FFTs in Graphics and Vision

Rotational and Reflective
Symmetry Detection

Outline

Representation Theory

Symmetry Detection

- Rotational Symmetry
- Reflective Symmetry

Representation Theory

Recall:

A group is a set of elements G with a binary operation (often denoted "·") such that for all $f, g, h \in G$, the following are satisfied:

- Closure:
$$g \cdot h \in G$$
- Associativity:
$$f \cdot (g \cdot h) = (f \cdot g) \cdot h$$
- Identity: There exists an identity element $1 \in G$ s.t.:
$$1 \cdot g = g \cdot 1 = g$$
- Inverse: Every element g has an inverse g^{-1} s.t.:
$$g \cdot g^{-1} = g^{-1} \cdot g = 1$$

Representation Theory

Observation 1:

Given a group $G = \{g_1, \cdots, g_n\}$, for any $g \in G$, the (set-theoretic) map that multiplies the elements of G on the left by g is invertible.

(The inverse is the map multiplying the elements of G on the left by g^{-1}.)

Representation Theory

Observation 1:

In particular, the set $\{g \cdot g_1, \cdots, g \cdot g_n\}$ is a re-ordering of the set $\{g_1, \cdots, g_n\}$.

Or more simply, $g \cdot G = G$.

Similarly, the set $\{g_1^{-1}, \cdots, g_n^{-1}\}$ is a re-ordering of the set $\{g_1, \cdots, g_n\}$.

Or more simply, $G^{-1} = G$.

Representation Theory

Recall:

A Hermitian inner product is a map from $V \times V$ into the complex numbers that is:

1. Linear: For all $u, v, w \in V$ and any scalar $\lambda \in \mathbb{C}$

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$
$$\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$$

2. Conjugate Symmetric: For all $v, w \in V$

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

3. Positive Definite: For all $v \in V$

$$\langle v, v \rangle \geq 0$$
$$\langle v, v \rangle = 0 \Leftrightarrow v = 0$$

Representation Theory

Observation 2:

Given a Hermitian inner-product space V, and vectors $\{v_1, \cdots, v_n\} \subset V$, the vector minimizing the sum of squared distances is the average:

$$\frac{1}{n}\sum_{k=1}^{n} v_k = \underset{v \in V}{\arg\min}\left(\sum_{k=1}^{n} \|v - v_k\|^2\right)$$

Representation Theory

Recall:

A unitary representation of a group G on a Hermitian inner-product space V is a map ρ that sends every element in G to an orthogonal transformation on V, satisfying:

$$\rho_{g \cdot h} = \rho_g \cdot \rho_h$$

for all $g, h \in G$.

Representation Theory

Definition:

A vector $v \in V$ is invariant under the action of G if:

$$\rho_g(v) = v$$

for all $g \in G$.

We denote by V_G the set of vectors in V that are invariant under the action of G:

$$V_G = \{ v \in V \mid \rho_g(v) = v, \forall g \in G \}$$

Representation Theory

Observation 3:

The set V_G is a vector sub-space of V.

If $v, w \in V_G$, then for any $g \in G$, we have:

$$\rho_g(v) = v \quad \text{and} \quad \rho_g(w) = w$$

And for all scalars α and β we have:

$$\begin{aligned}\rho_g(\alpha \cdot v + \beta \cdot w) &= \alpha \cdot \rho_g(v) + \beta \cdot \rho_g(w) \\ &= \alpha \cdot v + \beta \cdot w\end{aligned}$$

So $\alpha \cdot v + \beta \cdot w \in V_G$ as well.

Representation Theory

Observation 4:

Given a finite group G and given a vector $v \in V$, the average of v over G:

$$\text{Average}(v, G) = \frac{1}{|G|} \sum_{g \in G} \rho_g(v)$$

is invariant under the action of G.

Representation Theory

Observation 4:

Let h be any element in G.

We show that h maps the average back to itself:

$$\text{Average}(v, G) = \rho_h\big(\text{Average}(v, G)\big)$$

Representation Theory

Observation 4:

$$
\begin{aligned}
\text{Average}(v,G) &= \rho_h\big(\text{Average}(v,G)\big)\\
&= \rho_h\left(\frac{1}{|G|}\sum_{g\in G}\rho_g(v)\right)\\
&= \frac{1}{|G|}\sum_{g\in G}\rho_h\cdot\rho_g(v)\\
&= \frac{1}{|G|}\sum_{g\in G}\rho_{h\cdot g}(v)\\
&= \frac{1}{|G|}\sum_{g\in h\cdot G}\rho_g(v)\\
&= \frac{1}{|G|}\sum_{g\in G}\rho_g(v)\\
&= \text{Average}(v,G)
\end{aligned}
$$

Representation Theory

Observation 5:

Given a finite group G and given a vector $v \in V$, the average of v over G is the closest G-invariant vector to v:

$$\text{Average}(v, G) = \underset{v_0 \in V_G}{\arg\min}(\|v_0 - v\|^2)$$

Representation Theory

Observation 5:

$$\|v_0 - v\|^2 = \frac{1}{|G|}\sum_{g\in G}\left\|\rho_g(v_0) - v\right\|^2$$

$$= \frac{1}{|G|}\sum_{g\in G}\left\|v_0 - \rho_g^{-1}(v)\right\|^2$$

$$= \frac{1}{|G|}\sum_{g\in G}\left\|v_0 - \rho_{g^{-1}}(v)\right\|^2$$

$$= \frac{1}{|G|}\sum_{g\in G^{-1}}\left\|v_0 - \rho_g(v)\right\|^2$$

$$= \frac{1}{|G|}\sum_{g\in G}\left\|v_0 - \rho_g(v)\right\|^2$$

Representation Theory

Observation 5:

$$\|v_0 - v\|^2 = \frac{1}{|G|}\sum_{g\in G}\left\|v_0 - \rho_g\,(v)\right\|^2$$

Thus, v_0 is the G-invariant vector minimizing the squared distance to v if and only if it minimizes the sum of squared distances to the vectors:

$$\{\rho_{g_1}(v), \cdots, \rho_{g_n}(v)\}$$

So v_0 must be the average of these vectors:

$$v_0 = \frac{1}{|G|}\sum_{g\in G}\rho_g(v) = \text{Average}(v, G)$$

Representation Theory

Note:

Since the average map:

$$\text{Average}(v, G) = \frac{1}{|G|} = \sum_{g \in G} \rho_g(v)$$

is a linear map returning the closest G-invariant vector to v, it is the projection map from V to V_G.

Outline

Representation Theory

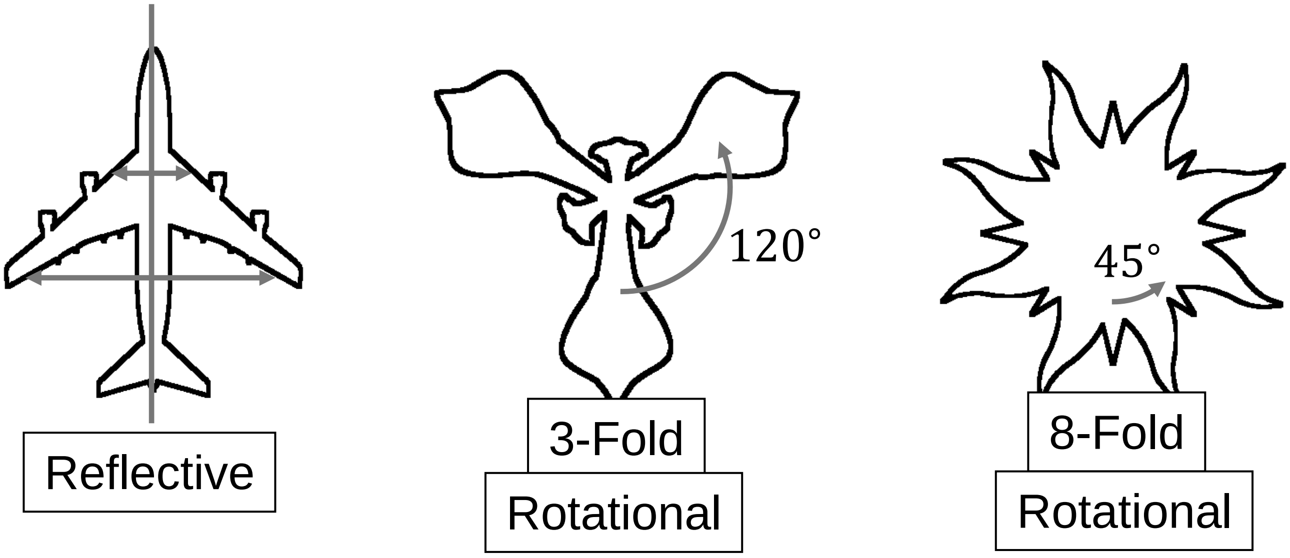
Symmetry Detection

- Rotational Symmetry
- Reflective Symmetry

Symmetry Detection

For functions on a circle, we defined measures of:

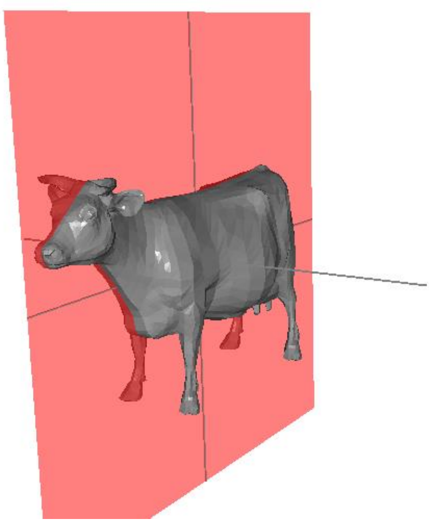
- Reflective Symmetry: for every axis of reflective symmetry.
- Rotational Symmetry: for every order of rotational symmetry.

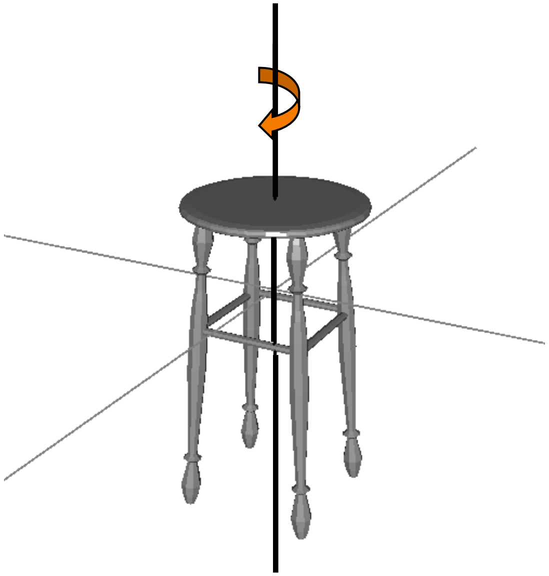


Symmetry Detection

For functions on a sphere, we would like to define a measure of:

- Reflective Symmetry: for *every* plane of reflective symmetry.
- Rotational Symmetry: for *every* axis through the origin and *every* order of rotational symmetry.





Symmetry Detection

Goal:

Reflective Symmetry:

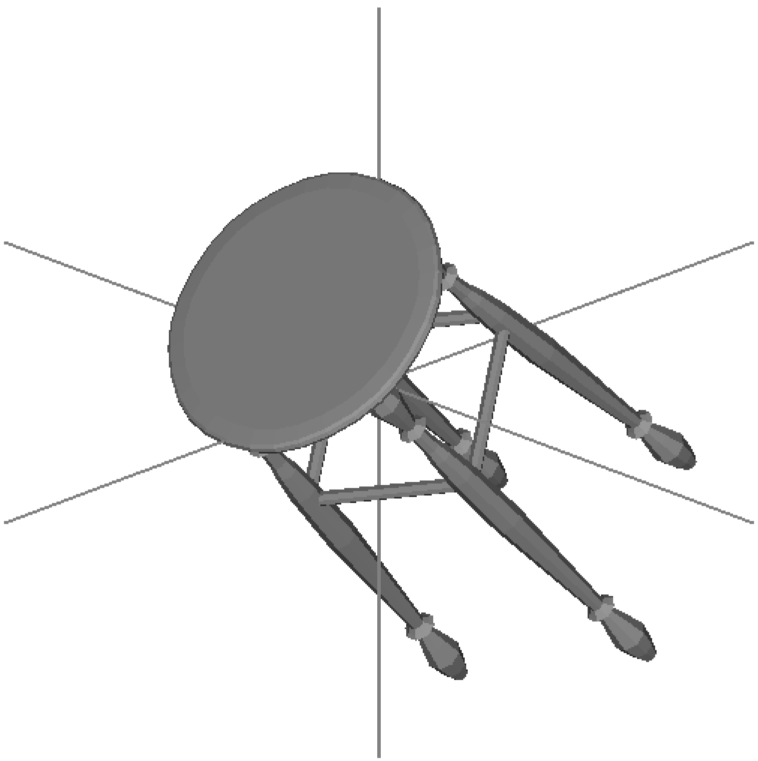
- Compute the spherical function giving the measure of reflective symmetry of every plane passing through the origin.

Rotational Symmetry:

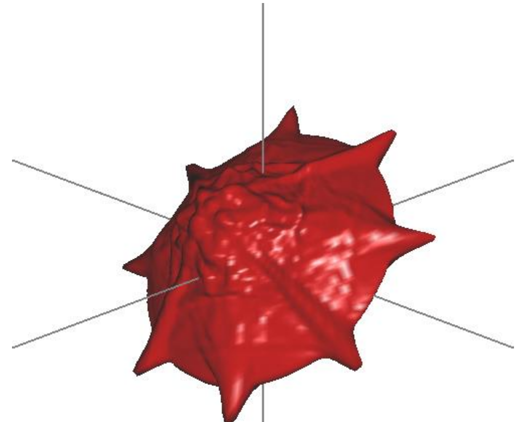
- For every order of rotational symmetry k:
  - Compute the spherical function giving the measure of k-fold symmetry about every axis through the origin.

Symmetry Detection

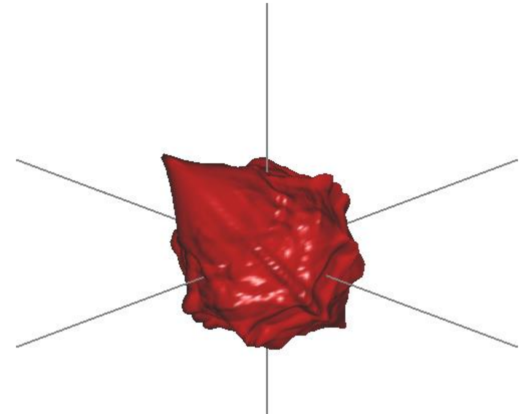
Goal:



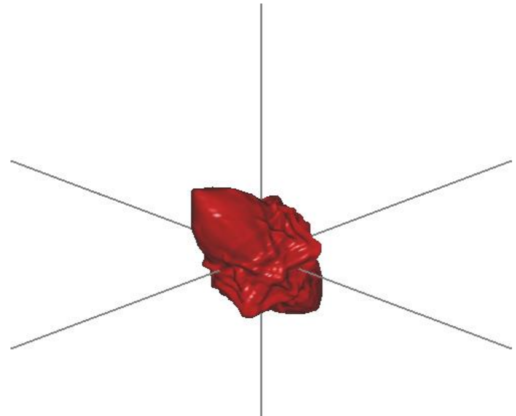
Model



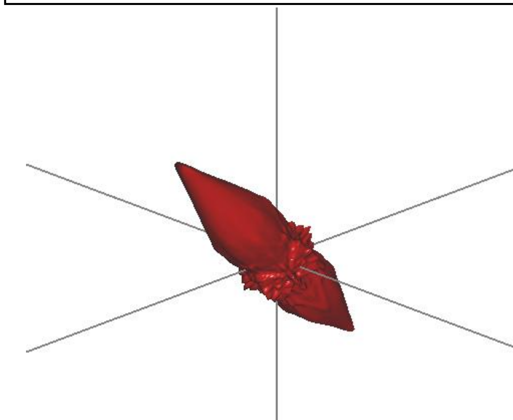
Reflective Symmetries



2-Fold
Rotational Symmetries



3-Fold
Rotational Symmetries

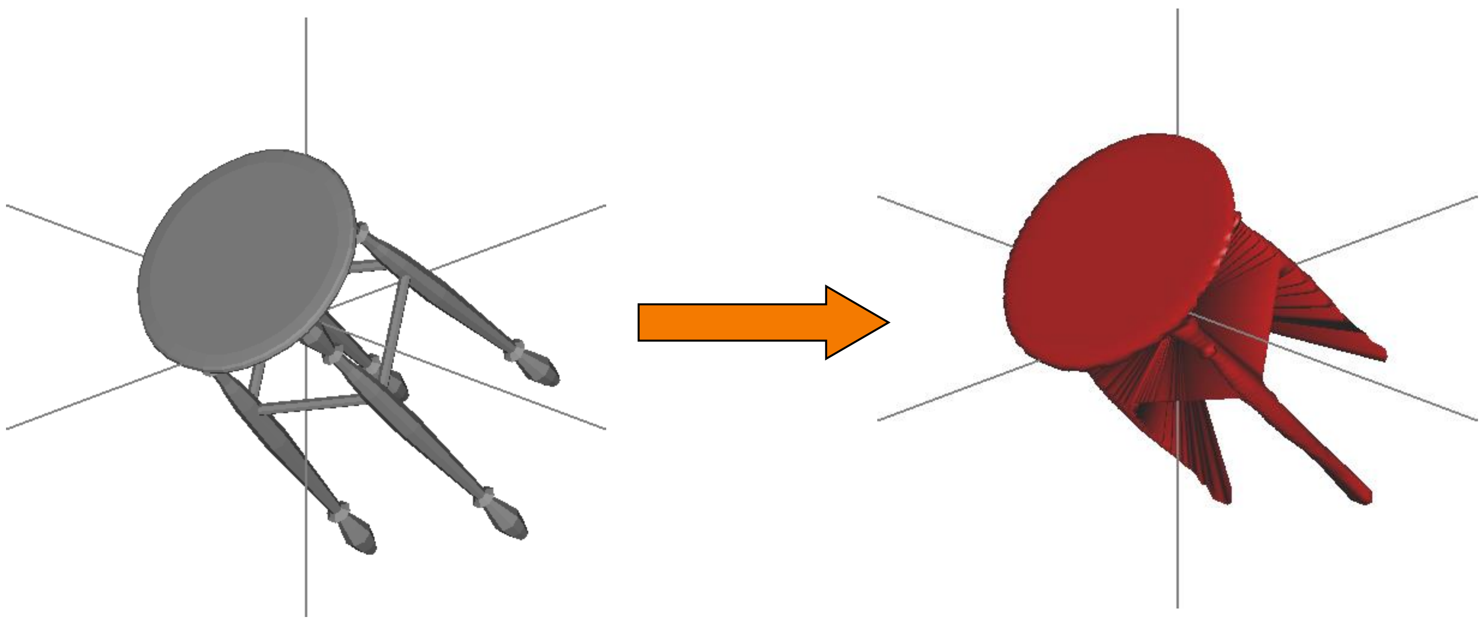


4-Fold
Rotational Symmetries

Symmetry Detection

Approach:

As in the 1D case, we will compute the symmetries of a shape by representing the shape by a spherical function.



Model

Spherical Extent Function

Symmetry Detection

Recall:

To measure a function's symmetry we:

- Associated a group G of transformations to each type of symmetry
- Defined the measure of symmetry as the size of the closest G-invariant function:

$$\text{Sym}^2(f,G) = \|\pi_G(f)\|^2$$

Since the nearest symmetric function is the average under the action of the group, we got:

$$\text{Sym}^2(f,G) = \left\| \frac{1}{|G|} \sum_{g \in G} \rho_g(f) \right\|^2$$

Outline

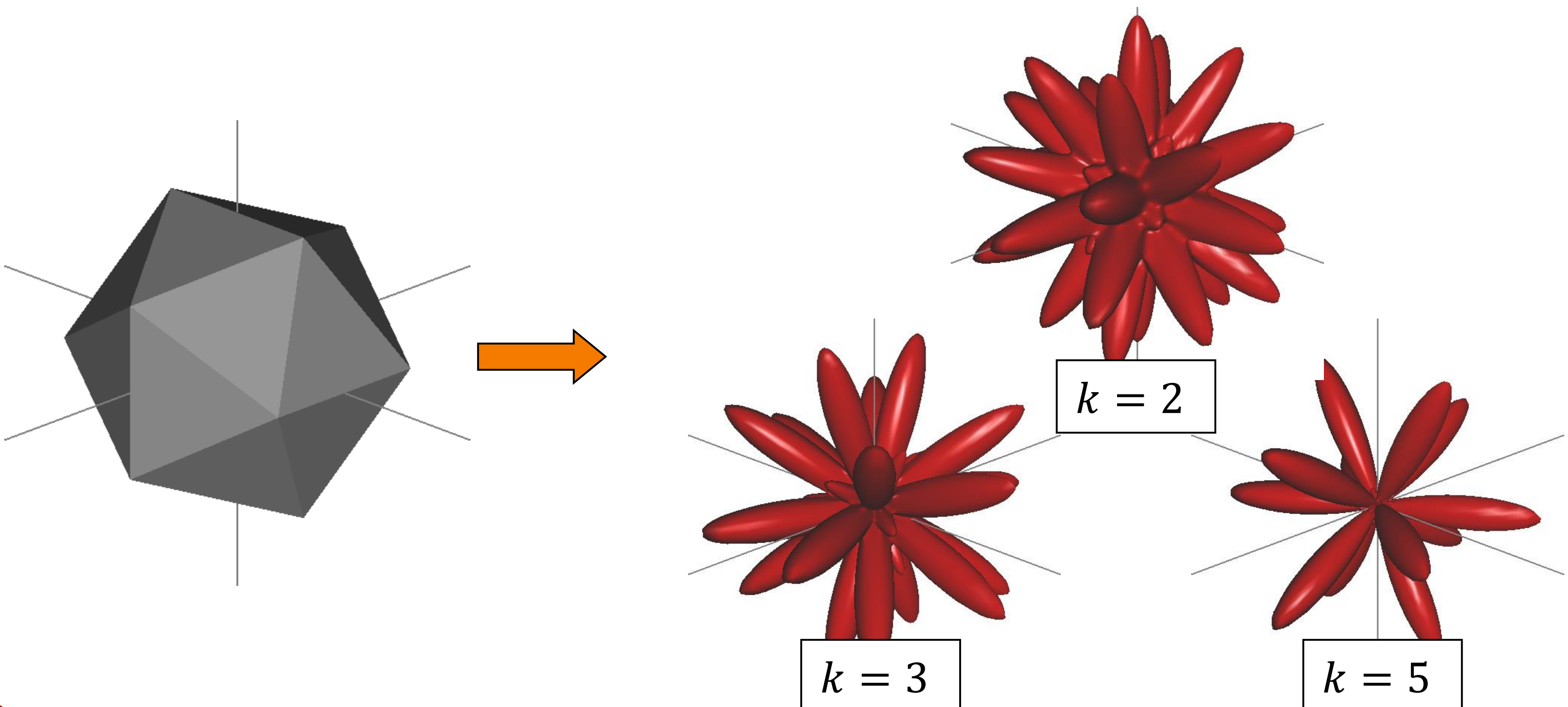
Representation Theory

Symmetry Detection

- Rotational Symmetry
- Reflective Symmetry

Rotational Symmetry

Given a function on the sphere, and given a fixed order of rotational symmetry k, define a function whose value at a point is the measure of k-fold rotational symmetry about the associated axis.



Rotational Symmetry

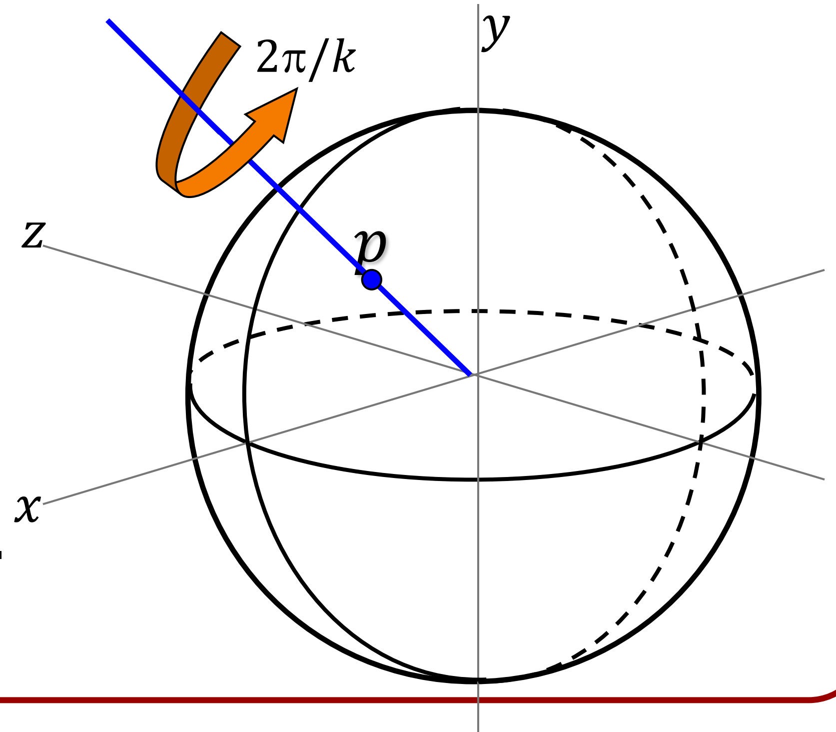
To do this, we need to associate a group to every axis passing through the origin.

We denote by $G_{p,k}$ the group of k-fold rotations about the axis through p.

The elements of the group are the rotations:

$$g_j = R\left(p, \frac{2j\pi}{k}\right)$$

corresponding to rotations about p by the angle $2j\pi/k$.



Rotational Symmetry

$$
\begin{aligned}
\mathrm{Sym}^2(f, G_{p,k}) &= \left\| \frac{1}{k} \sum_{j=0}^{k-1} \rho_{g_j}(f) \right\|^2 \\
&= \frac{1}{k^2} \left\langle \sum_{i=0}^{k-1} \rho_{g_i}(f), \sum_{j=0}^{k-1} \rho_{g_j}(f) \right\rangle \\
&= \frac{1}{k^2} \sum_{i,j=0}^{k-1} \left\langle \rho_{g_i}(f), \rho_{g_j}(f) \right\rangle \\
&= \frac{1}{k^2} \sum_{i,j=0}^{k-1} \left\langle \rho_{g_{i-j}}(f), f \right\rangle \\
&= \frac{1}{k} \sum_{j=0}^{k-1} \left\langle \rho_{g_j}(f), f \right\rangle
\end{aligned}
$$

Rotational Symmetry

$$\mathrm{Sym}^2(f, G_{p,k}) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle \rho_{g_j}(f), f \right\rangle$$

The measure of k-fold rotational symmetry about the axis p can be computed by taking the average of the dot-products of the function f with its k rotations about the axis p.

Rotational Symmetry

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle \rho_{g_j}(f), f \right\rangle$$

Computing the measures of rotational symmetry reduces to computing the correlation of f with itself:

$$D_{f,f}(R) = \langle f, \rho_R(f) \rangle$$

This is something that we can do using the Wigner D-transform from last lecture.

Rotational Symmetry

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle \rho_{g_j}(f), f \right\rangle$$

<u>Algorithm</u>:

Given a function f:

- Compute the correlation of f with itself (a.k.a. auto-correlation).
- For each order of symmetry k:
  - Compute the spherical function whose value at p is the average of the correlation values at rotations $R\left(p, \frac{2\pi j}{k}\right)$, with $0 \leq j < k$.

Rotational Symmetry

$$\mathrm{Sym}^2\left(f, G_{p,k}\right) = \frac{1}{k}\sum_{j=0}^{k-1}\left\langle \rho_{g_j}(f), f \right\rangle$$

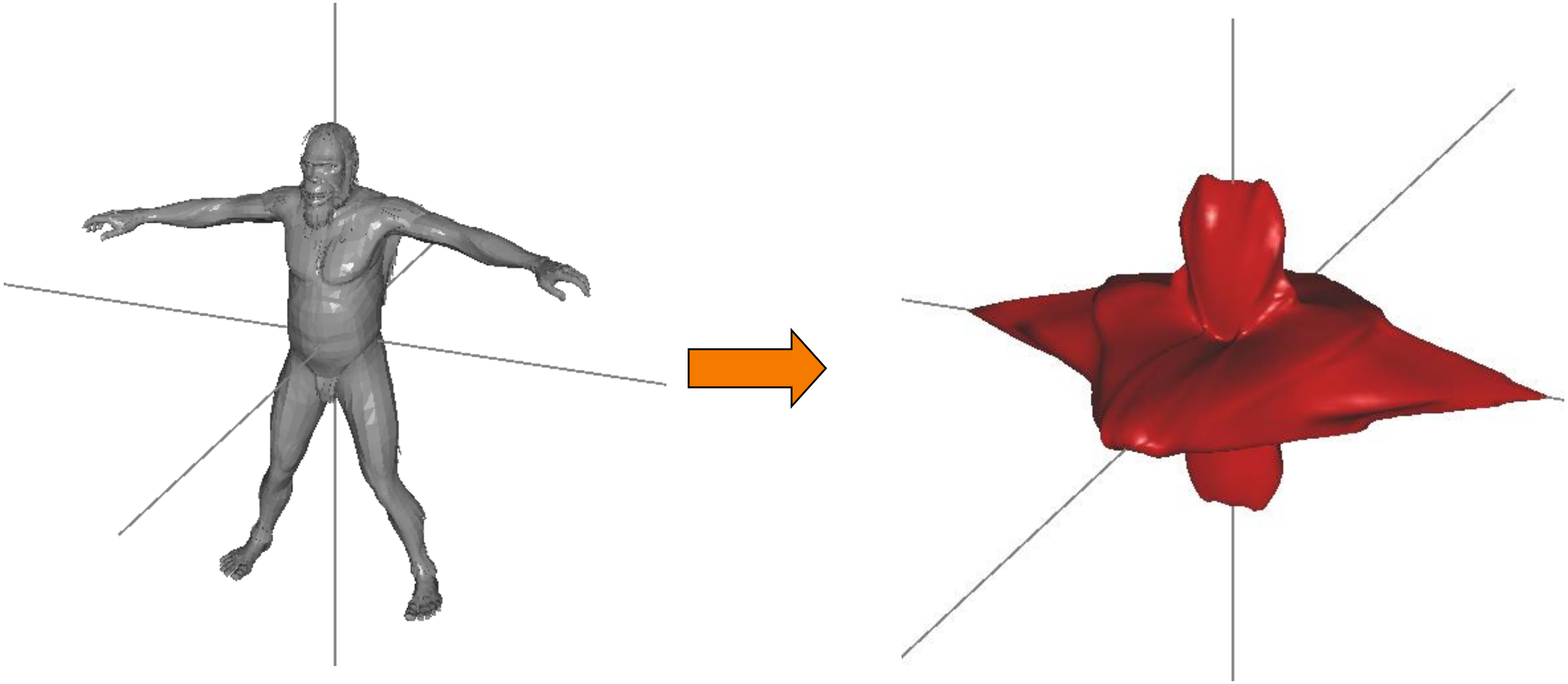
<u>Complexity</u>:

- Compute the auto-correlation: $O(n^3 \log^2 n)$
- For each order of symmetry k:
    - Compute the spherical function: $O(n^2 k)$

Giving a complexity of $O(n^2 K^2 + n^3 \log^2 n)$ to compute rotational symmetries through order K.

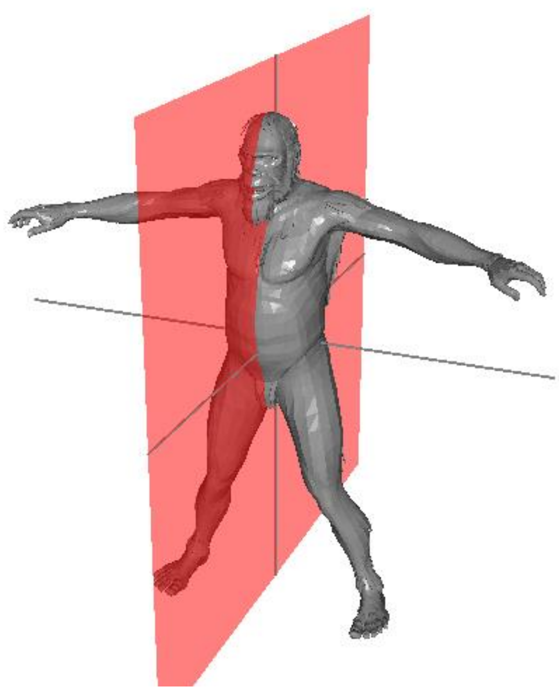
Outline

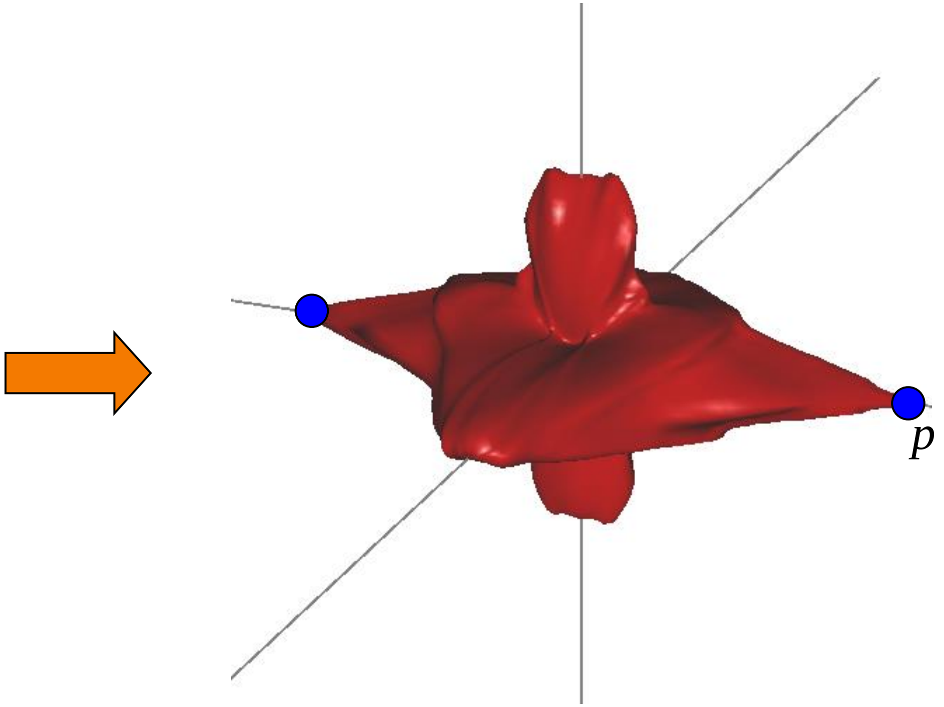
- Representation Theory
- Symmetry Detection
  - Rotational Symmetry
  - Reflective Symmetry



Reflective Symmetry

Given a spherical function f, we would like to compute a function whose value at a point p is the measure of reflective symmetry with respect to the plane perpendicular to p.

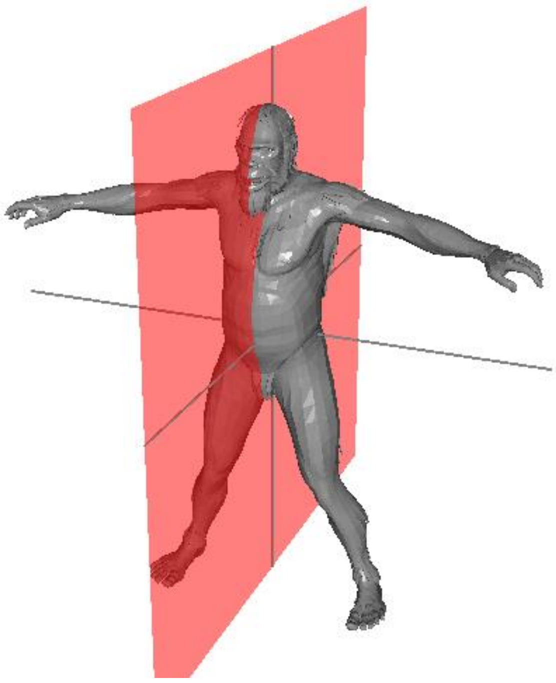


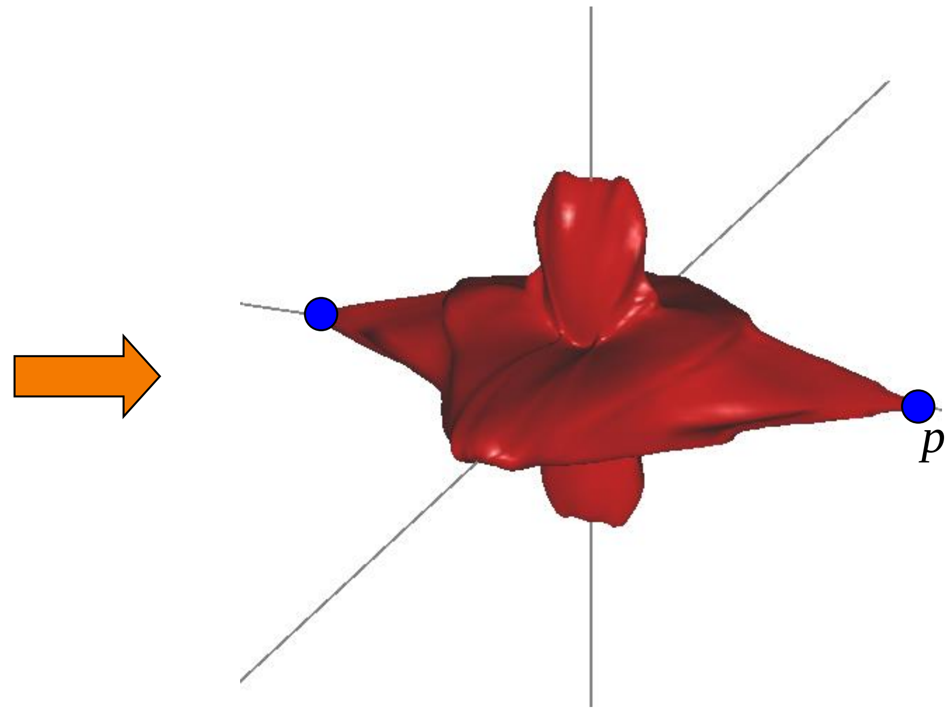


Reflective Symmetry

Reflections through the plane perpendicular to p correspond to a group with two elements:

$$G_p = \{\mathrm{Id}, \mathrm{Ref}_p\}$$





Reflective Symmetry

Reflections through the plane perpendicular to p correspond to a group with two elements:

$$G_p = \{\mathrm{Id}, \mathrm{Ref}_p\}$$

So the measure of reflective symmetry becomes:

$$\begin{aligned}\mathrm{Sym}^2(f, G_p) &= \frac{1}{2}\left(\langle f, f\rangle + \langle \rho_{\mathrm{Ref}_p}(f), f\rangle\right)\\ &= \frac{1}{2}\left(\|f\|^2 + \langle \rho_{\mathrm{Ref}_p}(f), f\rangle\right)\end{aligned}$$

Reflective Symmetry

How do we compute the dot-product of the function f with the reflection of f through the plane perpendicular to p?

Since reflections are not rotations we cannot use the auto-correlation (directly).

Reflective Symmetry

General Approach:

If we have two orthogonal transformations S and T, both with determinant -1, we can set R to be the transformation:

$$R = T \cdot S$$

Since S and T are both orthogonal, the product R must also be orthogonal.

Since both S and T have determinant -1, R must have determinant 1.

Reflective Symmetry

General Approach:

If we have two orthogonal transformations S and T, both with determinant -1, we can set R to be the transformation:

$$R = T \cdot S$$

Thus, R must be a rotation and we have:

$$T = R \cdot S^{-1}$$

$\Rightarrow$ Any orthogonal transformation T with det. -1 can be expressed as the product of some rotation R with (the inverse of) a fixed orthogonal transformation S with det. -1.

Reflective Symmetry

General Approach:

Compute the correlation of f with an orthogonal transformation with determinant -1, $\rho_S(f)$:

$$D_{\rho_S(f),f}(R) = \langle \rho_R\big(\rho_S(f)\big), f \rangle$$

Then we can get the dot-product of f with its reflection through the plane perpendicular to p:

$$\begin{aligned} \langle \rho_{\mathrm{Ref}_p}(f), f \rangle &= \langle \rho_{\mathrm{Ref}_p \cdot S^{-1}}\big(\rho_S(f)\big), f \rangle \\ &= D_{\rho_S(f),f}\big(\underbrace{\mathrm{Ref}_p \cdot S^{-1}}_{\text{Rotation}}\big) \end{aligned}$$

Reflective Symmetry

$$\mathrm{Sym}^2(f, G_p) = \frac{1}{2}\left(\|f\|^2 + D_{\rho_S(f),f}\left(\mathrm{Ref}_p \cdot S^{-1}\right)\right)$$

Algorithm:

Given a function f:

- Compute the correlation of f with $\rho_S(f)$
- Compute the spherical function whose value at p is the average of the size of f and the dot-product of f with the rotation of $\rho_S(f)$ by $\mathrm{Ref}_p \cdot S^{-1}$.

Reflective Symmetry

$$\mathrm{Sym}^2(f, G_p) = \frac{1}{2}\left(\|f\|^2 + D_{\rho_S(f),f}\left(\mathrm{Ref}_p \cdot S^{-1}\right)\right)$$

Complexity:

- Compute the correlation: $O(n^3 \log^2 n)$
- Compute the spherical function: $O(n^2)$

Giving a complexity of $O(n^3 \log^2 n)$ to compute all reflective symmetries.

Reflective Symmetry

There are many different choices for the reflection S we use to compute:

$$D_{\rho_S(f),f}(R)$$

A simple orthogonal transformation with determinant -1 is the antipodal map:

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

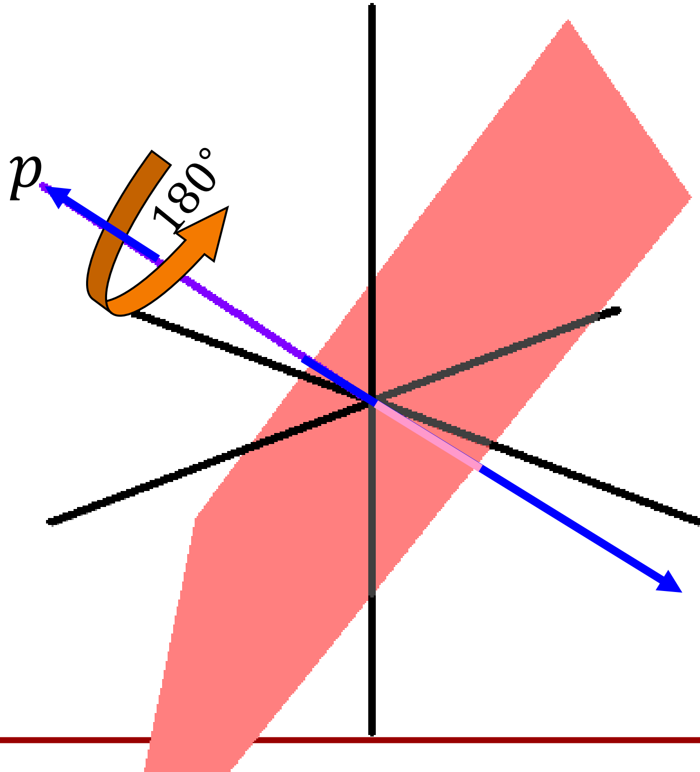
Note that $S = S^{-1}$.

Reflective Symmetry

The advantage of using the antipodal map is that it makes it easy to express $\text{Ref}_p \cdot S$.

Fixing a point p, the antipodal map S is the composition of two maps:

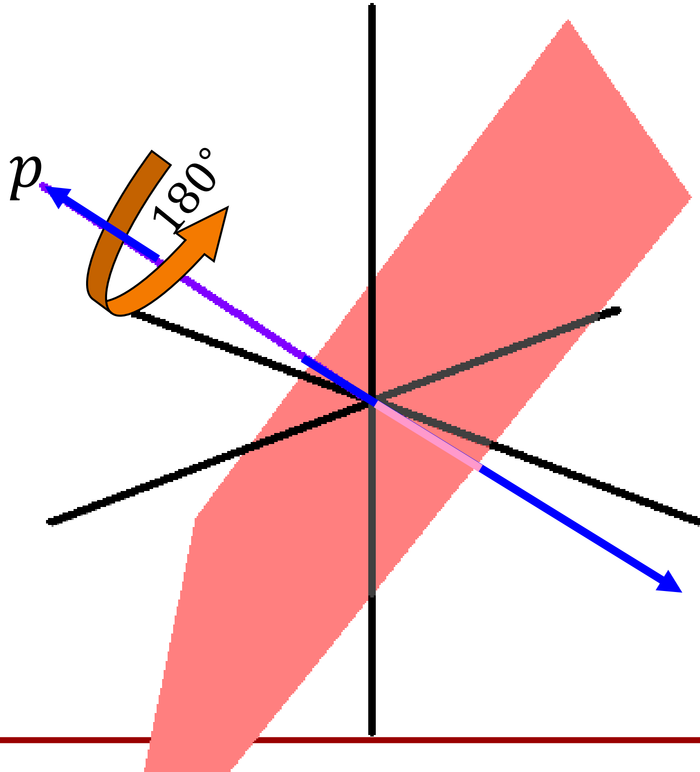
- A reflection through the plane perpendicular to p, and
- A rotation by $180°$ about the axis through p.



Reflective Symmetry

So a reflection through the plane perpendicular to p is the product of the antipodal map and a rotation by $180°$ around the axis through p:

$$\text{Ref}_p = R(p,\pi) \cdot S$$



Reflective Symmetry

Setting S to be the antipodal map, we get:

$$\mathrm{Sym}^2(f, G_p) = \frac{1}{2}\left(\|f\|^2 + \left\langle \rho_{R(p,\pi)}\big(\rho_S(f)\big), f \right\rangle\right)$$

Note that evaluating reflective symmetry only requires knowing the correlation values for 180° rotations.

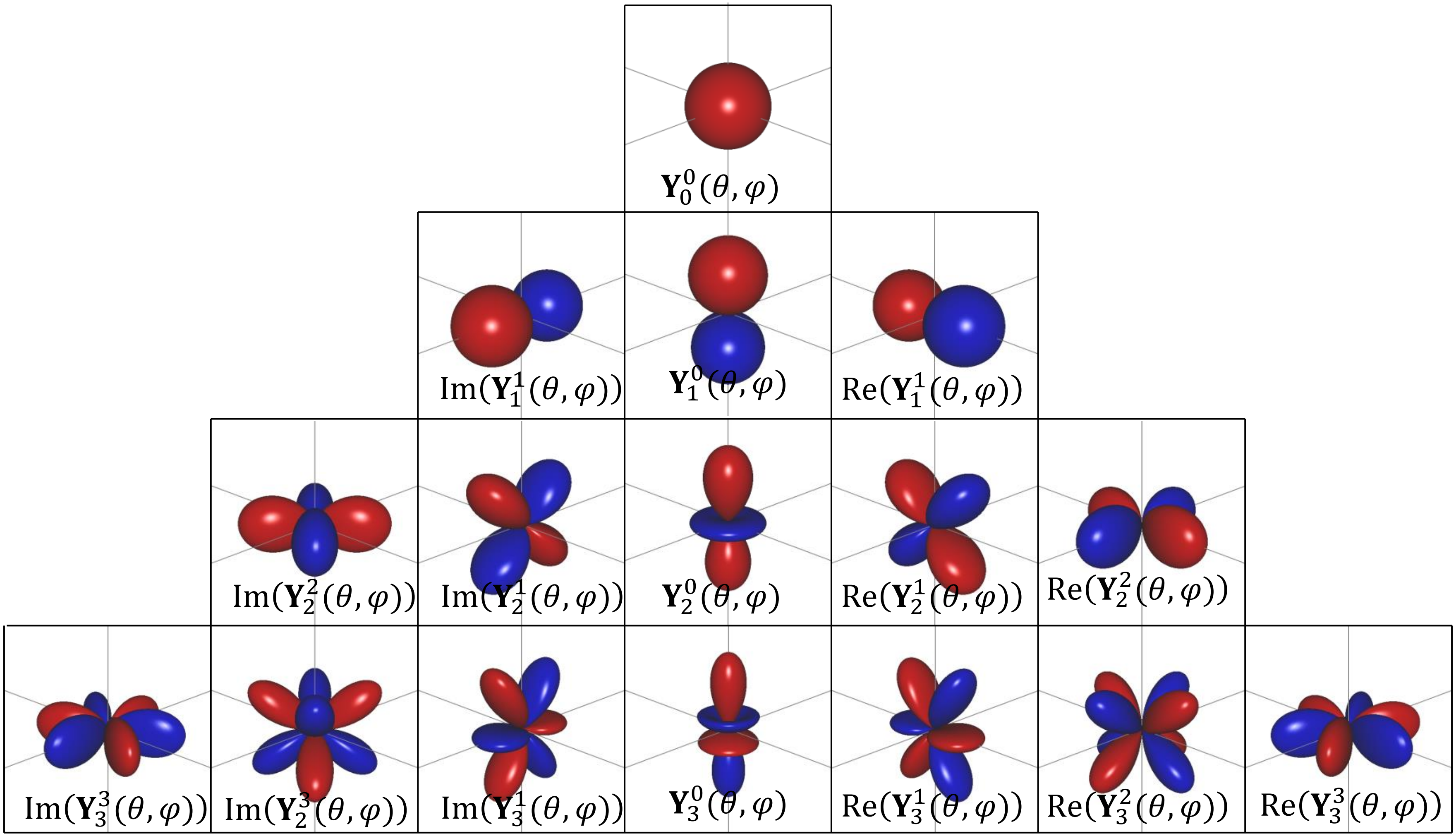
For computing reflective symmetries, the computation of the correlation is overkill as we don't use most of the correlation values.

Reflective Symmetry

Since the spherical harmonics of degree l are homogenous polynomials of degree l, we get a simple expression for $\rho_S(f)$:

$$\rho_S(f) = \sum_l (-1)^l \sum_{m=-l}^{l} \hat{f}_{lm} \cdot \mathbf{Y}_l^m$$

Reflective Symmetry



Reflective Symmetry

In particular, if f is antipodally symmetric:

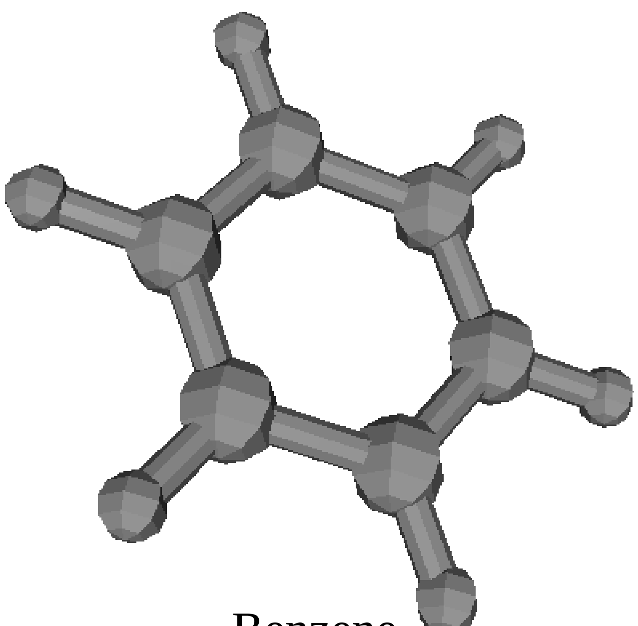
$$\rho_S(f) = f$$

we have:

$$\begin{aligned}\mathrm{Sym}^2(f, G_p) &= \frac{1}{2}\left(\|f\|^2 + \left\langle \rho_{R(p,\pi)}\left(\rho_S(f)\right), f \right\rangle\right) \\ &= \frac{1}{2}\left(\|f\|^2 + \left\langle \rho_{R(p,\pi)}(f), f\right\rangle\right) \\ &= \mathrm{Sym}^2\left(f, G_{p,2}\right)\end{aligned}$$

Reflective Symmetry

That is, if f is antipodally symmetric, the 2-fold rotational symmetries of f and the reflective symmetries of f are the same.



Benzene