Representing Meshes
Parametric Curves

Michael Kazhdan

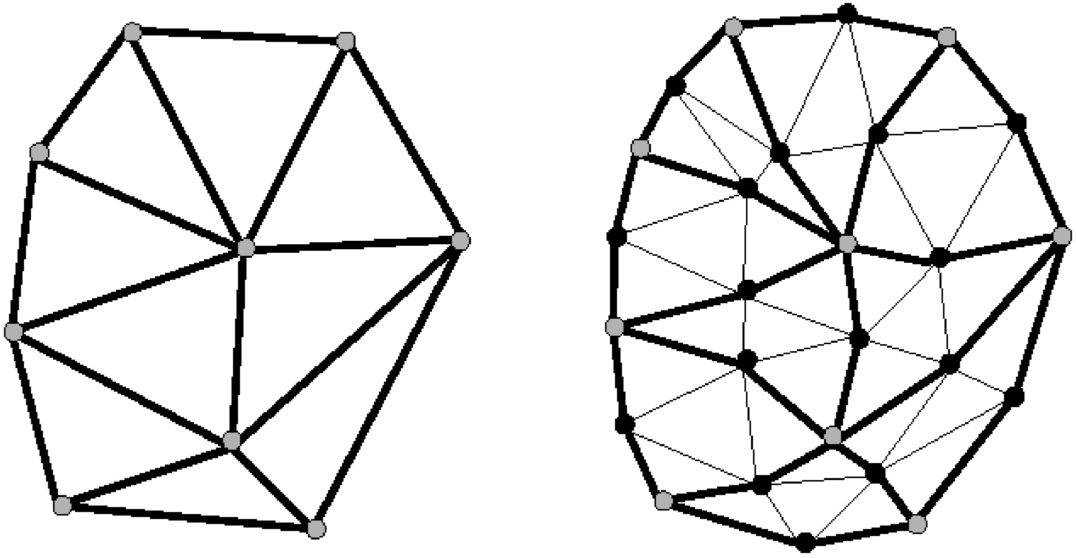
(601.457/657)

Outline

- Representing Meshes
- Parametric Curves

Key Questions

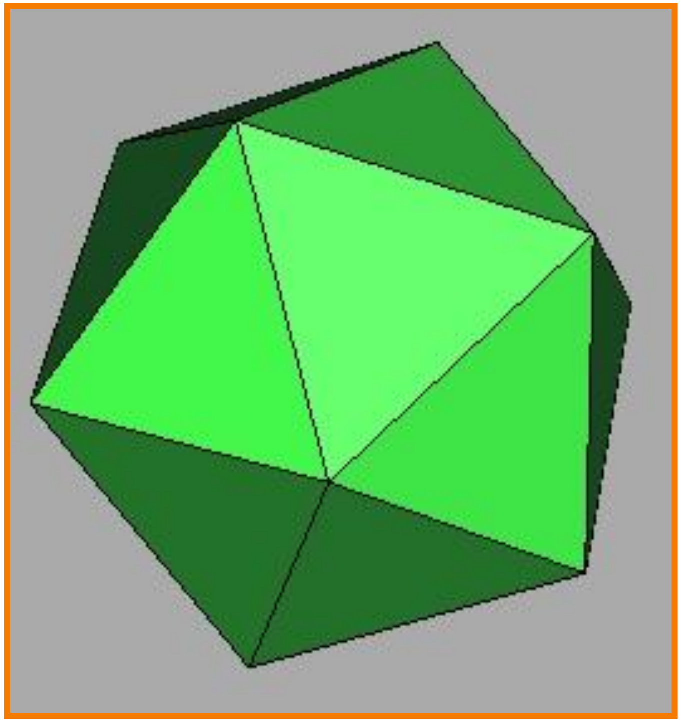
- How to refine the mesh?
  - Aim for properties like smoothness
- **How to store the mesh?**
  - Aim for efficiency in implementing subdivision rules



Zorin & Schroeder
SIGGRAPH 99
Course Notes

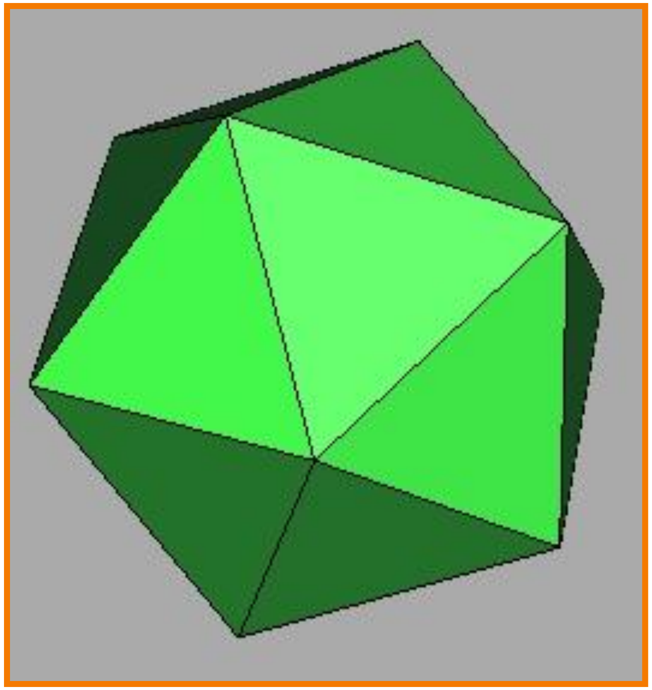
Polygon Meshes

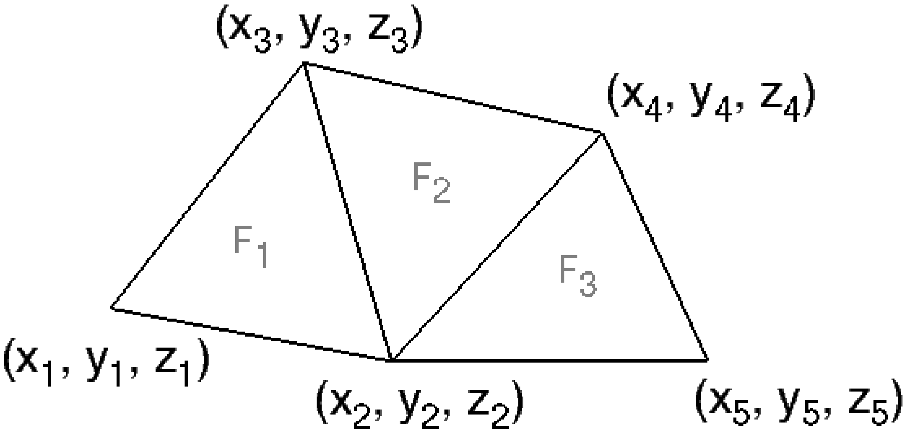
- Mesh Representations
  - Independent faces
  - Vertex and face tables
  - Adjacency lists
  - Winged-Edge



Independent Faces

- Each face lists vertex coordinates

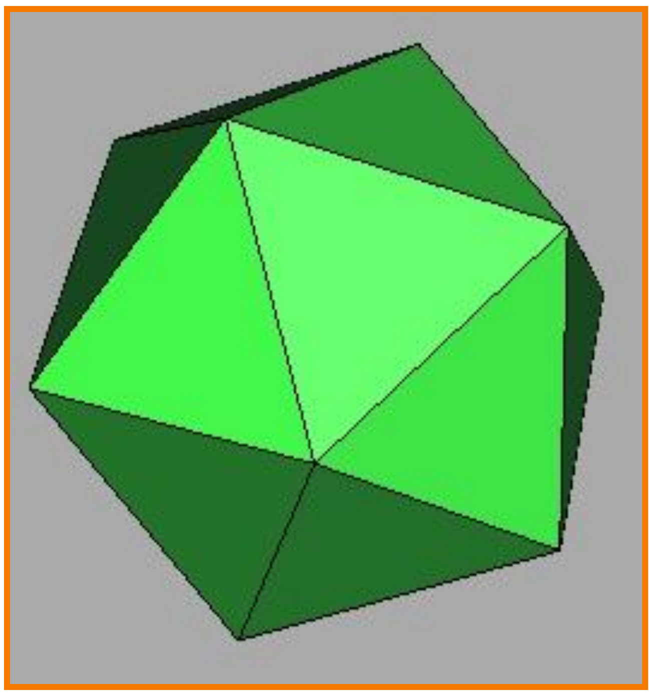


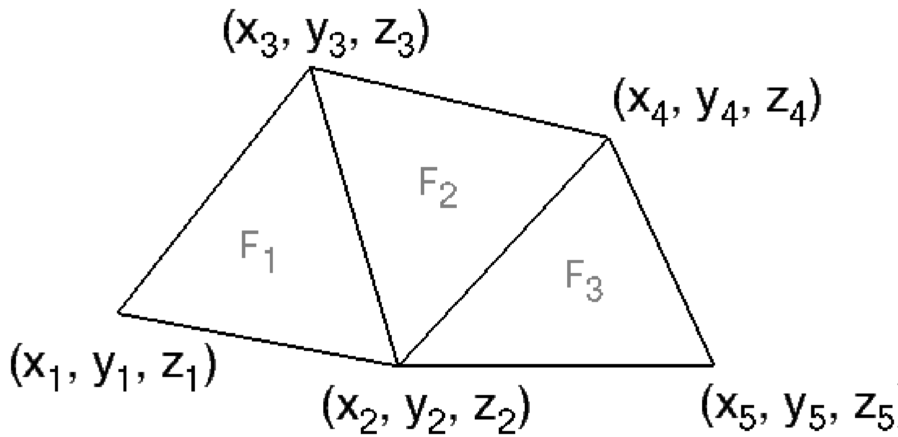


FACE TABLE	
F_1	(x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3)
F_2	(x_2, y_2, z_2) (x_4, y_4, z_4) (x_3, y_3, z_3)
F_3	(x_2, y_2, z_2) (x_5, y_5, z_5) (x_4, y_4, z_4)

Independent Faces

- Each face lists vertex coordinates
  - Redundant vertices

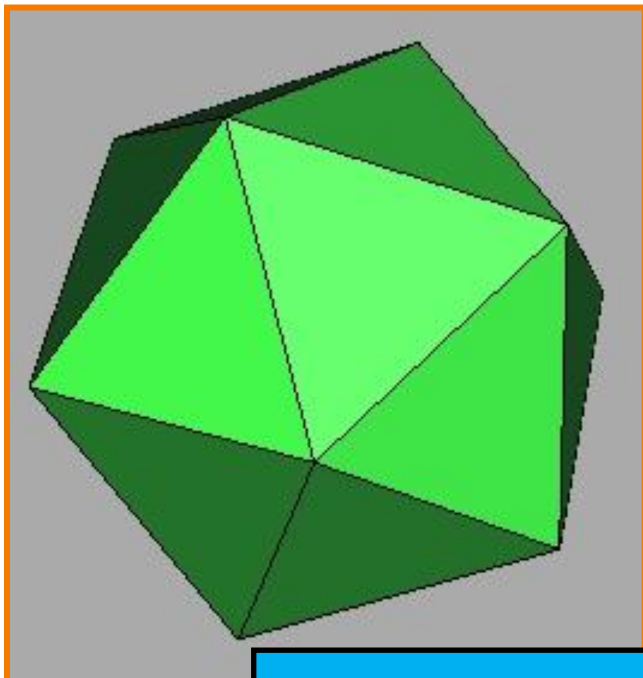


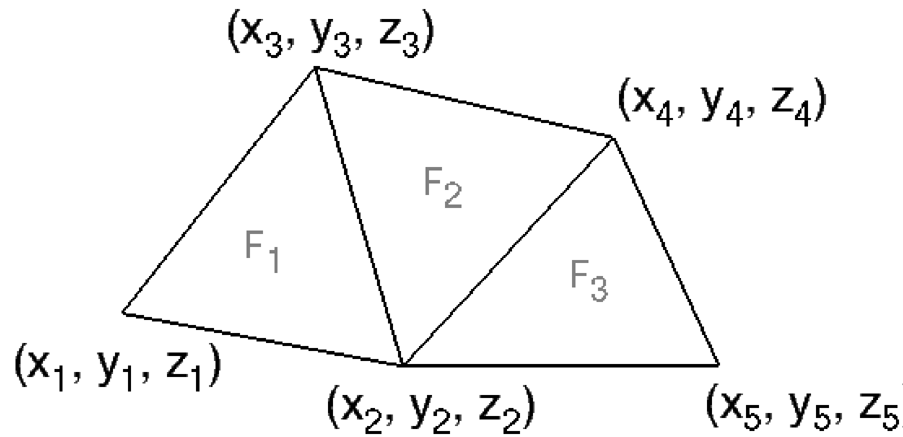


FACE TABLE			
F_1	(x_1, y_1, z_1)	(x_2, y_2, z_2)	(x_3, y_3, z_3)
F_2	(x_2, y_2, z_2)	(x_4, y_4, z_4)	(x_3, y_3, z_3)
F_3	(x_2, y_2, z_2)	(x_5, y_5, z_5)	(x_4, y_4, z_4)

Independent Faces

- Each face lists vertex coordinates
  - Redundant vertices



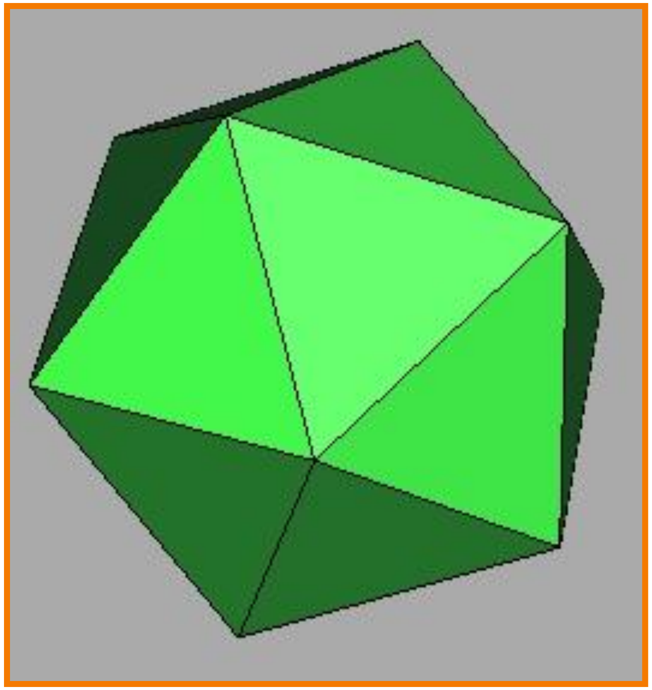


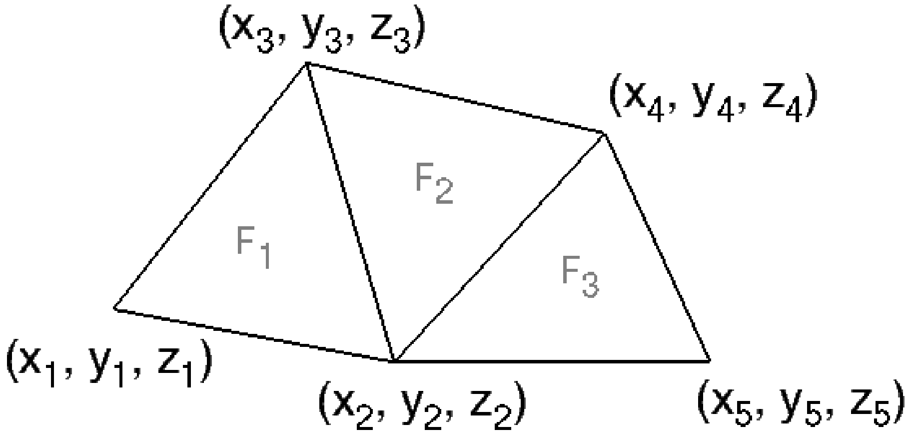
FACE TABLE			
F_1	(x_1, y_1, z_1)	(x_2, y_2, z_2)	(x_3, y_3, z_3)
F_2	(x_2, y_2, z_2)	(x_4, y_4, z_4)	(x_3, y_3, z_3)
F	[illegible]	[illegible] z_5)	(x_4, y_4, z_4)

⇒ Moving a vertex requires changing the coordinates of **each** instance.

Independent Faces

- Each face lists vertex coordinates
  - Redundant vertices
  - No (efficient/precise) vertex-adjacency info

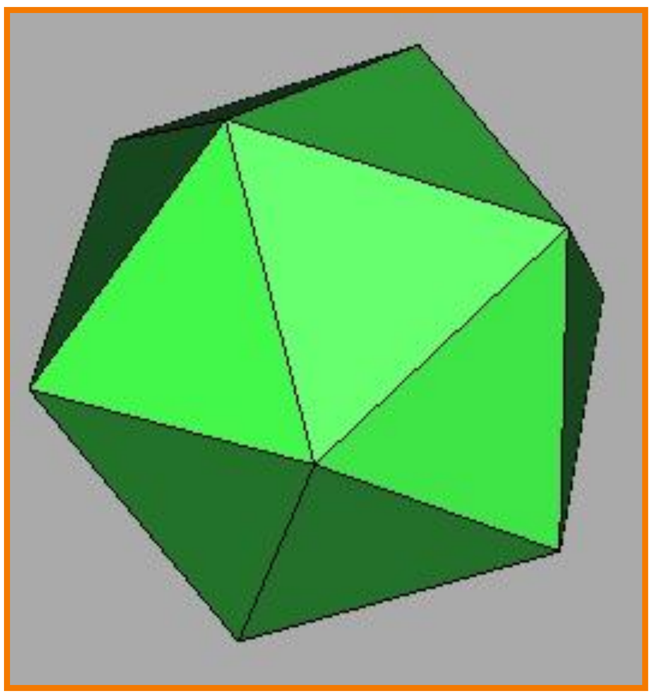


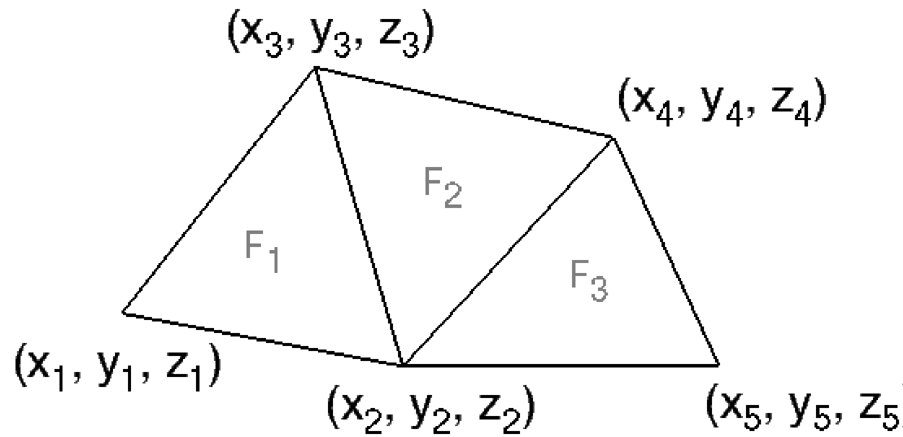


FACE TABLE	
F_1	(x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3)
F_2	(x_2, y_2, z_2) (x_4, y_4, z_4) (x_3, y_3, z_3)
F_3	(x_2, y_2, z_2) (x_5, y_5, z_5) (x_4, y_4, z_4)

Vertex and Face Tables

- Each face lists vertex references



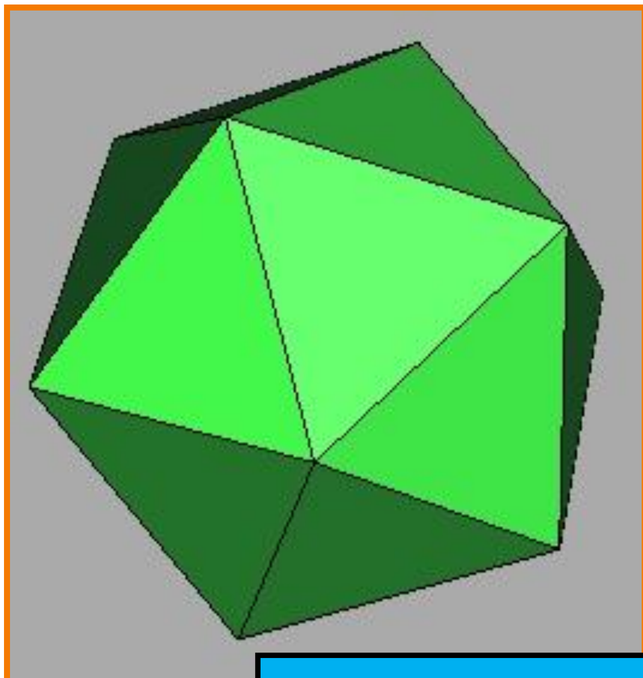


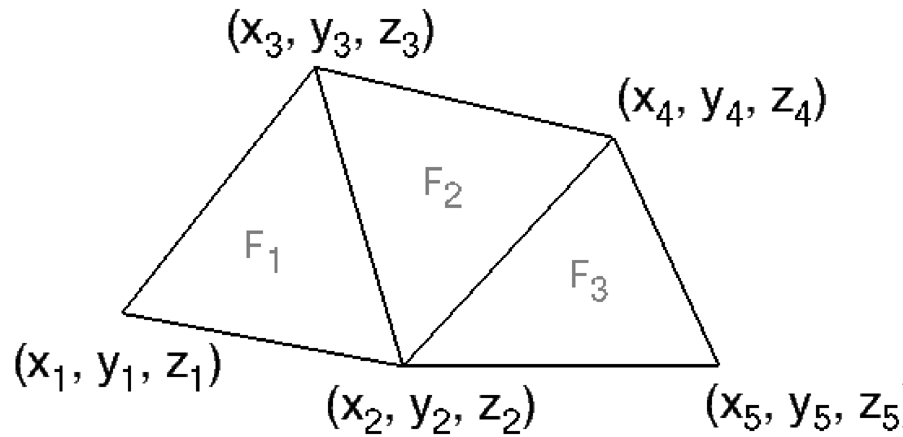
VERTEX TABLE			
V_1	X_1	Y_1	Z_1
V_2	X_2	Y_2	Z_2
V_3	X_3	Y_3	Z_3
V_4	X_4	Y_4	Z_4
V_5	X_5	Y_5	Z_5

FACE TABLE			
F_1	V_1	V_2	V_3
F_2	V_2	V_4	V_3
F_3	V_2	V_5	V_4

Vertex and Face Tables

- Each face lists vertex references
  - ✓ Shared vertices





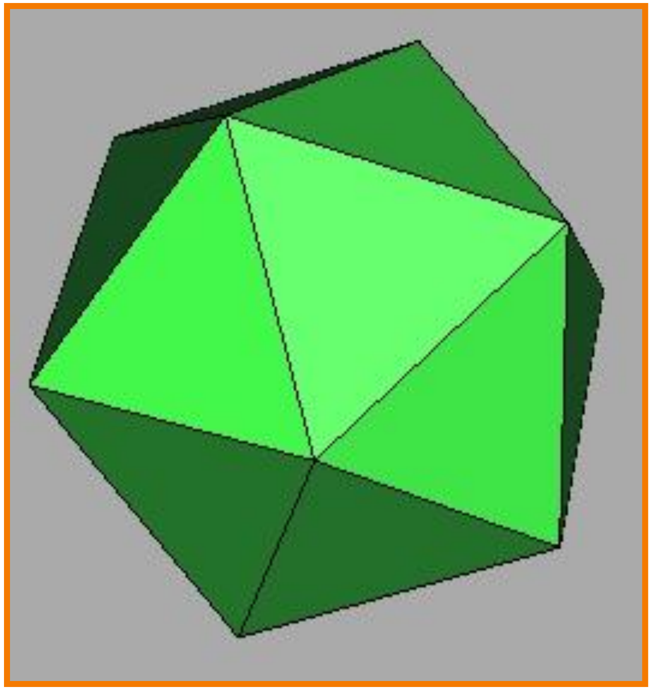
VERTEX TABLE			
V_1	X_1	Y_1	Z_1
V_2	X_2	Y_2	Z_2
V_3	X_3	Y_3	Z_3
V_4	X_4	Y_4	Z_4

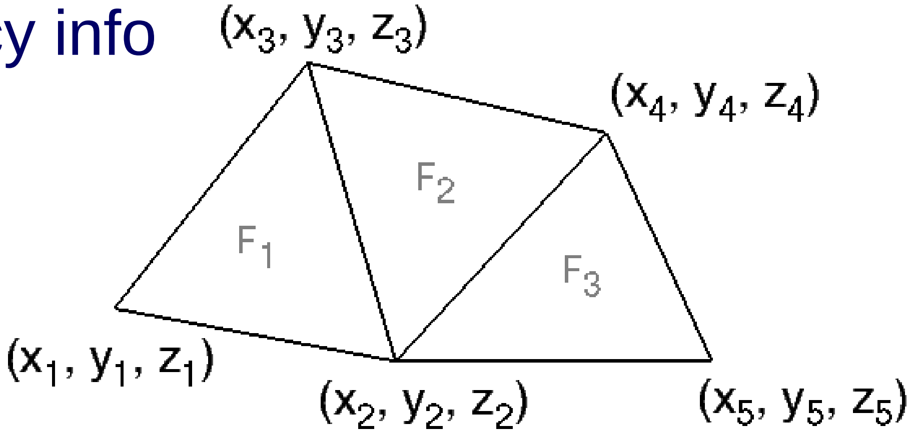
FACE TABLE			
F_1	V_1	V_2	V_3
F_2	V_2	V_4	V_3
F_3	V_2	V_5	V_4

⇒ Moving a vertex requires changing the coordinates of a **single** point.

Vertex and Face Tables

- Each face lists vertex references
  - ✓ Shared vertices
  - ✗ No (efficient) adjacency info



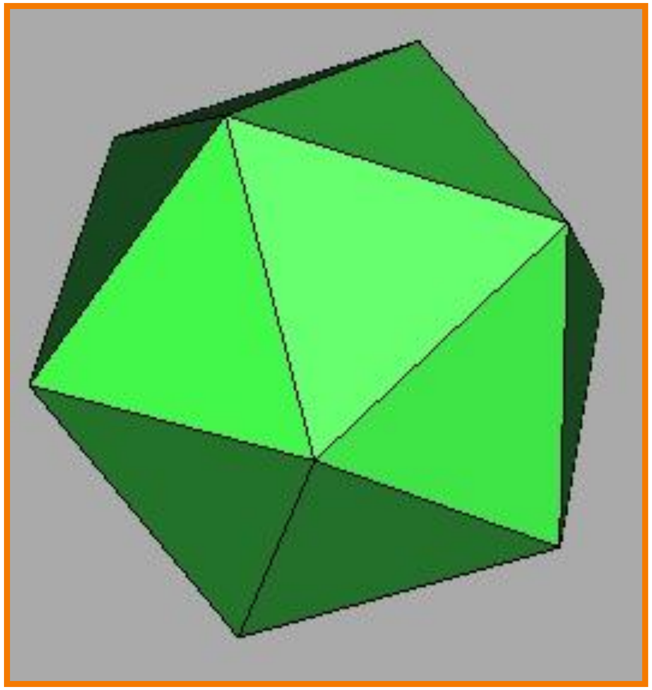


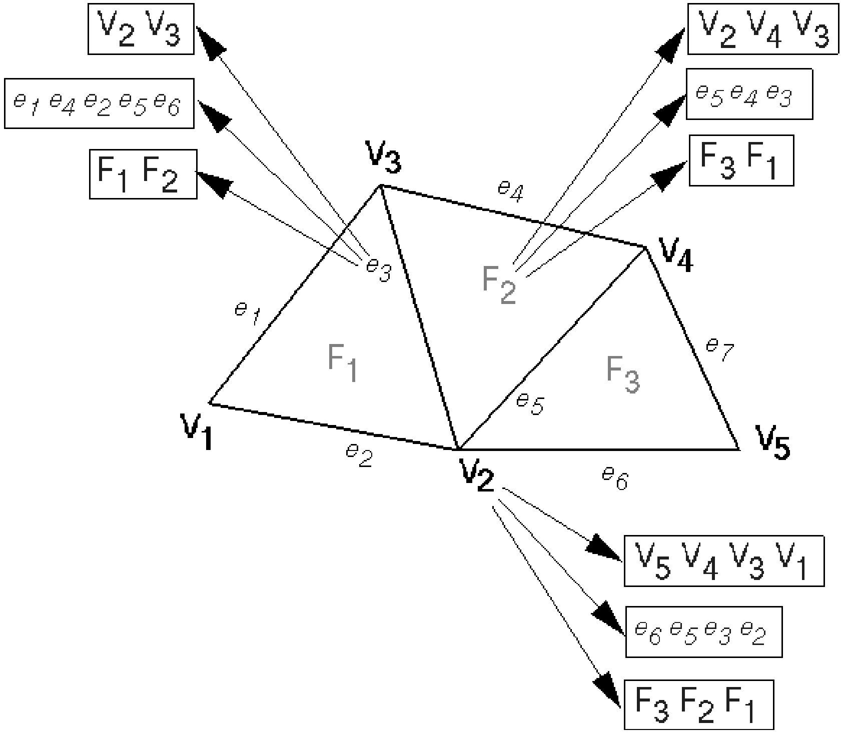
VERTEX TABLE			
V_1	X_1	Y_1	Z_1
V_2	X_2	Y_2	Z_2
V_3	X_3	Y_3	Z_3
V_4	X_4	Y_4	Z_4
V_5	X_5	Y_5	Z_5

FACE TABLE			
F_1	V_1	V_2	V_3
F_2	V_2	V_4	V_3
F_3	V_2	V_5	V_4

Adjacency Lists

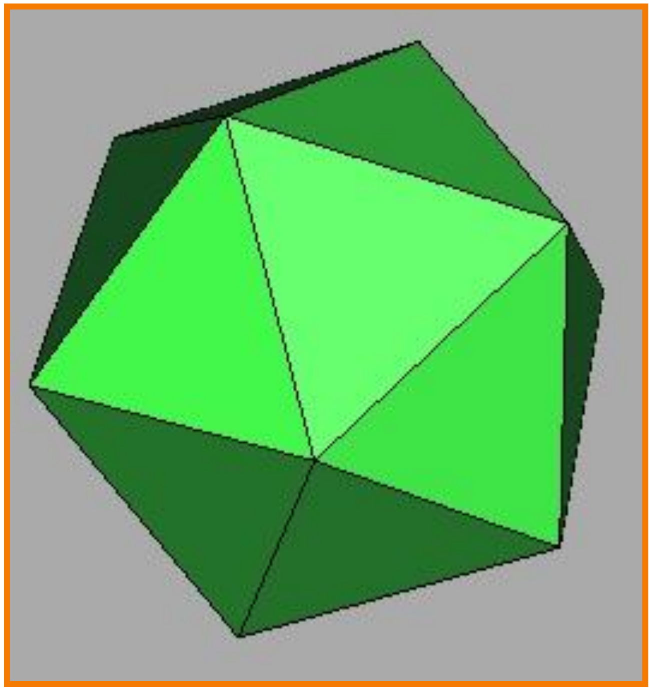
- Store all vertex, edge, and face adjacencies

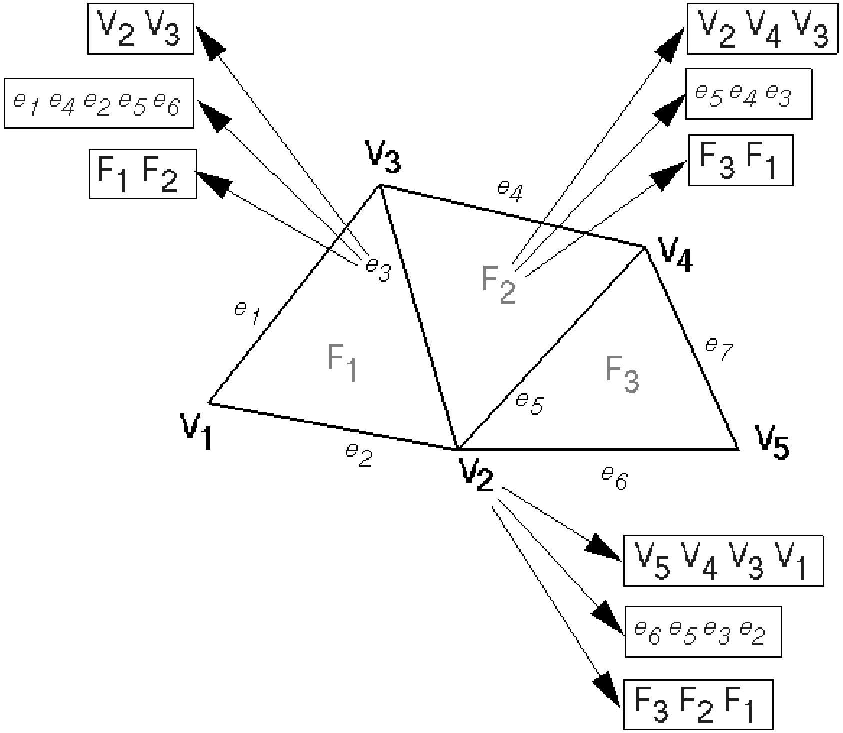




Adjacency Lists

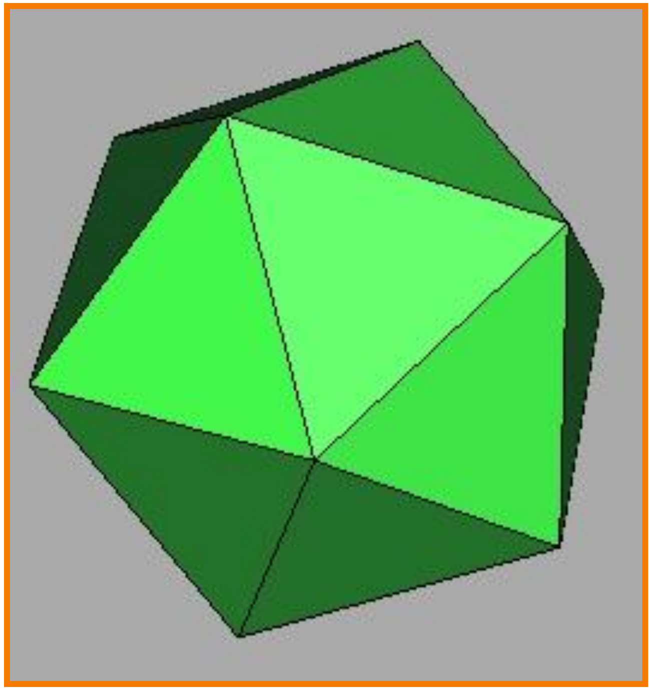
- Store all vertex, edge, and face adjacencies
  - ✓ Efficient adjacency info

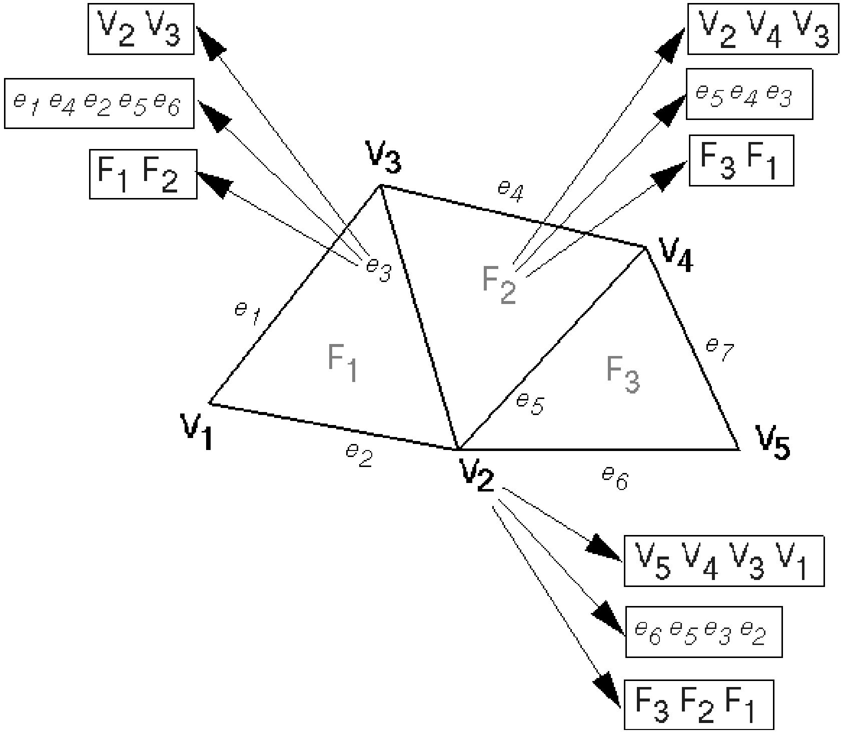




Adjacency Lists

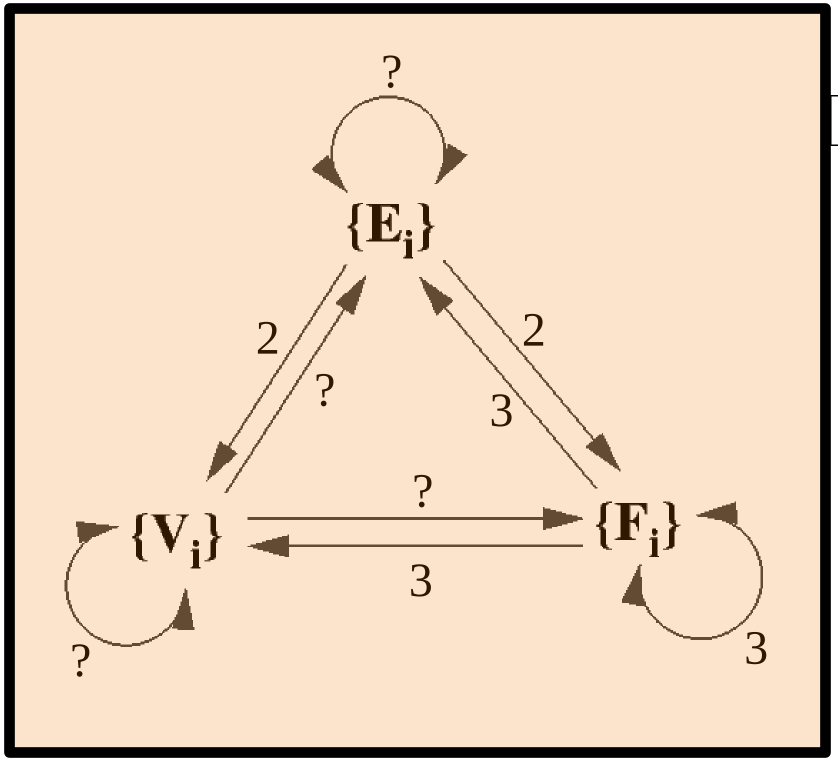
- Store all vertex, edge, and face adjacencies
  - ✓ Efficient adjacency info
  - ✗ Extra storage
  - ✗ Variable size arrays

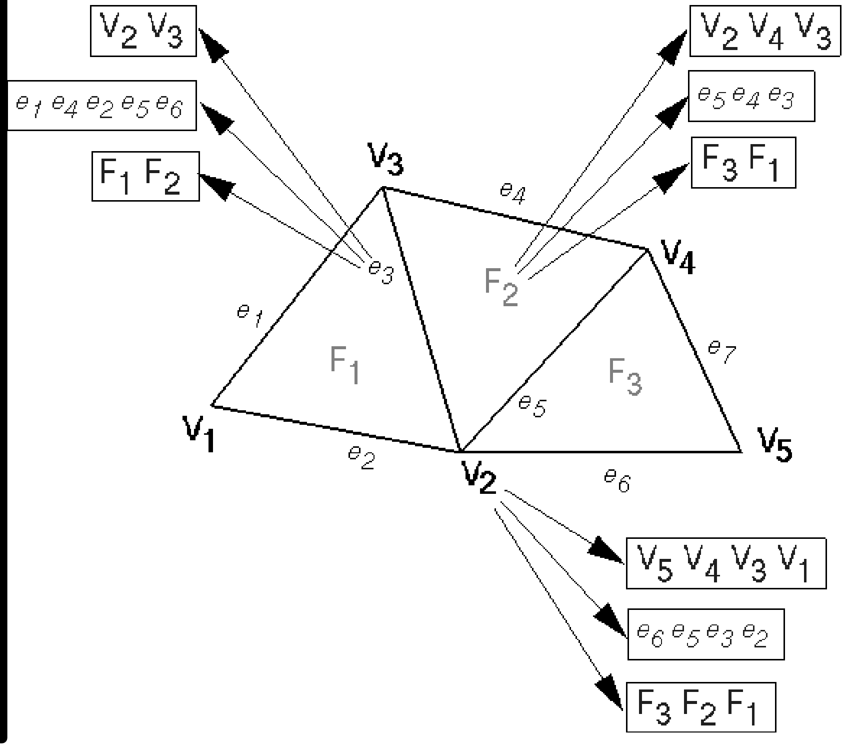




Partial Adjacency Lists

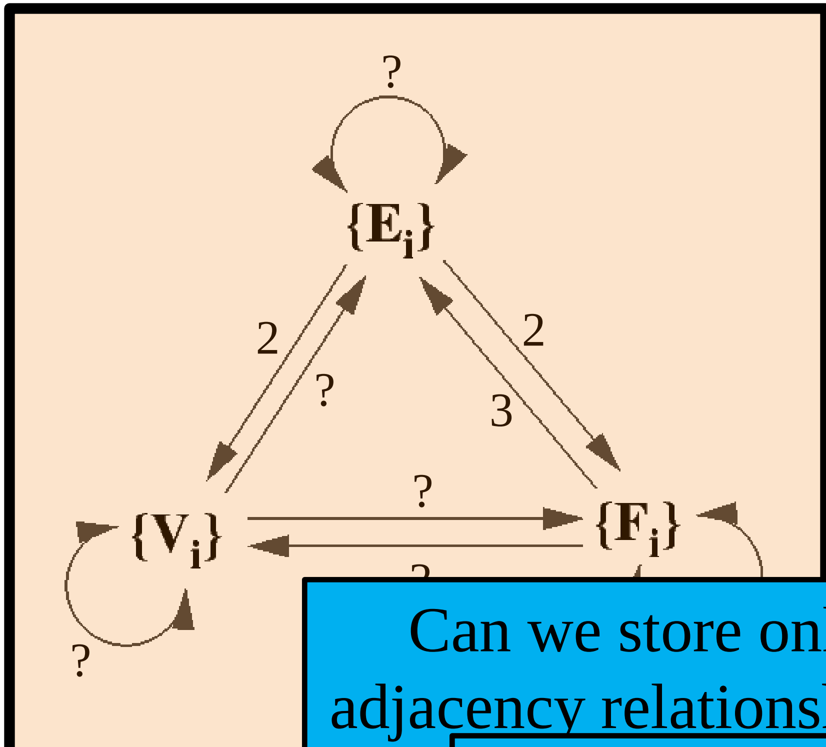
- Store all vertex, edge, and face adjacencies
  - ✓ Efficient adjacency info
  - ✗ Extra storage

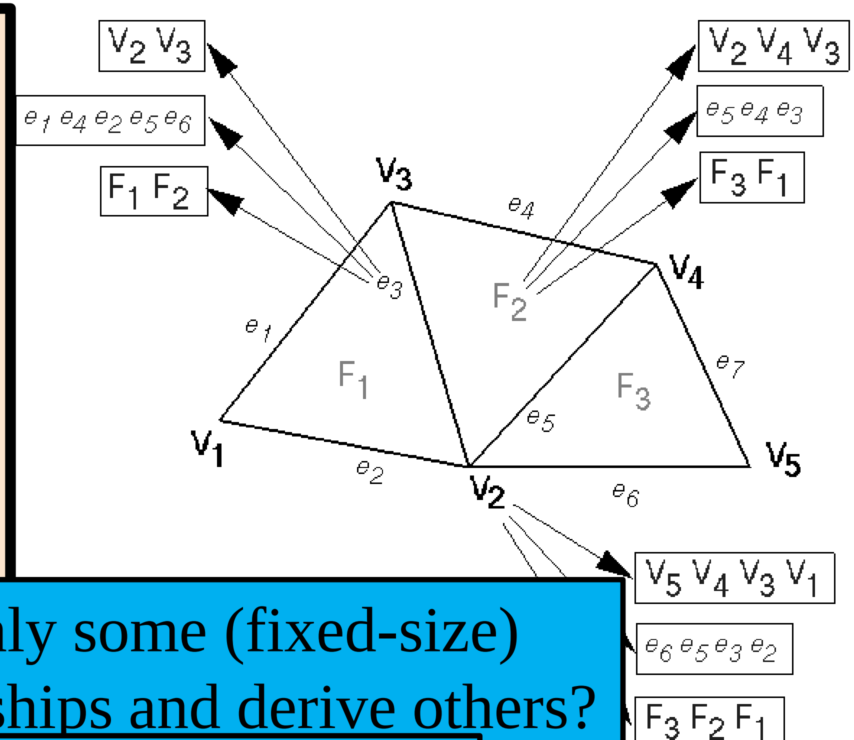




Partial Adjacency Lists

- Store all vertex, edge, and face adjacencies
  - ✓ Efficient adjacency info
  - ✗ Extra storage



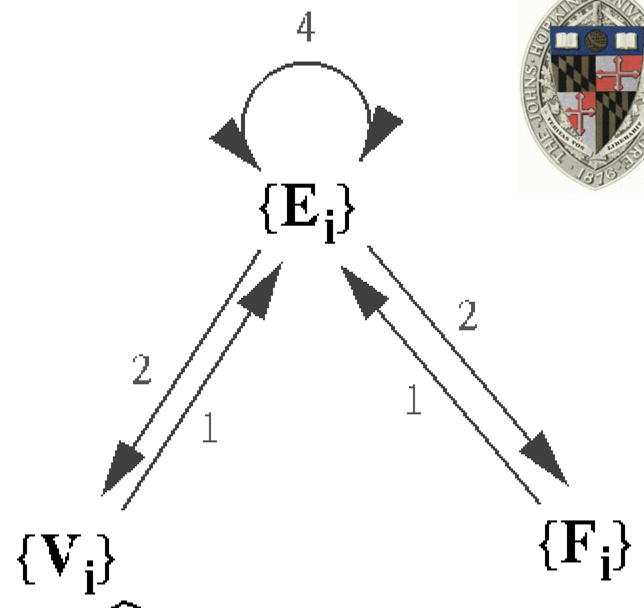


Can we store only some (fixed-size) adjacency relationships and derive others?

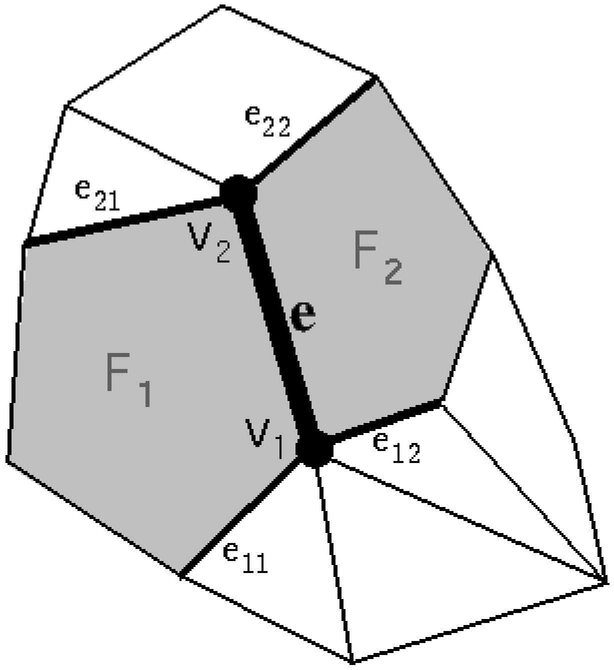
In general, redundancy creates opportunity for inconsistency.

Winged Edge

- Adjacency encoded in edges
  - All adjacencies in $O(1)$ time
  - Little extra storage
  - Fixed-size records
  - Supports polygonal faces

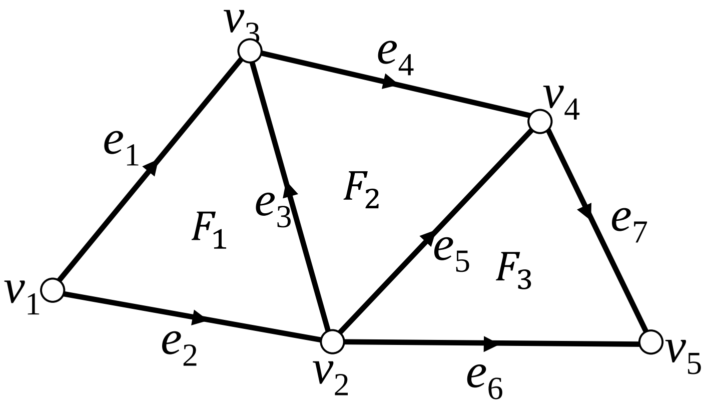


Each edge stores:
4 “wing” edges
2 vertices
2 faces
Each face stores:
1 (some) edge
Each vertex stores:
1 (some) edge



Winged Edge

- <u>Vertex table</u>:
  - A reference to some incident edge



VERTEX TABLE

V_1	X_1	Y_1	Z_1	$\boxed{e_1}$
V_2	X_2	Y_2	Z_2	$\boxed{e_6}$
V_3	X_3	Y_3	Z_3	$\boxed{e_3}$
V_4	X_4	Y_4	Z_4	$\boxed{e_5}$
V_5	X_5	Y_5	Z_5	$\boxed{e_6}$

EDGE TABLE

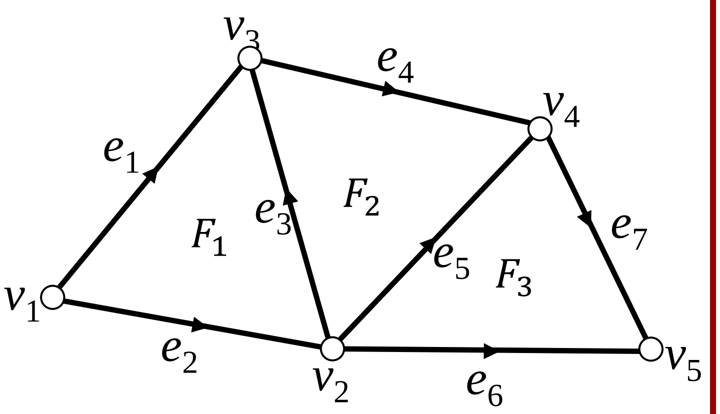
					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE

F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- Vertex table:
  - A reference to some incident edge
  - Vertex positions (and other attributes)



VERTEX TABLE

V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE

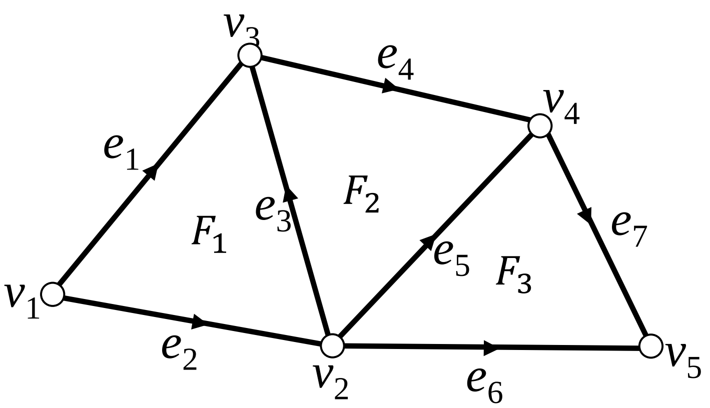
					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE

F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- Face table:
  - A reference to some incident edge
  - (And other attributes)



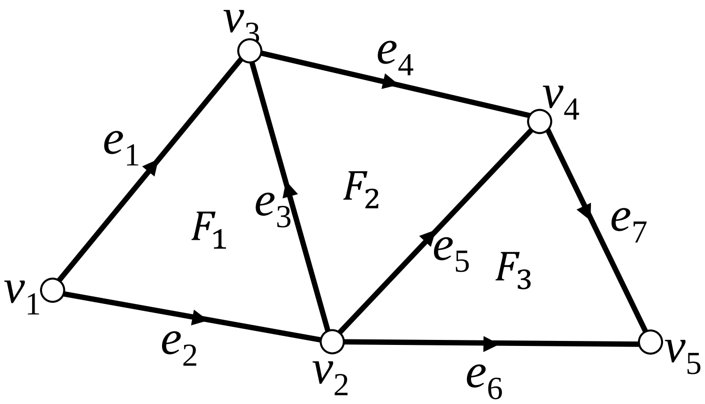
VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- <u>Edge table</u>:
  - References to **S**tart and **E**nd vertices (orientation arbitrary)



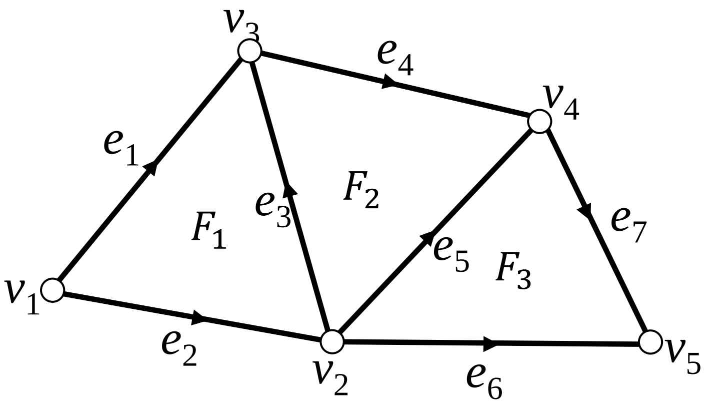
VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- Edge table:
  - References to **S**tart and **E**nd vertices (orientation arbitrary)
  - References to **L**eft and **R**ight faces



VERTEX TABLE

V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE

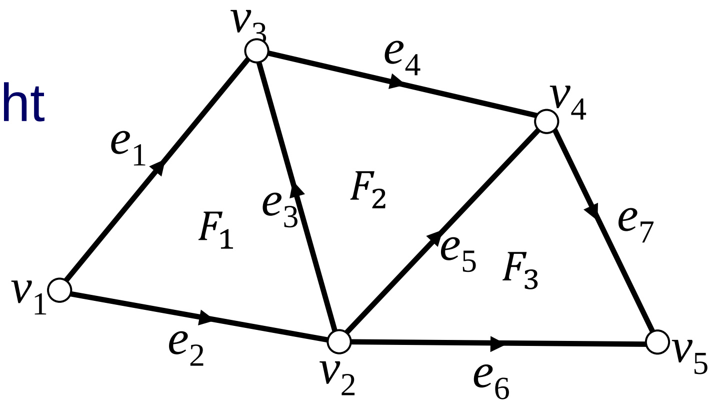
					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE

F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- Edge table:
  - References to **S**tart and **E**nd vertices (orientation arbitrary)
  - References to **L**eft and **R**ight faces
  - References to immediate **L**eft and **R**ight edges coming out of the **S**tart vertex



VERTEX TABLE

V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE

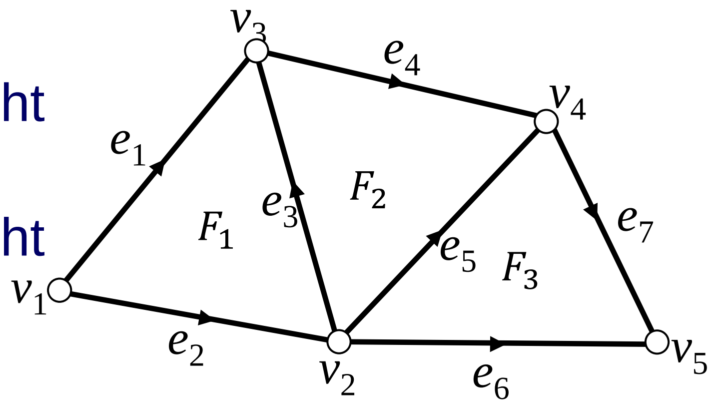
					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	$V3$	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE

F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- <u>Edge table</u>:
  - References to **S**tart and **E**nd vertices (orientation arbitrary)
  - References to **L**eft and **R**ight faces
  - References to immediate **L**eft and **R**ight edges coming out of the **S**tart vertex
  - References to immediate **L**eft and **R**ight edges coming out of the **E**nd vertex



VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

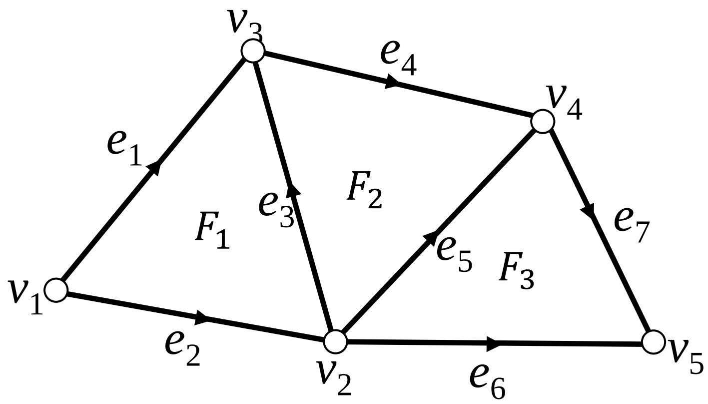
EDGE TABLE					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

Boundary edges:

- Have only one incident face



VERTEX TABLE

V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE

					S		E	
	S	E	L	R	L	R	L	R
$\boxed{e_1}$	V_1	V_3		F_1	e_2	e_2	e_4	e_3
$\boxed{e_2}$	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
$\boxed{e_4}$	$V3$	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
$\boxed{e_6}$	V_2	V_5	F_3		e_5	e_2	e_7	e_7
$\boxed{e_7}$	V_4	V_5		F_3	e_4	e_5	e_6	e_6

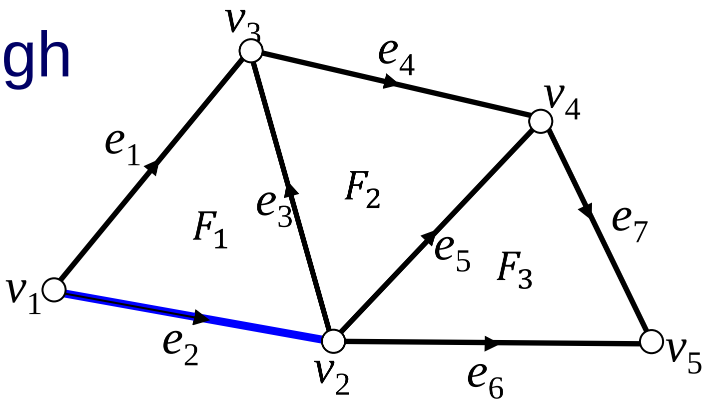
FACE TABLE

F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

Boundary edges:

- Have only one incident face
- Wing edges are defined as though the boundary was also a face



VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
$\boxed{e_2}$	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

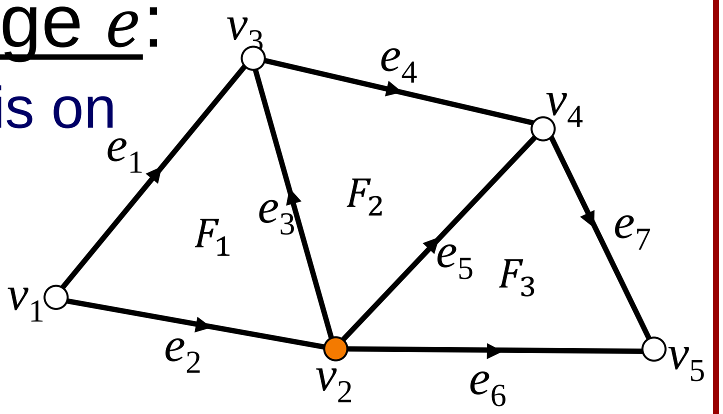
FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge (Example)

Find CCW edges adjacent to v_2.

<u>Note that given a vertex v on edge e</u>:

- If v is the **S**tart, the next CCW edge is on the **L**eft of e, coming out of the **S**tart.
- Otherwise it is on the **R**ight of e, coming out of the **E**nd.



VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

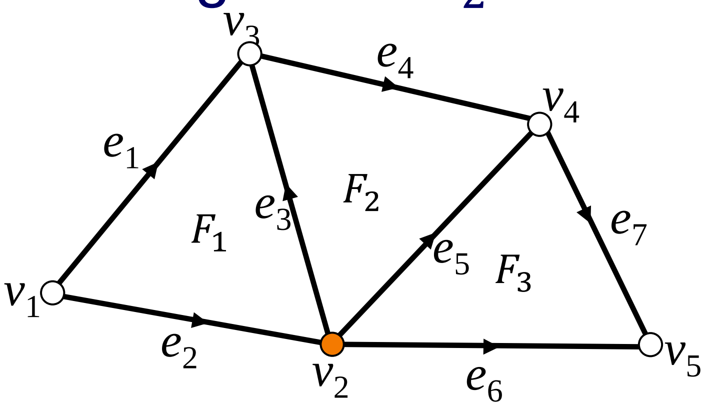
EDGE TABLE					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge (Example)

Find CCW edges adjacent to v_2:

- **Initialize**: Choose the only edge coming out of v_2
- **Do**: Iterate CCW around v_2
- **While**: Haven't cycled back to the start edge



VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

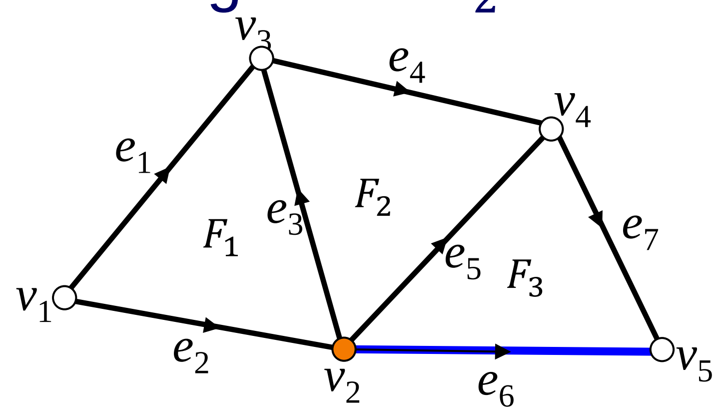
EDGE TABLE					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge (Example)

Find CCW edges adjacent to v_2:

- **Initialize**: Choose the only edge coming out of v_2
- **Do**: Iterate CCW around v_2
- **While**: Haven't cycled back to the start edge



VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

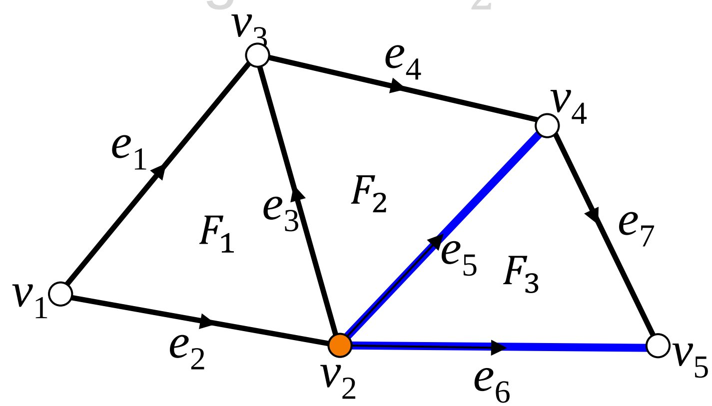
EDGE TABLE					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge (Example)

Find CCW edges adjacent to v_2:

- **Initialize**: Choose the only edge coming out of v_2
- **Do**: Iterate CCW around v_2
  - » If v_2 is the **S**tart...
  - » Otherwise...
- **While**: Haven't cycled back to the start edge



VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

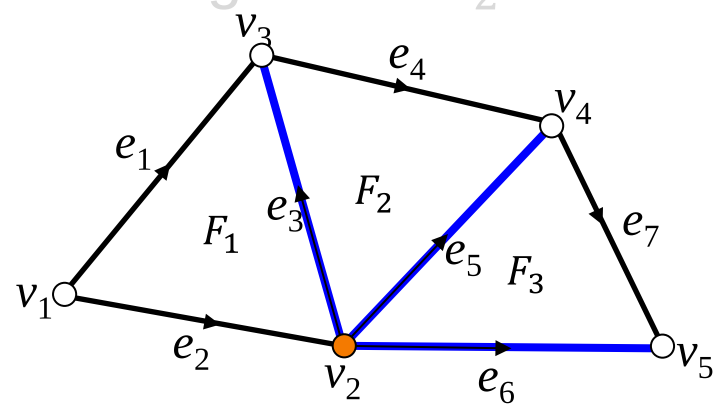
EDGE TABLE	S	E	L	R	S L	S R	E L	E R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge (Example)

Find CCW edges adjacent to v_2:

- **Initialize**: Choose the only edge coming out of v_2
- **Do**: Iterate CCW around v_2
  - » If v_2 is the **S**tart...
  - » Otherwise...
- **While**: Haven't cycled back to the start edge



VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

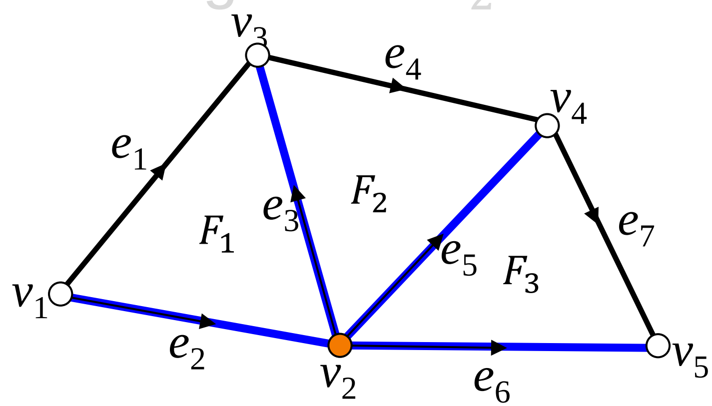
EDGE TABLE					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge (Example)

Find CCW edges adjacent to v_2:

- **Initialize**: Choose the only edge coming out of v_2
- **Do**: Iterate CCW around v_2
  - » If v_2 is the **S**tart...
  - » Otherwise...
- **While**: Haven't cycled back to the start edge



VERTEX TABLE

V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE

					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

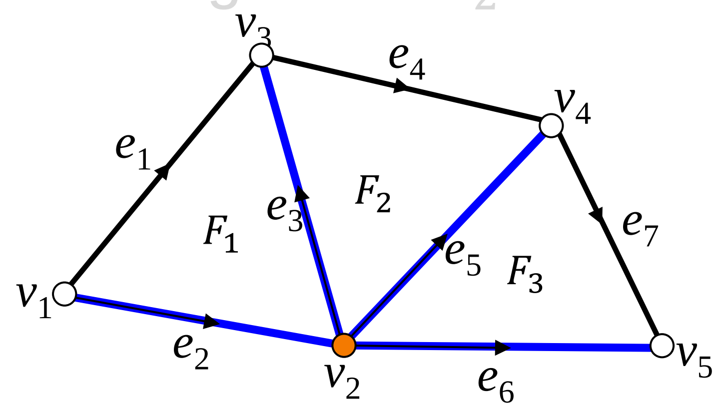
FACE TABLE

F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge (Example)

Find CCW edges adjacent to v_2:

- **Initialize**: Choose the only edge coming out of v_2
- **Do**: Iterate CCW around v_2
  - » If v_2 is the **S**tart...
  - » Otherwise...
- **While**: Haven't cycled back to the start edge



VERTEX TABLE

V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE

					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

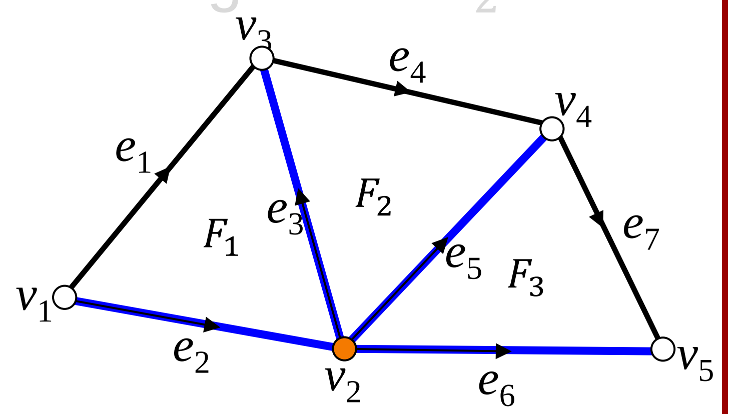
FACE TABLE

F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge (Example)

Find CCW edges adjacent to v_2:

- **Initialize**: Choose the only edge coming out of v_2
- **Do**: Iterate CCW around v_2
- **While**: Haven't cycled back to the start edge



VERTEX TABLE

V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE

					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

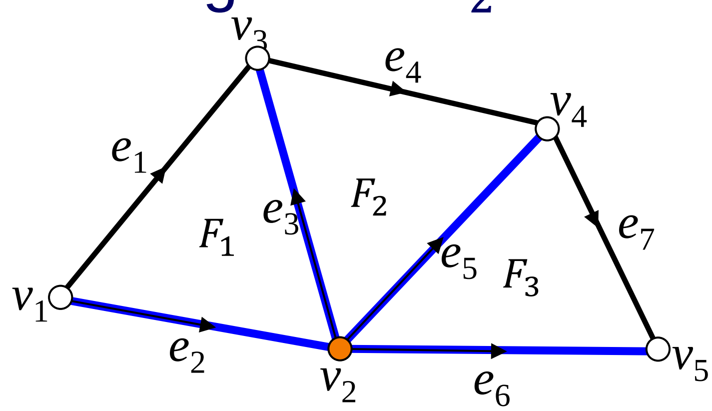
FACE TABLE

F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge (Example)

Find CCW edges adjacent to v_2:

- **Initialize**: Choose the only edge coming out of v_2
- **Do**: Iterate CCW around v_2
- **While**: Haven't cycled back to the start edge



VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE					S		E	
	S	E	L	R	L	R	L	R
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Computational complexity is proportional to the size of the output.
(Independent of the size of the mesh.)

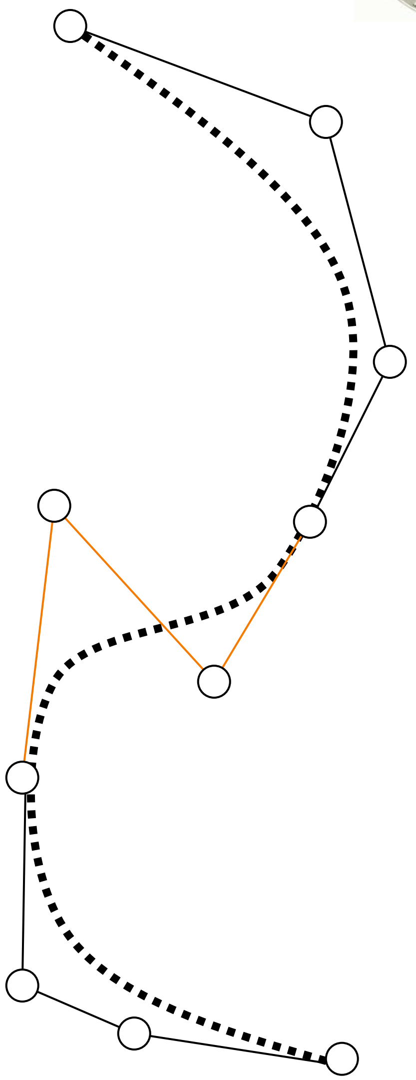
Outline

- Representing Meshes
- Parametric Curves

Parametric Curves

Given a 1D control lattice

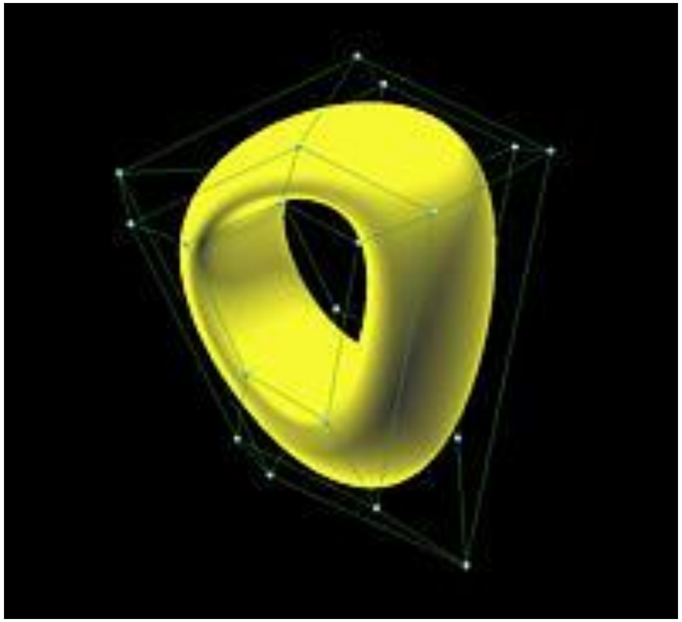
- Compute a smooth curve passing through/near the control points



Parametric Surfaces

Given a 2D control lattice

- Compute a smooth surface passing through/near the control points



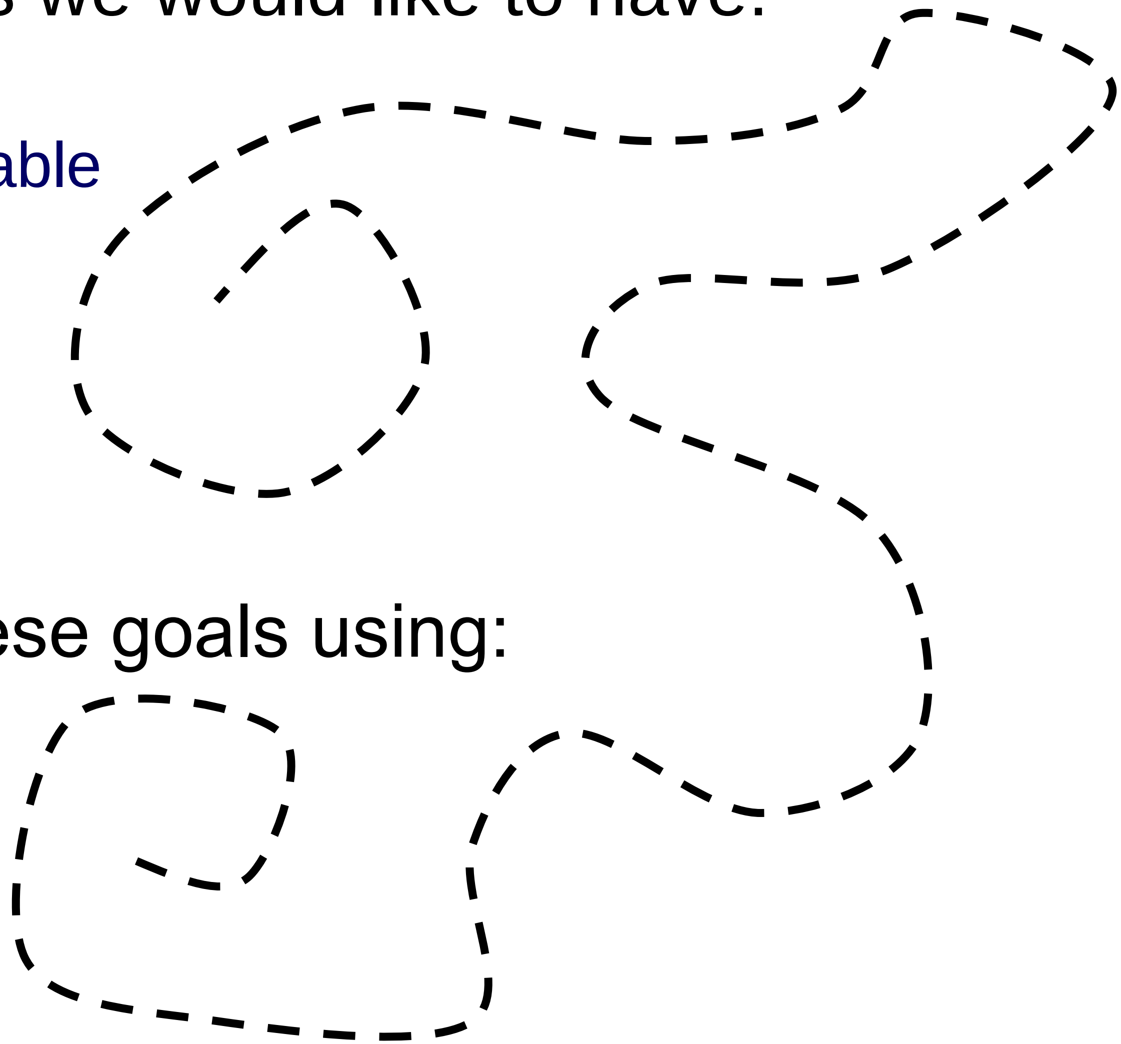
Courtesy of C.K. Shene

Very closely related to subdivision surfaces!

"Exact Evaluation Of Catmull-Clark Subdivision Surfaces At Arbitrary Parameter Values". [Stam, 1998]

Goals

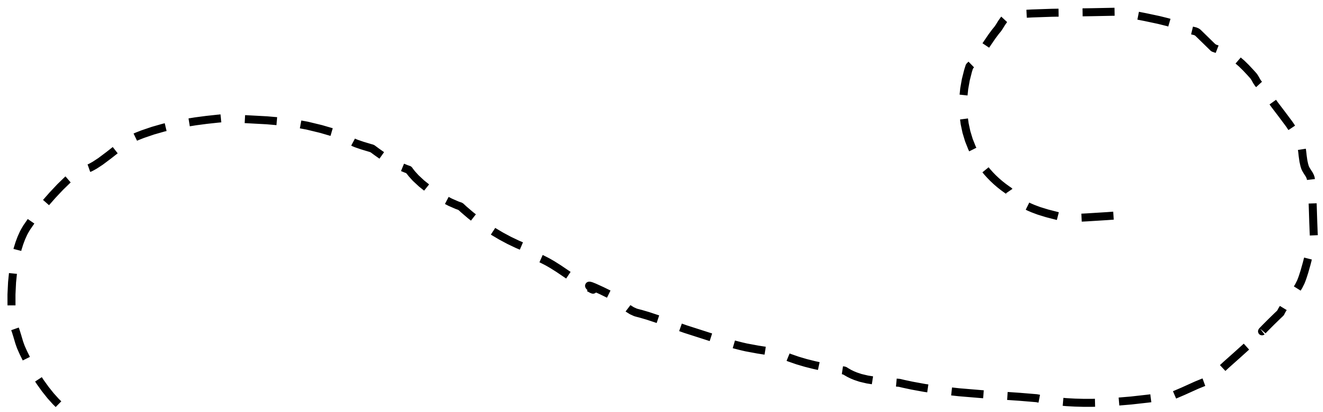
- Some attributes we would like to have:
  - Local support
  - Simple/predictable
  - Continuous



- We'll satisfy these goals using:
  - Piecewise
  - Polynomials

What is a Spline in CG?

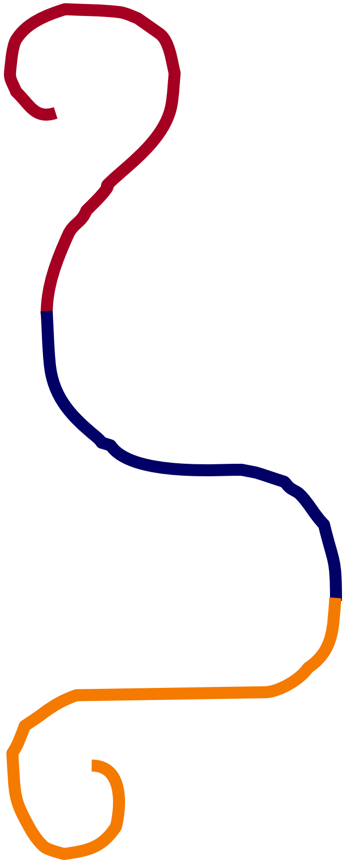
A spline is a *piecewise* *polynomial function* whose derivatives satisfy *continuity constraints* across curve boundaries.



What is a Spline in CG?

Piecewise: the spline is a collection of *parametric curves* segments joined together.

Polynomial functions: each segment is a *parametric polynomial curve*.

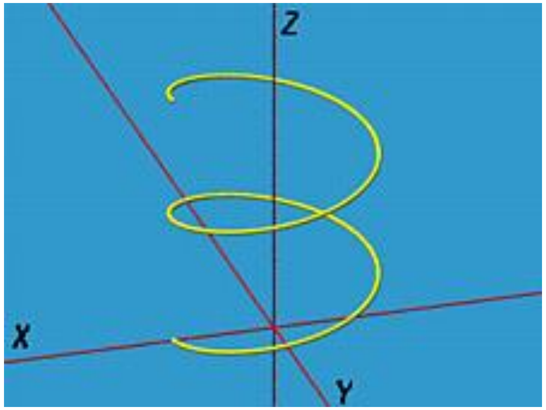


Parametric Curves

A <u>parametric curve</u> in d-dimensions is defined by a collection of coordinate functions in u giving the position of a point on the curve at each u value:

$$\Phi(u) = \left(x_1(u), \cdots, x_d(u)\right)$$

$\Phi(u) = (\cos u, \sin u, u)$



Courtesy of C.K. Shene

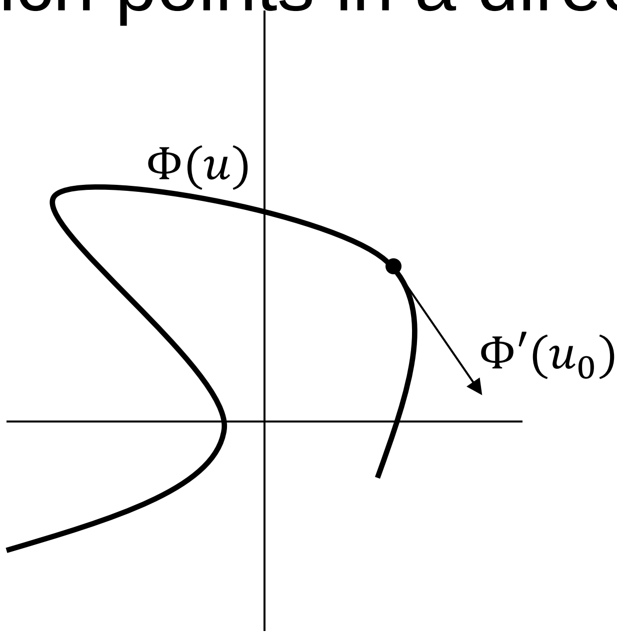
<u>Note</u>:
A parametric curve is **not** the graph of a function.

Derivatives

If $\Phi(u) = \big(x(u), y(u)\big)$ is the parametric equation of a curve, the parametric derivative of the curve at a point u_0 is the vector:

$$\Phi'(u_0) = \big(x'(u_0), y'(u_0)\big)$$

which points in a direction tangent to the curve.



<u>Note</u>:
The direction of the derivative is determined by the path that the curve traces out.

The magnitude of the parametric derivative is determined by the tracing speed.

Polynomials

A polynomial in the variable u is:

"An algebraic expression written as a sum of constants multiplied by different powers of a variable."

$$P(u) = a_0 + a_1 \cdot u + a_2 \cdot u^2 + \cdots + a_n \cdot u^n = \sum_{k=0}^{n} a_k \cdot u^k$$

The constant a_k is referred to as the <u>k-th coefficient</u> of the polynomial P.

A polynomial $P(u)$ has <u>degree</u> n if for all $k > n$, the coefficients of the polynomial satisfy $a_k = 0$.

Polynomials

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A polynomial of degree n has
$n + 1$ degrees of freedom

$\Updownarrow$

Knowing $n + 1$ pieces of information about a polynomial of degree n should give enough information to reconstruct the coefficients

Polynomials (Matrices)

$$P(u) = a_0 + a_1 \cdot u + a_2 \cdot u^2 + \cdots + a_n \cdot u^n = \sum_{k=0}^{n} a_k \cdot u^k$$

The polynomial P can be expressed as the matrix multiplication of a row vectors containing the powers of u and a column vector containing the coefficients:

$$P(u) = \begin{pmatrix} u^n & \cdots & u^0 \end{pmatrix} \cdot \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix}$$

Polynomials (1st Derivative Matrices)

$$P(u) = a_0 + a_1 \cdot u + a_2 \cdot u^2 + \cdots + a_n \cdot u^n = \sum_{k=0}^{n} a_k \cdot u^k$$

The derivative of the polynomial is:

$$P'(u) = a_1 + 2 \cdot a_2 \cdot u + \cdots + n \cdot a_n \cdot u^{n-1} = \sum_{k=1}^{n} k \cdot a_k \cdot u^{k-1}$$

$\Rightarrow$ The derivative of polynomial P can also be expressed as a matrix multiplication:

$$P'(u) = \begin{pmatrix} n \cdot u^{n-1} & (n-1) \cdot u^{n-2} & \cdots & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{pmatrix}$$

Polynomials (Matrices)

$$P(u) = \sum_{k=0}^{n} a_k \cdot u^k$$

Example:

Given the values of $P(u)$ at $n+1$ different locations:

$$p_0 = P(u_0), \cdots, p_n = P(u_n)$$

$$p_0 = \begin{pmatrix} u_0^n & \cdots & u_0^0 \end{pmatrix} \cdot \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix}, \cdots, p_n = \begin{pmatrix} u_n^n & \cdots & u_n^0 \end{pmatrix} \cdot \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix}$$

We can stack into one linear system:

$$\begin{pmatrix} p_0 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} u_0^n & \cdots & u_0^0 \\ \vdots & \ddots & \vdots \\ u_n^n & \cdots & u_n^0 \end{pmatrix} \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix}$$

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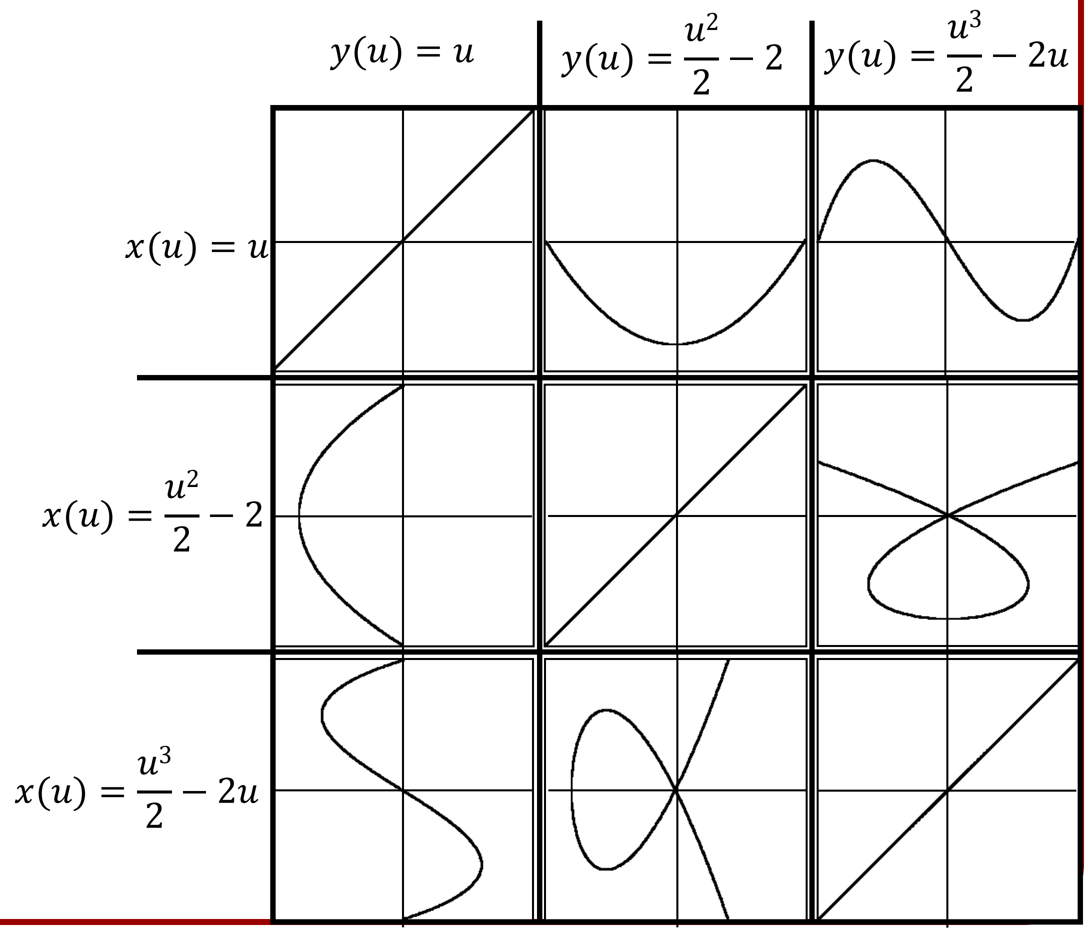
We can stack into one linear system, and invert to get the coefficients:

$$\begin{pmatrix} p_0 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} u_0^n & \cdots & u_0^0 \\ \vdots & \ddots & \vdots \\ u_n^n & \cdots & u_n^0 \end{pmatrix} \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix} = \begin{pmatrix} u_0^n & \cdots & u_0^0 \\ \vdots & \ddots & \vdots \\ u_n^n & \cdots & u_n^0 \end{pmatrix}^{-1} \begin{pmatrix} p_0 \\ \vdots \\ p_n \end{pmatrix}$$

Parametric Polynomial Curves

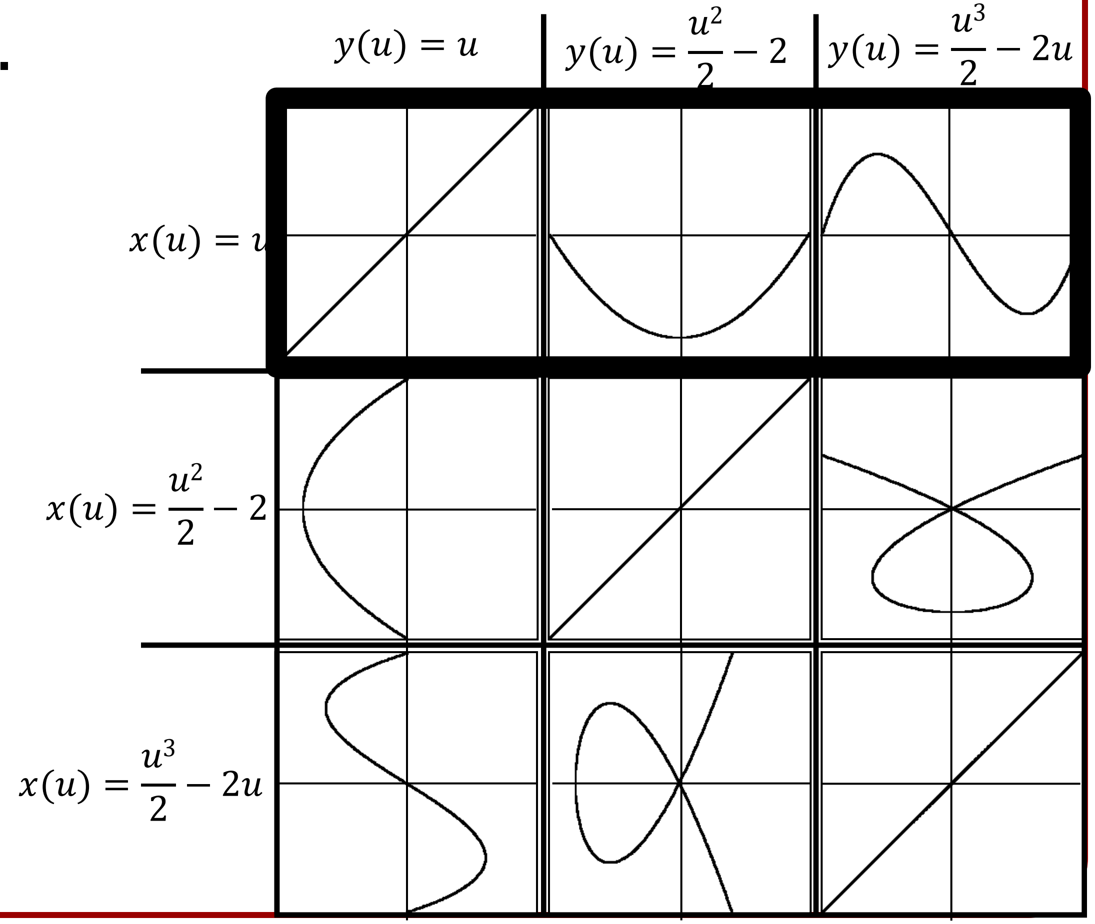
<u>Examples</u>:



Parametric Polynomial Curves

Examples:

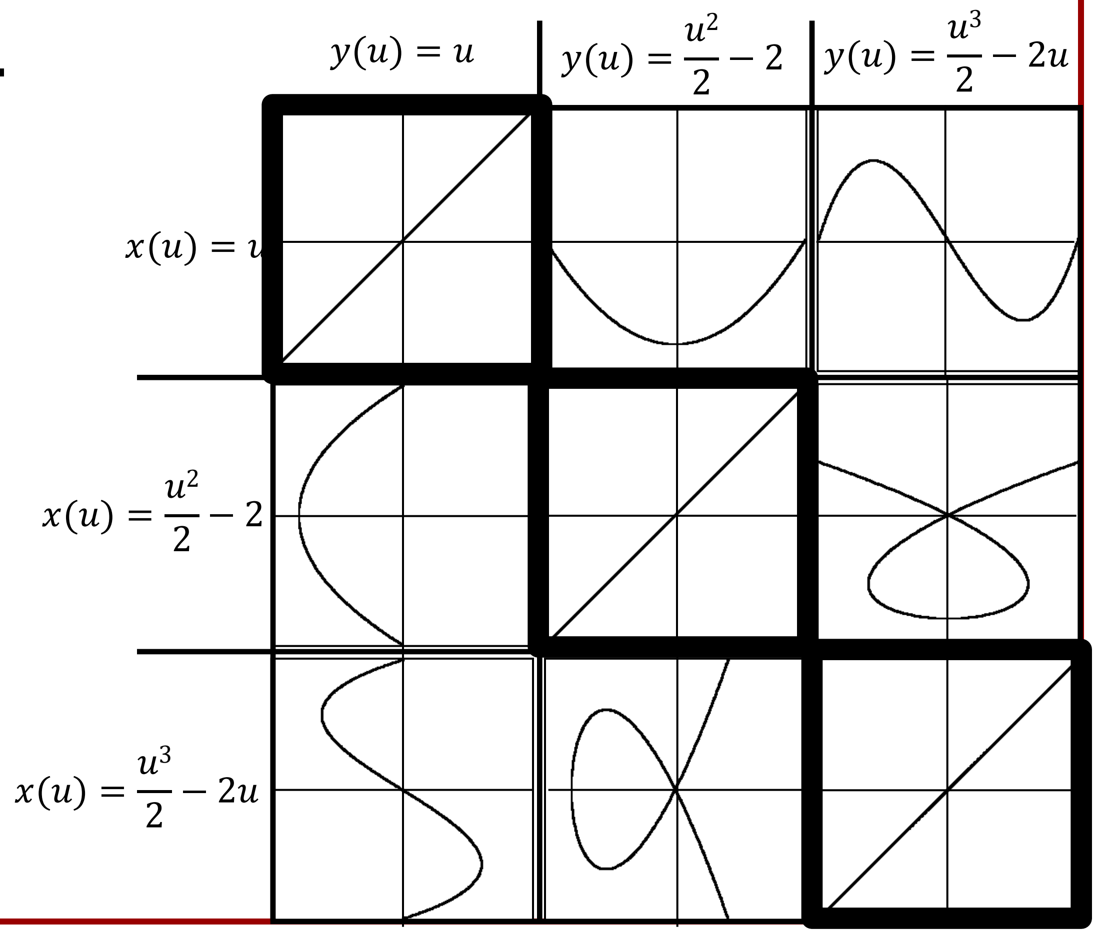
- When $x(u) = u$, the curve is the graph of $y(u)$.



Parametric Polynomial Curves

Examples:

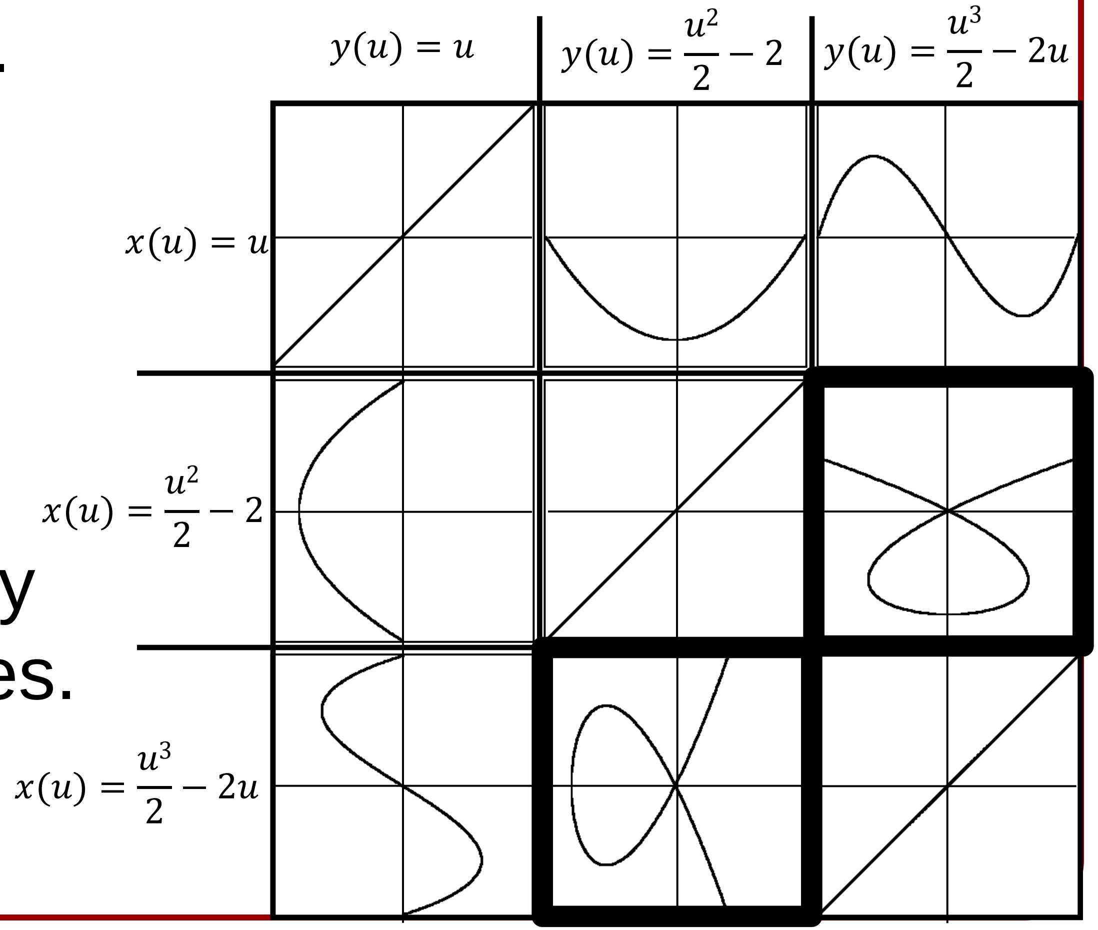
- When $x(u) = u$, the curve is the graph of $y(u)$.
- Different parametric equations can trace out the same curve.



Parametric Polynomial Curves

Examples:

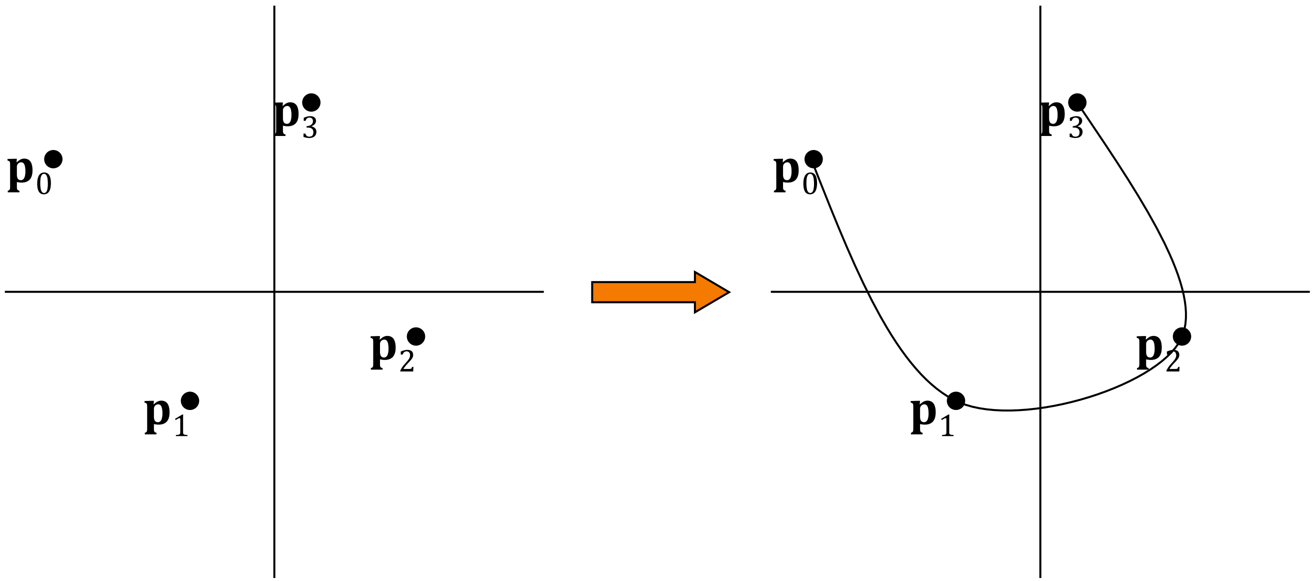
- When $x(u) = u$, the curve is the graph of $y(u)$.
- Different parametric equations can trace out the same curve.
- As the degree gets larger, the complexity of the curve increases.



Parametric Curves (in $\mathbb{R}^d$)

Goal:

Given a sequence of points, $\{\mathbf{p}_1, \cdots, \mathbf{p}_m\} \subset \mathbb{R}^d$, define a parametric curve that passes through/near the points



Parametric Curves (in $\mathbb{R}^d$)

Direct Approach:

Solve for the $d \times m$ coefficients of a parametric polynomial curve of degree $m - 1$, passing through the points.

Limitations:

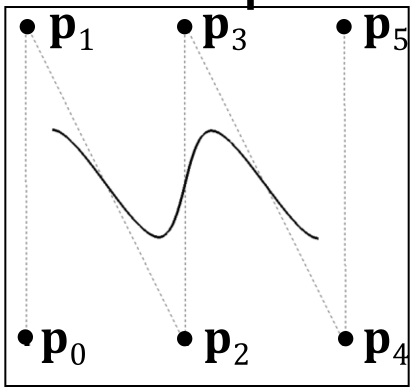
- No local control
- As the number of points increases:
  - The dimension increases and the curve oscillates more
  - Requires inverting a large linear system

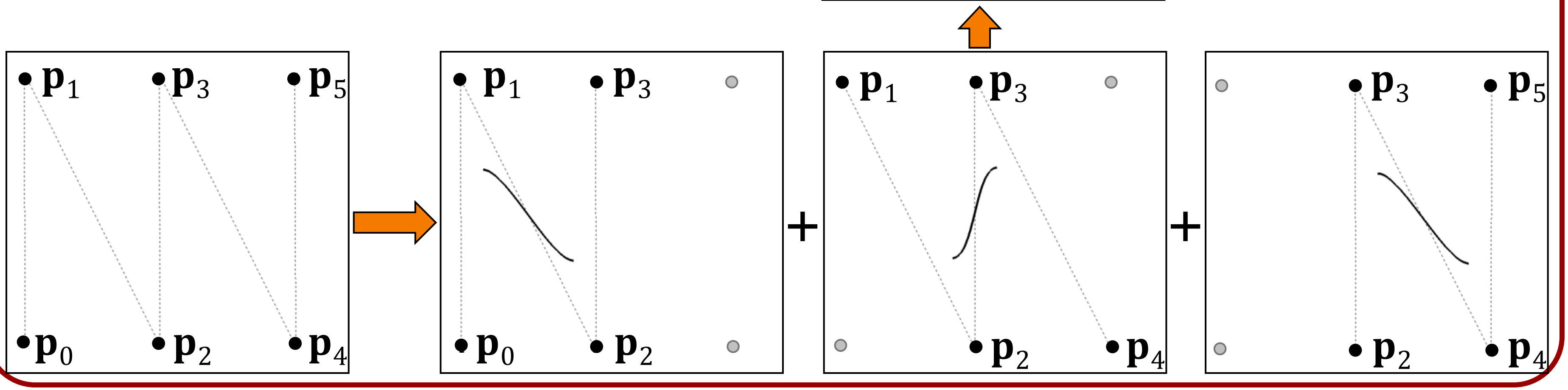
Polynomial Fitting Demo

Piecewise parametric polynomials

<u>Approach</u>:

Fit low-order polynomials to (overlapping) groups of points so that the combined curve passes through/near the points





Piecewise parametric polynomials

Approach:

Fit low-order polynomials to overlapping groups of points so that the combined curve passes through/near the points

Properties:

- Local Control:
  - » A curve segment only depends on its group of points
- Simplicity
  - » Curve segments are low-order polynomials
- Continuity/Smoothness
  - » How do we guarantee smoothness?

What is a Spline in CG?

Continuity:

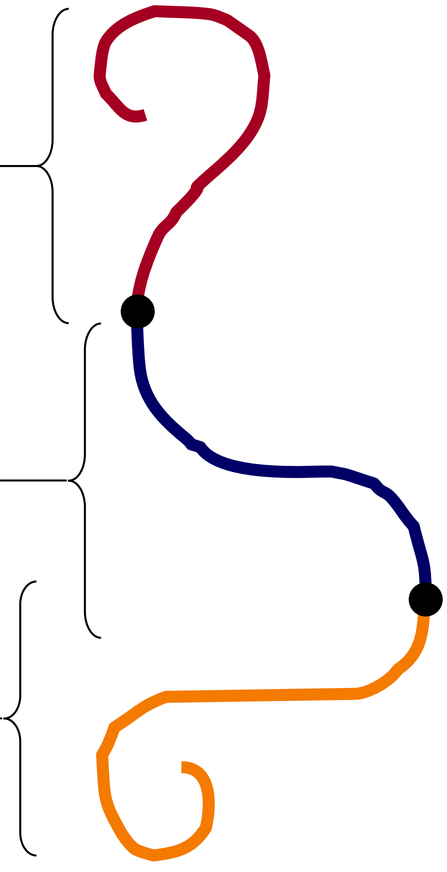
Within the parameterized domain, the polynomial functions are smooth.

The values/derivatives of the polynomials must match at the boundaries.

$\mathbf{P}_1(u) \quad u \in [0,1]$

$\mathbf{P}_2(u) \quad u \in [0,1]$

$\mathbf{P}_3(u) \quad u \in [0,1]$



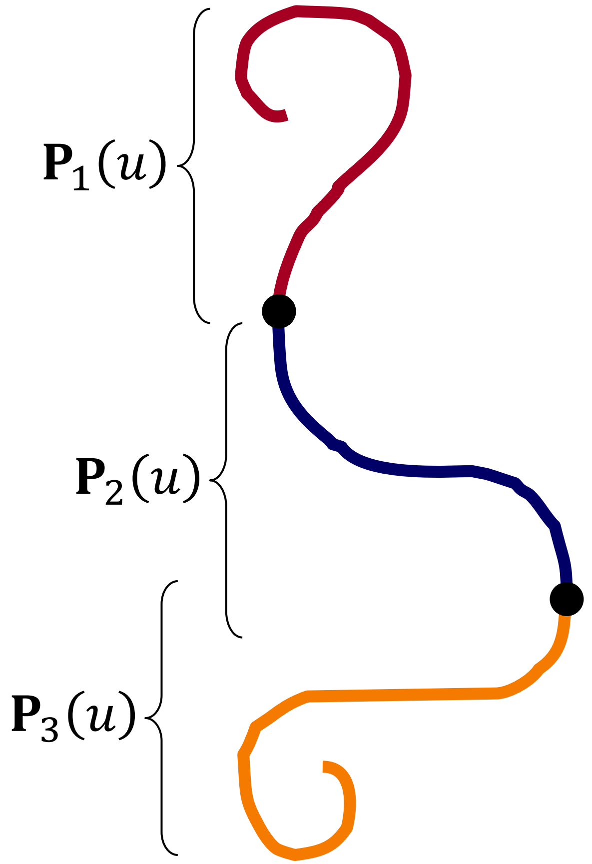
$$\boxed{\mathbf{P}_i(u) = \sum_{j=0}^{n} \mathbf{a}_{ij} \cdot u^j}$$

Continuity/Smoothness

<u>Continuity</u>:

Values/derivatives of the two curves are *equal* where they meet.

- C^0: function is continuous
  $$\Rightarrow \quad \mathbf{P}_i(1) = \mathbf{P}_{i+1}(0)$$
- C^1: function is continuous and 1st derivatives equal
  $$\Rightarrow C^0 \text{ and } \mathbf{P}_i'(1) = \mathbf{P}_{i+1}'(0)$$
- C^2: function is continuous and 1st and 2nd derivatives are equal
  $$\Rightarrow C^1 \text{ and } \mathbf{P}_i''(1) = \mathbf{P}_{i+1}''(0)$$
- C^k: function is continuous and …



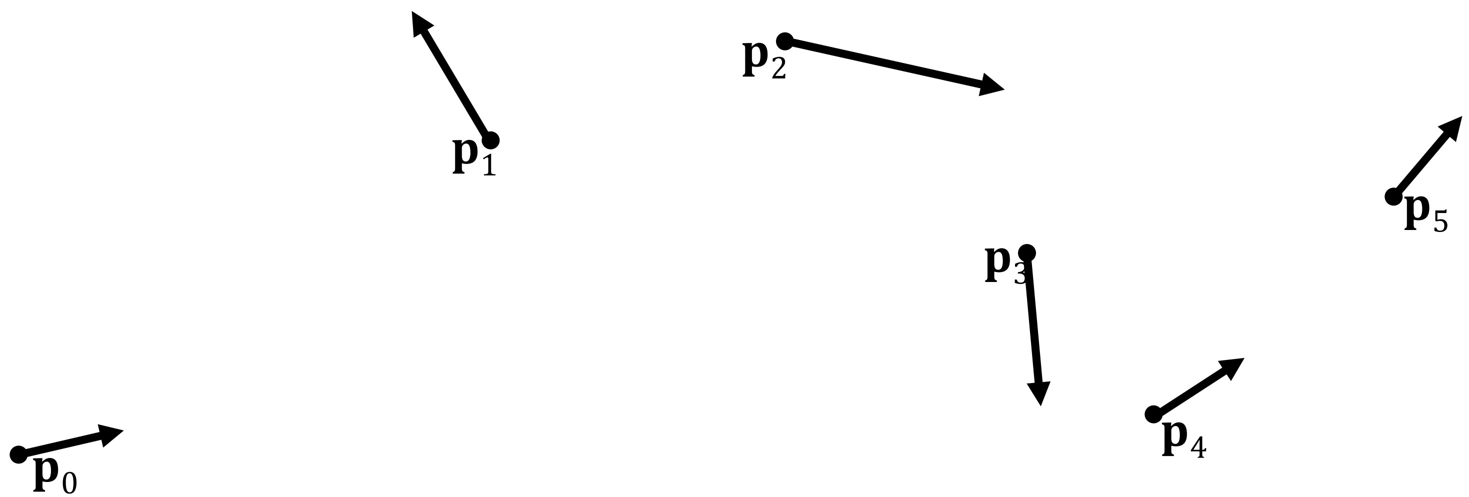
Overview

- What is a Spline?
- Specific Examples:
  - Hermite Splines

Specific Example: Hermite Splines

- Interpolating piecewise *cubic* polynomial, each specified by:
  - Start/end positions
  - Start/end tangents

- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.



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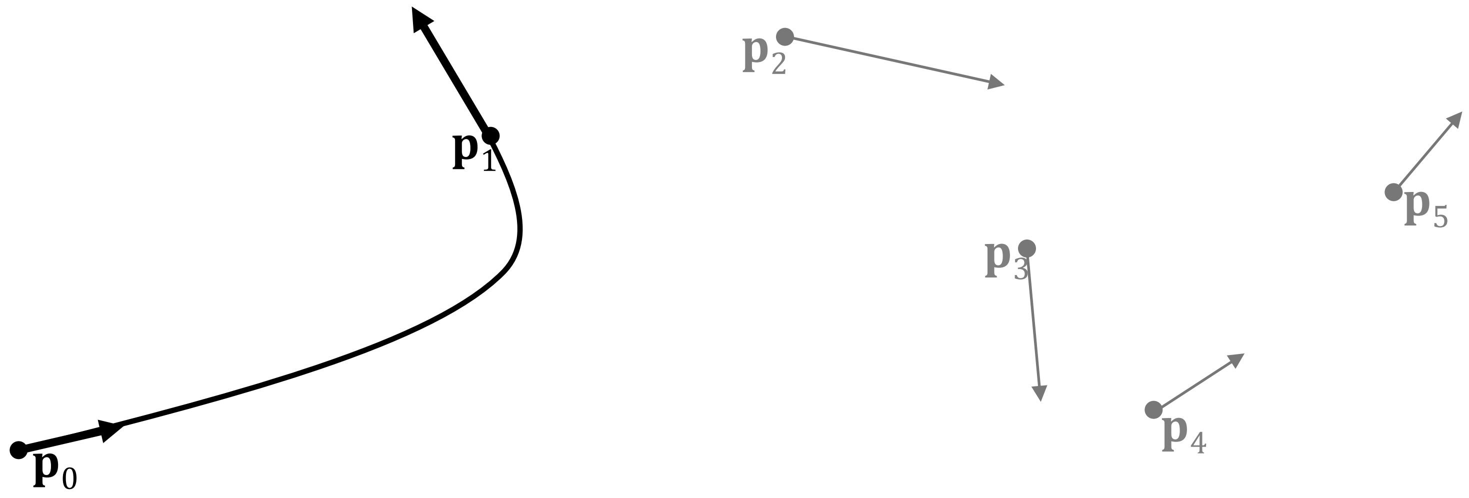

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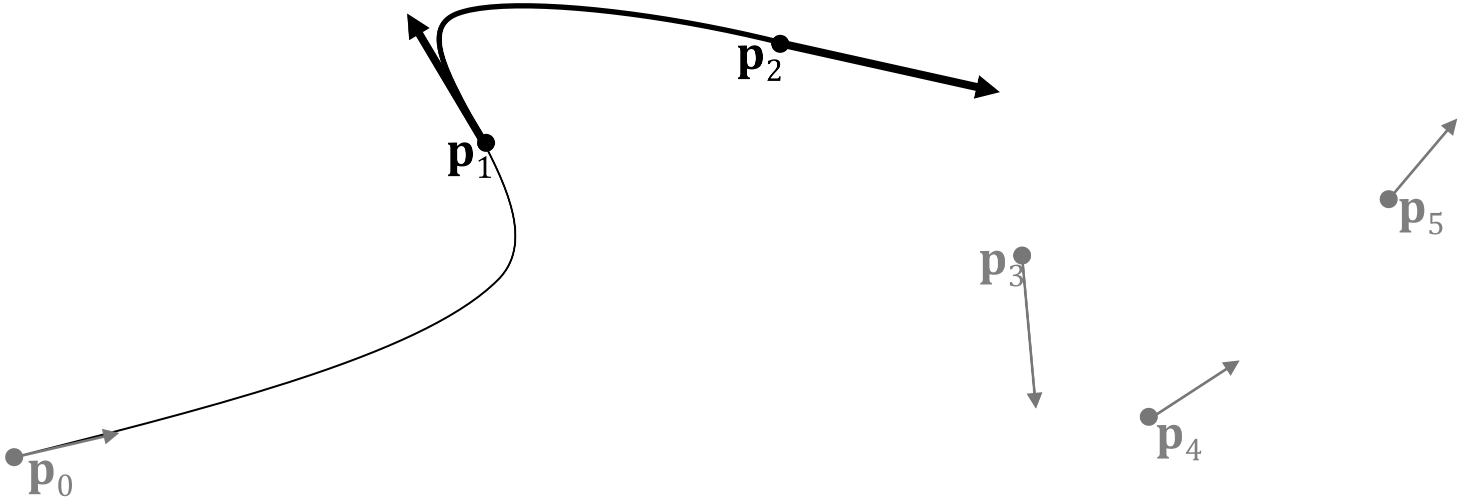

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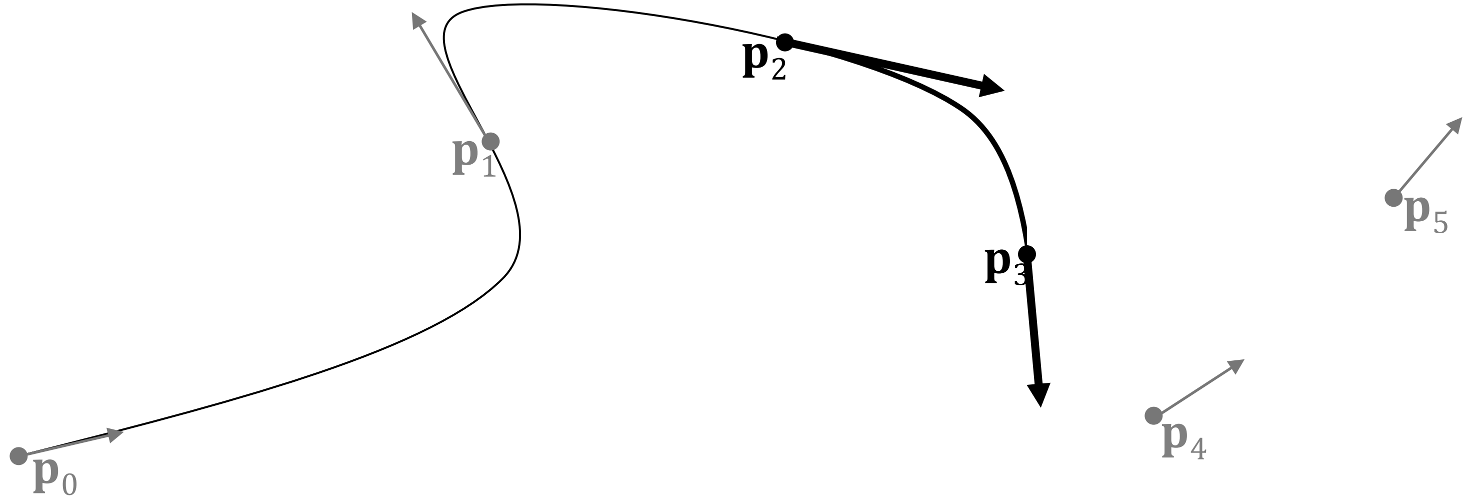

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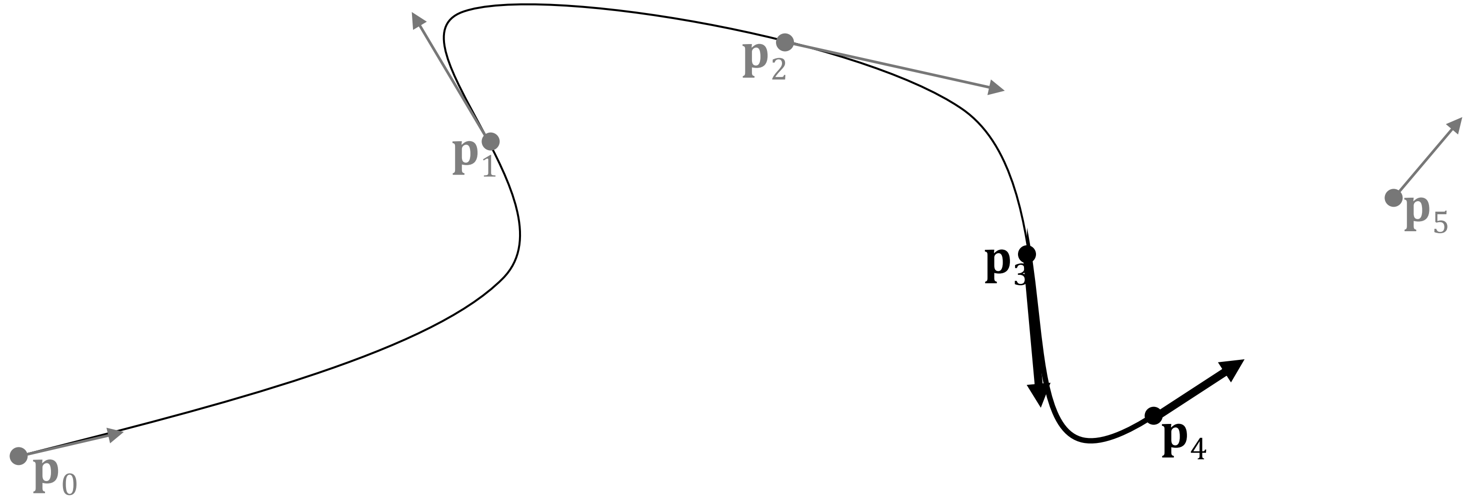

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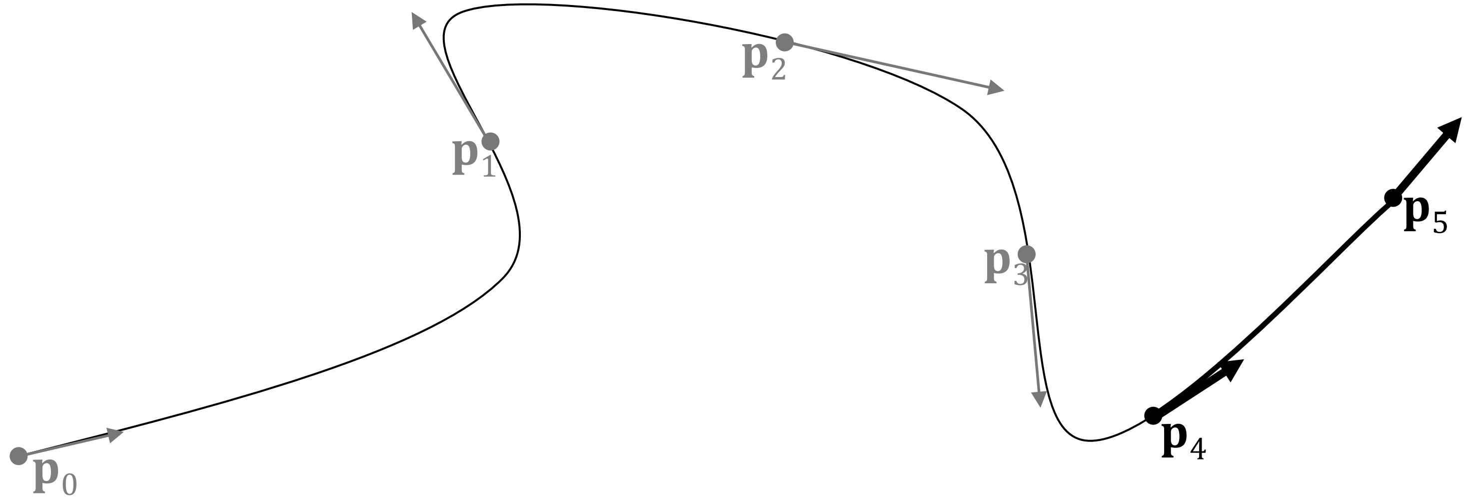

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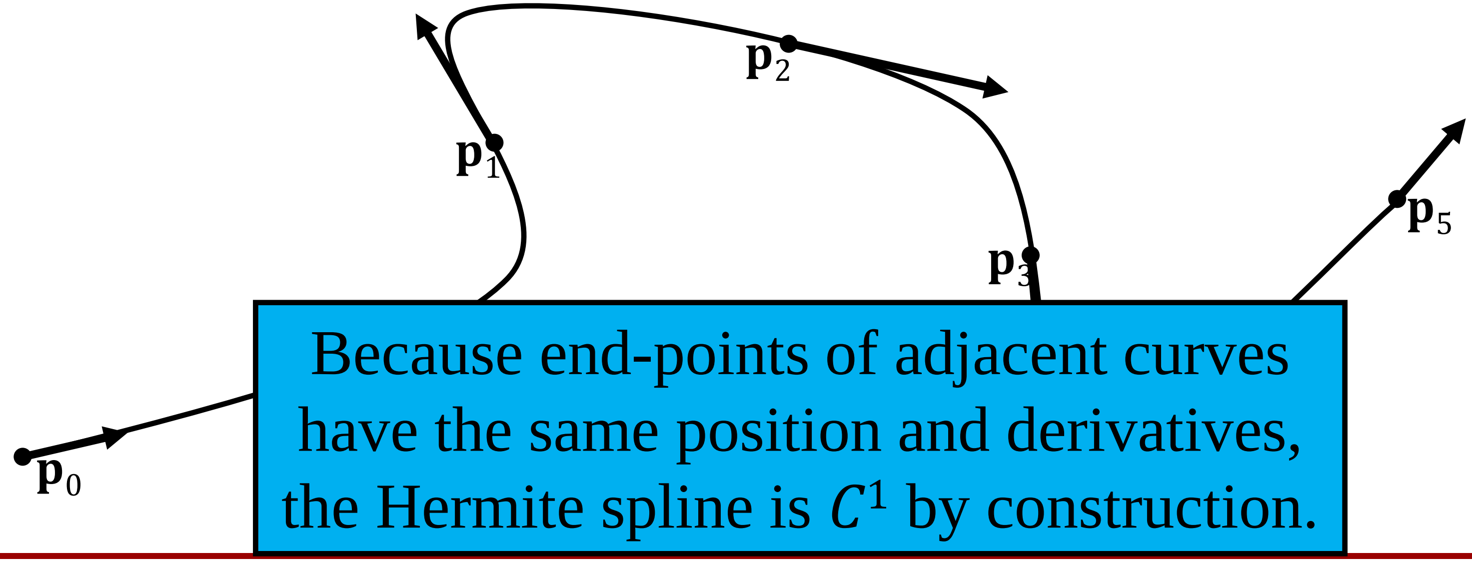
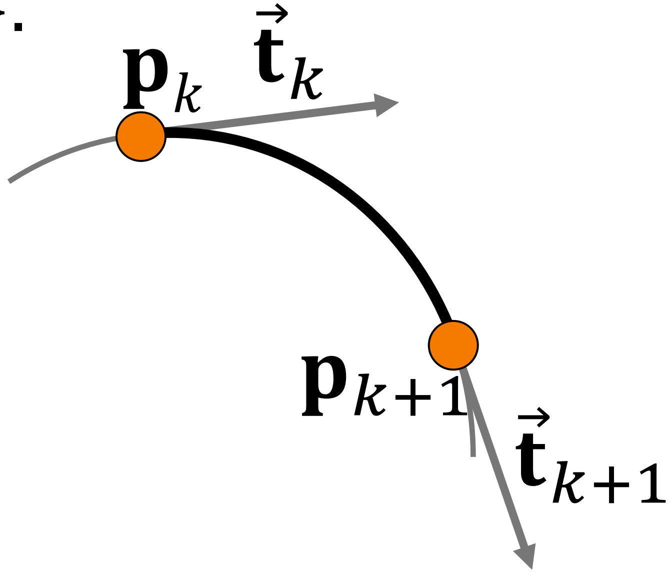


Because end-points of adjacent curves have the same position and derivatives, the Hermite spline is C^1 by construction.

Specific Example: Hermite Splines

- Let $\mathbf{P}_k(u) = \left(x_k(u), y_k(u)\right)$ with $0 \leq u \leq 1$ be the polynomial curve for the section between control points $\{\mathbf{p}_k, \vec{\mathbf{t}}_k\}$ and $\{\mathbf{p}_{k+1}, \vec{\mathbf{t}}_{k+1}\}$.
- Boundary conditions are:
  - $\mathbf{P}_k(0) = \mathbf{p}_k$
  - $\mathbf{P}_k(1) = \mathbf{p}_{k+1}$
  - $\mathbf{P}_k'(0) = \vec{\mathbf{t}}_k$
  - $\mathbf{P}_k'(1) = \vec{\mathbf{t}}_{k+1}$



- Solve for the coefficients of the polynomials $x_k(u)$ and $y_k(u)$ that satisfy the boundary conditions.

Note:
Four constraints $\Rightarrow$ we need a cubic polynomial.

Specific Example: Hermite Splines

Recall:

For a polynomial:

$$\mathbf{P}_k(u) = \mathbf{a} \cdot u^3 + \mathbf{b} \cdot u^2 + \mathbf{c} \cdot u + \mathbf{d}$$

we have:

$$\mathbf{P}'_k(u) = 3 \cdot \mathbf{a} \cdot u^2 + 2 \cdot \mathbf{b} \cdot u + \mathbf{c}$$

Using the matrix representation:

$$\mathbf{P}_k(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix} \qquad \mathbf{P}'_k(u) = \begin{pmatrix} 3 \cdot u^2 & 2 \cdot u & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

By abuse of notation, we will think of the coefficients **a**, **b**, **c**, and **d** as d-dimensional vectors rather than scalars so that $\mathbf{P}_k(u)$ is a function taking values in $\mathbb{R}^d$.

Specific Example: Hermite Splines

$$\mathbf{P}_k(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix} \qquad \mathbf{P}'_k(u) = \begin{pmatrix} 3 \cdot u^2 & 2 \cdot u & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

The values/derivatives at the end-points are:

$$\mathbf{p}_k = \mathbf{P}_k(0) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

Specific Example: Hermite Splines

$$\mathbf{P}_k(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix} \qquad \mathbf{P}'_k(u) = \begin{pmatrix} 3 \cdot u^2 & 2 \cdot u & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

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$$\mathbf{p}_{k+1} = \mathbf{P}_k(1) = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

Specific Example: Hermite Splines

$$\mathbf{P}_k(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix} \qquad \mathbf{P}_k'(u) = \begin{pmatrix} 3 \cdot u^2 & 2 \cdot u & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

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Specific Example: Hermite Splines

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Specific Example: Hermite Splines

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Combining into a single matrix gives:

$$\begin{pmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \vec{\mathbf{t}}_k \\ \vec{\mathbf{t}}_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

Specific Example: Hermite Splines

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Inverting, we get:

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \vec{\mathbf{t}}_k \\ \vec{\mathbf{t}}_{k+1} \end{pmatrix}$$

Specific Example: Hermite Splines

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Specific Example: Hermite Splines

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \vec{\mathbf{t}}_k \\ \vec{\mathbf{t}}_{k+1} \end{pmatrix}$$

Using the fact that:

$$\mathbf{P}_k(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

We get:

$$\mathbf{P}_k(u) = \underset{\text{parameters}}{\begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}} \underset{\mathbf{M}_{\text{Hermite}}}{\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}} \underset{\text{boundary info}}{\begin{pmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \vec{\mathbf{t}}_k \\ \vec{\mathbf{t}}_{k+1} \end{pmatrix}}$$

Specific Example: Hermite Splines

$$\mathbf{P}_k(u) = \left(\begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \vec{\mathbf{t}}_k \\ \vec{\mathbf{t}}_{k+1} \end{pmatrix}$$

Multiplying out and rearranging terms, we get:

$$\begin{aligned} \mathbf{P}_k(u) = & \ (2u^3 - 3u^2 + 1) \cdot \mathbf{p}_k \\ & + (-2u^3 + 3u^2) \cdot \mathbf{p}_{k+1} \\ & + (u^3 - 2u^2 + u) \cdot \vec{\mathbf{t}}_k \\ & + (u^3 - u^2) \cdot \vec{\mathbf{t}}_{k+1} \end{aligned}$$

Specific Example: Hermite Splines

$$\mathbf{P}_k(u) = (2u^3 - 3u^2 + 1) \cdot \mathbf{p}_k + (-2u^3 + 3u^2) \cdot \mathbf{p}_{k+1} + (u^3 - 2u^2 + u) \cdot \vec{\mathbf{t}}_k + (u^3 - u^2) \cdot \vec{\mathbf{t}}_{k+1}$$

Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
- $H_3(u) = u^3 - u^2$

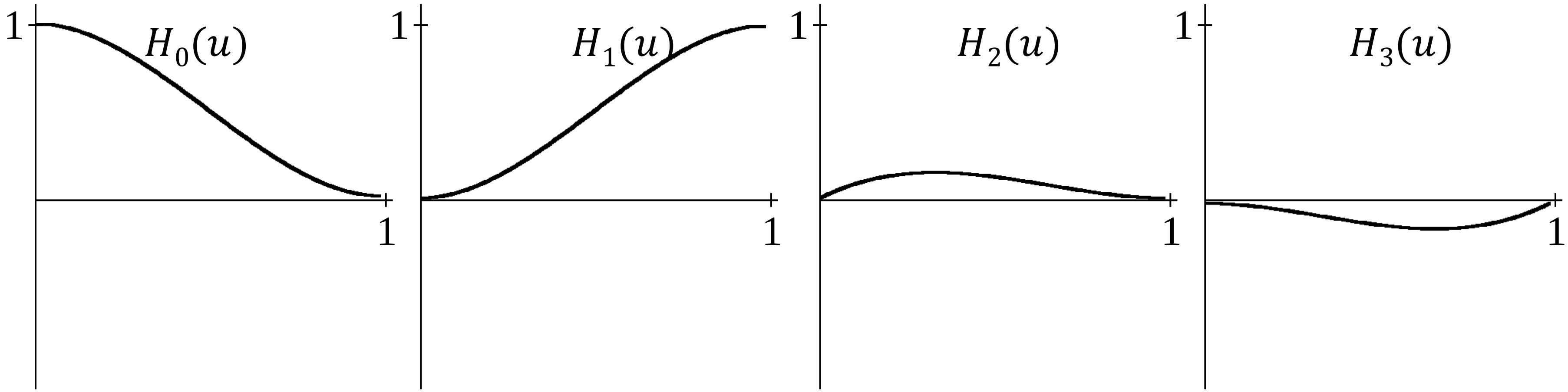
we can write $\mathbf{P}_k(u)$ as:

$$\mathbf{P}_k(u) = H_0(u) \cdot \mathbf{p}_k + H_1(u) \cdot \mathbf{p}_{k+1} + H_2(u) \cdot \vec{\mathbf{t}}_k + H_3(u) \cdot \vec{\mathbf{t}}_{k+1}$$

Specific Example: Hermite Splines

Setting:

$$\left.\begin{array}{l} \circ\ H_0(u) = 2u^3 - 3u^2 + 1 \\ \circ\ H_1(u) = -2u^3 + 3u^2 \\ \circ\ H_2(u) = u^3 - 2u^2 + u \\ \circ\ H_3(u) = u^3 - u^2 \end{array}\right\} \text{Blending Functions}$$



$$\mathbf{P}_k(u) = H_0(u) \cdot \mathbf{p}_k + H_1(u) \cdot \mathbf{p}_{k+1} + H_2(u) \cdot \vec{\mathbf{t}}_k + H_3(u) \cdot \vec{\mathbf{t}}_{k+1}$$

Specific Example: Hermite Splines

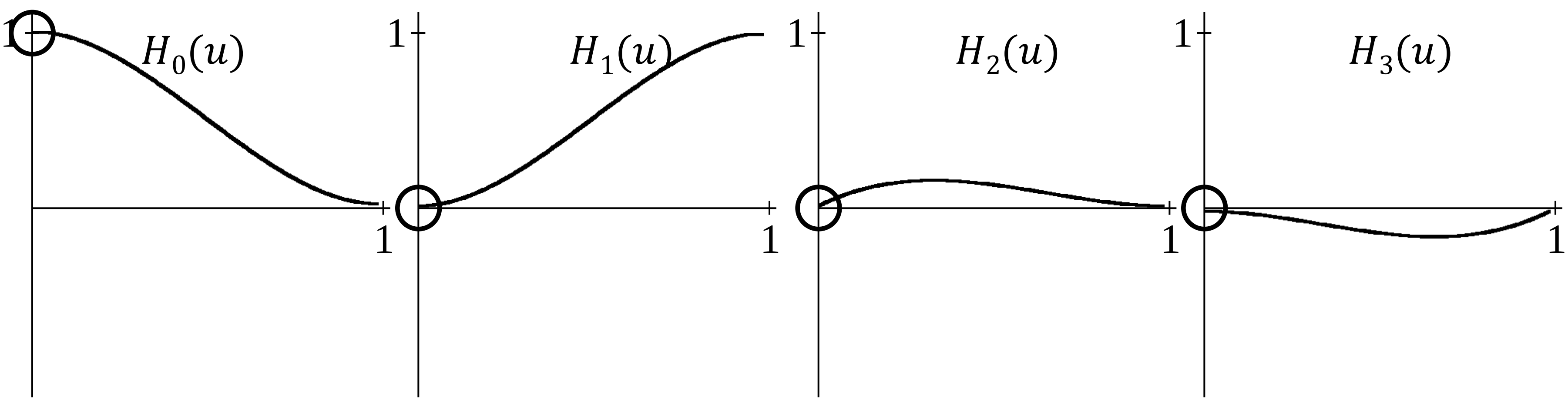
Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
- $H_3(u) = u^3 - u^2$

When $u = 0$:

- $H_0(u) = 1$
- $H_1(u) = 0$
- $H_2(u) = 0$
- $H_3(u) = 0$

So $\mathbf{P}_k(0) = \mathbf{p}_k$



$$\mathbf{P}_k(u) = H_0(u) \cdot \mathbf{p}_k + H_1(u) \cdot \mathbf{p}_{k+1} + H_2(u) \cdot \vec{\mathbf{t}}_k + H_3(u) \cdot \vec{\mathbf{t}}_{k+1}$$

Specific Example: Hermite Splines

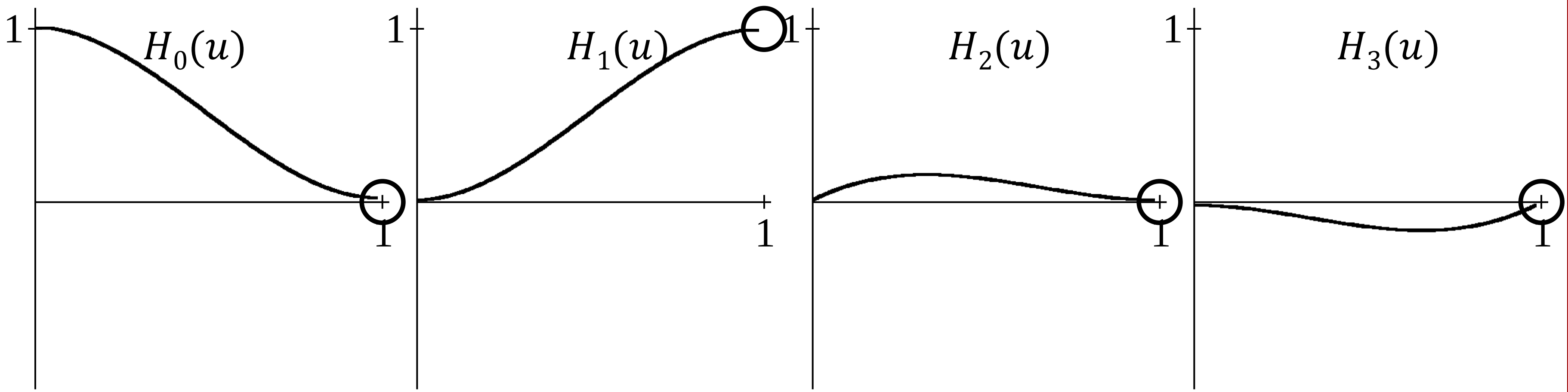
Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
- $H_3(u) = u^3 - u^2$

When $u = 1$:

- $H_0(u) = 0$
- $H_1(u) = 1$
- $H_2(u) = 0$
- $H_3(u) = 0$

So $\mathbf{P}_k(1) = \mathbf{p}_{k+1}$



$$\mathbf{P}_k(u) = H_0(u) \cdot \mathbf{p}_k + H_1(u) \cdot \mathbf{p}_{k+1} + H_2(u) \cdot \vec{\mathbf{t}}_k + H_3(u) \cdot \vec{\mathbf{t}}_{k+1}$$

Specific Example: Hermite Splines

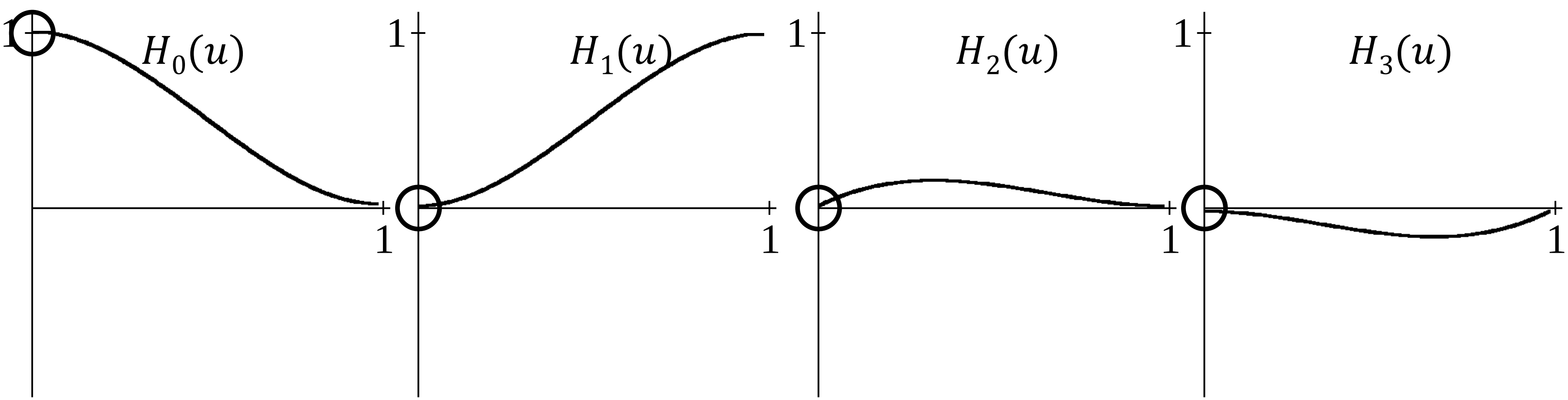
Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
- $H_3(u) = u^3 - u^2$

When $u = 0$:

- $H_0'(u) = 0$
- $H_1'(u) = 0$
- $H_2'(u) = 1$
- $H_3'(u) = 0$

So $\mathbf{P}_k'(0) = \vec{\mathbf{t}}_k$



$$\mathbf{P}_k'(u) = H_0'(u) \cdot \mathbf{p}_k + H_1'(u) \cdot \mathbf{p}_{k+1} + H_2'(u) \cdot \vec{\mathbf{t}}_k + H_3'(u) \cdot \vec{\mathbf{t}}_{k+1}$$

Specific Example: Hermite Splines

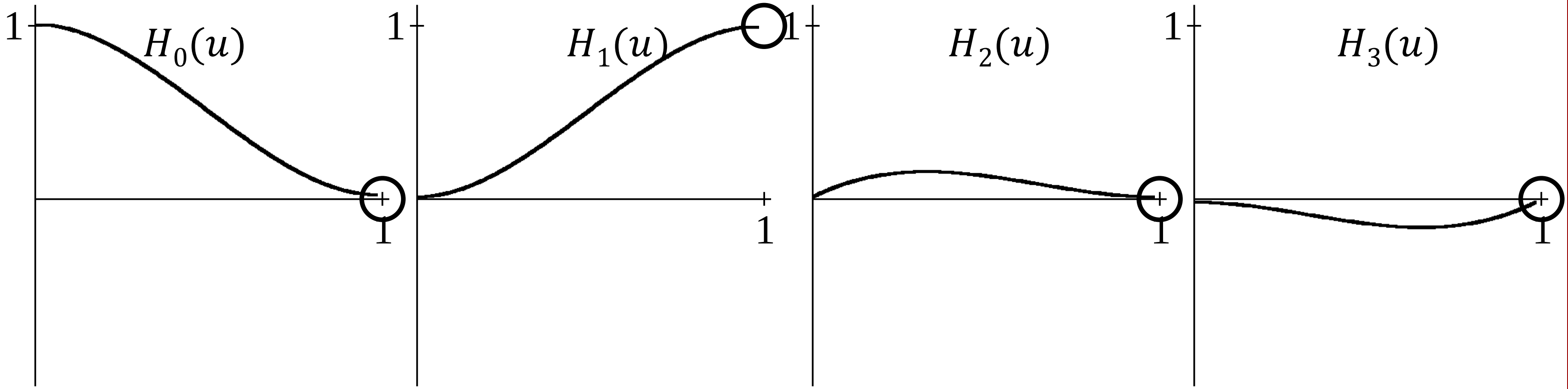
Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
- $H_3(u) = u^3 - u^2$

When $u = 1$:

- $H_0'(u) = 0$
- $H_1'(u) = 0$
- $H_2'(u) = 0$
- $H_3'(u) = 1$

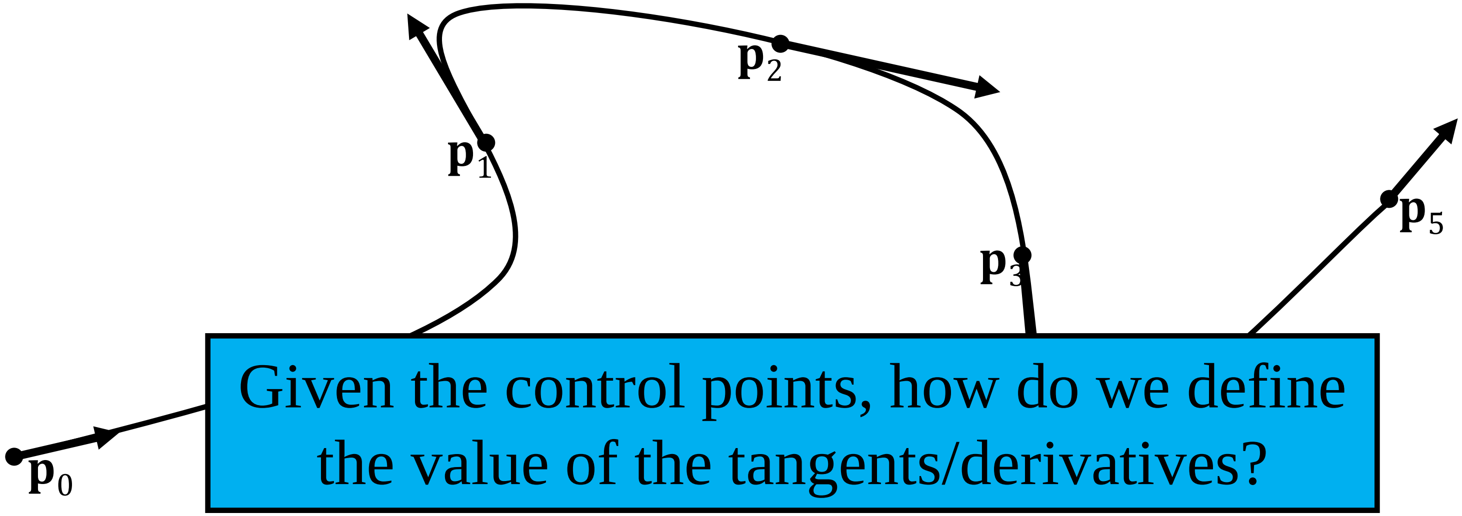
So $\mathbf{P}_k'(1) = \vec{\mathbf{t}}_{k+1}$



$$\mathbf{P}_k'(u) = H_0'(u) \cdot \mathbf{p}_k + H_1'(u) \cdot \mathbf{p}_{k+1} + H_2'(u) \cdot \vec{\mathbf{t}}_k + H_3'(u) \cdot \vec{\mathbf{t}}_{k+1}$$

Specific Example: Hermite Splines

- Interpolating piecewise *cubic* polynomial, each specified by:
  - Start/end positions
  - Start/end tangents
- Iteratively construct the curve between adjacent end points that interpolate positions and tangents.



Given the control points, how do we define the value of the tangents/derivatives?