Quaternions and Exponentials

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(601.457/657)

Recall

We saw two different methods for interpolating/approximating between rotations:

- Normalization: (SVD)
  Blend as 3×3 matrices and then map to the closest rotation.
  - ✗ Requires SVD
  - ✗ Works in a 9-dimensional space
- Parameterization: (Euler angles)
  Compute the parameter values, blend those, and then evaluate at the blended values.
  - ✗ Parameterization is not uniform
    (e.g. dense sampling near poles)

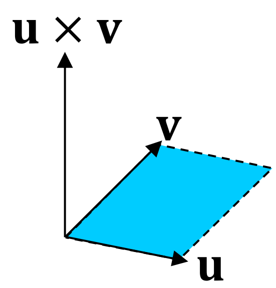
Overview

- Math review
  - Cross products
  - Symmetric matrices
  - Complex numbers
  - The exponential map
- Quaternions
- The exponential map

Cross Product

Given vectors $\mathbf{u} = (u_1, u_2, u_3)^\top$ and $\mathbf{v} = (v_1, v_2, v_3)^\top$ in 3D, the cross product of $\mathbf{u}$ and $\mathbf{v}$ is:

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$



Properties:

- The cross product is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
- The vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ align with the right-hand rule.
- The length of the cross product is equal to the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$.
- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
- $(t\mathbf{u}) \times \mathbf{v} = t(\mathbf{u} \times \mathbf{v})$

(Skew) Symmetric Matrices

A matrix $\mathbf{M}$ is <u>symmetric</u> if:

$$\mathbf{M}_{ij} = \mathbf{M}_{ji} \quad \Leftrightarrow \quad \mathbf{M} = \mathbf{M}^\top$$

A matrix $\mathbf{M}$ is <u>skew-symmetric</u> if:

$$\mathbf{M}_{ij} = -\mathbf{M}_{ji} \quad \Leftrightarrow \quad \mathbf{M} = -\mathbf{M}^\top$$

The space of (skew) symmetric matrices is closed under addition and scaling:

- If $\mathbf{A} = \mathbf{A}^\top$ and $\mathbf{B} = \mathbf{B}^\top$, then $(\mathbf{A} + \mathbf{B}) = (\mathbf{A} + \mathbf{B})^\top$.
- If $\mathbf{A} = -\mathbf{A}^\top$ and $\mathbf{B} = -\mathbf{B}^\top$, then $(\mathbf{A} + \mathbf{B}) = -(\mathbf{A} + \mathbf{B})^\top$.
- If $\mathbf{A} = \mathbf{A}^\top$ then $(\alpha\mathbf{A}) = (\alpha\mathbf{A})^\top$.
- If $\mathbf{A} = -\mathbf{A}^\top$ then $(\alpha\mathbf{A}) = -(\alpha\mathbf{A})^\top$.

Complex Numbers

Complex numbers are extensions of the real numbers, incorporating an imaginary value:

$$a + ib$$

We add complex numbers together by summing the real and imaginary components:

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

Squaring the imaginary component gives:

$$i^2 = -1$$

The product of two complex numbers is:

$$(a_1 + ib_1) \times (a_2 + ib_2)$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Complex Numbers

- Given a complex number $c = a + ib$
  - The conjugate of c is:
$$\bar{c} = a - ib$$
  - The (squared) norm of c is the real value:
$$|c|^2 = a^2 + b^2 = c \cdot \bar{c}$$
  - The norm of the product is the product of the norms:
$$|c_1 \cdot c_2| = |c_1| \cdot |c_2|$$
  - The reciprocal of c (assuming $c \neq 0$) is defined by dividing the conjugate of c by the square norm:
$$\frac{1}{c} = \frac{1}{c} \cdot \frac{\bar{c}}{\bar{c}} = \frac{\bar{c}}{|c|^2}$$

The Exponential Map

The *exponential* is a map from real values to positive real values:

$$\exp\colon \mathbb{R} \to \mathbb{R}^{>0}$$

The inverse is the *logarithm*, taking positive real values to real values:

$$\ln\colon \mathbb{R}^{>0} \to \mathbb{R}$$

The Exponential Map

Properties:

- $\exp(0) = 1$
- $\left.\frac{\partial \exp(t\alpha)}{\partial t}\right|_{t=0} = \alpha$
- $\ln\big(\exp(t)\big) = t$

The Exponential Map

Taylor Expansion:

We can approximate the exponential map by its Taylor Expansion around $s = 0$:

$$\exp(s) = 1 + s + \frac{1}{2!}s^2 + \cdots + \frac{1}{n!}s^n + \cdots$$

We can approximate the logarithm map by its Taylor Expansion around $s = 1$:

$$\ln(s) = (s-1) - \frac{(s-1)^2}{2} + \cdots + (-1)^{n+1}\frac{(s-1)^n}{n} + \cdots$$

Overview

- Math review
- Quaternions
- The exponential map

Quaternions

Normalization:

- Treat rotations as living in a linear space
- Blend rotations
- Map the blend to the closest rotation

Goal:

- Find a linear space making it easy to map the blend to the closest rotation

Quaternions

Quaternions are extensions of complex numbers, with three imaginary values instead of one:

$$a + ib + jc + kd$$

Like the complex numbers, we can add quaternions together by summing the individual components:

$$\begin{array}{c} (a_1 + ib_1 + jc_1 + kd_1) \\ +(a_2 + ib_2 + jc_2 + kd_2) \\ \hline = (a_1 + a_2) + i(b_1 + b_2) + j(c_1 + c_2) + k(d_1 + d_2) \end{array}$$

Quaternions

Quaternions are extensions of complex numbers, with three imaginary values instead of one:

$$a + ib + jc + kd$$

Like the imaginary component of complex numbers, squaring the components gives:

$$i^2 = j^2 = k^2 = -1$$

The multiplication rules are more complex:

$$\begin{array}{lll} ij = k & ik = -j & jk = i \\ ji = -k & ki = j & kj = -i \end{array}$$

<u>Note</u>:
Multiplication of quaternions is not commutative – the result is order-dependent.

Quaternions

More generally, the product of two quaternions is:

$$\begin{array}{r} (a_1 + ib_1 + jc_1 + kd_1) \\ \times\,(a_2 + ib_2 + jc_2 + kd_2) \\ \hline \end{array}$$

$$\begin{aligned} = \;& (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) \\ +i\,& (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2) \\ +j\,& (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2) \\ +k& (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2) \end{aligned}$$

$$i^2 = j^2 = k^2 = -1$$

$$\begin{array}{lll} ij = k & ik = -j & jk = i \\ ji = -k & ki = j & kj = -i \end{array}$$

Quaternions

As with complex numbers, for $q = a + ib + jc + kd$:

- The conjugate is:

$$\bar{q} = a - ib - jc - kd$$

- The (squared) norm is:

$$|q|^2 = a^2 + b^2 + c^2 + d^2 = q \cdot \bar{q}$$

- The norm of the products is the product of the norms:

$$|q_1 \cdot q_2| = |q_1| \cdot |q_2|$$

- The reciprocal is defined by dividing the conjugate by the square norm:

$$\frac{1}{q} = \frac{1}{q} \cdot \frac{\bar{q}}{\bar{q}} = \frac{\bar{q}}{|q|^2}$$

Quaternions

One way to express a quaternion is as a pair consisting of a scalar (the real coefficient) and a 3D vector (the imaginary coefficients):

$$q = (\alpha, \mathbf{w}) \quad \text{with } \alpha = a \text{ and } \mathbf{w} = (b, c, d)^{\top}$$

In this representation, multiplication becomes:

$$\begin{aligned} q_1 \cdot q_2 &\equiv (\alpha_1, \mathbf{w}_1) \cdot (\alpha_2, \mathbf{w}_2) \\ &= (\alpha_1 \cdot \alpha_2 - \langle \mathbf{w}_1, \mathbf{w}_2 \rangle, \alpha_1 \cdot \mathbf{w}_2 + \alpha_2 \cdot \mathbf{w}_1 + \mathbf{w}_1 \times \mathbf{w}_2) \end{aligned}$$

$$\begin{aligned} q_1 \cdot q_2 = (&a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) \\ &+ i\,(a_1 b_2 + a_2 b_1 + c_1 d_2 - c_2 d_1) \\ &+ j\,(a_1 c_2 + a_2 c_1 - b_1 d_2 + b_2 d_1) \\ &+ k(a_1 d_2 + a_2 d_1 + b_1 c_2 - b_2 c_1) \end{aligned}$$

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This is the (only) part that is order-dependent.

$$\begin{aligned} q_1 \cdot q_2 & \ldots d_1 d_2) \\ & +i\,(a_1 b_2 + a_2 b_1 + c_1 d_2 - c_2 d_1) \\ & +j\,(a_1 c_2 + a_2 c_1 - b_1 d_2 + b_2 d_1) \\ & +k(a_1 d_2 + a_2 d_1 + b_1 c_2 - b_2 c_1) \end{aligned}$$

Quaternions as Transformations

We can also think of points in 3D as (purely imaginary) quaternions:

$$(x, y, z) \rightarrow ix + jy + kz = (0, \mathbf{w})$$

Given a **quaternion** q and an imaginary quaternion (3D point) p, consider the map:

$$q(p) = qp\bar{q}$$

Quaternions as Transformations

$$q(p) = qp\bar{q}$$

$$q_1 \cdot q_2 = (\alpha_1 \cdot \alpha_2 - \langle \mathbf{w}_1, \mathbf{w}_2 \rangle, \alpha_1 \cdot \mathbf{w}_2 + \alpha_2 \cdot \mathbf{w}_1 + \mathbf{w}_1 \times \mathbf{w}_2)$$

<u>Claim</u>:

1. The map takes imaginary quaternions (points in 3D) to imaginary quaternions (points in 3D)

$$\begin{aligned} q(p) &= (\alpha_q, \mathbf{w}_q) \cdot (0, \mathbf{w}_p) \cdot (\alpha_q, -\mathbf{w}_q) \\ &= (-\langle \mathbf{w}_q, \mathbf{w}_p \rangle, \alpha_q \mathbf{w}_p + \mathbf{w}_q \times \mathbf{w}_p) \cdot (\alpha_q, -\mathbf{w}_q) \\ &= (-\alpha_q \langle \mathbf{w}_q, \mathbf{w}_p \rangle + \alpha_q \langle \mathbf{w}_p, \mathbf{w}_q \rangle + \langle \mathbf{w}_q \times \mathbf{w}_p, \mathbf{w}_q \rangle, \dots) \\ &= (0, \dots) \end{aligned}$$

Quaternions as Transformations

$$q(p) = qp\bar{q}$$

$$q_1 \cdot q_2 = (\alpha_1 \cdot \alpha_2 - \langle \mathbf{w}_1, \mathbf{w}_2 \rangle, \alpha_1 \cdot \mathbf{w}_2 + \alpha_2 \cdot \mathbf{w}_1 + \mathbf{w}_1 \times \mathbf{w}_2)$$

Claim:

1. The map takes 3D points to 3D points

2. The map is linear

$$\begin{aligned} q(a \cdot p_1 + b \cdot p_2) &= q(a \cdot p_1 + b \cdot p_2)\bar{q} \\ &= a \cdot qp_1\bar{q} + b \cdot qp_2\bar{q} \\ &= a \cdot q(p_1) + b \cdot q(p_2) \end{aligned}$$

Quaternions as Transformations

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$$q_1 \cdot q_2 = (\alpha_1 \cdot \alpha_2 - \langle \mathbf{w}_1, \mathbf{w}_2 \rangle, \alpha_1 \cdot \mathbf{w}_2 + \alpha_2 \cdot \mathbf{w}_1 + \mathbf{w}_1 \times \mathbf{w}_2)$$

Claim:

1. The map takes 3D points to 3D points
2. The map is linear
3. If $|q| = 1$, the map is norm-preserving

$$\begin{aligned} |q(p)| &= |qp\bar{q}| \\ &= |q||p||\bar{q}| \\ &= |p| \end{aligned}$$

Quaternions as Transformations

$$q(p) = qp\bar{q}$$

$$q_1 \cdot q_2 = (\alpha_1 \cdot \alpha_2 - \langle \mathbf{w}_1, \mathbf{w}_2 \rangle, \alpha_1 \cdot \mathbf{w}_2 + \alpha_2 \cdot \mathbf{w}_1 + \mathbf{w}_1 \times \mathbf{w}_2)$$

Claim:

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$$\Downarrow$$

When q is a unit quaternion, the map $p \to qp\bar{q}$ is an orthogonal transformation (specifically, a rotation).

Unit Quaternions and Rotations

If $q = a + ib + jc + kd$ is a unit quaternion ($|q| = 1$), we can associate q with the rotation:

$$\mathbf{R}(q) = \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2bd + 2ac \\ 2bc + 2ad & a^2 - b^2 + c^2 - d^2 & 2cd - 2ab \\ 2bd - 2ac & 2cd + 2ab & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$$

Note that all of the terms are quadratic.

$\Downarrow$

The rotation associated with q is the same as the rotation associated with $-q$.

Unit Quaternions and Rotations

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Because q is a unit quaternion, we have:

$$|q|^2 = \|(\alpha, \mathbf{w})\|^2 = \alpha^2 + \|\mathbf{w}\|^2 = 1$$

Or equivalently, if we set $\mathbf{v} = \mathbf{w}/\|\mathbf{w}\|$, there exists θ such that:

$$q = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\mathbf{v}\right)$$

Unit Quaternions and Rotations

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Because q is a unit quaternion, we have:

$$q = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\mathbf{v}\right)$$

It turns out that q corresponds to the rotation whose:

- axis of rotation is $\mathbf{v}$, and
- angle of rotation is θ.

Unit Quaternions and Rotations

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Because q is a unit quaternion, we have:

In particular, if we express rotations in the axis-angle representation, we can compute the composition by multiplying quaternions.

It turns out that q corresponds to the rotation whose:

- axis of rotation is $\mathbf{v}$, and
- angle of rotation is θ.

Quaternions

Instead of blending rotations $\{\mathbf{R}_0, \ldots, \mathbf{R}_{n-1}\}$ and then normalizing using SVD, we can blend the quaternions and then normalize them:

- For each $\mathbf{R}_i$, compute the quaternion rep. $(\alpha_i, \mathbf{w}_i)$

Quaternions

Instead of blending rotations $\{\mathbf{R}_0, \ldots, \mathbf{R}_{n-1}\}$ and then normalizing using SVD, we can blend the quaternions and then normalize them:

- For each $\mathbf{R}_i$, compute the quaternion rep. $(\alpha_i, \mathbf{w}_i)$
- Interpolate/Approximate the quaternions:

Quaternions

Instead of blending rotations $\{\mathbf{R}_0, \ldots, \mathbf{R}_{n-1}\}$ and then normalizing using SVD, we can blend the quaternions and then normalize them:

- For each $\mathbf{R}_i$, compute the quaternion rep. $(\alpha_i, \mathbf{w}_i)$
- Interpolate/Approximate the quaternions:
  - » Linear Interpolation:

$$\alpha_k(t) = (1-t)\alpha_k + t\alpha_{k+1}$$
$$\mathbf{w}_k(t) = (1-t)\mathbf{w}_k + t\mathbf{w}_{k+1}$$

Quaternions

Instead of blending rotations $\{\mathbf{R}_0, \ldots, \mathbf{R}_{n-1}\}$ and then normalizing using SVD, we can blend the quaternions and then normalize them:

- For each $\mathbf{R}_i$, compute the quaternion rep. $(\alpha_i, \mathbf{w}_i)$
- Interpolate/Approximate the quaternions:
  - » Linear Interpolation
  - » Catmull-Rom Interpolation:

$$\alpha_k(t) = CR_0(t)\alpha_{k-1} + CR_1(t)\alpha_k + CR_2(t)\alpha_{k+1} + CR_3(t)\alpha_{k+2}$$
$$\mathbf{w}_k(t) = CR_0(t)\mathbf{w}_{k-1} + CR_1(t)\mathbf{w}_k + CR_2(t)\mathbf{w}_{k+1} + CR_3(t)\mathbf{w}_{k+2}$$

Quaternions

Instead of blending rotations $\{\mathbf{R}_0, \ldots, \mathbf{R}_{n-1}\}$ and then normalizing using SVD, we can blend the quaternions and then normalize them:

- For each $\mathbf{R}_i$, compute the quaternion rep. $(\alpha_i, \mathbf{w}_i)$
- Interpolate/Approximate the quaternions:
  - » Linear Interpolation
  - » Catmull-Rom Interpolation
  - » Uniform Cubic B-Spline Approximation:

    $\alpha_k(t) = B_{0,3}(t)\alpha_{k-1} + B_{1,3}(t)\alpha_k + B_{2,3}(t)\alpha_{k+1} + B_{3,3}(t)\alpha_{k+2}$

    $\mathbf{w}_k(t) = B_{0,3}(t)\mathbf{w}_{k-1} + B_{1,3}(t)\mathbf{w}_k + B_{2,3}(t)\mathbf{w}_{k+1} + B_{3,3}(t)\mathbf{w}_{k+2}$

Quaternions

Instead of blending rotations $\{\mathbf{R}_0, \ldots, \mathbf{R}_{n-1}\}$ and then normalizing using SVD, we can blend the quaternions and then normalize them:

- For each $\mathbf{R}_i$, compute the quaternion rep. $(\alpha_i, \mathbf{w}_i)$
- Interpolate/Approximate the quaternions:
  - » Linear Interpolation
  - » Catmull-Rom Interpolation
  - » Uniform Cubic B-Spline Approximation
- Set the value of the in-between rotation to be the normalized quaternion:

$$q_k(t) = \frac{\big(\alpha_k(t), \mathbf{w}_k(t)\big)}{\big\|\big(\alpha_k(t), \mathbf{w}_k(t)\big)\big\|}$$

Quaternions

Instead of blending rotations $\{\mathbf{R}_0, \ldots, \mathbf{R}_{n-1}\}$ and then normalizing using SVD, we can blend the quaternions and then normalize them:

- For each $\mathbf{R}_i$, compute the quaternion rep. $(\alpha_i, \mathbf{w}_i)$
- Interpolate/Approximate the quaternions:

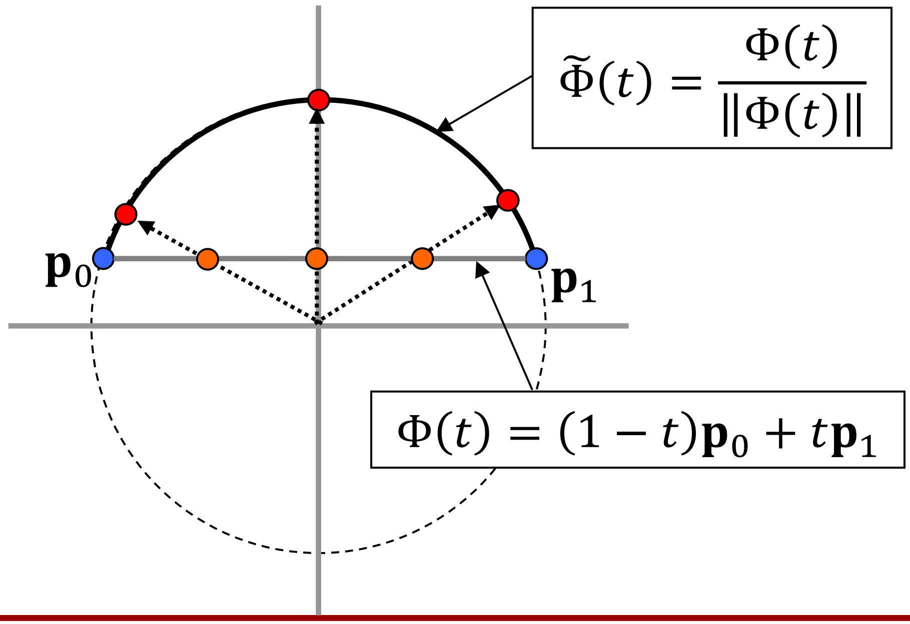
Note:
- Using SVD, we interpolated in the $(9 = 3 \times 3)$-dimensional space of matrices and then normalized.
- With quaternions we interpolate in the 4-dimensional space of quaternions and normalize.

$$q_k(t) = \frac{\big(\alpha_k(t), \mathbf{w}_k(t)\big)}{\big\|\big(\alpha_k(t), \mathbf{w}_k(t)\big)\big\|}$$

Quaternions

As with points on the circle/sphere, this type of interpolation/approximation has the limitation:

- Uniform sampling in quaternion space does not result in uniform sampling in rotation space.



Quaternions

As with points on the circle/sphere, this type of interpolation/approximation has the limitation:

- Uniform sampling in quaternion space does not result in uniform sampling in rotation space.

Additionally, since $\mathbf{R}(-q) = \mathbf{R}(q)$ there are two different quaternions we can associate with a rotation, so the mapping is not unique.

Quaternions

Aside:

- In animations/games, the orientation of the camera is the result of the composition of many rotations.
- Due to numerical imprecision, the composition of these rotations may not itself be a rotation.
- To avoid distortion, we need to “snap” the composition to the closest rotation.

This is easily done using quaternions to represent the camera’s orientation.

Overview

- Math review
- Quaternions
- The exponential map

The Exponential Map

Parametrization:

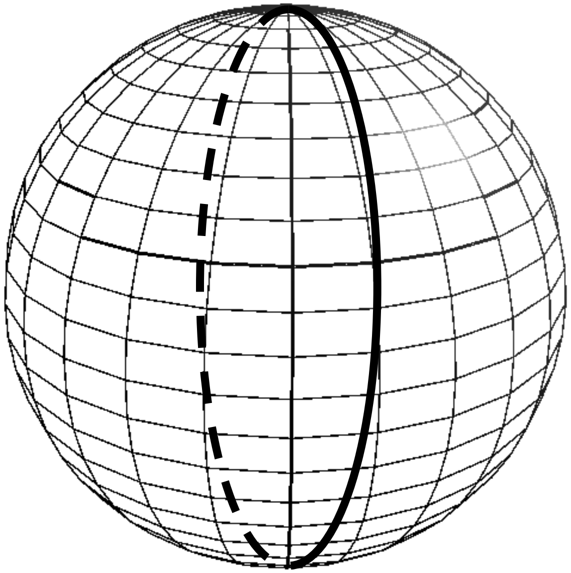
- Parameterize rotations in a linear space
- Blend parameters
- Evaluate the parameterization at the blend

Goal:

- Find a canonical way to parametrize rotations so that there is little distortion

Geodesics

Given a surface $\mathcal{S}(u,v)$ a *geodesic* is a curve that is (locally) the shortest path between two points.



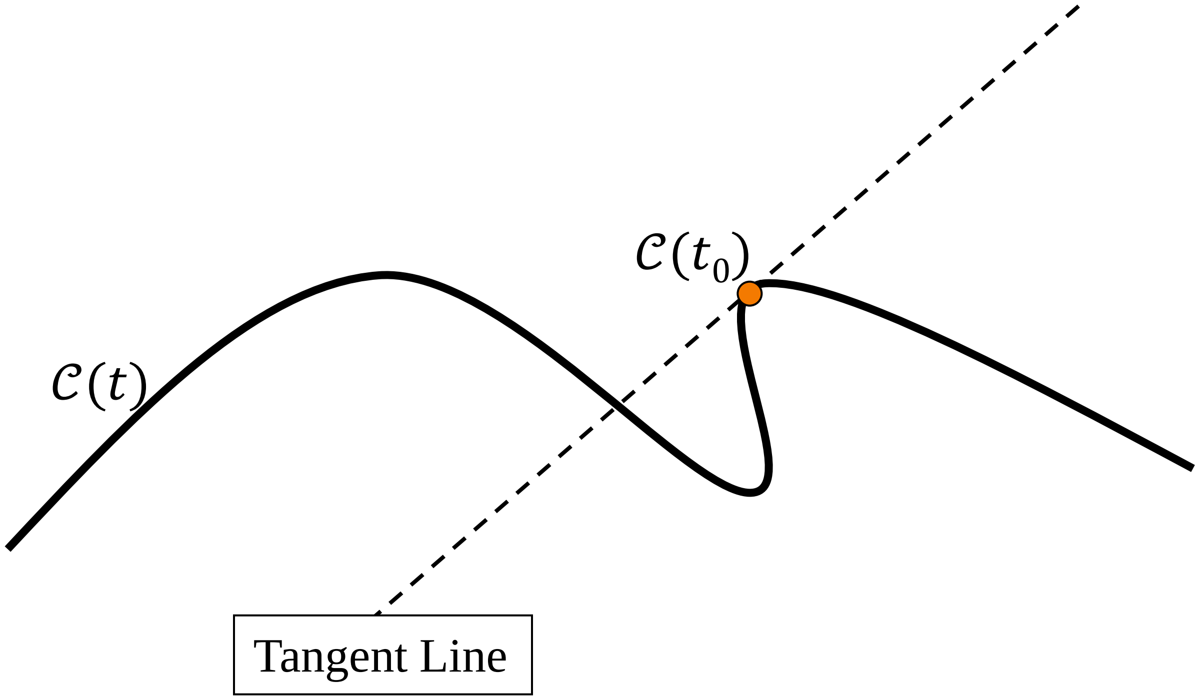
$$\mathcal{S}(u,v) = (\cos u \cos v\,, \sin u\,, \cos u \sin v)$$

Geodesics

Given a manifold (a d-dimensional surface) a *geodesic* is a curve that is (locally) the shortest path between two points.

Tangent Spaces

Given a curve $\mathcal{C}(t)$, the *tangent line* to the curve at a point $\mathbf{p}_0 = \mathcal{C}(t_0)$ is the line that most closely approximates the curve $\mathcal{C}(t)$ at the point $\mathbf{p}_0$.



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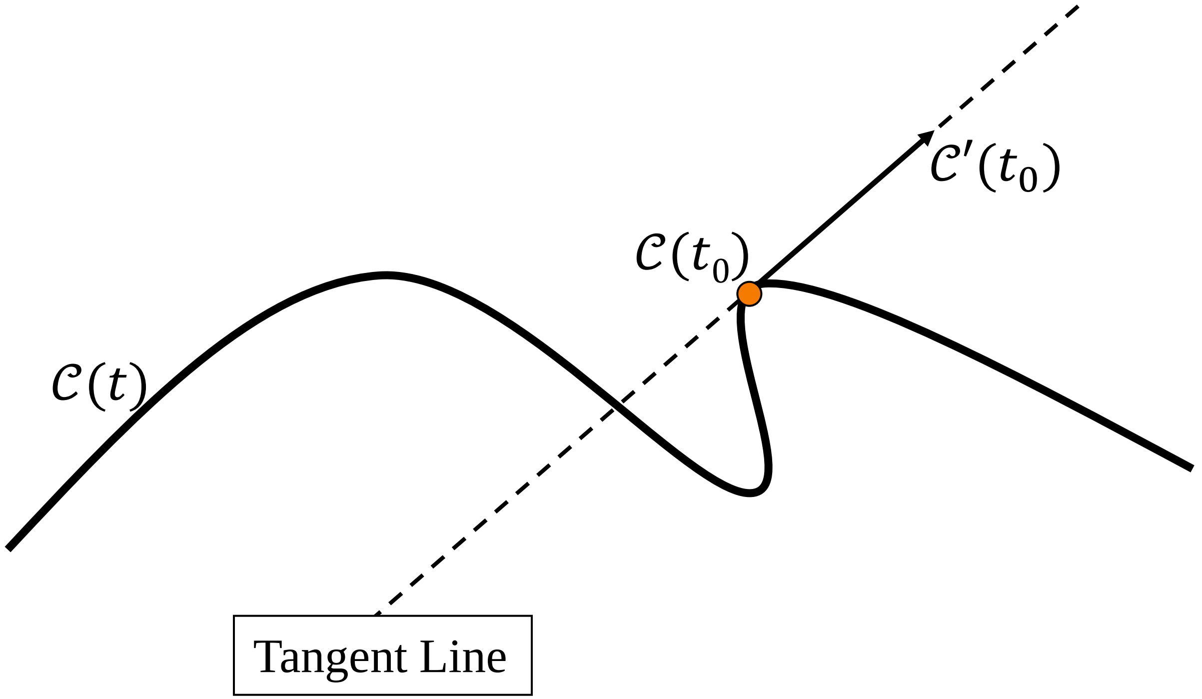
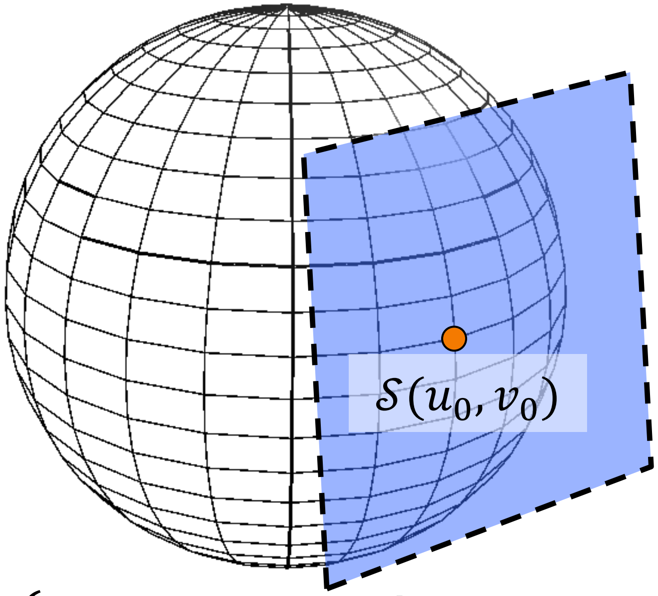
This is the line through $\mathbf{p}_0$ with direction $\mathcal{C}'(t_0)$.



Tangent Spaces

Given a surface $\mathcal{S}(u,v)$ the *tangent plane* to the curve at a point $\mathbf{p}_0 = \mathcal{S}(u_0,v_0)$ is the plane that most closely approximates $\mathcal{S}(u,v)$ at the point $\mathbf{p}_0$.



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Tangent Spaces

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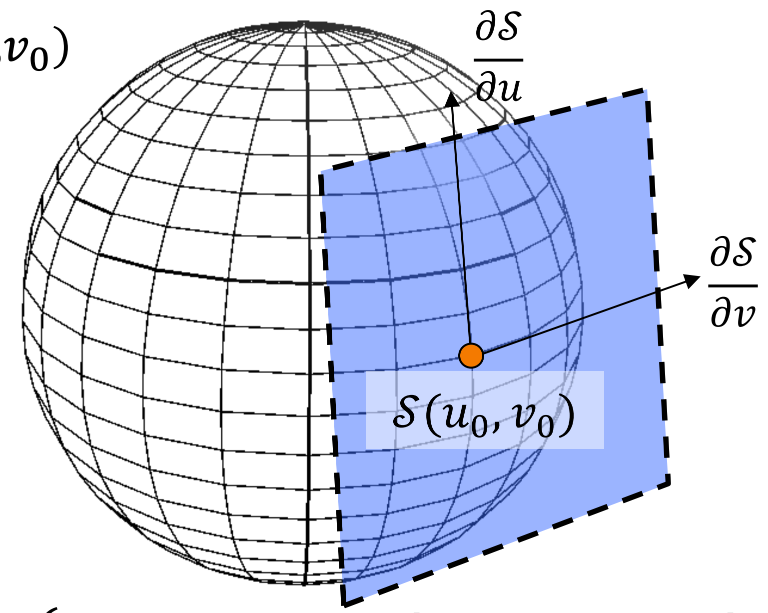
This is the plane through $\mathbf{p}_0$, spanned by:

$$\left.\frac{\partial \mathcal{S}(u,v)}{\partial u}\right|_{(u_0,v_0)} \quad \text{and} \quad \left.\frac{\partial \mathcal{S}(u,v)}{\partial v}\right|_{(u_0,v_0)}$$



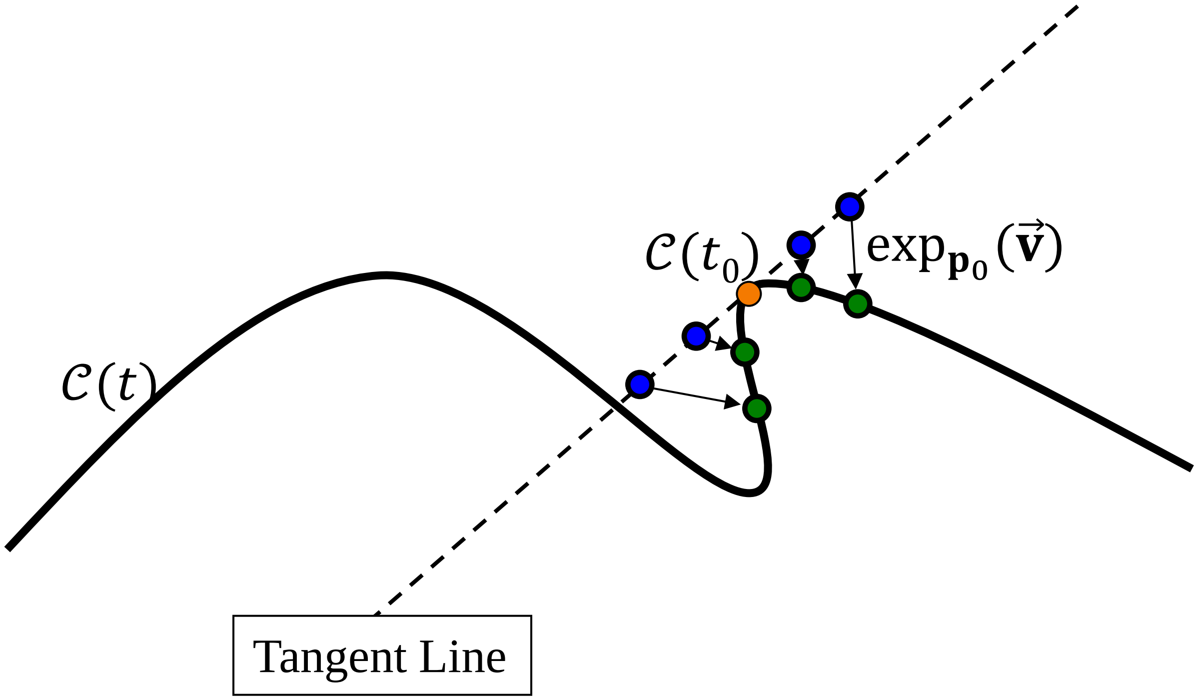
$$\mathcal{S}(u,v) = (\cos u \cos v, \sin u, \cos u \sin v)$$

Tangent Spaces

Given a manifold (a d-dimensional surface) the *tangent space* to the manifold at a point $\mathbf{p}_0$ on the manifold is the d-dimensional plane that describes the directions you can "move" from $\mathbf{p}_0$ while still staying close to the manifold.

The Exponential Map

Given a curve $\mathcal{C}(t)$, the *exponential* at $\mathbf{p}_0 = \mathcal{C}(t_0)$ is a map that sends points in the tangent space of $\mathbf{p}_0$ to the curve $\mathcal{C}(t)$.



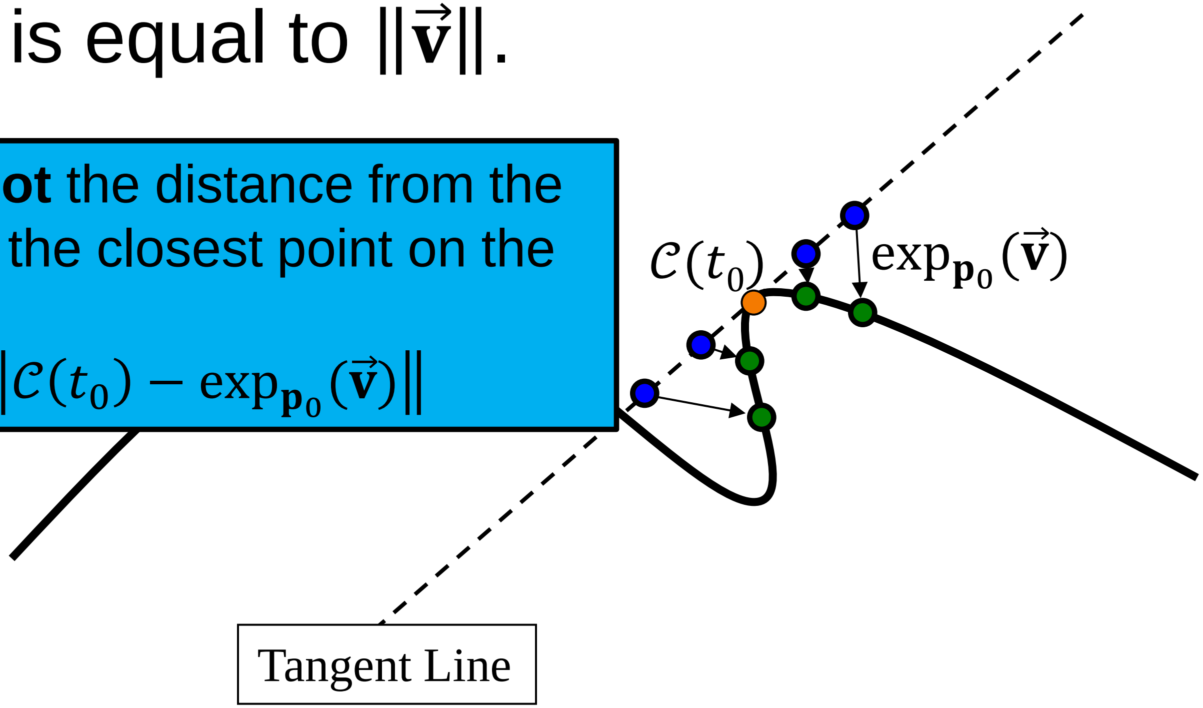
The Exponential Map

Given a curve $\mathcal{C}(t)$, the *exponential* at $\mathbf{p}_0 = \mathcal{C}(t_0)$ is a map that sends points in the tangent space of $\mathbf{p}_0$ to the curve $\mathcal{C}(t)$.

The distance **along the curve** from $\mathbf{p}_0$ to point $\exp_{\mathbf{p}_0}(\vec{\mathbf{v}})$ is equal to $\|\vec{\mathbf{v}}\|$.

Note: This is **not** the distance from the tangent line to the closest point on the curve:

$$\|\vec{\mathbf{v}}\| \neq \left\|\mathcal{C}(t_0) - \exp_{\mathbf{p}_0}(\vec{\mathbf{v}})\right\|$$



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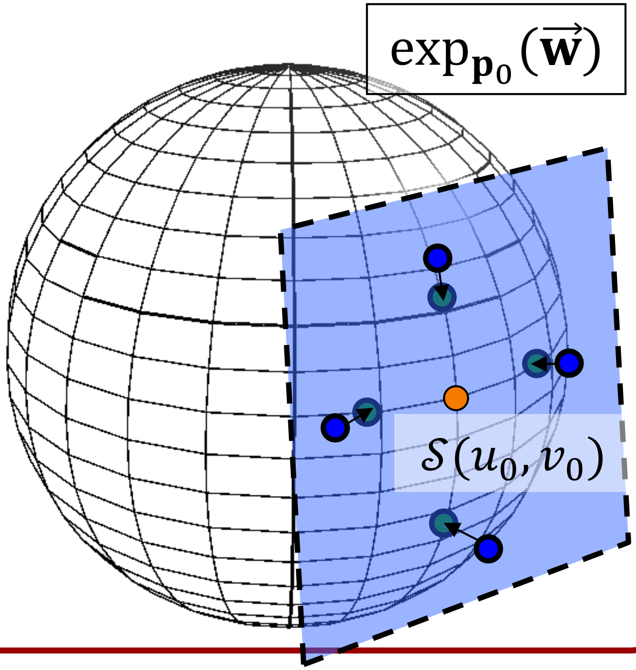
Given a surface $\mathcal{S}(u,v)$, the *exponential* at the point $\mathbf{p}_0 = \mathcal{S}(u_0,v_0)$ is a map that sends points in the tangent plane of $\mathbf{p}_0$ to the surface $\mathcal{S}(u,v)$.



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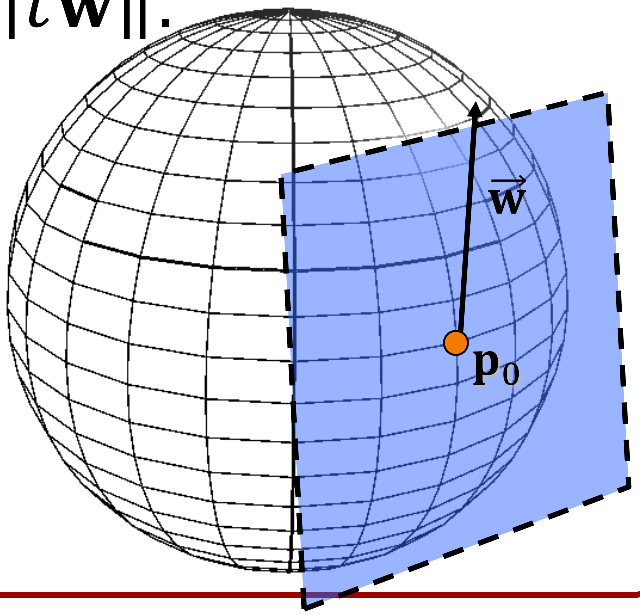
Fixing a vector $\vec{\mathbf{w}}$ in the tangent space at $\mathbf{p}_0$, the curve $\exp_{\mathbf{p}_0}(t\vec{\mathbf{w}})$ follows the geodesic leaving $\mathbf{p}_0$ in direction $\vec{\mathbf{w}}$, with length equal to $\|t\vec{\mathbf{w}}\|$.



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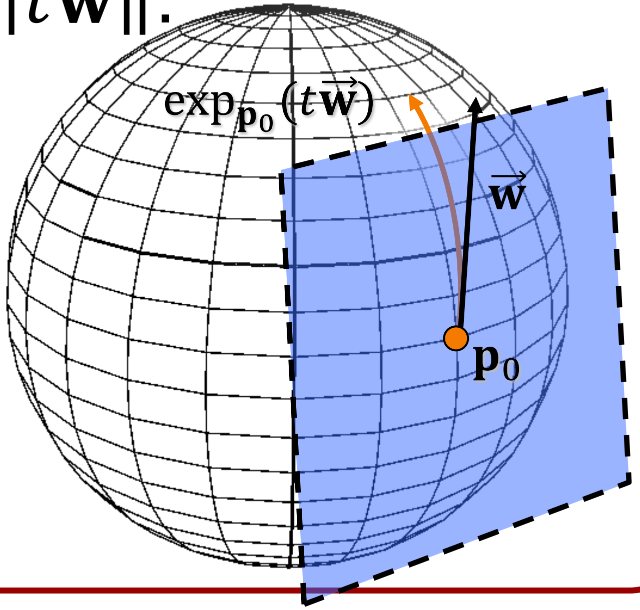

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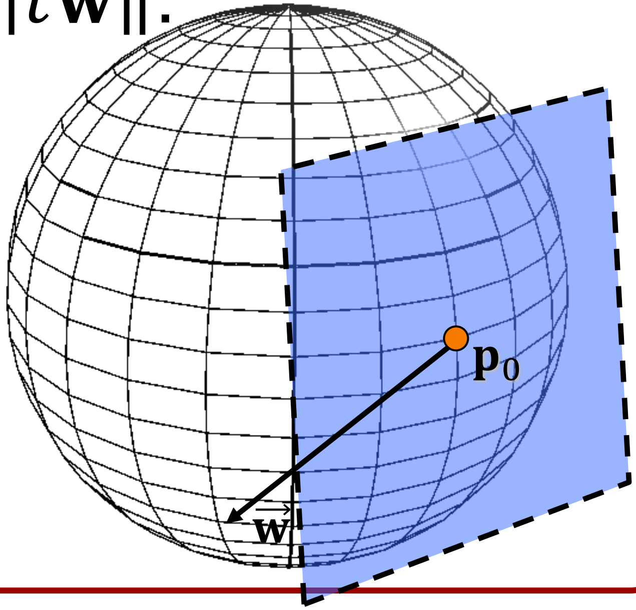

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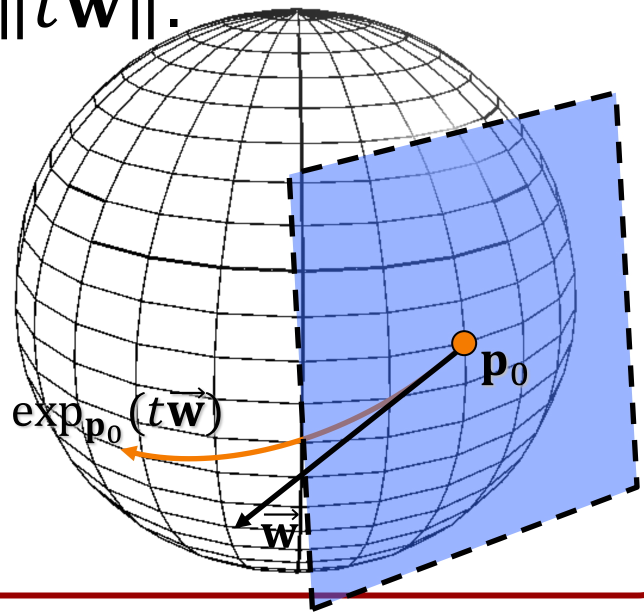

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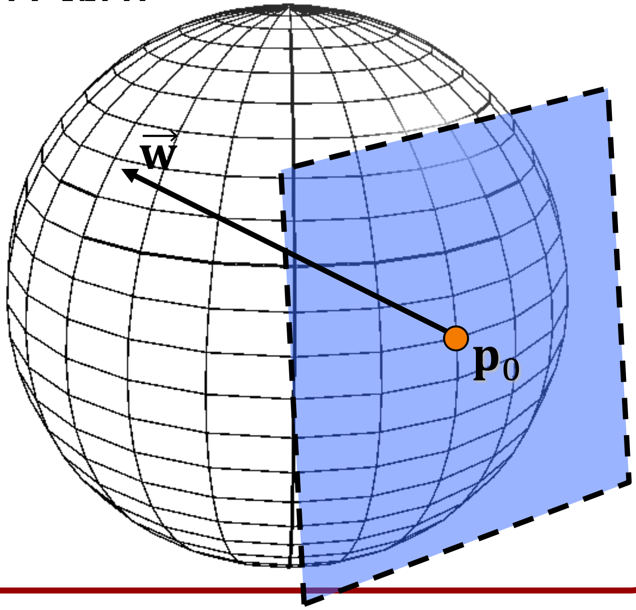

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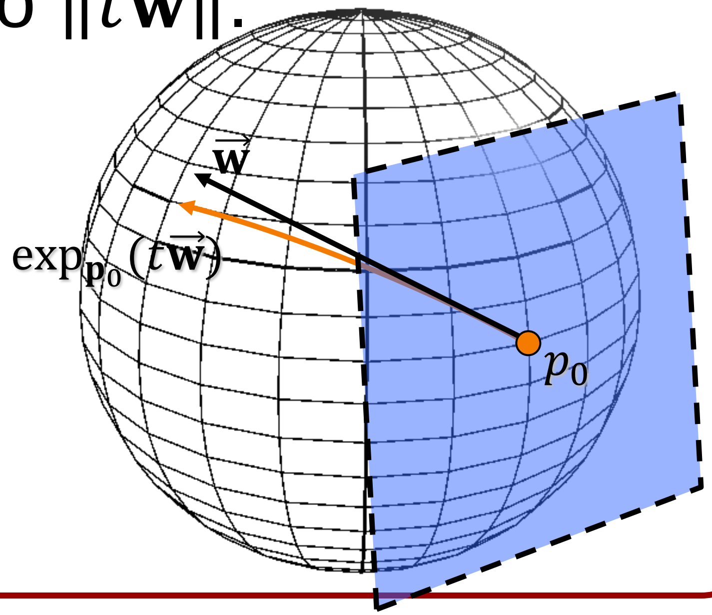


The Exponential Map

Given a manifold (a d-dimensional surface), the *exponential* at point $\mathbf{p}_0$ on the manifold is a map that sends points in the tangent plane of $\mathbf{p}_0$ to the manifold.

Fixing a vector $\vec{\mathbf{w}}$ in the tangent space at $\mathbf{p}_0$, the curve $\exp_{\mathbf{p}_0}(t\vec{\mathbf{w}})$ follows the geodesic leaving $\mathbf{p}_0$ in direction $\vec{\mathbf{w}}$, with length equal to $\|t\vec{\mathbf{w}}\|$.

Answers the question:
Starting at a point $\mathbf{p}_0$, if we "walk" along the manifold in direction $\vec{\mathbf{w}}$ for time t, where do we end up?

The Logarithm Map

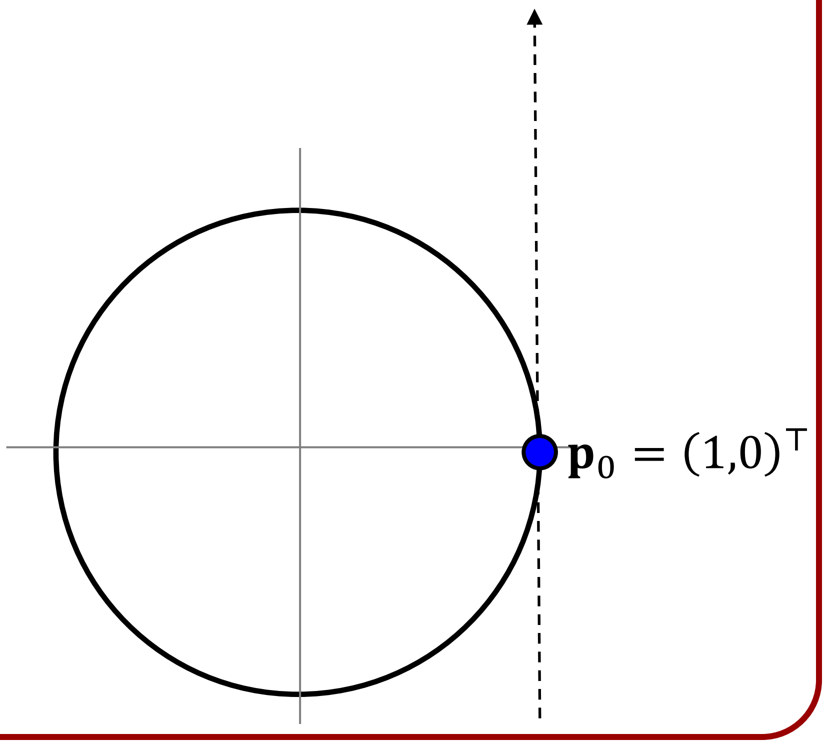
For a point $\mathbf{p}_0$ on a curve/surface/manifold, the *logarithm* is the inverse of the exponential, sending points on the curve/surface/manifold back into the tangent space of $\mathbf{p}_0$.

Answers the question:
Given a starting point $\mathbf{p}_0$, and some other point $\mathbf{p}$ on the manifold, what direction (and how long) do we need to walk from $\mathbf{p}_0$ to get to $\mathbf{p}$?

The Exponential Map

Example:

Let $\mathcal{C}$ be the unit circle, the tangent line at the point $\mathbf{p}_0 = (1,0)^\top$ is the vertical line through $\mathbf{p}_0$.

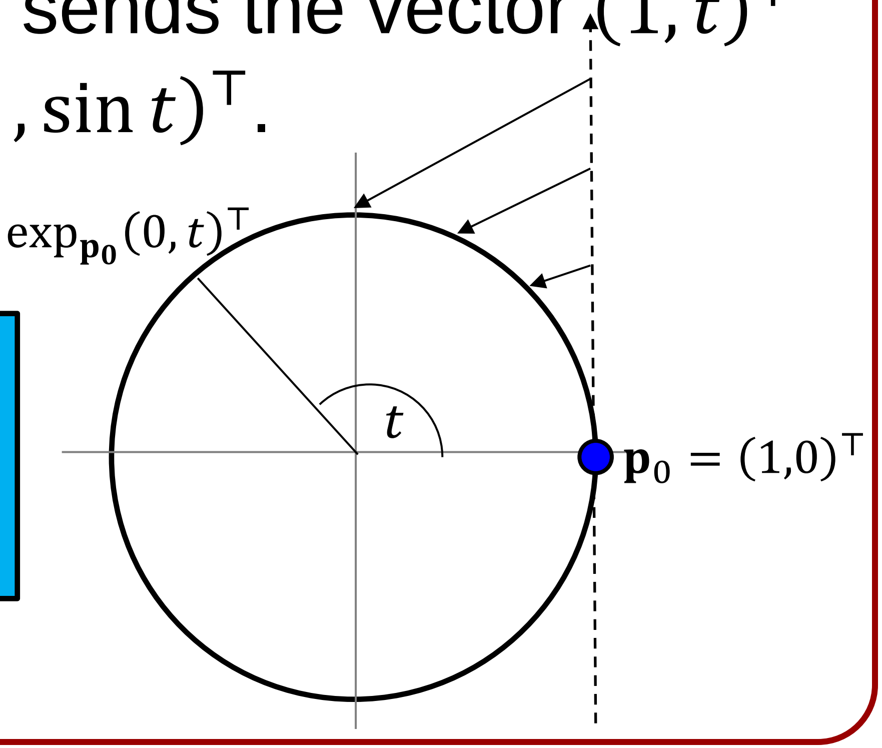


The Exponential Map

Example:

Let $\mathcal{C}$ be the unit circle, the tangent line at the point $\mathbf{p}_0 = (1,0)^\top$ is the vertical line through $\mathbf{p}_0$.

The exponential map $\exp_{\mathbf{p}_0}$ sends the vector $(1,t)^\top$ on the tangent line to $(\cos t, \sin t)^\top$.



Note:
The exponential map is many-to-one:

$$\exp_{\mathbf{p}_0}(0,t)^\top = \exp_{\mathbf{p}_0}(0,t+2k\pi)^\top$$

so the logarithm is not unique.

The Exponential Map

Fact:

1. The tangent space to the manifold of $(n \times n)$ rotations at the identity is the is space of $(n \times n)$ skew-symmetric matrices.

2. The exponential map at the identity, $\exp_{\mathbf{id.}}$, sends skew-symmetric matrices to rotations.

The Exponential Map

How do we compute the exponential map?

The Exponential Map

How do we compute the exponential map?

It is difficult to find a closed form solution, but for matrices we can use a Taylor series approximation:

$$\exp_{\mathbf{id.}}(\boldsymbol{S}) = \mathbf{id.} + \boldsymbol{S} + \frac{1}{2!}\boldsymbol{S}^2 + \cdots + \frac{1}{n!}\boldsymbol{S}^n + \cdots$$

In a similar manner, we can define the logarithm:

$$\ln_{\mathbf{id.}}(\mathbf{R}) = (\mathbf{R} - \mathbf{id.}) - \frac{(\mathbf{R}-\mathbf{id.})^2}{2} + \cdots + (-1)^{n+1}\frac{(\mathbf{R}-\mathbf{id.})^n}{n} + \cdots$$

The Exponential Map

Properties:

- $\exp_{\mathbf{id.}}(0) = \mathbf{id}.$
  - If we start at the identity and don't go anywhere, we are still at the identity.

- $\left.\frac{\partial \exp_{\mathbf{id.}}(t\boldsymbol{S})}{\partial t}\right|_{t=0} = \boldsymbol{S}$
  - If we follow the geodesic from the identity in direction $\boldsymbol{S}$, then we start by going in direction $\boldsymbol{S}$.

- $\ln_{\mathbf{id.}}(\exp_{\mathbf{id.}}\boldsymbol{S}) = \boldsymbol{S}$
  - The direction we need to travel from the identity to end up at the rotation we would get to by walking in direction $\boldsymbol{S}$ is itself $\boldsymbol{S}$.

Rotation Interpolation/Approximation

Given a collection of rotations $\{\mathbf{R}_0, \ldots, \mathbf{R}_{n-1}\}$ we can generate a curve passing through/near the matrices:

- For each $\mathbf{R}_i$, compute the logarithm $\mathbf{S}_i = \ln_{\mathbf{id.}}(\mathbf{R}_i)$

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- Interpolate/Approximate the logarithms:
  - » Linear Interpolation:

$$\mathbf{S}_k(t) = (1-t)\mathbf{S}_k + t\mathbf{S}_{k+1}$$

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- Interpolate/Approximate the logarithms:
  - » Linear Interpolation:
  - » Catmull-Rom Interpolation:

$$\mathbf{S}_k(t) = CR_0(t)\mathbf{S}_{k-1} + CR_1(t)\mathbf{S}_k + CR_1(t)\mathbf{S}_{k+1} + CR_1(t)\mathbf{S}_{k+2}$$

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- Interpolate/Approximate the logarithms:
  - » Linear Interpolation:
  - » Catmull-Rom Interpolation:
  - » Uniform Cubic B-Spline Approximation:

$$\mathbf{S}_k(t) = B_{0,3}(t)\mathbf{S}_{k-1} + B_{1,3}(t)\mathbf{S}_k + B_{2,3}(t)\mathbf{S}_{k+1} + B_{3,3}(t)\mathbf{S}_{k+2}$$

Rotation Interpolation/Approximation

Given a collection of rotations $\{\mathbf{R}_0, \ldots, \mathbf{R}_{n-1}\}$ we can generate a curve passing through/near the matrices:

- For each $\mathbf{R}_i$, compute the logarithm $\mathbf{S}_i = \ln_{\mathbf{id.}}(\mathbf{R}_i)$
- Interpolate/Approximate the logarithms:
  - » Linear Interpolation:
  - » Catmull-Rom Interpolation:
  - » Uniform Cubic B-Spline Approximation:
- Set the value of the in-between rotation to be the exponent of the blended logarithms:

$$\mathbf{R}_k(t) = \exp_{\mathbf{id.}}\left(\mathbf{S}_k(t)\right)$$

Rotation Interpolation/Approximation

Given a collection of rotations $\{\mathbf{R}_0, \ldots, \mathbf{R}_{n-1}\}$ we can generate a curve passing through/near the matrices:

- For each $\mathbf{R}_i$, compute the logarithm $\mathbf{S}_i = \ln_{\mathbf{id.}}(\mathbf{R}_i)$
- Interpolate/Approximate the logarithms:

Note:
Since the logarithm of rotations is a skew-symmetric matrix, and since skew-symmetric matrices are closed under addition and scaling, the weighted average $\mathbf{S}_k(t)$ is also skew-symmetric, so its exponent will be a rotation.

Warning:
Is taking the exponential/logarithm with respect to the identity the right thing to do? (e.g. Maybe we should take it with respect to some other point.)

Summary

To define in-between frames for an animation, we need to interpolate/approximate the transformations specified in the key-frames.

- For translation, we can use splines
- For rotations, we need to ensure that the in-between transformations are also rotations:
  - Euler angles
  - Exponential map

    } In-between transformations are guaranteed to be rotations

  - SVD
  - Quaternions

    } Normalize in-between transformations to turn them into the nearest rotations