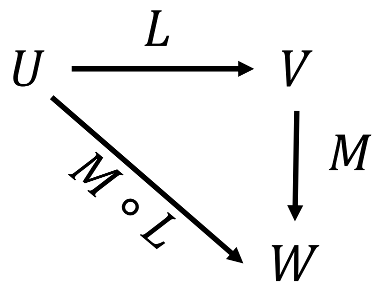


Given vector spaces U, V, and W:

If we have linear maps $L \in \mathrm{Hom}(U, V)$ and $M \in \mathrm{Hom}(V, W)$, the composition $M \circ L$ is also a linear map.

$\Rightarrow M \circ L \in \mathrm{Hom}(U, W)$

Linear Maps (Example 1)



Given vector spaces U, V, and W, and given $L \in \mathrm{Hom}(U, V)$:

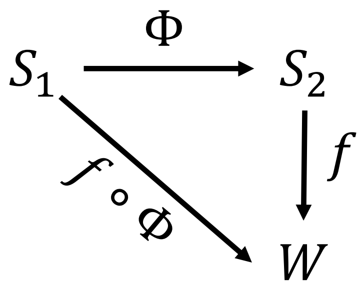
The pull-back:

$$L^*: \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W)$$
$$M \mapsto M \circ L$$

is a linear map between vector spaces:

$$(\alpha \cdot M + \beta \cdot N) \circ L = \alpha \cdot (M \circ L) + \beta \cdot (N \circ L)$$

Linear Maps (Example 2)



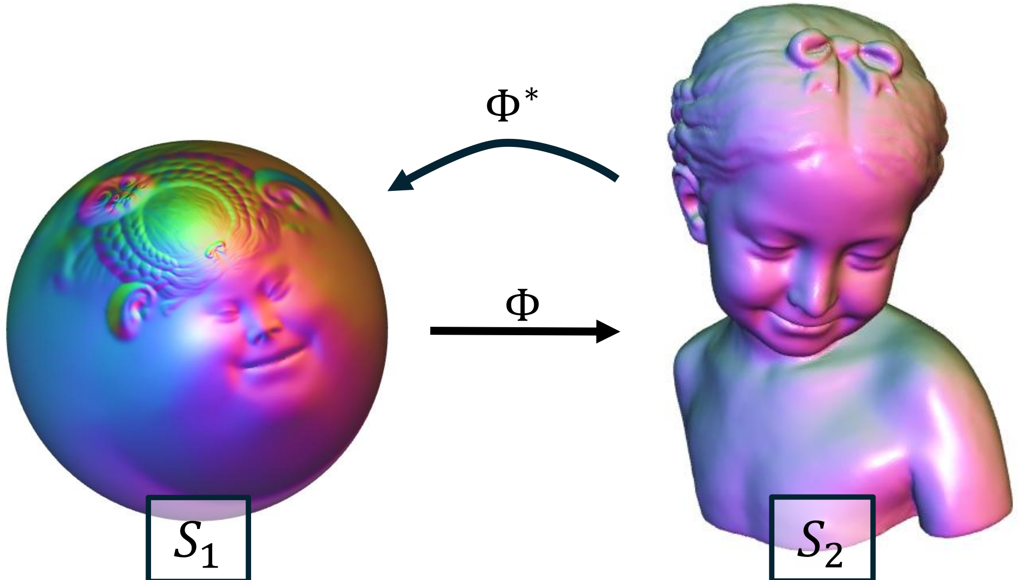
Given sets S_1 and S_2, and a vector space W:

Recall that the set of W-valued functions S_1^W and S_2^W are vector spaces.

For any function $\Phi: S_1 \rightarrow S_2$, the pull-back:

$$\Phi^*: S_2^W \rightarrow S_1^W$$
$$f \mapsto f \circ \Phi$$

is a linear map.



Linear Maps (Example 3)

If V is the space of bounded-degree polynomials, the operator taking the k-th derivative is a linear operator.

If $P(s)$ and $Q(s)$ are polynomials:

$$\frac{d}{ds}(\alpha \cdot P + \beta \cdot Q) = \alpha \cdot \frac{dP}{ds} + \beta \cdot \frac{dQ}{ds}$$

$$\frac{d^2}{ds^2}(\alpha \cdot P + \beta \cdot Q) = \alpha \cdot \frac{d^2P}{ds^2} + \beta \cdot \frac{d^2Q}{ds^2}$$

$$\vdots$$

Note that integration is also a linear map on the space of finite-degree polynomials.

But it raises the degree of the polynomial.
So it's not a linear map on the space polynomials whose degree is bounded by a constant.

Vector Space Bases

The *span of vectors* $\{v_1, \ldots, v_n\}$, denoted $\mathrm{Span}(v_1, \ldots, v_n)$, is the subset of vectors in V that is the linear combinations of $v_1, \ldots, v_n$:

$$v \in \mathrm{Span}(v_1, \ldots, v_n) \quad \Leftrightarrow \quad v = \mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_n \cdot v_n$$

for all $\mathbf{a}_i \in \mathbb{R}$.

This it itself a subspace.

The vectors are *linearly independent* if, for $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}$:

$$\mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_n \cdot v_n = 0$$

is true if and only if $\mathbf{a}_i = 0$ for all $1 \leq i \leq n$.

Vector Space Basis

We say that the set $\{v_1, \dots, v_n\}$ is a *basis* if for any non-zero $v_0 \in V$, the set $\{v_0, v_1, \dots, v_n\}$ is *not* linearly independent:

$$\mathbf{a}_0 \cdot v_0 + \mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_n \cdot v_n = 0$$

with $\mathbf{a}_i \neq 0$ for some $1 \leq i \leq n$.

$$\Updownarrow$$

$$\mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_n \cdot v_n = -\mathbf{a}_0 \cdot v_0$$

Note that $\mathbf{a}_0 \neq 0$, otherwise $v_1, \dots, v_n$ would be linearly dependent.

$$\Updownarrow$$

$$-\frac{\mathbf{a}_1}{\mathbf{a}_0} \cdot v_1 - \cdots - \frac{\mathbf{a}_n}{\mathbf{a}_0} \cdot v_n = v_0$$

$\Leftrightarrow v_0 \in \mathrm{Span}(v_1, \dots, v_n)$

Vector Space Basis

If $\{v_1, \ldots, v_n\} \subset V$ is a basis, we say that V *has dimension* n.

For a vector space V, we denote its dimension as $\dim(V)$.

For vector spaces V and W, the dimension of the space of homomorphisms from V to W is:

$$\dim\big(\mathrm{Hom}(V, W)\big) = \dim(V) \times \dim(W)$$

Throughout the course, we will assume that all vector spaces are finite-dimensional.

Vector Space Basis

Given a choice of basis $\{v_1, \dots, v_n\}$ for V, for any $v \in V$, there exist $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^n$ such that*:

$$v = \mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_n \cdot v_n$$

We refer to $\mathbf{a} = (\mathbf{a}_1, \cdots, \mathbf{a}_n)^\top \in \mathbb{R}^n$ as the *column vector representation* of v w.r.t. (with respect to) $\{v_1, \dots, v_n\}$.

The coefficients are unique**.

A choice of a basis defines an identification of V with $\mathbb{R}^n$.

*Otherwise the set $\{v_1, \dots, v_n\}$ is not a basis

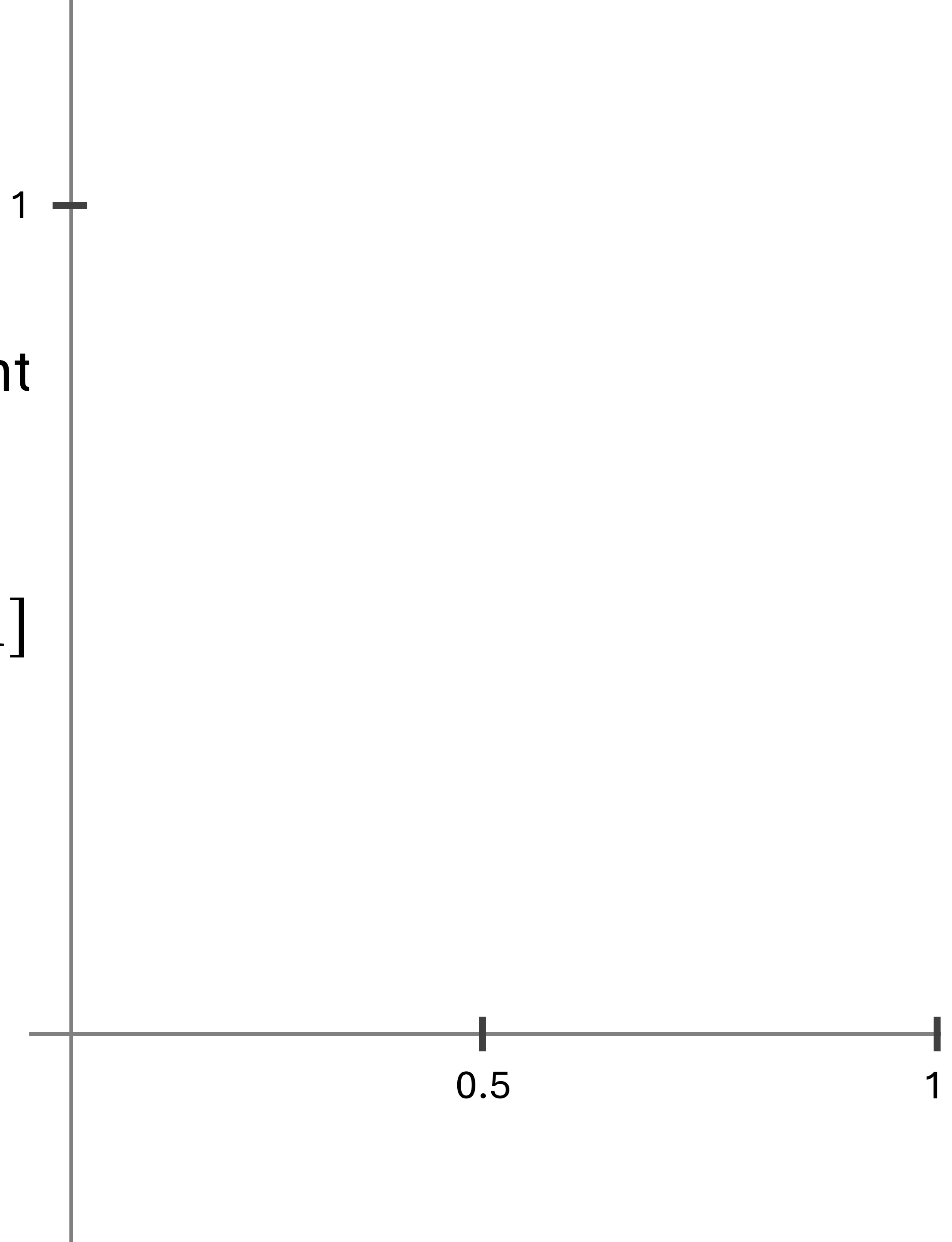
**Otherwise the vectors $v_1, \dots, v_n$ are linearly independent

Vector Space Basis

Different bases can be used to represent the same vector space.

Let $\mathbb{P}_2$ be quadratic polynomials on $[0,1]$

$$\mathbb{P}_2 = \{a + bs + cs^2 | a, b, c \in \mathbb{R}\}$$



Vector Space Basis

Different bases can be used to represent the same vector space.

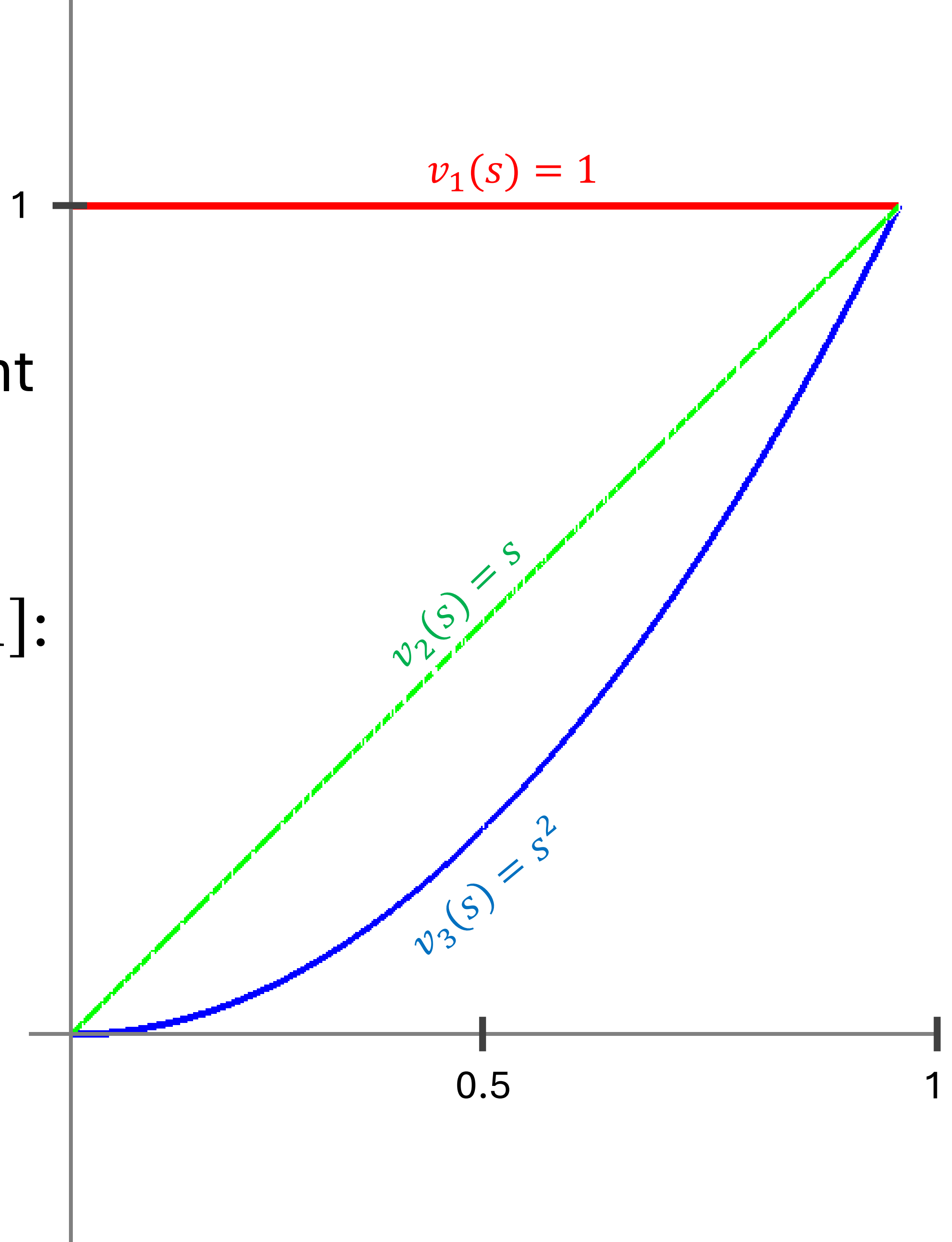
Let $\mathbb{P}_2$ be quadratic polynomials on $[0,1]$:

$$\mathbb{P}_2 = \{a + bs + cs^2 | a, b, c \in \mathbb{R}\}$$

We could use the monomial basis.

$$v_1 = 1$$
$$v_2 = s$$
$$v_3 = s^2$$

- ✓ Easy to differentiate
- ✗ Hard to fit to known values at 0, 0.5, and 1



Vector Space Basis

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Let $\mathbb{P}_2$ be quadratic polynomials on $[0,1]$:

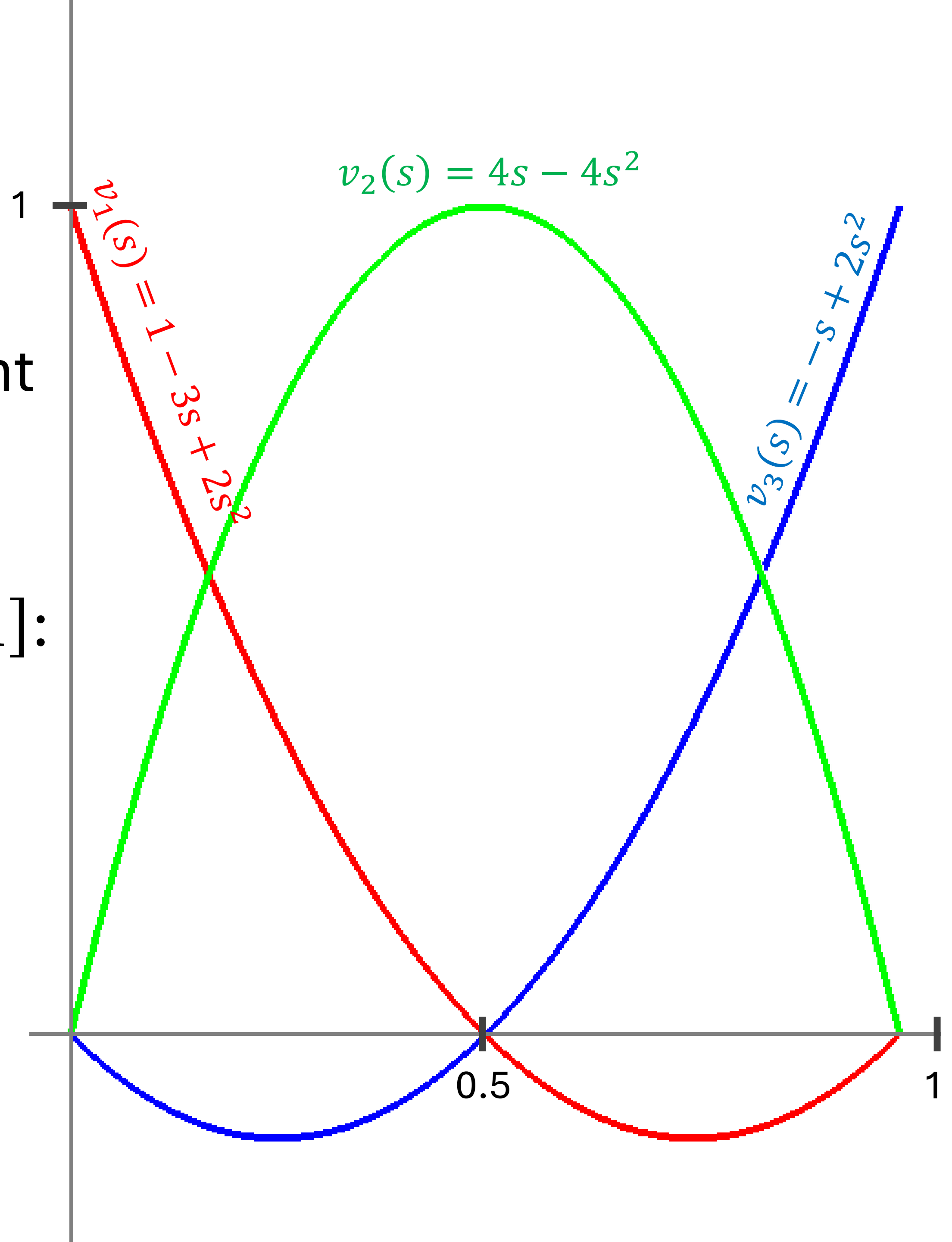
$$\mathbb{P}_2 = \{a + bs + cs^2 | a, b, c \in \mathbb{R}\}$$

We could use the monomial basis.

Or we could use the Lagrange basis.

$$v_1 = 1 - 3s + 2s^2$$
$$v_2 = 4s - 4s^2$$
$$v_3 = -s + 2s^2$$

- ✗ Hard to differentiate
- ✓ Easy to fit to known values at 0, 0.5, and 1



Matrices

$$V \xrightarrow{L} W$$

Given a linear map $L: V \to W$ and bases $\{v_1, \dots, v_n\} \subset V$ and $\{w_1, \dots, w_m\} \subset W$:

For every v_j, with $1 \leq j \leq n$, we have $L(v_j) \in W$.

$\Rightarrow$ We can express:

$$L(v_j) = \mathbf{L}_{1j} \cdot w_1 + \cdots + \mathbf{L}_{mj} \cdot w_m$$

with $\mathbf{L}_{ij} \in \mathbb{R}$.

We refer to $\mathbf{L} \in \mathbb{R}^{m \times n}$ as the *matrix representation* of L w.r.t. $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$.

Matrices

$$V \xrightarrow{L} W$$

$$\boxed{L(v_j) = \mathbf{L}_{1j} \cdot w_1 + \cdots + \mathbf{L}_{mj} \cdot w_m}$$

Given a linear map $L: V \to W$ and bases $\{v_1, \dots, v_n\} \subset V$ and $\{w_1, \dots, w_m\} \subset W$:

For a vector $v \in V$, let $\mathbf{a} \in \mathbb{R}^n$ be the coefficients of v.

By linearity we have:

$$\begin{aligned} L(v) &= \mathbf{a}_1 \cdot L(v_1) + \cdots + \mathbf{a}_n \cdot L(v_n) \\ &= \mathbf{a}_1 \cdot (\mathbf{L}_{11} \cdot w_1 + \cdots + \mathbf{L}_{m1} \cdot w_m) + \cdots + \mathbf{a}_n \cdot (\mathbf{L}_{1n} \cdot w_1 + \cdots + \mathbf{L}_{mn} \cdot w_m) \\ &= (\mathbf{a}_1 \cdot \mathbf{L}_{11} + \cdots + \mathbf{a}_n \cdot \mathbf{L}_{1n}) \cdot w_1 + \cdots + (\mathbf{a}_1 \cdot \mathbf{L}_{m1} + \cdots + \mathbf{a}_n \cdot \mathbf{L}_{mn}) \cdot w_m \\ &= \mathbf{b}_1 \cdot w_1 + \cdots \mathbf{b}_n \cdot w_n \end{aligned}$$

with $\mathbf{b} = \mathbf{L} \cdot \mathbf{a}$.

$\Rightarrow$ If $\mathbf{a} \in \mathbb{R}^n$ are the coefficients of $v \in V$ w.r.t. $\{v_1, \dots, v_n\}$, then the coefficients of $L(v)$ w.r.t. $\{w_1, \dots, w_n\}$ are $\mathbf{b} = \mathbf{L} \cdot \mathbf{a} \in \mathbb{R}^m$

Similarly, composition of linear operators is consistent with matrix multiplication

Outline

Recall

Vector spaces

Dual spaces

- Dual linear maps
- Dual bases
- Matrix transpose

Dual Spaces

Recall that for vector spaces V and W, the space of linear maps between V and W is denoted $\mathrm{Hom}(V, W)$ and has dimension:

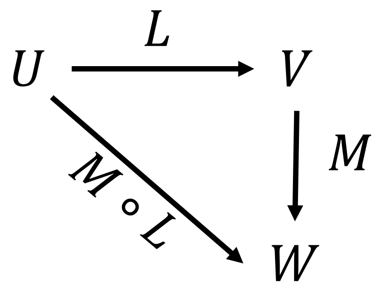
$$\dim\big(\mathrm{Hom}(V, W)\big) = \dim(V) \times \dim(W)$$

In the case that $W = \mathbb{R}$ we call the space of linear maps the *dual space* of V and denote it:

$$V^* \equiv \mathrm{Hom}(V, \mathbb{R})$$

⇒ The space has dimension $\dim(V^*) = \dim(V)$.

Dual Linear Maps



Recall:

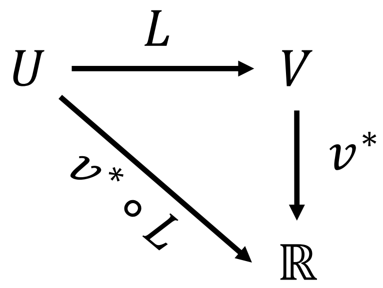
If we have vector spaces U and V, and W, and linear maps $L \in \mathrm{Hom}(U, V)$ and $M \in \mathrm{Hom}(V, W)$, then the composition $M \circ L$ is also a linear map…

… and the pull-back:

$$L^*: \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W)$$
$$M \mapsto M \circ L$$

is a linear map.

Dual Linear Maps



In the case that $W = \mathbb{R}$:

If we have vector spaces U and V, and ~~W~~ $\mathbb{R}$, and linear maps $L \in \mathrm{Hom}(U, V)$ and $M \in$ ~~$\mathrm{Hom}(V, W)$~~, then the composition $M \circ L$ is also a linear map…

$\mathrm{Hom}(V, \mathbb{R}) = V^*$

… and the pull-back:

$$L^*: V^* \to U^*$$
$$v^* \mapsto v^* \circ L$$

is a linear map.

Dual Linear Maps

$$U \xrightarrow{L} V$$

$$U^* \xleftarrow{L^*} V^*$$

In the case that $W = \mathbb{R}$:

If we have vector spaces U and V, and ~~W~~ $\mathbb{R}$, and linear maps $L \in \mathrm{Hom}(U, V)$ and $M \in$ ~~$\mathrm{Hom}(V, W)$~~ $\mathrm{Hom}(V, \mathbb{R}) = V^*$, then the composition $M \circ L$ is also a linear map...

... and the pull-back:

$$L^*: V^* \to U^*$$
$$v^* \mapsto v^* \circ L$$

is a linear map.

We call the map $L^* \in \mathrm{Hom}(V^*, U^*)$ the *dual of* L.

Dual Linear Maps

$$U \xrightarrow{\;L\;} V \xrightarrow{\;M\;} W$$

$$U^* \xleftarrow{\;L^*\;} V^* \xleftarrow{\;M^*\;} W^*$$

Given vector spaces U, V, and W, and linear maps $L \in \mathrm{Hom}(U, V)$ and $M \in \mathrm{Hom}(V, W)$, for all $u \in U$ and $w^* \in W^*$, we have:

$$\begin{aligned} [(L \circ M)^*(w^*)](u) &= w^*\big((L \circ M)(u)\big) \\ &= w^*\big(L\big(M(u)\big)\big) \\ &= \big(L^*(w^*)\big)\big(M(u)\big) \\ &= \Big(M^*\big(L^*(w^*)\big)\Big)(u) \\ &= \big((M^* \circ L^*)(w^*)\big)(u) \end{aligned}$$

$$\Downarrow$$

$$(L \circ M)^* = M^* \circ L^*$$

Dual Basis

Given a vector space V with basis $\{v_1, \dots, v_n\}$, we can define the *canonical* dual basis $\{v_1^*, \dots, v_n^*\}$ for V^* with $v_i^*: V \to \mathbb{R}$:

$$v_i^*(v_j) = \delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Dual Basis

$$v_i^*(v_j) = \delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

If we write out $v \in V$ in the basis $\{v_1, \ldots, v_n\}$:

$$v = \mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_n \cdot v_n$$

then:

$$\begin{aligned} v_i^*(v) &= v_i^*(\mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_n \cdot v_n) \\ &= \mathbf{a}_1 \cdot v_i^*(v_1) + \cdots + \mathbf{a}_n \cdot v_i^*(v_n) \\ &= \mathbf{a}_i \end{aligned}$$

$\Rightarrow v_i^* : V \to \mathbb{R}$ returns the i-th coefficient of v w.r.t. $\{v_1, \ldots, v_n\}$.

Dual Dual Basis

For a vector space V, we can also define the *dual of the dual* space:

$$V^{**} \equiv \mathrm{Hom}(V^*, \mathbb{R})$$

Given a vector $v \in V$, we can think of it as a linear map from V^* to $\mathbb{R}$:

$$v: V^* \to \mathbb{R}$$
$$v^* \mapsto v^*(v)$$

⇒ This defines an isomorphism $V \simeq V^{**}$.

⇒ If $\{v_1, \dots, v_n\}$ is a basis for V then $\{v_1^{**}, \dots, v_n^{**}\} = \{v_1, \dots, v_n\}$

$$v_i^{**}(v_j^*) \equiv v_j^*(v_i) = \delta_{ji}$$

Note that $v^* \in V^*$ is **not** the dual vector of $v \in V$, that phrasing doesn't make sense.
It does make sense to say that $\{v_1^*, \dots, v_n^*\}$ is the dual basis of $\{v_1, \dots, v_n\}$.

Dual Basis

$$v_i^*(v_j) = \delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Given a vector space V with basis $\{v_1, \ldots, v_n\}$, for $v \in V$ and $v^* \in V^*$, we can write:

$$v = \mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_n \cdot v_n$$
$$v^* = \mathbf{b}_1 \cdot v_1^* + \cdots + \mathbf{b}_n \cdot v_n^*$$

for some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

$\Rightarrow$ Evaluating v^* at v gives:

$$\begin{aligned} v^*(v) &= \sum_{i=1}^{n} \mathbf{b}_i \cdot v_i^* \left(\sum_{j=1}^{n} \mathbf{a}_j \cdot v_j \right) \\ &= \sum_{i,j=1}^{n} \mathbf{b}_i \cdot \mathbf{a}_j \cdot v_i^*(v_j) \\ &= \sum_{i=1}^{n} \mathbf{b}_i \cdot \mathbf{a}_i \end{aligned}$$

Dual Basis

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$$v = \mathbf{a}_1 \cdot v_1 + \cdots + \mathbf{a}_n \cdot v_n$$
$$v^* = \mathbf{b}_1 \cdot v_1^* + \cdots + \mathbf{b}_n \cdot v_n^*$$

for some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

$\Rightarrow$ Evaluating v^* at v gives:

$$v^*(v) = \sum_{i=1}^{n} \mathbf{b}_i \cdot v_i^* \left(\sum_{j=1}^{n} \mathbf{a}_j \cdot v_j \right)$$

W.r.t. a basis, evaluation is the same as a Euclidean dot-product.

$$= \sum_{i=1}^{n} \mathbf{b}_i \cdot \mathbf{a}_i = \mathbf{b}^\top \cdot \mathbf{a} = \mathbf{a}^\top \cdot \mathbf{b}$$

$$V \xrightarrow{L} W$$

Dual Basis

Recall:

Given vector spaces V and W with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$, the matrix expression $\mathbf{L} \in \mathbb{R}^{m \times n}$ for L with respect to this basis satisfies:

$$L(v_j) = \mathbf{L}_{1j} \cdot w_1 + \cdots + \mathbf{L}_{mj} \cdot w_m$$

$\Rightarrow$ The (i, j)-th coefficient of $\mathbf{L}$ satisfies:

$$\mathbf{L}_{ij} = w_i^* \left(L(v_j) \right)$$

Dual Basis

For the space of quadratic polynomials on the unit interval, $\mathbb{P}_2$:

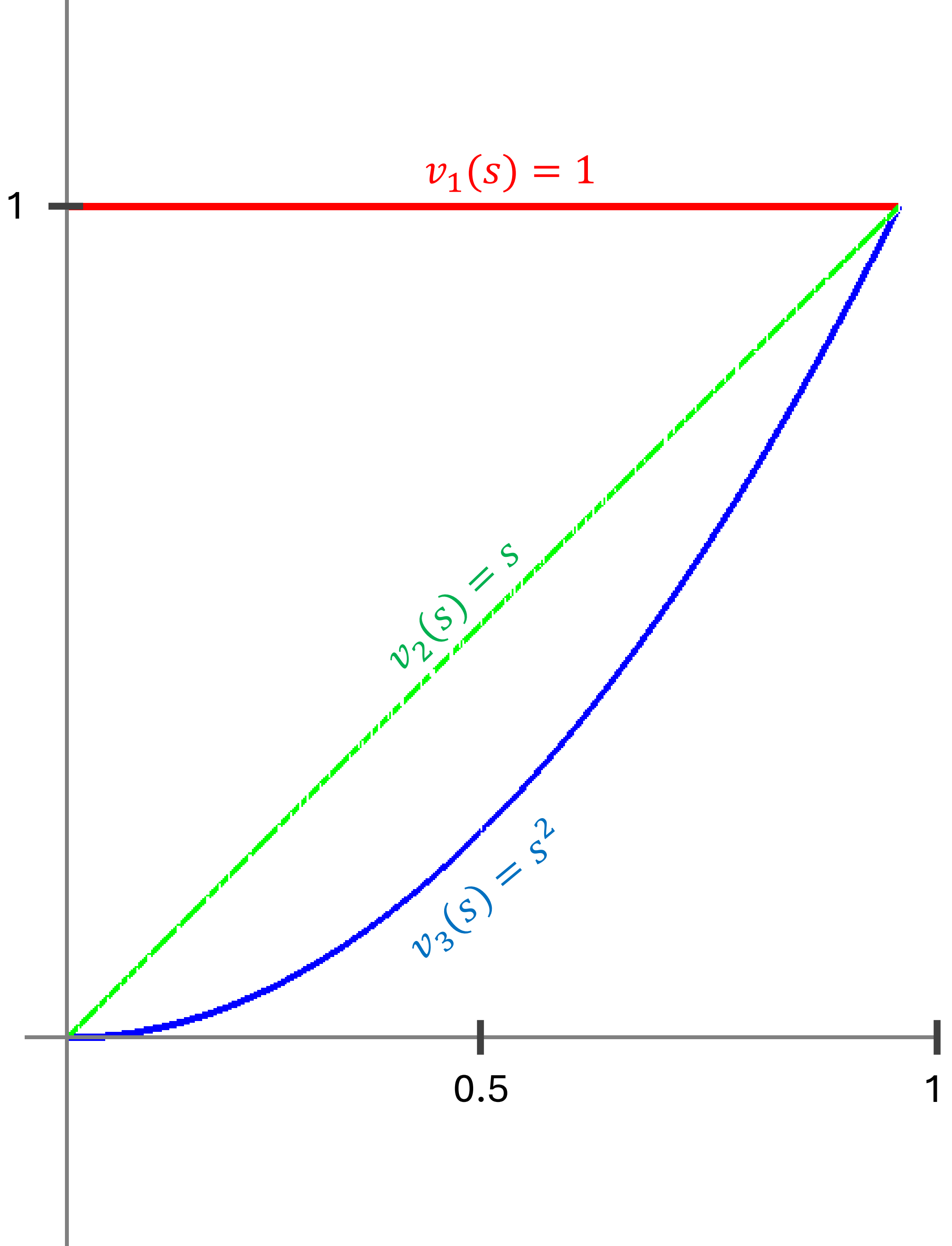
If we use the monomial basis:

$$v_1 = 1$$
$$v_2 = s$$
$$v_3 = s^2$$

$\Rightarrow$ The dual basis consists of functions

$$\{v_i^*: \mathbb{P}_2 \to \mathbb{R}\}$$

returning the i-th coefficient of the polynomial.



Dual Basis

For the space of quadratic polynomials on the unit interval, $\mathbb{P}_2$:

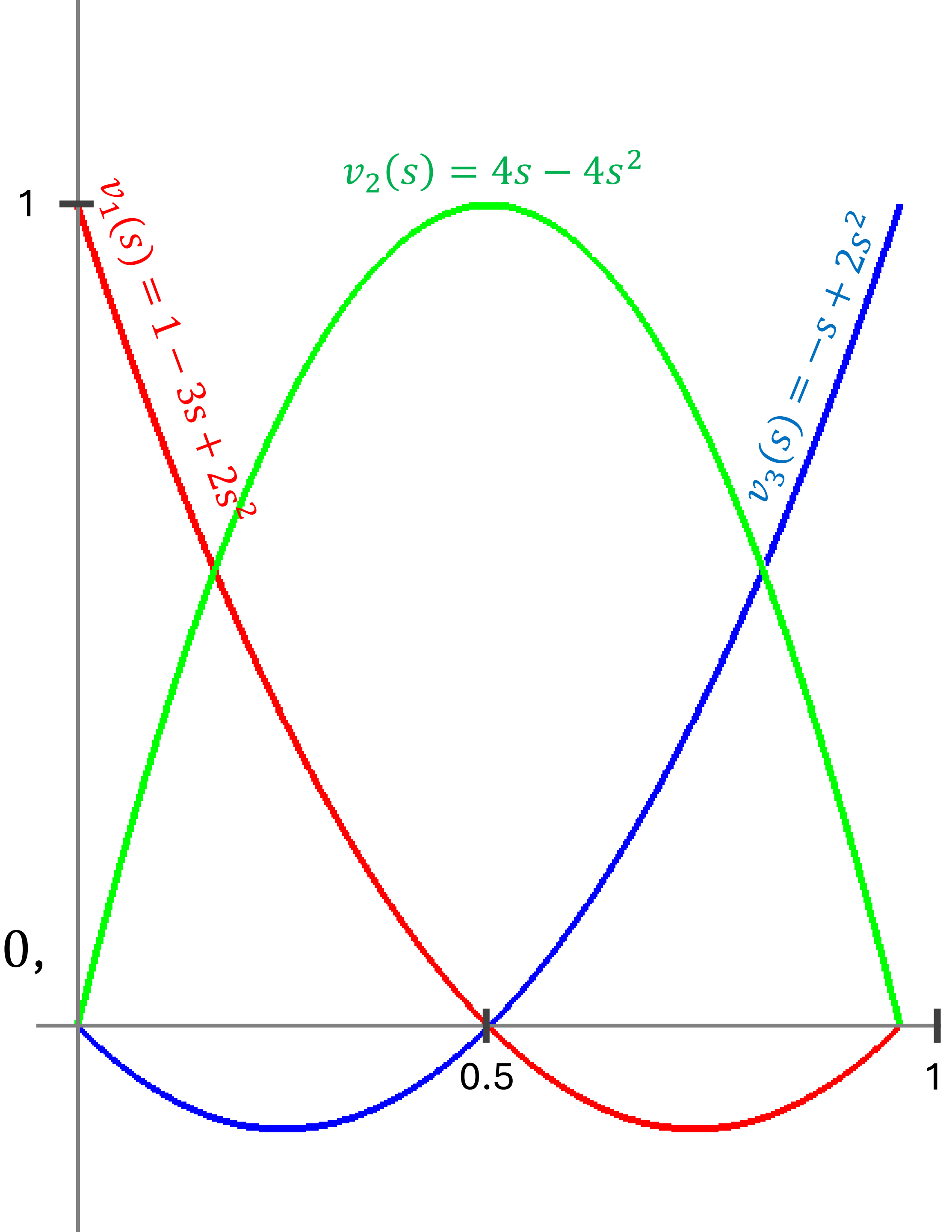
If we use the monomial basis:

$$v_1 = 1 - 3s + 2s^2$$
$$v_2 = 4s - 4s^2$$
$$v_3 = -s + 2s^2$$

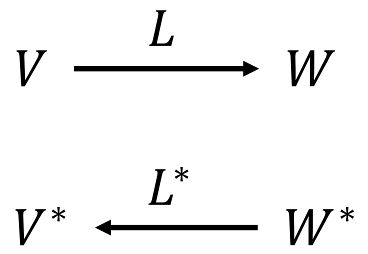
$\Rightarrow$ The dual basis consists of functions

$$\{v_i^* : \mathbb{P}_2 \to \mathbb{R}\}$$

returning the value of the polynomial at 0, 0.5, and 1 (respectively).



Matrix Transpose



Given vector spaces V and W, bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$, and a linear map $L \in \mathrm{Hom}(V, W)$:

⇒ We can express L as a matrix $\mathbf{L} \in \mathbb{R}^{m \times n}$

⇒ We have dual bases $\{v_1^*, \ldots, v_n^*\}$ and $\{w_1^*, \ldots, w_m^*\}$

⇒ We have dual dual bases $\{v_1^{**}, \ldots, v_n^{**}\}$ and $\{w_1^{**}, \ldots, w_m^{**}\}$

To get the (i, j)-th coefficient of the matrix expression for $L^* \in \mathrm{Hom}(W^*, V^*)$ w.r.t. the bases, we evaluate:

$$\begin{aligned} \mathbf{L}_{ij}^* &= v_i^{**}\left(L^*\left(w_j^*\right)\right) \\ &= \left(L^*\left(w_j^*\right)\right)(v_i) \\ &= w_j^*\left(L(v_i)\right) \\ &= \mathbf{L}_{ji} \end{aligned}$$

Matrix Transpose

$$V \xrightarrow{L} W$$

$$V^* \xleftarrow{L^*} W^*$$

Given vector spaces V and W, bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$, and a linear map $L \in \mathrm{Hom}(V, W)$:

⇒ We can express L as a matrix $\mathbf{L} \in \mathbb{R}^{m \times n}$

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To get the (i, j)-th coefficient of the matrix expression for $L^* \in \mathrm{Hom}(W^*, V^*)$ w.r.t. the bases, we evaluate:

$$\mathbf{L}_{ij}^* = v_i^{**}\left(L^*\left(w_j^*\right)\right)$$

⇒ The matrix expression for the dual is the transpose:

$$\mathbf{L}^* = \mathbf{L}^\top$$

$$= \mathbf{L}_{ji}$$