FFTs in Graphics and Vision

Characters of Representations

Outline

- Math Review
- Characters

Math Review

Notation:

Given vector spaces V and W, we define $V \oplus W$ to be the direct sum of the vector spaces:

$$V \oplus W = \{(v,w) | v \in V \text{ and } w \in W\}$$

- Given $v \in V$, $w \in W$, and a scalar $\alpha \in \mathbb{C}$:

$$\alpha(v,w) \equiv (\alpha v, \alpha w)$$

- Given $v_1, v_2 \in V$, and $w_1, w_2 \in W$:

$$(v_1, w_1) + (v_2, w_2) \equiv (v_1 + v_2, w_1 + w_2)$$

Math Review

Notation:

Given vector spaces V and W, we define $V \oplus W$ to be the direct sum of the vector spaces:

$$V \oplus W = \{(v, w) | v \in V \text{ and } w \in W\}$$

Given bases $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$, $V \oplus W$ is a $(n + m)$-dimensional space obtained by "stacking" coefficients:

$$(a_1, \dots, a_{n+m}) \mapsto \left(\sum_{i=1}^{n} a_i \cdot \mathbf{v}_i , \sum_{i=1}^{m} a_{n+i} \cdot \mathbf{w}_i \right)$$

Math Review

Notation:

Given linear maps $\mathcal{L}: V \to V$ and $\mathcal{M}: W \to W$, we define $\mathcal{L} \oplus \mathcal{M}$ to be the map:

$$\mathcal{L} \oplus \mathcal{M}: V \oplus W \to V \oplus W$$
$$(v, w) \mapsto \big(\mathcal{L}(v), \mathcal{M}(w)\big)$$

Given bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, with $\mathbf{L}$ and $\mathbf{M}$ the associated matrices, $\mathcal{L} \oplus \mathcal{M}$ is represented by the block-diagonal matrix:

$$\begin{pmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{M} \end{pmatrix} \in \mathbb{C}^{(n+m)\times(n+m)}$$

Math Review

<u>Definition</u>:

Given a matrix $\mathbf{M} \in \mathbb{C}^{n\times n}$, the *trace* of $\mathbf{M}$ is the sum of the diagonal entries:

$$\text{Tr}(M) = \sum_{i=1}^{n} \mathbf{M}_{ii}$$

Math Review

Properties:

1. $\mathrm{Tr}(a \cdot \mathbf{M}) = a \cdot \mathrm{Tr}(\mathbf{M})$
2. $\mathrm{Tr}(\mathbf{M}) = \mathrm{Tr}(\mathbf{M}^t)$
3. $\mathrm{Tr}(\overline{\mathbf{M}}) = \overline{\mathrm{Tr}(\mathbf{M})}$
4. If $\mathbf{M}$ is a unitary matrix, then:
$$\mathrm{Tr}(\mathbf{M}^{-1}) = \mathrm{Tr}\left(\overline{\mathbf{M}^t}\right) = \overline{\mathrm{Tr}(\mathbf{M})}$$
5. $\mathrm{Tr}(\mathbf{M}) = \mathrm{Tr}(\mathbf{N}^{-1} \cdot \mathbf{M} \cdot \mathbf{N}) \quad \forall \mathbf{N} \in GL(n)$

Math Review

$$\mathrm{Tr}(\mathbf{M}) = \mathrm{Tr}(\mathbf{N}^{-1} \cdot \mathbf{M} \cdot \mathbf{N}) \quad \forall \mathbf{N} \in GL(n)$$

<u>Properties</u>:

If $\mathcal{L}: V \to V$ is a linear transformation, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ a basis for V, and $\mathbf{M} \in \mathbb{C}^{n \times n}$ the representation of $\mathcal{L}$ in the basis, then the trace of $\mathbf{M}$ is independent of the choice of basis.

$\Rightarrow$ The “trace of a linear operator” is well-defined without a matrix representation.

Math Review

Properties:

Given vector spaces V and W and linear maps $\mathcal{L}: V \to V$ and $\mathcal{M}: W \to W$ we have:

$$\mathrm{Tr}(\mathcal{L} \oplus \mathcal{M}) = \mathrm{Tr}(\mathcal{L}) + \mathrm{Tr}(\mathcal{M})$$

Math Review

Notation:

Given the space of n-dimensional vectors, we denote by $\mathbf{e}^i \in \mathbb{C}^n$ the vector with a "1" in the i-th entry and "0" everywhere else:

$$\mathbf{e}^i_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Given the space of $n \times n$ matrices, we denote by $\mathbf{E}^{ij} \in \mathbb{C}^{n\times n}$ the matrix with "1" in the i-th row and j-th column and "0" everywhere else:

$$\mathbf{E}^{ij}_{ab} = \delta_{ia} \cdot \delta_{jb}$$

Math Review

Notation:

Given a matrix $\mathbf{N} \in \mathbb{C}^{n \times n}$, we have:

$$\begin{aligned}\left(\mathbf{E}^{ij} \cdot \mathbf{N}\right)_{ab} &= \sum_{c=0}^{n} \mathbf{E}_{ac}^{ij} \cdot \mathbf{N}_{cb} \\ &= \sum_{c=0}^{n} \delta_{ia} \cdot \delta_{jc} \cdot \mathbf{N}_{cb} \\ &= \delta_{ia} \cdot \mathbf{N}_{jb}\end{aligned}$$

This is the matrix whose i-th row contains the j-th row of $\mathbf{N}$.

Math Review

Notation:

Given matrices $\mathbf{M}, \mathbf{N} \in \mathbb{C}^{n\times n}$, we have:

$$\begin{aligned}\left(\mathbf{M}\cdot\mathbf{E}^{ij}\cdot\mathbf{N}\right)_{ab} &= \sum_{c=0}^{n}\mathbf{M}_{ac}\cdot\left(\mathbf{E}^{ij}\cdot\mathbf{N}\right)_{cb} \\ &= \sum_{c=0}^{n}\mathbf{M}_{ac}\cdot\delta_{ic}\cdot\mathbf{N}_{jb} \\ &= \mathbf{M}_{ai}\cdot\mathbf{N}_{jb}\end{aligned}$$

This is the matrix whose (a,b)-th entry is the product of the (a,i)-th entry of $\mathbf{M}$ and the (j,b)-th entry of $\mathbf{N}$.

Functions on Groups

Note:

Given a representation (ρ, V) and given a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ we can represent each ρ_g as a matrix $\mathbf{M}^\rho(g)$, with coefficients that are functions:

$$\mathbf{M}_{ij}^\rho : G \to \mathbb{C}$$

Since $\mathbf{M}^\rho(g)$ is unitary, we have:

$$\mathbf{M}^\rho(g^{-1}) = \left(\mathbf{M}^\rho(g)\right)^{-1} = \overline{\left(\mathbf{M}^\rho(g)\right)^t}$$

Functions on Groups

<u>Notation</u>:

Given the space of complex-valued functions on G, we can define a scalar product by setting:

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \cdot \overline{\psi(g)}$$

for any functions $\phi, \psi: G \to \mathbb{C}$.

Math Review

Definition:

Given representations (ρ_1, V) and (ρ_2, W) of a group G, a linear map $\mathcal{L}: V \to W$ is G-*linear* if:

$$\rho_2(g) \circ \mathcal{L} = \mathcal{L} \circ \rho_1(g) \quad \forall g \in G$$

$$\Updownarrow$$

$$\mathcal{L} = \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g) \quad \forall g \in G$$

Math Review

Definition:

Given representations (ρ_1, V) and (ρ_2, W), we say the two representations are *isomorphic* if there exists a G-linear isomorphism:

$$\mathcal{L}: V \to W$$

Math Review

Schur's Lemma:

Given irreducible representations (ρ_1, V) and (ρ_2, W) of a group G, if $\mathcal{L}: V \rightarrow W$ is G-linear then:

1. If $\mathcal{L}$ is not an isomorphism, then $\mathcal{L} = 0$
2. If $V = W$ and $\rho_1 = \rho_2$, then $\mathcal{L} = \lambda \cdot \mathrm{Id}$.

Math Review

Maschke's Theorem:

If W is a sub-representation of V, then the space $W^\perp$ will also be a sub-representation of V.

Corollary:

Given a representation (ρ, V) we can decompose V into a direct-sum of irreducible representations:

$$V = \bigoplus_i V_i$$

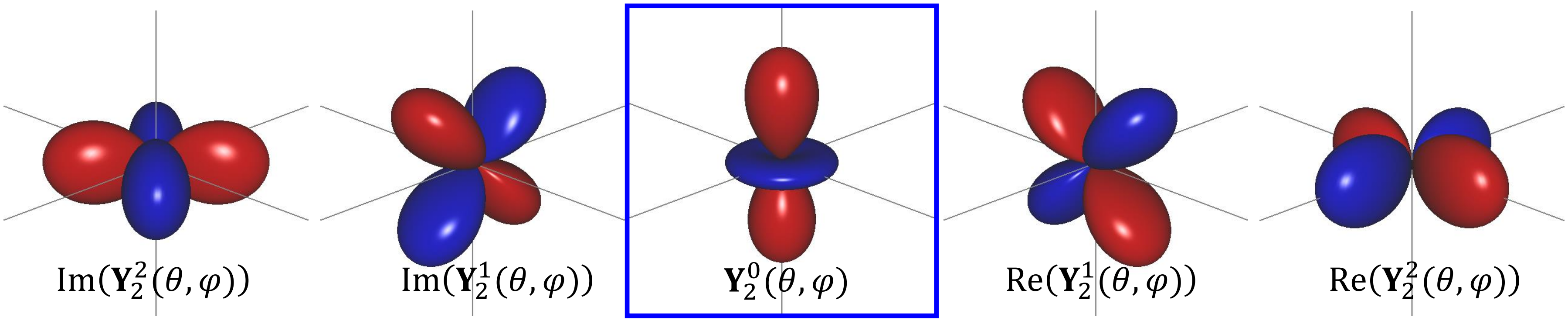
Note that an irreducible representation may occur with multiplicity (i.e. V_i may be isomorphic to V_j).

Math Review

Recall:

Convolving with the l-th zonal harmonic is the same as scaling the l-th spherical frequency:

$$\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_{l'}^m}, \rangle = \delta_{l,l'} \cdot \lambda_l \cdot \mathbf{Y}_l^m$$



Math Review

Recall:

Convolving with the l-th zonal harmonic is the same as scaling the l-th spherical frequency:

$$\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_{l'}^m}, \rangle = \delta_{l,l'} \cdot \lambda_l \cdot \mathbf{Y}_l^m$$

We had asserted that:

$$\lambda_l = \sqrt{\frac{4\pi}{2l+1}}$$

Math Review

Recall:

Given the spherical harmonics, we defined the Wigner-D functions to be:

$$\mathbf{D}_l^{m,m'}(R) = \langle R(\mathbf{Y}_l^m), \mathbf{Y}_l^{m'} \rangle$$

We had asserted that the Wigner-D functions:

1. Form an orthogonal basis
2. For a fixed l, form a representation of $SO(3)$.

Outline

- Math Review
- Characters

G-Linear Maps by Averaging

Given representations (ρ_1, V) and (ρ_2, W) of a group G, and given a linear map $\mathcal{L}: V \to W$, we can construct a G-linear map $\mathcal{L}^0$ by averaging:

$$\mathcal{L}^0 = \frac{1}{|G|} \sum_{g \in G} \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g)$$

G-Linear Maps by Averaging

<u>Proof</u>:

For any $h \in G$ we have:

$$\begin{aligned}
\rho_2(h^{-1}) \circ \mathcal{L}^0 \circ \rho_1(h) &= \rho_2(h^{-1}) \circ \left(\frac{1}{|G|} \sum_{g \in G} \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g) \right) \circ \rho_1(h) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_2(h^{-1} \cdot g^{-1}) \circ \mathcal{L} \circ \rho_1(g \cdot h) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_2\big((g \cdot h)^{-1}\big) \circ \mathcal{L} \circ \rho_1(g \cdot h) \\
&= \frac{1}{|G|} \sum_{g \in Gh} \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g) \\
&= \mathcal{L}^0
\end{aligned}$$

G-Linear Maps by Averaging

$$\mathcal{L}^0 = \frac{1}{|G|}\sum_{g\in G}\rho_2(g^{-1})\circ\mathcal{L}\circ\rho_1(g)$$

By Schur's Lemma:

1. If $\mathcal{L}^0$ is not an isomorphism, then $\mathcal{L}^0 = 0$.
2. If $V = W$ and $\rho_1 = \rho_2$, then $\mathcal{L}^0 = \lambda \cdot \mathrm{Id}$.
   Taking the trace:

$$\mathrm{Tr}(\mathcal{L}^0) = \mathrm{Tr}\left(\frac{1}{|G|}\sum_{g\in G}\rho_1(g^{-1})\circ\mathcal{L}\circ\rho_1(g)\right)$$

$$\mathrm{Tr}(\lambda\cdot\mathrm{Id.}) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}\left(\rho_1^{-1}(g)\circ\mathcal{L}\circ\rho_1(g)\right)$$

$$\lambda\cdot\mathrm{Tr}(\mathrm{Id.}) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(\mathcal{L})$$

$$\lambda\cdot\dim(V) = \mathrm{Tr}(\mathcal{L})$$

G-Linear Maps by Averaging

$$\mathcal{L}^0 = \frac{1}{|G|}\sum_{g\in G} \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g)$$

By Schur's Lemma:

1. If $\mathcal{L}^0$ is not an isomorphism, then $\mathcal{L}^0 = 0$.
2. If $V = W$ and $\rho_1 = \rho_2$, then $\mathcal{L}^0 = \frac{\mathrm{Tr}(\mathcal{L})}{\dim(V)} \cdot \mathrm{Id}$.

Coefficient Orthogonality

$$\mathcal{L}^0 = \frac{1}{|G|}\sum_{g\in G}\rho_2(g^{-1})\circ\mathcal{L}\circ\rho_1(g)$$

Choosing bases and taking $\mathcal{L} = \mathbf{E}^{ij}$ we get:

1. If $\mathcal{L}^0$ is not an isomorphism, then $\mathcal{L}^0 = 0$.

$$0 = \frac{1}{|G|}\sum_{g\in G}\rho_2(g^{-1})\circ\mathbf{E}^{ij}\circ\rho_1(g)$$

$$\Downarrow$$

$$(0)_{ab} = \frac{1}{|G|}\sum_{g\in G}\mathbf{M}^{\rho_2}_{ai}(g^{-1})\cdot\mathbf{M}^{\rho_1}_{jb}(g)$$

$$= \frac{1}{|G|}\sum_{g\in G}\overline{\mathbf{M}^{\rho_2}_{ia}(g)}\cdot\mathbf{M}^{\rho_1}_{jb}(g)$$

$$\Updownarrow$$

$$0 = \langle\mathbf{M}^{\rho_1}_{jb},\mathbf{M}^{\rho_2}_{ia}\rangle$$

Coefficient Orthogonality

$$\mathcal{L}^0 = \frac{1}{|G|}\sum_{g\in G}\rho_2(g^{-1})\circ\mathcal{L}\circ\rho_1(g)$$

Choosing a basis and taking $\mathcal{L} = \mathbf{E}^{ij}$ we get:

2. If $V = W$ and $\rho_1 = \rho_2$, then $\mathcal{L}^0 = \frac{\mathrm{Tr}(\mathbf{E}^{ij})}{\dim(V)}\cdot \mathrm{Id.}$:

$$\frac{\mathrm{Tr}(\mathbf{E}^{ij})}{\dim(V)}\cdot \mathrm{Id.} = \frac{1}{|G|}\sum_{g\in G}\rho(g^{-1})\circ\mathbf{E}^{ij}\circ\rho(g)$$

$$\Downarrow$$

$$\left(\frac{\delta_{ij}}{\dim(V)}\cdot \mathrm{Id.}\right)_{ab} = \frac{1}{|G|}\sum_{g\in G}\overline{\mathbf{M}^{\rho}_{ia}(g)}\cdot\mathbf{M}^{\rho}_{jb}(g)$$

$$\Updownarrow$$

$$\frac{\delta_{ij}\cdot\delta_{ab}}{\dim(V)} = \langle \mathbf{M}^{\rho}_{jb}, \mathbf{M}^{\rho}_{ia}\rangle$$

Characters

Definition:

Given a representation (ρ, V) of a group G, the *character of the representation* is a map:

$$\chi_\rho : G \to \mathbb{C}$$
$$g \mapsto \mathrm{Tr}(\rho_g)$$

Characters

Claim:

Given irreducible representations (ρ_1, V_1) and (ρ_2, V_2), let $\chi_1, \chi_2: G \to \mathbb{C}$ be their characters.

1. If the representations are not isomorphic:
$$\langle \chi_1, \chi_2 \rangle = 0$$
2. If the representations are isomorphic:
$$\langle \chi_1, \chi_2 \rangle = 1$$

Characters

$$0 = \langle \mathbf{M}_{jb}^{\rho_1}, \mathbf{M}_{ia}^{\rho_2} \rangle$$

1. Not isomorphic: $\langle \chi_1, \chi_2 \rangle = 0$

<u>Proof</u>:

$$\begin{aligned}\langle \chi_1, \chi_2 \rangle &= \langle \mathrm{Tr}(\rho_1), \mathrm{Tr}(\rho_2) \rangle \\ &= \left\langle \sum_{i=1}^{n_1} \mathbf{M}_{ii}^{\rho_1}, \sum_{j=1}^{n_2} \mathbf{M}_{jj}^{\rho_2} \right\rangle \\ &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \langle \mathbf{M}_{ii}^{\rho_1}, \mathbf{M}_{jj}^{\rho_2} \rangle \\ &= 0\end{aligned}$$

Characters

$$\frac{\delta_{ij} \cdot \delta_{ab}}{\dim(V)} = \langle \mathbf{M}^{\rho}_{jb}, \mathbf{M}^{\rho}_{ia} \rangle$$

2. Isomorphic: $\langle \chi_1, \chi_2 \rangle = 1$

<u>Proof</u>:

$\exists$ isomorphism $\mathcal{L}: V_1 \to V_2$ s.t. $\rho_1 = \mathcal{L}^{-1} \circ \rho_2 \circ \mathcal{L}$.

Rewriting the inner-product we get:

$$\begin{aligned}
\langle \chi_1, \chi_2 \rangle &= \langle \mathrm{Tr}(\rho_1), \mathrm{Tr}(\rho_2) \rangle \\
&= \langle \mathrm{Tr}(\mathcal{L}^{-1} \circ \rho_2 \circ \mathcal{L}), \mathrm{Tr}(\rho_2) \rangle \\
&= \langle \mathrm{Tr}(\rho_2), \mathrm{Tr}(\rho_2) \rangle \\
&= \sum_{i,j=1}^{n} \left\langle \mathbf{M}^{\rho_2}_{ii}, \mathbf{M}^{\rho_2}_{jj} \right\rangle = \sum_{i,j=1}^{n} \frac{\delta_{ij} \cdot \delta_{ij}}{\dim(V_2)} \\
&= \sum_{i=1}^{n} \frac{1}{\dim(V_2)} = 1
\end{aligned}$$

Characters

Implications:

Given a representation (ρ, V) of a group G and given some irreducible representation (ρ', V') we would like to know "how many times" the irreducible representation ρ' occurs in ρ.

Characters

Implications:

How many times does ρ' occurs in ρ?

$$V = \bigoplus_i V_i$$

$$\Downarrow$$

$$\chi_\rho = \sum_i \chi_{\rho_i}$$

$$\Downarrow$$

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \sum_i \langle \chi_{\rho_i}, \chi_{\rho'} \rangle$$

$$= \sum_i \begin{cases} 1 & \text{if } (\rho_i, V_i) \approx (\rho', V') \\ 0 & \text{otherwise} \end{cases}$$

Characters

Example:

We know that if G is the group of 2D rotations, $G = SO(2)$, and V is the space of functions on a circle, we have:

$$V = \bigoplus_{k=-\infty}^{\infty} V_k$$

where V_k is the 1D space of functions spanned by the complex exponentials:

$$V_k = \text{Span}\{e^{ik\theta}\}$$

Characters

Example:

Using the fact that $\{e^{ik\theta}\}$ is a basis for V_k, we can express $\rho_k(g)$ as a (1×1) matrix w.r.t. this basis.

Denoting by g_ϕ the rotation by ϕ degrees, we get:

$$\rho_k(g_\phi) = \left(e^{-ik\phi}\right)$$

So the character of this representation is:

$$\chi_{\rho_k}(g_\phi) = \mathrm{Tr}\left(e^{-ik\phi}\right) = e^{-ik\phi}$$

Characters

<u>Example</u>:

We know that if G is the group of 3D rotations, $G = SO(3)$, and V is the space of functions on a circle, we have:

$$V = \bigoplus_{l=0}^{\infty} V_l$$

where V_l is the $(2l+1)$-dimensional space of functions spanned by the spherical harmonics:

$$V_l = \text{Span}\{\mathbf{Y}_l^{-l}, \cdots, \mathbf{Y}_l^{l}\}$$

Characters

Example:

Using the spherical harmonic basis we get:

$$\rho_l(R)\left[\sum_{m=-l}^{m} a_m \cdot \mathbf{Y}_l^m\right] = \sum_{m=-l}^{m} a_m \sum_{m\prime=-l}^{l} \langle R(\mathbf{Y}_l^m), \mathbf{Y}_l^{m'}\rangle \cdot \mathbf{Y}_l^{m'}$$

$$= \sum_{m=-l}^{m} a_m \sum_{m\prime=-l}^{l} \mathbf{D}_l^{m,m'}(R) \cdot \mathbf{Y}_l^{m'}$$

So the character of this representation is:

$$\chi_{\rho_l}(R) = \mathrm{Tr}\begin{pmatrix} \mathbf{D}_l^{-l,-l}(R) & \cdots & \mathbf{D}_l^{-l,l}(R) \\ \vdots & \ddots & \vdots \\ \mathbf{D}_l^{l,-l}(R) & \cdots & \mathbf{D}_l^{l,l}(R) \end{pmatrix} = \sum_{m=-l}^{l} \mathbf{D}_l^{m,m}(R)$$

Characters

Application:

$$\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_l^m} \rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta, \phi)$$

What is λ_l?

Characters

$$\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_l^m} \rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Since $\mathbf{Y}_l^0$ is real-valued, we can re-write this as:

$$\langle \mathbf{Y}_l^m, \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right) \rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Characters

$$\langle \mathbf{Y}_l^m, \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right)\rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Recall that:

$$\frac{\delta_{ij} \cdot \delta_{ab}}{\dim(V_l)} = \langle \mathbf{M}_{jb}^{\rho_l}, \mathbf{M}_{ia}^{\rho_l}\rangle$$

$$\mathbf{M}_{mm'}^{\rho_l}(R) = \langle R(\mathbf{Y}_l^m), \mathbf{Y}_l^{m'}\rangle$$

Setting $i,j = m$, $a,b = 0$, and writing the dot-product as an integral:

$$\frac{1}{2l+1} = \langle \mathbf{M}_{m0}^{\rho_l}, \mathbf{M}_{m0}^{\rho_l}\rangle$$

$$= \frac{1}{|SO(3)|}\int_{R\in SO(3)} \overline{\langle R(\mathbf{Y}_l^m), \mathbf{Y}_l^0\rangle} \cdot \langle R(\mathbf{Y}_l^m), \mathbf{Y}_l^0\rangle dR$$

Characters

$$\langle \mathbf{Y}_l^m, \rho_{R(\theta,\phi)}(\mathbf{Y}_l^0) \rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Since rotations are orthogonal:

$$\begin{aligned}
\frac{1}{2l+1} &= \frac{1}{|SO(3)|}\int_{R\in SO(3)} \overline{\langle R(\mathbf{Y}_l^m), \mathbf{Y}_l^0 \rangle} \cdot \langle R(\mathbf{Y}_l^m), \mathbf{Y}_l^0 \rangle \, dR \\
&= \frac{1}{|SO(3)|}\int_{R\in SO(3)} \overline{\langle \mathbf{Y}_l^m, R^{-1}(\mathbf{Y}_l^0) \rangle} \cdot \langle \mathbf{Y}_l^m, R^{-1}(\mathbf{Y}_l^0) \rangle dR \\
&= \frac{1}{|SO(3)|}\int_{R\in SO(3)} \overline{\langle \mathbf{Y}_l^m, R(\mathbf{Y}_l^0) \rangle} \cdot \langle \mathbf{Y}_l^m, R(\mathbf{Y}_l^0) \rangle dR
\end{aligned}$$

Characters

$$\langle \mathbf{Y}_l^m, \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right)\rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Factoring the integral we get:

$$\frac{1}{2l+1} = \frac{1}{|SO(3)|}\int_{R\in SO(3)} \overline{\langle \mathbf{Y}_l^m, R(\mathbf{Y}_l^0)\rangle} \cdot \langle \mathbf{Y}_l^m, R(\mathbf{Y}_l^0)\rangle dR$$

$$\Downarrow$$

$$\frac{1}{2l+1} = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \overline{\langle \mathbf{Y}_l^m, R_{\theta,\phi,\psi}(\mathbf{Y}_l^0)\rangle} \cdot \langle \mathbf{Y}_l^m, R_{\theta,\phi,\psi}\left(\mathbf{Y}_l^0\right)\rangle \, d\psi \sin(\phi)\; d\phi \, d\theta$$

Characters

$$\langle \mathbf{Y}_l^m, \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right)\rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Using the invariance of the zonal harmonics to rotations about the y-axis:

$$\frac{1}{2l+1} = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\overline{\langle \mathbf{Y}_l^m, R_{\theta,\phi,\psi}(\mathbf{Y}_l^0)\rangle}\cdot\langle \mathbf{Y}_l^m, R_{\theta,\phi,\psi}\left(\mathbf{Y}_l^0\right)\rangle\, d\psi\,\sin(\phi)\; d\phi\, d\theta$$

$$\Downarrow$$

$$\frac{1}{2l+1} = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\overline{\langle \mathbf{Y}_l^m, R_{\theta,\phi}(\mathbf{Y}_l^0)\rangle}\cdot\langle \mathbf{Y}_l^m, R_{\theta,\phi}\left(\mathbf{Y}_l^0\right)\rangle\, d\psi\,\sin(\phi)\; d\phi\, d\theta$$

Characters

$$\langle \mathbf{Y}_l^m, \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right)\rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Since the integrand is independent of ψ:

$$\frac{1}{2l+1} = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\overline{\langle \mathbf{Y}_l^m, R_{\theta,\phi}(\mathbf{Y}_l^0)\rangle}\cdot\langle \mathbf{Y}_l^m, R_{\theta,\phi}\left(\mathbf{Y}_l^0\right)\rangle\, d\psi \sin(\phi)\ d\phi\, d\theta$$

$$\Downarrow$$

$$\frac{1}{2l+1} = \frac{2\pi}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\overline{\langle \mathbf{Y}_l^m, R_{\theta,\phi}(\mathbf{Y}_l^0)\rangle}\cdot\langle \mathbf{Y}_l^m, R_{\theta,\phi}\left(\mathbf{Y}_l^0\right)\rangle \sin(\phi)\ d\phi\, d\theta$$

Characters

$$\langle \mathbf{Y}_l^m, \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right)\rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Plugging in the equation for zonal convolution:

$$\frac{1}{2l+1} = \frac{2\pi}{8\pi^2}\int_0^{2\pi}\int_0^{\pi} \overline{\langle \mathbf{Y}_l^m, R_{\theta,\phi}(\mathbf{Y}_l^0)\ \rangle} \cdot \langle \mathbf{Y}_l^m, R_{\theta,\phi}\left(\mathbf{Y}_l^0\right)\rangle \sin(\phi)\ d\phi\, d\theta$$

$$\Downarrow$$

$$\frac{1}{2l+1} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \overline{\lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)} \cdot \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi) \sin(\phi)\ d\phi\, d\theta$$

$$\Downarrow$$

$$\frac{4\pi}{2l+1} = \|\lambda_l\|^2$$

$$\Downarrow$$

$$\lambda_l = \zeta \cdot \sqrt{\frac{4\pi}{2l+1}} \quad \text{with } \zeta \in \mathbb{C} \text{ and } \|\zeta\| = 1$$

Characters

$$\lambda_l = \zeta \cdot \sqrt{\frac{4\pi}{2l+1}} \quad \text{with } \zeta \in \mathbb{C} \text{ and } \|\zeta\| = 1$$

Taking $m = 0$ and $\theta, \phi = 0$, we get:

$$\langle \mathbf{Y}_l^m, \rho_{R(\theta,\phi)}(\mathbf{Y}_l^0) \rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

$$\Downarrow$$

$$1 = \lambda_l \cdot \mathbf{Y}_l^0(0,0)$$

Since the convention is for the zonal harmonics to be real and positive at the north pole:

$$\lambda_l = \sqrt{\frac{4\pi}{2l+1}}$$

Wigner-D Functions

$$\mathbf{M}_{mn}^{\rho_l}(R) = \langle R(\mathbf{Y}_l^m), \mathbf{Y}_l^n \rangle = \mathbf{D}_l^{mn}(R)$$

For $l \neq l'$:

We know that $V_l = \{\mathbf{Y}_l^{-l}, \cdots, \mathbf{Y}_l^l\}$ and $V_{l'} = \left\{\mathbf{Y}_{l'}^{-l'}, \cdots, \mathbf{Y}_{l'}^{l'}\right\}$ are not isomorphic (e.g. they have different dimensions).

$$\Downarrow$$

$$\left\langle \mathbf{M}_{mn}^{\rho_l}, \mathbf{M}_{m'n'}^{\rho_{l'}} \right\rangle = 0$$

$$\Downarrow$$

$$\left\langle \mathbf{D}_l^{mn}, \mathbf{D}_{l'}^{m'n'} \right\rangle = 0$$

Wigner-D Functions

$$\mathbf{M}_{mn}^{\rho_l}(R) = \langle R(\mathbf{Y}_l^m), \mathbf{Y}_l^n \rangle = \mathbf{D}_l^{mn}(R)$$

For $l \neq l'$:

$$\left\langle \mathbf{D}_l^{mn}, \mathbf{D}_{l'}^{m'n'} \right\rangle = 0$$

For $l = l'$:

$$\left\langle \mathbf{M}_{mn}^{\rho_l}, \mathbf{M}_{m'n'}^{\rho_l} \right\rangle = \frac{\delta_{mm'} \cdot \delta_{nn'}}{2l+1}$$

$$\Downarrow$$

$$\left\langle \mathbf{D}_l^{mn}, \mathbf{D}_l^{m'n'} \right\rangle = \frac{\delta_{mm'} \cdot \delta_{nn'}}{2l+1}$$

Wigner-D Functions

$$\mathbf{M}_{mn}^{\rho_l}(R) = \langle R(\mathbf{Y}_l^m), \mathbf{Y}_l^n \rangle = \mathbf{D}_l^{mn}(R)$$

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For $l = l'$:

$$\left\langle \mathbf{D}_l^{mn}, \mathbf{D}_l^{m'n'} \right\rangle = \frac{\delta_{mm'} \cdot \delta_{nn'}}{2l+1}$$

Putting this together, we get:

$$\left\langle \mathbf{D}_l^{mn}, \mathbf{D}_{l'}^{m'n'} \right\rangle = \frac{\delta_{ll'} \cdot \delta_{mm'} \cdot \delta_{nn'}}{2l+1}$$

$\Rightarrow$ The Wigner-D functions form an orthogonal basis.

Wigner-D Functions

Given a frequency l, and the spherical harmonic basis for the l-th frequency, $V_l = \left\{\mathbf{Y}_l^{-l}, \ldots, \mathbf{Y}_l^{l}\right\}$, we can express the representation (ρ_l, V_l) in matrix form as:

$$\rho_l(R) = \mathbf{D}_l(R)$$

with $\mathbf{D}_l(R) \in \mathbb{C}^{(2l+1)\times(2l+1)}$ the matrix whose entries are the Wigner-D functions.

Wigner-D Functions

Given a rotation $R \in SO(3)$ and a Wigner-D function $\mathbf{D}_l^{mm'}$ we have:

$$\left(R\left(\mathbf{D}_l^{mm'}\right)\right)(S) = \mathbf{D}_l^{mm'}(R^{-1} \cdot S)$$
$$= \left(\mathbf{D}_l(R^{-1} \cdot S)\right)_{mm'}$$
$$= \left(\mathbf{D}_l(R^{-1}) \cdot \mathbf{D}_l(S)\right)_{mm'}$$
$$= \sum_{k=-l}^{l} \mathbf{D}_l^{mk}(R^{-1}) \cdot \mathbf{D}_l^{km'}(S)$$

Or in other words:

$$R(\mathbf{D}_l^{mm'}) = \sum_{k=-l}^{l} \mathbf{D}_l^{mk}(R^{-1}) \cdot \mathbf{D}_l^{km'}$$

Wigner-D Functions

$$R(\mathbf{D}_l^{mm'}) = \sum_{k=-l}^{l} \mathbf{D}_l^{mk}(R^{-1}) \cdot \mathbf{D}_l^{km'}$$

⇒ The rotation of a Wigner-D function of frequency l is a linear combination of Wigner-D functions of frequency l.

⇒ The Wigner-D functions of frequency l form a representation of $SO(3)$.