The Procrustes Method and 3D Scanning

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(601.457/657)

The Digital Michelangelo Project: 3D Scanning of Large Statues, Levoy *et al*. 2000

Recall

Any $n \times n$ matrix $\mathbf{M}$ can be expressed in terms of its Singular Value Decomposition as:

$$\mathbf{M} = \mathbf{U} \cdot \mathbf{D} \cdot \mathbf{V}^\top$$

where:

- $\mathbf{U}$ and $\mathbf{V}$ are $n \times n$ orthogonal matrices (i.e. det. ± 1)
- $\mathbf{D}$ is an $n \times n$ diagonal matrix (i.e. off-diagonals are 0) with non-negative entries that are monotonically decreasing

Recall

Given a matrix:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \cdots & \mathbf{M}_{n1} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{1n} & \cdots & \mathbf{M}_{nn} \end{pmatrix}$$

The trace is the sum of the diagonal entries:

$$\text{Trace}(\mathbf{M}) = \sum_i \mathbf{M}_{ii}$$

Recall

1. Given matrices $\mathbf{P}$ and $\mathbf{Q}$, we have:
$$(\mathbf{P}\cdot\mathbf{Q})^\top = \mathbf{Q}^\top\cdot\mathbf{P}^\top$$

2. Given a square matrix $\mathbf{P}$, we have:
$$\text{Trace}(\mathbf{P}) = \text{Trace}(\mathbf{P}^\top)$$

3. Given an $n\times m$ matrix $\mathbf{P}$ and an $m\times n$ matrix $\mathbf{Q}$, we have:
$$\text{Trace}(\mathbf{P}\cdot\mathbf{Q}) = \text{Trace}(\mathbf{Q}\cdot\mathbf{P})$$

4. Given vectors $\mathbf{v}$ and $\mathbf{w}$, we have:
$$\|\mathbf{v}\pm\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \pm 2\langle\mathbf{v},\mathbf{w}\rangle$$

Recall

Given a point-set $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{R}^m$,we denote by $\mathbf{P} = (\mathbf{p}_1 | \cdots | \mathbf{p}_n) \in \mathbb{R}^{m \times n}$ the matrix whose columns are the points $\{\mathbf{p}_i\}$.

Given a transformation $\mathbf{M} \in \mathbb{R}^{m \times m}$, the matrix defined by the transformed points is:

$$(\mathbf{M}(\mathbf{p}_1) | \cdots | \mathbf{M}(\mathbf{p}_n)) = \mathbf{M} \cdot \mathbf{P}$$

Recall

Given $\mathbf{P} \in \mathbb{R}^{n\times m}$ and $\mathbf{Q} \in \mathbb{R}^{m\times l}$, the (i,j)-th entry of $\mathbf{P}\cdot\mathbf{Q}$ is the dot-product of the i-th row of $\mathbf{P}$ and j-th column of $\mathbf{Q}$:

$$(\mathbf{P}\cdot\mathbf{Q})_{ij} = \sum_{k=1}^{m} \mathbf{P}_{ik}\cdot\mathbf{Q}_{kj}$$

$\Rightarrow$ Given $\mathbf{P} = (\mathbf{p}_1|\cdots|\mathbf{p}_n), \mathbf{Q} = (\mathbf{q}_1|\cdots|\mathbf{q}_n) \in \mathbb{R}^{m\times n}$:

$$\mathbf{P}^\top\cdot\mathbf{Q} = \begin{pmatrix} \langle\mathbf{p}_1,\mathbf{q}_1\rangle & \cdots & \langle\mathbf{p}_1,\mathbf{q}_n\rangle \\ \vdots & \ddots & \vdots \\ \langle\mathbf{p}_n,\mathbf{q}_1\rangle & \cdots & \langle\mathbf{p}_n,\mathbf{q}_n\rangle \end{pmatrix}$$

$\Rightarrow$ In particular, we have:

$$\text{Trace}(\mathbf{P}^\top\cdot\mathbf{Q}) = \sum_{i=1}^{n}\langle\mathbf{p}_i,\mathbf{q}_i\rangle$$

Recall

We denote by $O(m)$ the group of *orthogonal* $m \times m$ matrices (i.e. rotations and reflections):

$$(\mathbf{O}^\top \cdot \mathbf{O}) = \mathbf{Id}. \qquad \forall \mathbf{O} \in O(m)$$

$\Rightarrow$ The determinant of any orthogonal matrix is ± 1:

$$\det(\mathbf{O}) = \pm 1 \qquad \forall \mathbf{O} \in O(m)$$

We denote by $SO(m) \subset O(m)$ the of orthonormal $m \times m$ matrices (i.e. just rotations):

$$SO(m) = \{\mathbf{O} \in O(m) | \det(\mathbf{O}) = 1\}$$

Recall

If $\mathbf{O} \in O(m)$ is an orthogonal transformation:

$$(\mathbf{O}^{\top} \cdot \mathbf{O}) = \mathbf{Id}.$$

$\Leftrightarrow$ The columns vectors of $\mathbf{O}$ are unit-length:

$$\sum_{j=1}^{m} \mathbf{O}_{ij}^2 = 1 \quad \forall\, 1 \leq i \leq m$$

$\Rightarrow |\mathbf{O}_{ij}| \leq 1$

Recall

1. Given a function $F(\mathbf{p})$, the point $\mathbf{p}$ is an extremum of F if the gradient of F vanishes at $\mathbf{p}$.

2. If $F(\mathbf{p}) = \|\mathbf{p}\|^2$ then:

$$\begin{aligned}\nabla F &= \nabla\left(p_x^2 + p_y^2 + p_z^2\right) \\ &= \left(2p_x, 2p_y, 2p_z\right) \\ &= 2\mathbf{p}\end{aligned}$$

3. If $F(\mathbf{p}) = \langle \mathbf{p}, \mathbf{q} \rangle$ then:

$$\begin{aligned}\nabla_{\mathbf{p}} F &= \nabla_{\mathbf{p}}\left(p_x q_x + p_y q_y + p_z q_z\right) \\ &= \left(q_x, q_y, q_z\right) \\ &= \mathbf{q}\end{aligned}$$

Recall

1. Given two real values $a, b \in \mathbb{R}$, we have:
$$|a \cdot b| = |a| \cdot |b|$$

2. Given two real values $a, b \in \mathbb{R}$, we have:
$$|a + b| \leq |a| + |b|$$

Claim

Given a diagonal matrix $\mathbf{D} \in \mathbb{R}^m$, the orthogonal transformation $\mathbf{O} \in O(m)$ maximizing the trace:

$$\text{Trace}(\mathbf{O} \cdot \mathbf{D})$$

is the matrix:

$$\mathbf{O} = \text{sign}(\mathbf{D}) = \begin{pmatrix} \text{sign}(\mathbf{D}_{11}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \text{sign}(\mathbf{D}_{nn}) \end{pmatrix}$$

This gives:

$$\begin{aligned} \text{Trace}(\mathbf{O} \cdot \mathbf{D}) &= \text{sign}(\mathbf{D}_{11})\mathbf{D}_{11} + \cdots + \text{sign}(\mathbf{D}_{nn})\mathbf{D}_{nn} \\ &= |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}| \end{aligned}$$

Claim

Given a diagonal matrix $\mathbf{D} \in \mathbb{R}^m$, the orthogonal transformation $\mathbf{O} \in O(m)$ maximizing the trace:

$$\text{Trace}(\mathbf{O} \cdot \mathbf{D})$$

is the matrix:

$$\mathbf{O} = \text{sign}(\mathbf{D}) = \begin{pmatrix} \text{sign}(\mathbf{D}_{11}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \text{sign}(\mathbf{D}_{nn}) \end{pmatrix}$$

Will show that for any orthogonal $\mathbf{O}$:

$$\begin{aligned} \text{Trace}(\mathbf{O} \cdot \mathbf{D}) &\leq \text{Trace}(\text{sign}(\mathbf{D}) \cdot \mathbf{D}) \\ &= |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}| \end{aligned}$$

Proof

$$\text{Trace}(\mathbf{O}\cdot\mathbf{D}) \le |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}|$$

Setting:

$$\mathbf{O} = \begin{pmatrix} \mathbf{O}_{11} & \cdots & \mathbf{O}_{n1} \\ \vdots & \ddots & \vdots \\ \mathbf{O}_{1n} & \cdots & \mathbf{O}_{nn} \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} \mathbf{D}_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{D}_{nn} \end{pmatrix}$$

we get:

$$\begin{aligned} \text{Trace}(\mathbf{O}\cdot\mathbf{D}) &= \mathbf{O}_{11}\mathbf{D}_{11} + \cdots + \mathbf{O}_{nn}\mathbf{D}_{nn} \\ &\le |\mathbf{O}_{11}\mathbf{D}_{11} + \cdots + \mathbf{O}_{nn}\mathbf{D}_{nn}| \\ &\le |\mathbf{O}_{11}\mathbf{D}_{11}| + \cdots + |\mathbf{O}_{nn}\mathbf{D}_{nn}| \\ &= |\mathbf{O}_{11}||\mathbf{D}_{11}| + \cdots + |\mathbf{O}_{nn}||\mathbf{D}_{nn}| \end{aligned}$$

Proof

$$\text{Trace}(\mathbf{O}\cdot\mathbf{D}) \le |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}|$$

Setting:

$$\mathbf{O} = \begin{pmatrix} \mathbf{O}_{11} & \cdots & \mathbf{O}_{n1} \\ \vdots & \ddots & \vdots \\ \mathbf{O}_{1n} & \cdots & \mathbf{O}_{nn} \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} \mathbf{D}_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{D}_{nn} \end{pmatrix}$$

we get:

$$\text{Trace}(\mathbf{O}\cdot\mathbf{D}) \le |\mathbf{O}_{11}||\mathbf{D}_{11}| + \cdots + |\mathbf{O}_{nn}||\mathbf{D}_{nn}|$$

Since $\mathbf{O}$ is orthogonal, we have $|\mathbf{O}_{ii}| \le 1$:

$$\text{Trace}(\mathbf{O}\cdot\mathbf{D}) \le |\mathbf{D}_{11}| + \cdots + |\mathbf{D}_{nn}|$$

(Orthogonal) Procrustes Method

Goal:

Given points $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{R}^m$ and $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\} \subset \mathbb{R}^m$, find the **translation** $\boldsymbol{\delta} \in \mathbb{R}^m$ and **orthogonal transform** $\mathbf{O} \in O(m)$ that best aligns $\{\mathbf{p}_i\}$ to $\{\mathbf{q}_i\}$.

That is, find $\boldsymbol{\delta}$ and $\mathbf{O}$ minimizing the alignment energy:

$$E(\boldsymbol{\delta}, \mathbf{O}) = \sum_{i=1}^{n} \|\mathbf{O}(\mathbf{p}_i + \boldsymbol{\delta}) - \mathbf{q}_i\|^2$$

(Orthogonal) Procrustes Method

$\|\mathbf{v} \pm \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \pm 2\langle \mathbf{v}, \mathbf{w} \rangle$

Goal:

1. Find the **translation** $\boldsymbol{\delta} \in \mathbb{R}^m$ minimizing:*

$$E(\boldsymbol{\delta}) = \sum_{i=1}^{n} \|(\mathbf{p}_i + \boldsymbol{\delta}) - \mathbf{q}_i\|^2$$

$$= \sum_{i=1}^{n} \|(\mathbf{p}_i - \mathbf{q}_i) + \boldsymbol{\delta}\|^2$$

$$= \sum_{i=1}^{n} \left(\|\mathbf{p}_i - \mathbf{q}_i\|^2 + \|\boldsymbol{\delta}\|^2 + 2\langle \mathbf{p}_i - \mathbf{q}_i, \boldsymbol{\delta} \rangle\right)$$

*We'll see why we can ignore orthogonal transformations shortly.

(Orthogonal) Procrustes Method

$\nabla_{\mathbf{p}} \|\mathbf{p}\|^2 = 2\mathbf{p}$ and $\nabla_{\mathbf{p}} \langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{q}$

Goal:

$$E(\boldsymbol{\delta}) = \sum_{i=1}^{n} \left(\|\mathbf{p}_i - \mathbf{q}_i\|^2 + \|\boldsymbol{\delta}\|^2 + 2\langle \mathbf{p}_i - \mathbf{q}_i, \boldsymbol{\delta} \rangle \right)$$

1. Find the **translation** $\boldsymbol{\delta} \in \mathbb{R}^m$ minimizing $E(\boldsymbol{\delta})$.*

   Taking the gradient gives:

$$\nabla E(\boldsymbol{\delta}) = \sum_{i=1}^{n} 2\boldsymbol{\delta} + 2(\mathbf{p}_i - \mathbf{q}_i)$$

*We'll see why we can ignore orthogonal transformations shortly.

(Orthogonal) Procrustes Method

Goal:

$$\nabla E(\boldsymbol{\delta}) = \sum_{i=1}^{n} 2\boldsymbol{\delta} + 2(\mathbf{p}_i - \mathbf{q}_i)$$

1. Find the **translation** $\boldsymbol{\delta} \in \mathbb{R}^m$ minimizing $E(\boldsymbol{\delta})$.*

   The minimizing translation must satisfy:

$$\nabla E(\boldsymbol{\delta}) = 0$$
$$\Downarrow$$
$$\boldsymbol{\delta} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{q}_i - \mathbf{p}_i)$$

*We'll see why we can ignore orthogonal transformations shortly.

(Orthogonal) Procrustes Method

Goal:

$$\nabla E(\boldsymbol{\delta}) = \sum_{i=1}^{n} 2\boldsymbol{\delta} + 2(\mathbf{p}_i - \mathbf{q}_i)$$

1. Find the **translation** $\boldsymbol{\delta} \in \mathbb{R}^m$ minimizing $E(\boldsymbol{\delta})$.*

   The minimizing translation must satisfy:

$$\nabla E(\boldsymbol{\delta}) = 0$$

$$\Downarrow$$

$$\boldsymbol{\delta} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{q}_i - \mathbf{p}_i) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{q}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{p}_i$$

*We'll see why we can ignore orthogonal transformations shortly.

The minimizing translation takes the center of mass of $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ to the center of mass of $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$.

Goa

⇓

The point-sets are translationally aligned when their centers of mass coincide.

⇓

If the centers are both at the origin, the point-sets are translationally aligned

1. Fin ... $(\boldsymbol{\delta})$.*

Th

Note:

If a point-set is translated so its center of mass is at the origin, any linear transformation (e.g. rotation) of the point-set will still have its center of mass at the origin.

⇓

If the point-sets are centered at the origin, they are optimally translationally aligned regardless of the rotation.

(Orthogonal) Procrustes Method

$$\|\mathbf{v} \pm \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \pm 2\langle \mathbf{v}, \mathbf{w}\rangle$$

Goal:

2. Find the **transform** $\mathbf{O} \in O(m)$ minimizing:

$$E(\mathbf{O}) = \sum_{i=1}^{n} \|\mathbf{O}(\mathbf{p}_i) - \mathbf{q}_i\|^2$$

$$= \sum_{i=1}^{n} \|\mathbf{O}(\mathbf{p}_i)\|^2 + \|\mathbf{q}_i\|^2 - 2\langle \mathbf{O}(\mathbf{p}_i), \mathbf{q}_i\rangle$$

$$= \sum_{i=1}^{n} \|\mathbf{p}_i\|^2 + \|\mathbf{q}_i\|^2 - 2\langle \mathbf{O}(\mathbf{p}_i), \mathbf{q}_i\rangle$$

Minimizing $E(\mathbf{O})$ is the same as maximizing:

$$\tilde{E}(\mathbf{O}) = \sum_{i=1}^{n} \langle \mathbf{O}(\mathbf{p}_i), \mathbf{q}_i\rangle$$

(Orthogonal) Procrustes Method

<u>Goal</u>:

2. Find the **transform** $\mathbf{O} \in O(m)$ maximizing:

$$\tilde{E}(\mathbf{O}) = \sum_{i=1}^{n} \langle \mathbf{O}(\mathbf{p}_i), \mathbf{q}_i \rangle$$

- Set $\mathbf{P} = (\mathbf{p}_1 | \cdots | \mathbf{p}_n)$ and $\mathbf{Q} = (\mathbf{q}_1 | \cdots | \mathbf{q}_n)$.
- Use the facts that:

$$\mathbf{O} \cdot \mathbf{P} = \big(\mathbf{O}(\mathbf{p}_1) | \cdots \big| \mathbf{O}(\mathbf{p}_n)\big)$$

$$\mathrm{Trace}(\mathbf{P}^\top \cdot \mathbf{Q}) = \sum_{i=1}^{n} \langle \mathbf{p}_i, \mathbf{q}_i \rangle$$

$$\Downarrow$$

$$\tilde{E}(\mathbf{O}) = \mathrm{Trace}\big((\mathbf{O} \cdot \mathbf{P})^\top \cdot \mathbf{Q}\big)$$

(Orthogonal) Procrustes Method

$$(\mathbf{A} \cdot \mathbf{B})^\top = \mathbf{B}^\top \cdot \mathbf{A}^\top$$
$$\text{Trace}(\mathbf{A} \cdot \mathbf{B}) = \text{Trace}(\mathbf{B} \cdot \mathbf{A})$$

Goal:

2. Find the **transform** $\mathbf{O} \in O(m)$ maximizing:

$$\begin{aligned}\tilde{E}(\mathbf{O}) &= \text{Trace}\big((\mathbf{O} \cdot \mathbf{P})^\top \cdot \mathbf{Q}\big) \\ &= \text{Trace}(\mathbf{P}^\top \cdot \mathbf{O}^\top \cdot \mathbf{Q}) \\ &= \text{Trace}\big((\mathbf{Q} \cdot \mathbf{P}^\top) \cdot \mathbf{O}^\top\big)\end{aligned}$$

Massage so we are trying to maximize $\text{Trace}(\mathbf{O} \cdot \mathbf{D})$, for some diagonal $\mathbf{D}$.

(Orthogonal) Procrustes Method

$$\text{Trace}(\mathbf{A}^\top) = \text{Trace}(\mathbf{A})$$
$$(\mathbf{A}\cdot\mathbf{B})^\top = \mathbf{B}^\top\cdot\mathbf{A}^\top$$

Goal:

2. Find the **transform** $\mathbf{O} \in O(m)$ maximizing:

$$\begin{aligned}\tilde{E}(\mathbf{O}) &= \text{Trace}\left((\mathbf{O}\cdot\mathbf{P})^\top\cdot\mathbf{Q}\right)\\ &= \text{Trace}(\mathbf{P}^\top\cdot\mathbf{O}^\top\cdot\mathbf{Q})\\ &= \text{Trace}\left((\mathbf{Q}\cdot\mathbf{P}^\top)\cdot\mathbf{O}^\top\right)\\ &= \text{Trace}\left(\left((\mathbf{Q}\cdot\mathbf{P}^\top)\cdot\mathbf{O}^\top\right)^\top\right)\\ &= \text{Trace}(\mathbf{O}\cdot(\mathbf{Q}\cdot\mathbf{P}^\top)^\top)\end{aligned}$$

Massage so we are trying to maximize $\text{Trace}(\mathbf{O}\cdot\mathbf{D})$, for some diagonal $\mathbf{D}$.

(Orthogonal) Procrustes Method

$$(\mathbf{A}\cdot\mathbf{B})^\top = \mathbf{B}^\top\cdot\mathbf{A}^\top$$

<u>Goal</u>:

2. Find the **transform** $\mathbf{O} \in O(m)$ maximizing:

$$\begin{aligned}\tilde{E}(\mathbf{O}) &= \text{Trace}\left((\mathbf{O}\cdot\mathbf{P})^\top\cdot\mathbf{Q}\right)\\ &= \text{Trace}(\mathbf{P}^\top\cdot\mathbf{O}^\top\cdot\mathbf{Q})\\ &= \text{Trace}\left((\mathbf{Q}\cdot\mathbf{P}^\top)\cdot\mathbf{O}^\top\right)\\ &= \text{Trace}\left(\left((\mathbf{Q}\cdot\mathbf{P}^\top)\cdot\mathbf{O}^\top\right)^\top\right)\\ &= \text{Trace}(\mathbf{O}\cdot(\mathbf{Q}\cdot\mathbf{P}^\top)^\top)\\ &= \text{Trace}\left(\mathbf{O}\cdot(\mathbf{P}\cdot\mathbf{Q}^\top)\right)\end{aligned}$$

Massage so we are trying to maximize $\text{Trace}(\mathbf{O}\cdot\mathbf{D})$, for some diagonal $\mathbf{D}$.

(Orthogonal) Procrustes Method

$$\text{Trace}(\mathbf{A}\cdot\mathbf{B}) = \text{Trace}(\mathbf{B}\cdot\mathbf{A})$$

Goal:

2. Find the **transform** $\mathbf{O} \in O(m)$ maximizing:

$$\tilde{E}(\mathbf{O}) = \text{Trace}\big(\mathbf{O}\cdot(\mathbf{P}\cdot\mathbf{Q}^\top)\big)$$

Compute the singular value decomposition:

$$\mathbf{P}\cdot\mathbf{Q}^\top = \mathbf{U}\cdot\mathbf{D}\cdot\mathbf{V}^\top$$

with $\mathbf{U}$ and $\mathbf{V}$ orthogonal and $\mathbf{D}$ diagonal.

$$\Downarrow$$

$$\begin{aligned}\tilde{E}(\mathbf{O}) &= \text{Trace}(\mathbf{O}\cdot\mathbf{U}\cdot\mathbf{D}\cdot\mathbf{V}^\top)\\ &= \text{Trace}(\mathbf{V}^\top\cdot\mathbf{O}\cdot\mathbf{U}\cdot\mathbf{D})\\ &= \text{Trace}\big((\mathbf{V}^\top\cdot\mathbf{O}\cdot\mathbf{U})\cdot\mathbf{D}\big)\end{aligned}$$

Massage so we are trying to maximize $\text{Trace}(\mathbf{O}\cdot\mathbf{D})$, for some diagonal $\mathbf{D}$.

(Orthogonal) Procrustes Method

Goal:

2. Find the **transform** $\mathbf{O} \in O(m)$ maximizing:

$$\tilde{E}(\mathbf{O}) = \text{Trace}\left((\mathbf{V}^\top \cdot \mathbf{O} \cdot \mathbf{U}) \cdot \mathbf{D}\right)$$

Since $\mathbf{V}^\top \cdot \mathbf{O} \cdot \mathbf{U}$ is orthogonal, this is maximized if:

$$\mathbf{V}^\top \cdot \mathbf{O} \cdot \mathbf{U} = \text{sign}(\mathbf{D})$$
$$\Downarrow$$
$$\mathbf{O} = \mathbf{V} \cdot \text{sign}(\mathbf{D}) \cdot \mathbf{U}^\top$$

Since the diagonal entries of $\mathbf{D}$ are non-negative, this gives:

$$\mathbf{O} = \mathbf{V} \cdot \mathbf{U}^\top$$

(Orthonormal) Procrustes Method

$$\mathbf{O} = \mathbf{V} \cdot \mathrm{sign}(\mathbf{D}) \cdot \mathbf{U}^\top$$

In practice, we often want the best *orthonormal* transformation, $\mathbf{O} \in SO(m)$, not just orthogonal transformation. This requires:

$$\begin{aligned} 1 &= \det(\mathbf{O}) \\ &= \det(\mathbf{V}) \cdot \mathrm{sign}(\mathbf{D}) \cdot \det(\mathbf{U}^\top) \\ &= \det(\mathbf{V}) \cdot \det(\mathbf{U}^\top) \\ &= \det(\mathbf{V} \cdot \mathbf{U}^\top) \end{aligned}$$

(Orthonormal) Procrustes Method

For $\mathbf{O}$ to be orthornormal, we require:

$$1 = \det(\mathbf{V}) \cdot \mathrm{sign}(\mathbf{D}) \cdot \det(\mathbf{U}^\top) = \det(\mathbf{V} \cdot \mathbf{U}^\top)$$

but are only guaranteed that $\det(\mathbf{V} \cdot \mathbf{U}^\top) = \pm 1$.

⇒ If $\mathbf{V} \cdot \mathbf{U}^\top$ has determinant -1, we can make $\mathbf{O}$ have determinant 1, by replacing $\mathrm{sign}(\mathbf{D}) = \mathbf{Id}$. with some other diagonal matrix:

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{\Sigma}_{nn} \end{pmatrix}$$

with $\mathbf{\Sigma}_{ii} = \pm 1$, and an odd number of the diagonal entries equal to -1.

(Orthonormal) Procrustes Method

$$\mathbf{O} = \mathbf{V} \cdot \mathbf{\Sigma} \cdot \mathbf{U}^\top$$

Recall that our goal is to maximize

$$\begin{aligned}\tilde{E}(\mathbf{O}) &= \text{Trace}\left((\mathbf{V}^\top \cdot \mathbf{O} \cdot \mathbf{U}) \cdot \mathbf{D}\right) \\ &= \text{Trace}\left((\mathbf{V}^\top \cdot \mathbf{V} \cdot \mathbf{\Sigma} \cdot \mathbf{U}^\top \cdot \mathbf{U}) \cdot \mathbf{D}\right) \\ &= \text{Trace}(\mathbf{\Sigma} \cdot \mathbf{D}) \\ &= \mathbf{\Sigma}_{00} \cdot \mathbf{D}_{00} + \cdots + \mathbf{\Sigma}_{nn} \cdot \mathbf{D}_{nn}\end{aligned}$$

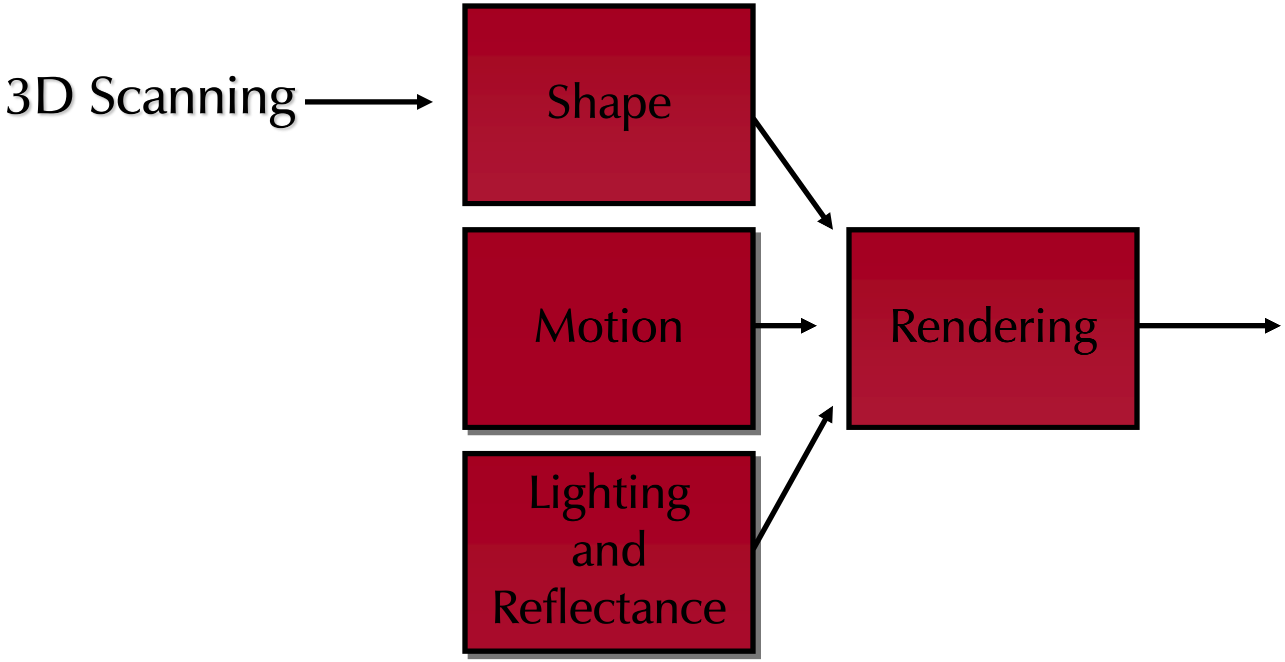
Since the $\mathbf{D}_{ii}$ are positive, since $\mathbf{\Sigma}_{ii} = \pm 1$, and since we require an odd number of $\mathbf{\Sigma}_{ii}$ to be negative, the sum is maximized when we set $\mathbf{\Sigma}_{nn} = \det(\mathbf{V} \cdot \mathbf{U}^\top)$.

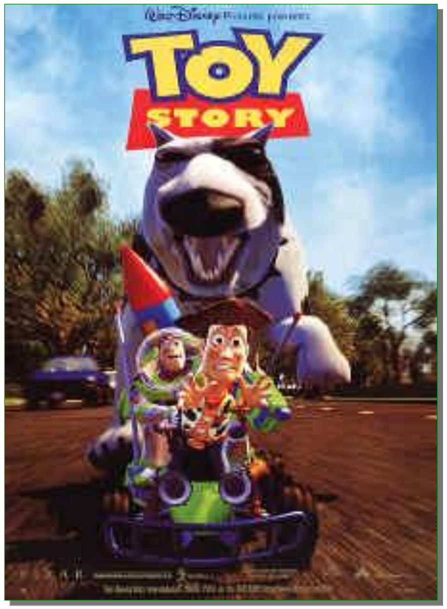
3D Scanning

Lecture courtesy of

Szymon Rusinkiewicz
Princeton University

Computer Graphics Pipeline





- Human time = expensive
- Sensors = cheap
  - Computer graphics increasingly relies on measurements of the real world

3D Scanning Applications

- Computer graphics
- Product inspection
- Robot navigation
- Product design
- Archaeology
- Clothes fitting

Industrial Inspection

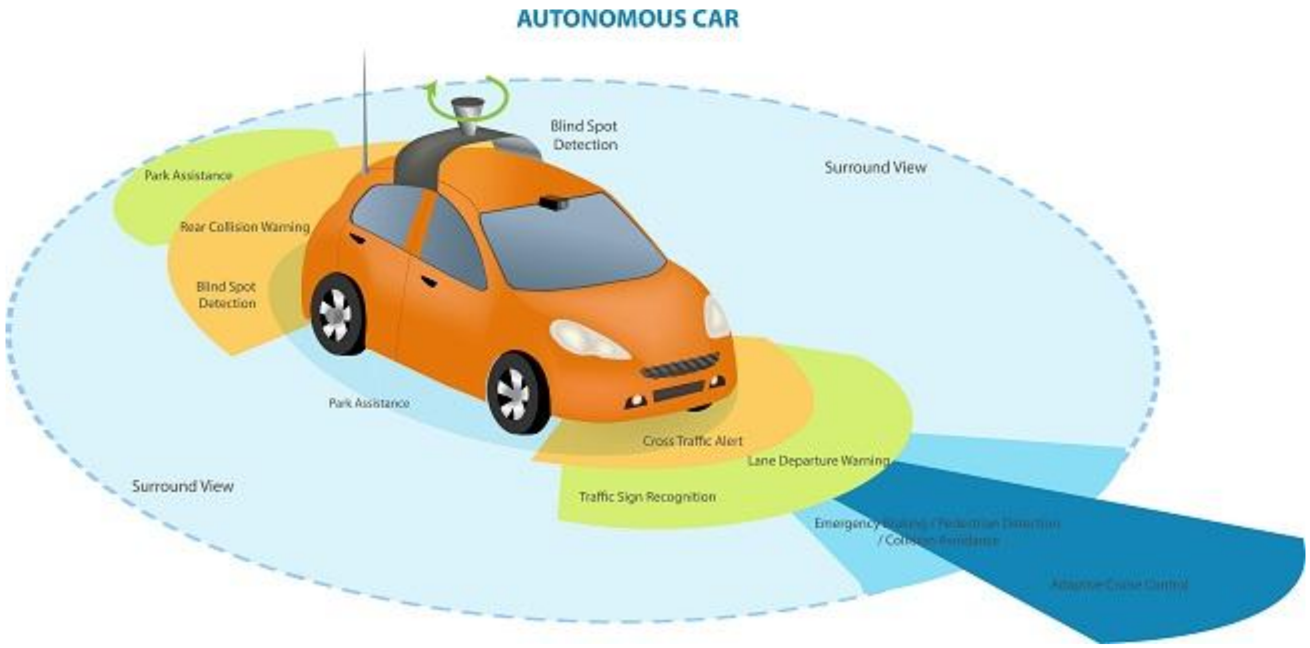
- Are manufactured parts within a tolerance?

Clothing

- Scan a person, custom-fit clothing
  - U.S. Army; booths in malls

Driving

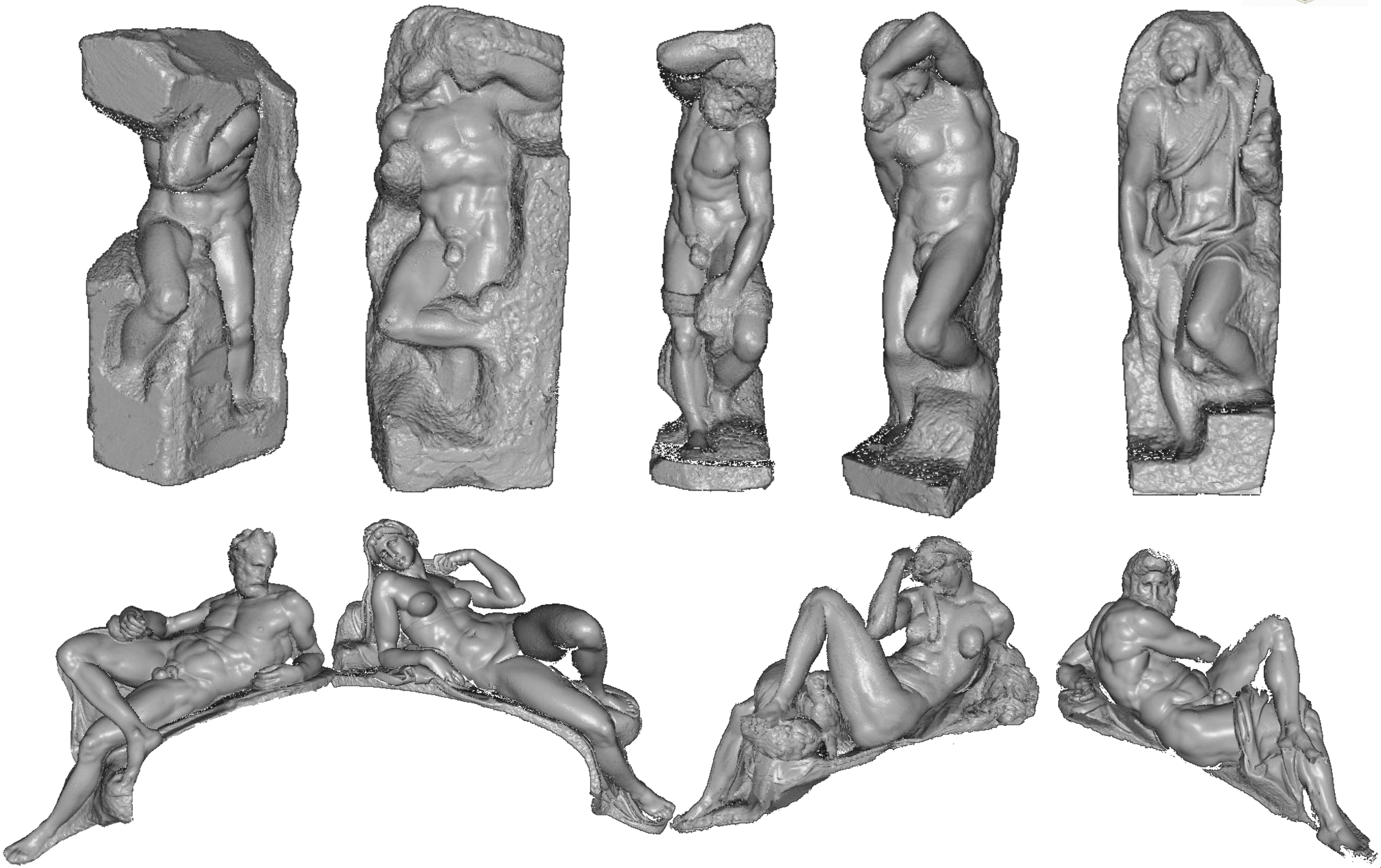
- Autonomous navigation
  - Collision avoidance



https://www.geotab.com/blog/crash-avoidance/

The Digital Michelangelo Project

The Digital Michelangelo Project



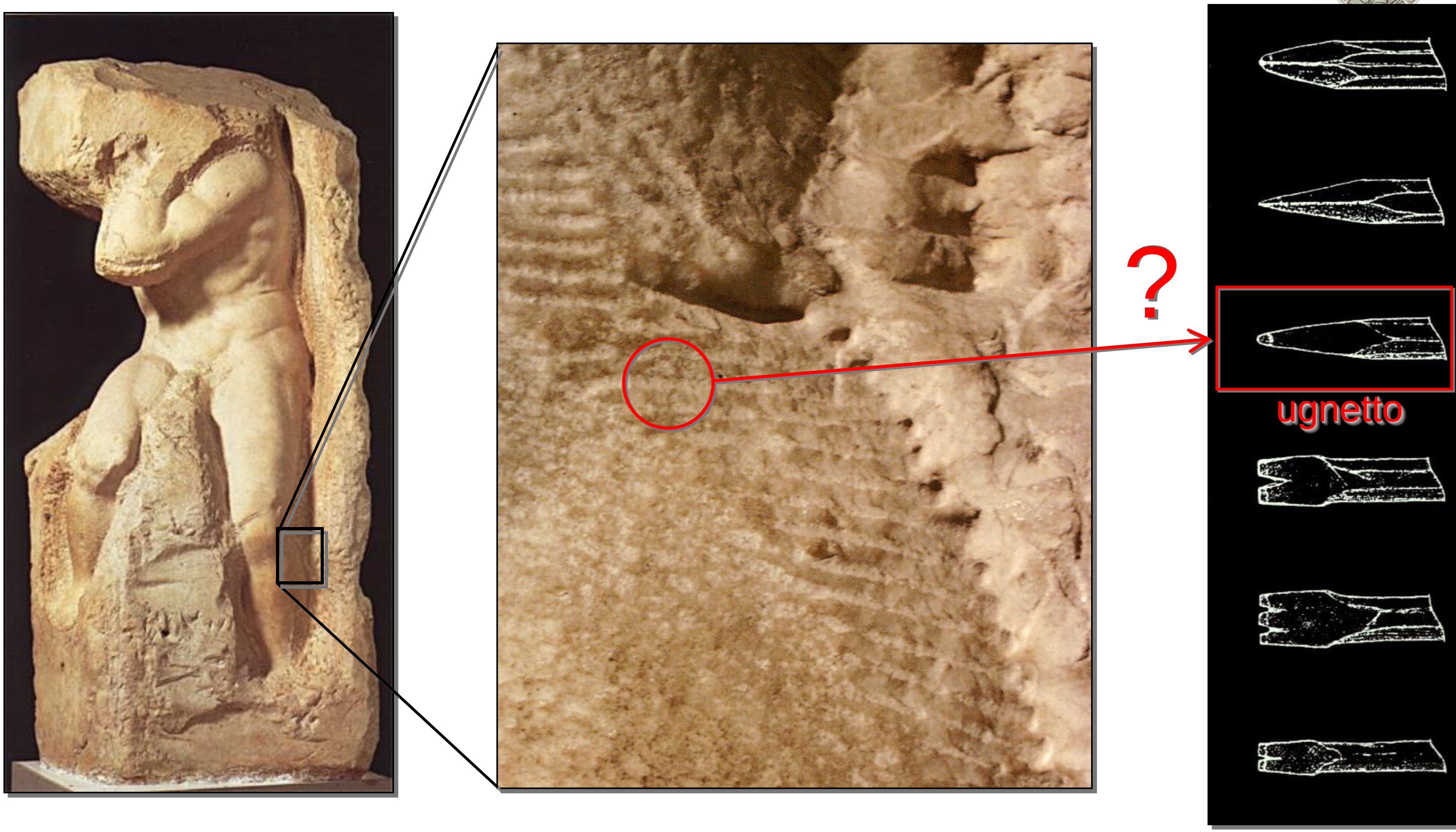
Why Scan Sculptures?

- Virtual museums
- Controlled interaction (lighting, proximity, etc.)
- Study working techniques
- Cultural heritage preservation

Goals

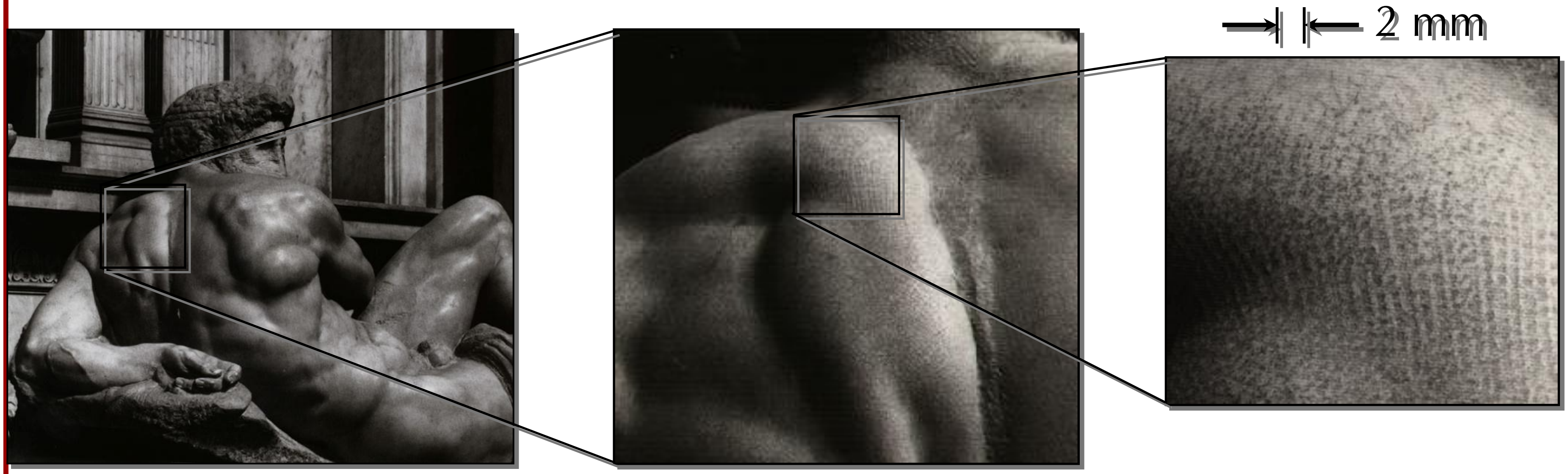
- Scan 10 sculptures by Michelangelo
- High-resolution (i.e. quarter-millimeter) resolution

Why Capture Chisel Marks?



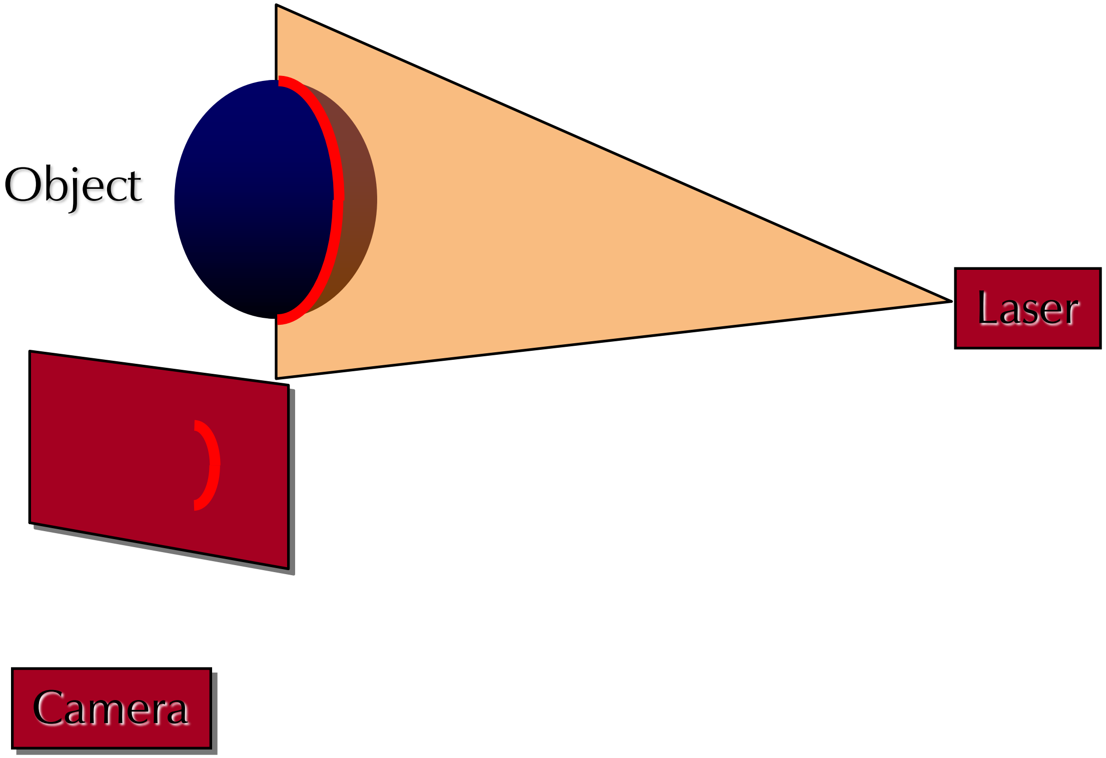
Atlas (Accademia)

Why Capture Chisel Mark Geometry?



Day (Medici Chapel)

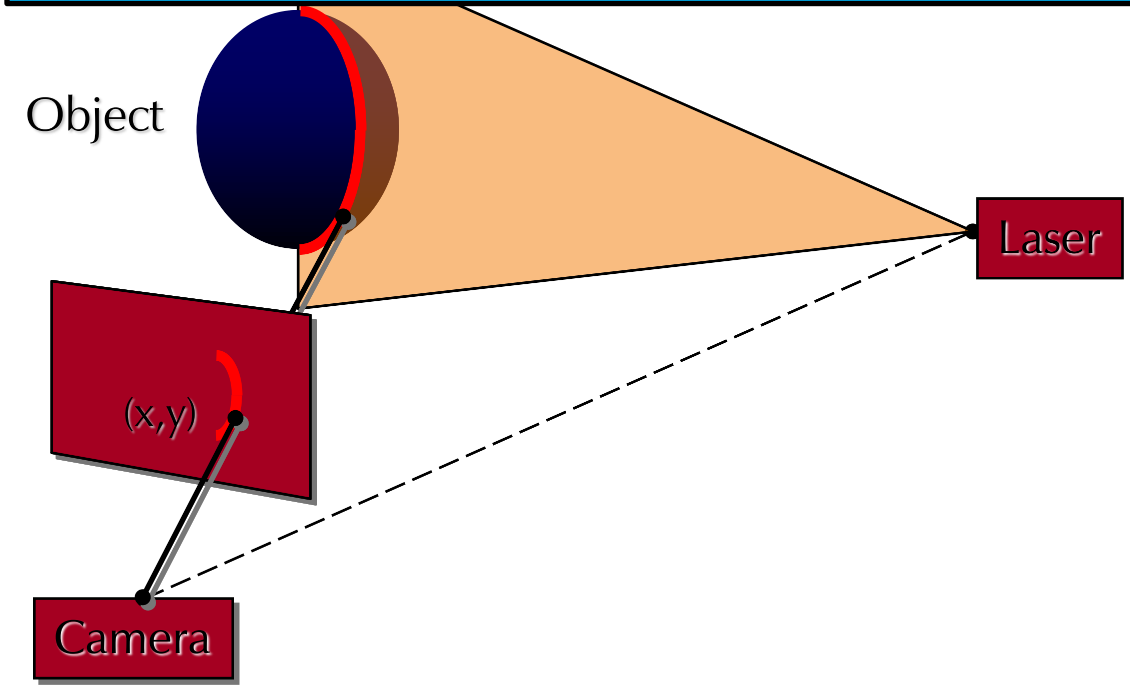
Triangulation



- Project laser stripe onto object
- Detect laser stripe in image

Triangulation

Gives the depth of the point (x, y) with respect to the camera.

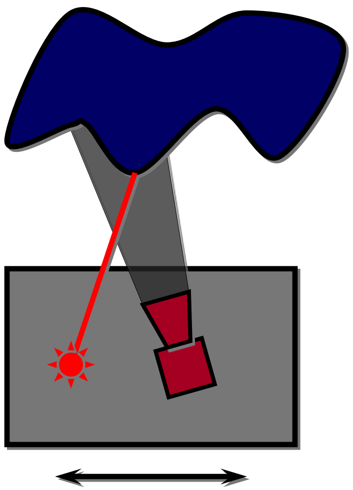


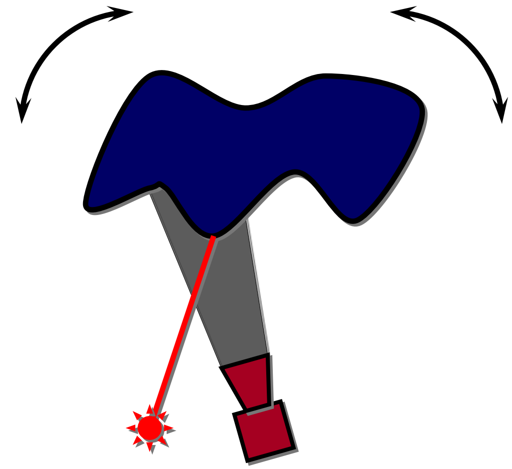
- Project laser stripe onto object
- Detect laser stripe in image
- Get depth from ray-plane triangulation

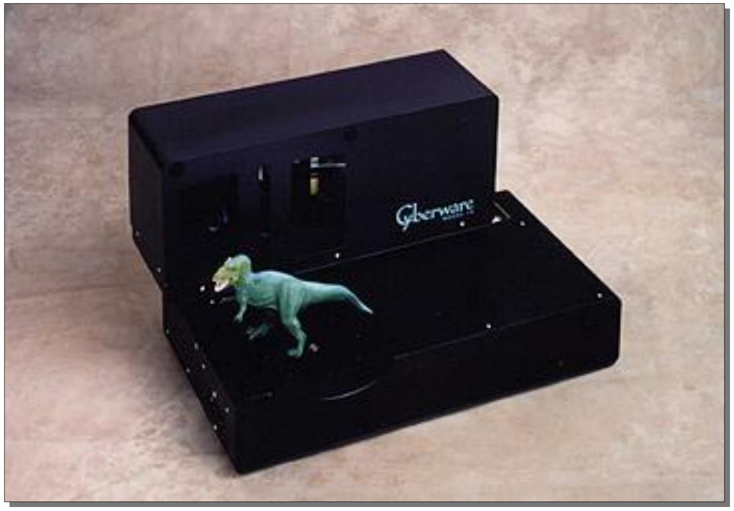
Triangulation: Moving the Camera and Illumination

- ✗ Moving independently leads to problems with calibration
- ✓ Most scanners mount camera and light source rigidly, move them as a unit

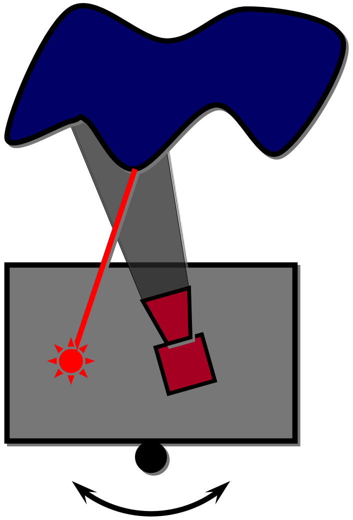
Triangulation: Moving the Camera and Illumination

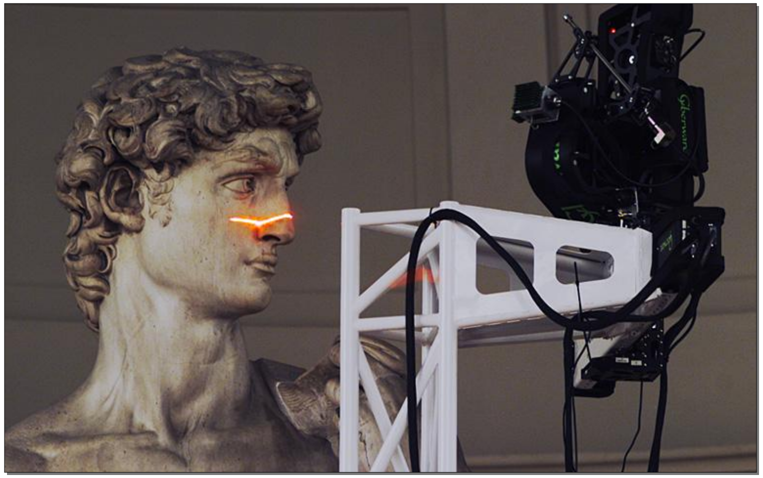




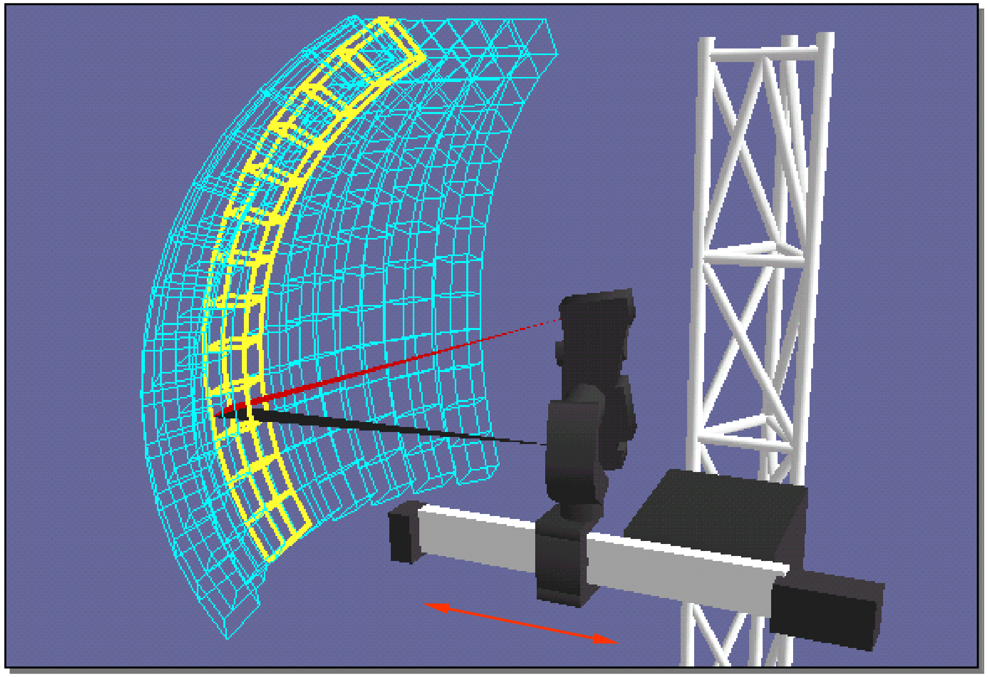


Triangulation: Moving the Camera and Illumination



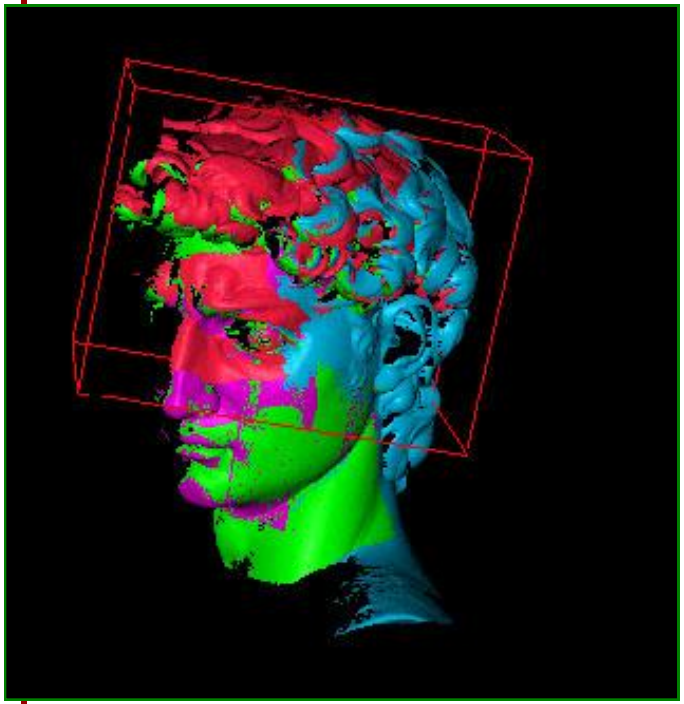


Scanning a Large Object



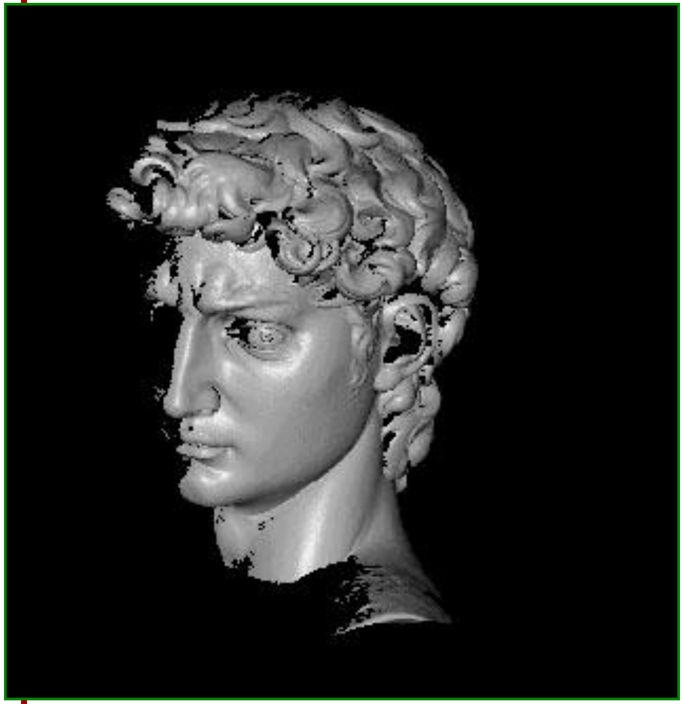
- Calibrated motions
  - pitch (yellow)
  - pan (blue)
  - horizontal translation (orange)
- Uncalibrated motions
  - vertical translation
  - rolling the gantry
  - remounting the scan head

Range Processing Pipeline



- Steps
  1. Manual initial alignment
  2. Automatic ICP to an existing scan
  3. Global relaxation to diffuse error
  4. Merging using volumetric method

Range Processing Pipeline



- Steps
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Iterative Closest Point (ICP)

Goal:

Given two point-sets $\mathbf{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{R}^d$, and $\mathbf{Q} = \{\mathbf{q}_1, \ldots, \mathbf{q}_m\} \subset \mathbb{R}^d$, find:

- The correspondence, $\Phi: \{1, \ldots, n\} \to \{1, \ldots, m\}$,
- The translation $\boldsymbol{\delta} \in \mathbb{R}^d$, and
- The rotation $\mathbf{O} \in SO(d)$

that minimize the sum of squared distances:

$$E(\Phi, \boldsymbol{\delta}, \mathbf{O}) = \sum_{i=1}^{n} \left\| \mathbf{O}(\mathbf{p}_i + \boldsymbol{\delta}) - \mathbf{q}_{\Phi(i)} \right\|^2$$

Iterative Closest Point (ICP)

$$E(\Phi, \boldsymbol{\delta}, \mathbf{O}) = \sum_{i=1}^{n} \left\| \mathbf{O}(\mathbf{p}_i + \boldsymbol{\delta}) - \mathbf{q}_{\Phi(i)} \right\|^2$$

<u>Approach</u>:

1. Create a sequence of correspondences, translations, and rotations:
   $$\{\{\Phi_0, \boldsymbol{\delta}_0, \mathbf{O}_0\}, \{\Phi_1, \boldsymbol{\delta}_1, \mathbf{O}_1\}, \cdots\}$$
   that monotonically reduces the sum of squared distances.

2. Alternately solve for the correspondence Φ_i and the transformation $\{\boldsymbol{\delta}_i, \mathbf{O}_i\}$.

Iterative Closest Point (ICP)

Algorithm:

0. Initialize $k = 0, \boldsymbol{\delta}_k = \vec{0}$, and $\mathbf{O}_k = \mathbf{Id}$.

1. Fix $\boldsymbol{\delta}_k$ and $\mathbf{O}_k$, and set Φ_{k+1} to minimize:

$$E(\Phi_{k+1}, \boldsymbol{\delta}_k, \mathbf{O}_k) = \sum_{i=1}^{n} \left\| \mathbf{O}_k(\mathbf{p}_i + \boldsymbol{\delta}_k) - \mathbf{q}_{\Phi_{k+1}(i)} \right\|^2$$

$\Rightarrow \Phi_{k+1}$ is the nearest-neighbor map:

$$\Phi_{k+1}(i) = \underset{j \in \{1,\ldots,m\}}{\arg\min} \left\| \mathbf{O}_k(\mathbf{p}_i + \boldsymbol{\delta}_k) - \mathbf{q}_j \right\|^2$$

Iterative Closest Point (ICP)

Algorithm:

0. Initialize $k = 0, \boldsymbol{\delta}_k = \vec{0}$, and $\mathbf{O}_k = \mathbf{Id}$.

1. Fix $\boldsymbol{\delta}_k$ and $\mathbf{O}_k$, and set Φ_{k+1} to minimize:

$$E(\Phi_{k+1}, \boldsymbol{\delta}_k, \mathbf{O}_k) = \sum_{i=1}^{n} \left\| \mathbf{O}_k(\mathbf{p}_i + \boldsymbol{\delta}_k) - \mathbf{q}_{\Phi_{k+1}(i)} \right\|^2$$

2. Fix Φ_{k+1}, and set $\boldsymbol{\delta}_{k+1}$ and $\mathbf{O}_{k+1}$ to minimize:

$$E(\Phi_{k+1}, \boldsymbol{\delta}_{k+1}, \mathbf{O}_{k+1}) = \sum_{i=1}^{n} \left\| \mathbf{O}_{k+1}(\mathbf{p}_i + \boldsymbol{\delta}_{k+1}) - \mathbf{q}_{\Phi_{k+1}(i)} \right\|^2$$

3. Update $k = k + 1$. Goto step 1.

Iterative Closest Point (ICP)

Algorithm:

0. Initialize $k = 0, \boldsymbol{\delta}_k = \vec{0}$, and $\mathbf{O}_k = \mathbf{Id}$.

1. Fix $\boldsymbol{\delta}_k$ and $\mathbf{O}_k$, and set Φ_{k+1} to minimize:

$$E(\Phi_{k+1}, \boldsymbol{\delta}_k, \mathbf{O}_k) = \sum_{i=1}^{n} \left\| \mathbf{O}_k(\mathbf{p}_i + \boldsymbol{\delta}_k) - \mathbf{q}_{\Phi_{k+1}(i)} \right\|^2$$

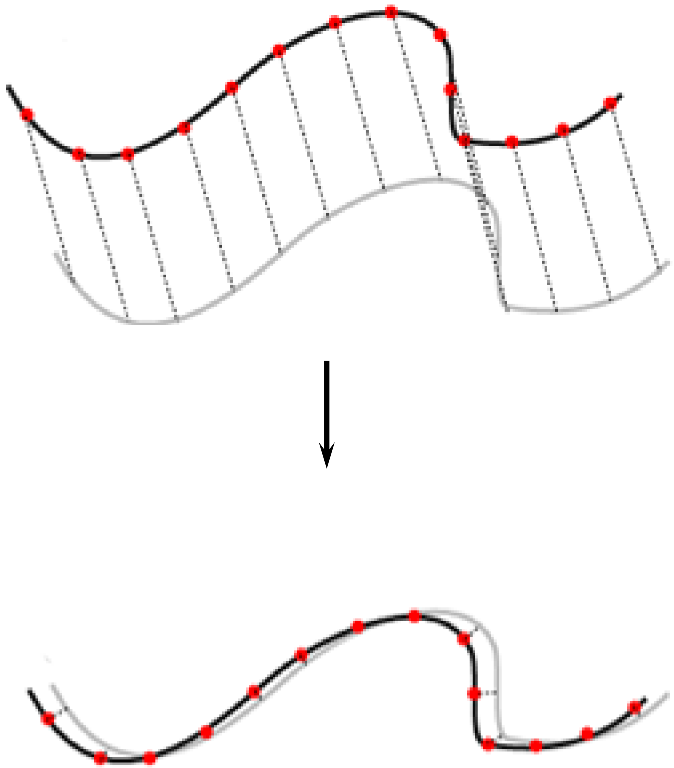
2. Fix Φ_{k+1}, and set $\boldsymbol{\delta}_{k+1}$ and $\mathbf{O}_{k+1}$ to minimize:

$$E(\Phi_{k+1}, \boldsymbol{\delta}_{k+1}, \mathbf{O}_{k+1}) = \sum_{i=1}^{n} \left\| \mathbf{O}_{k+1}(\mathbf{p}_i + \boldsymbol{\delta}_{k+1}) - \mathbf{q}_{\Phi_{k+1}(i)} \right\|^2$$

3.

Since the two steps reduce the same energy, the sum of squared distances reduces monotonically.

Iterative Closest Point (ICP)



- Steps
  1. Manual initial alignment
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  3. Global relaxation to diffuse error
  4. Merging using volumetric method

<u>ICP</u><u>(*Scan1* , *Scan2*)</u> :

1. For each point on *Scan1*, find the nearest point on *Scan2*.
2. Translate/Rotate *Scan1* to minimize the distance between corresponding points.
3. Go to step 1

Range Processing Pipeline

- Steps
  1. Manual initial alignment
  2. Automatic ICP to an existing scan
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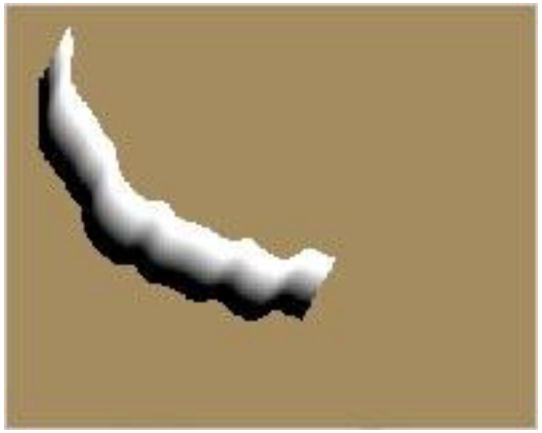
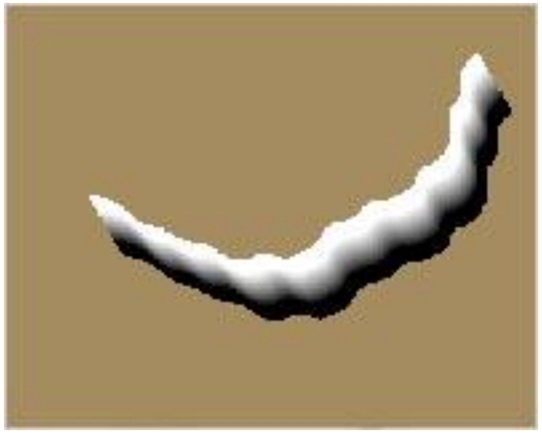
Range Processing Pipeline



+



- Steps
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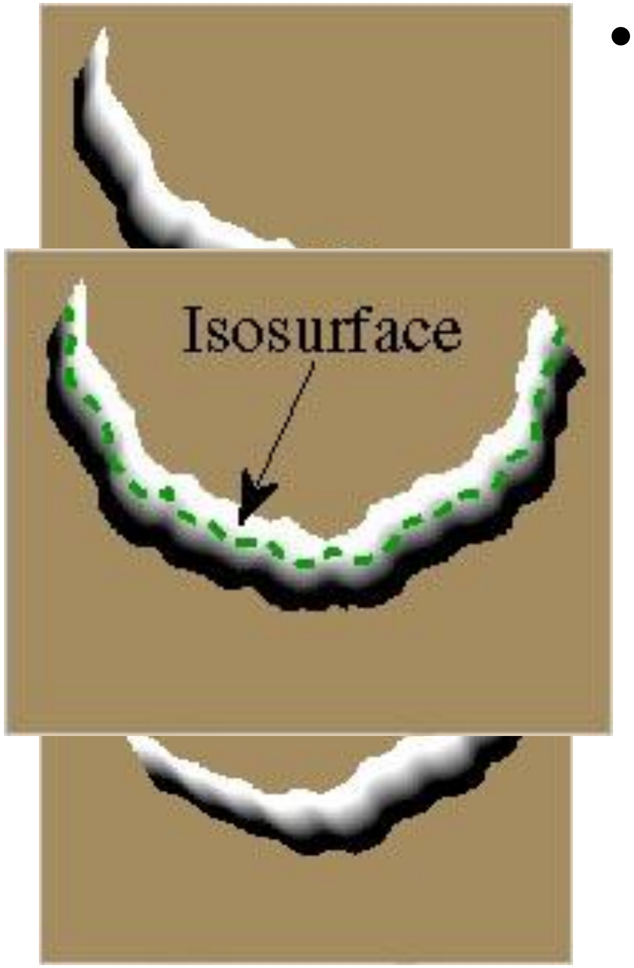
Range Processing Pipeline



- Steps
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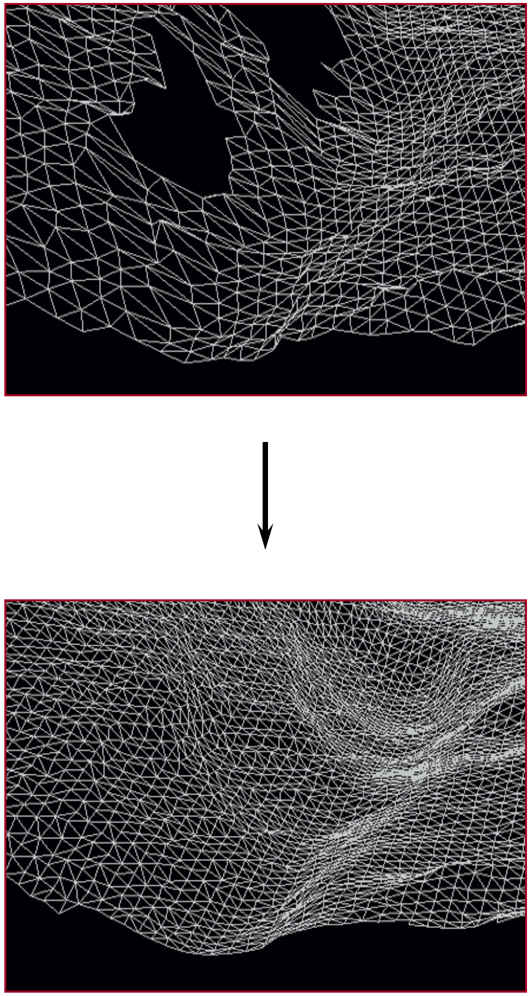
Range Processing Pipeline



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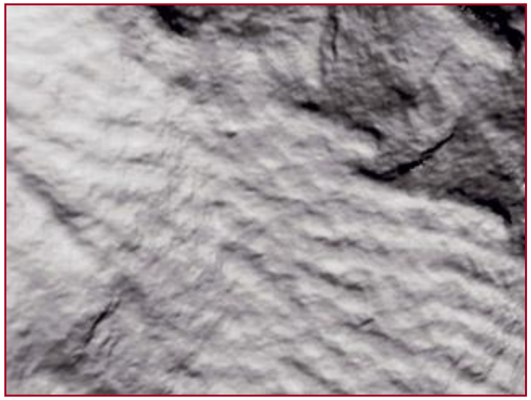
Range Processing Pipeline



- Steps
  1. Manual initial alignment
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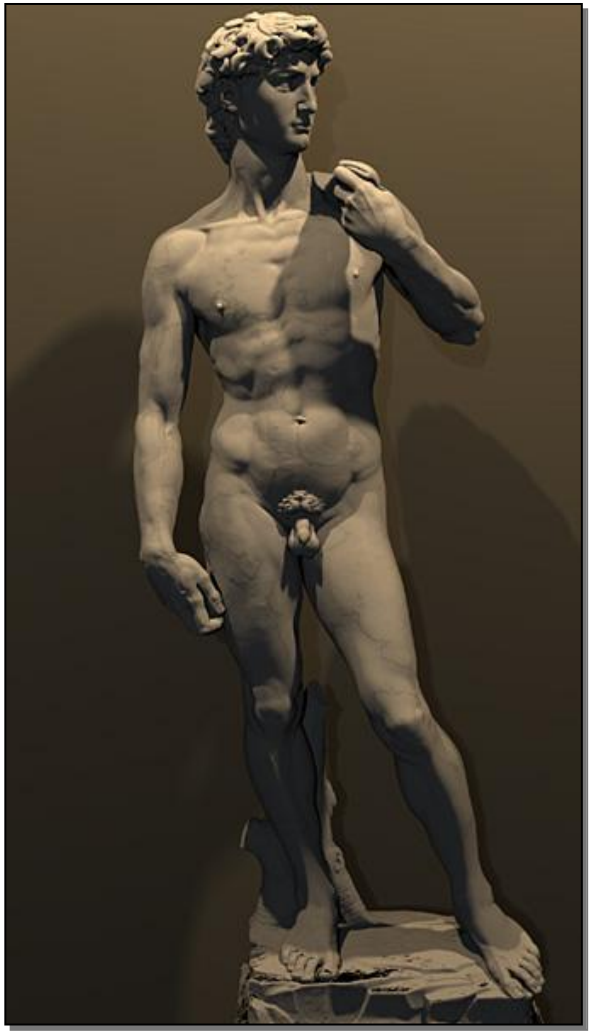
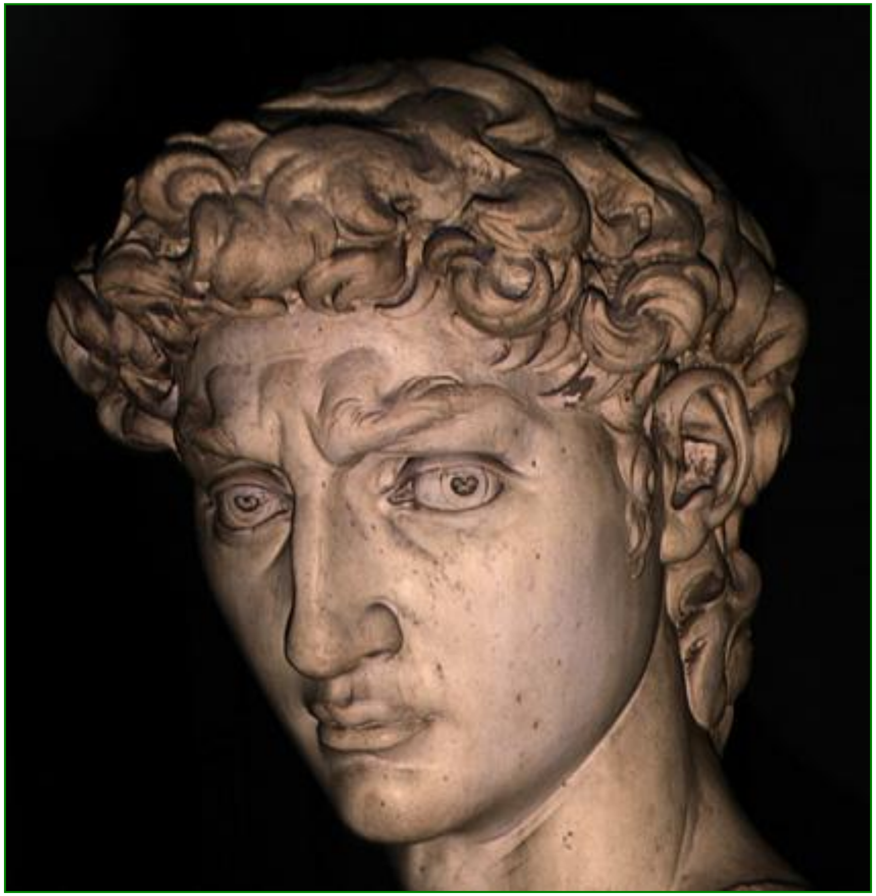
  4. Merging using volumetric method

Statistics About the Scan of David

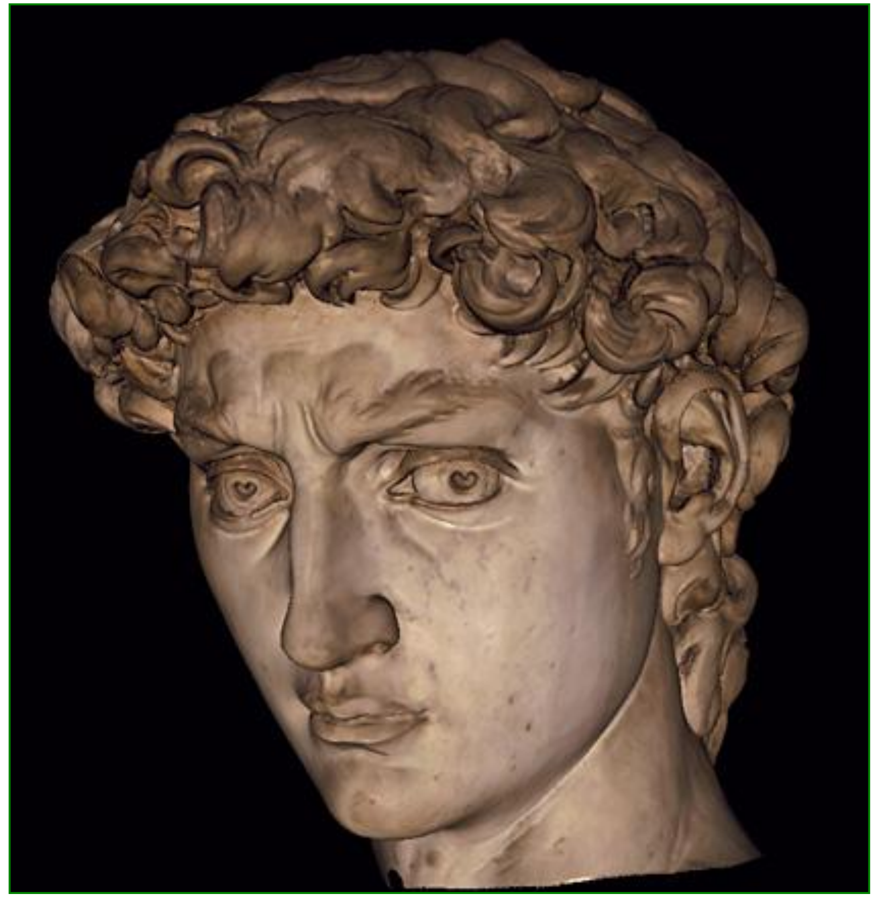


- 480 individually aimed scans
- 0.3 mm sample spacing
- 2 billion polygons
- 7,000 color images
- 32 gigabytes
- 30 nights of scanning
- 22 people

Head of Michelangelo's David



Photograph



1.0 mm computer model