FFTs in Graphics and Vision

Rotational and Reflective Symmetry Detection
Outline

Representation Theory

Symmetry Detection
  • Rotational Symmetry
  • Reflective Symmetry
Representation Theory

Recall:

A group is a set of elements $G$ with a binary operation (often denoted “$\cdot$”) such that for all $f, g, h \in G$, the following are satisfied:

- **Closure:**
  \[ g \cdot h \in G \]

- **Associativity:**
  \[ f \cdot (g \cdot h) = (f \cdot g) \cdot h \]

- **Identity:** There exists an identity element $1 \in G$ s.t.:
  \[ 1 \cdot g = g \cdot 1 = g \]

- **Inverse:** Every element $g$ has an inverse $g^{-1}$ s.t.:
  \[ g \cdot g^{-1} = g^{-1} \cdot g = 1 \]
Representation Theory

Observation 1:

Given a group \( G = \{g_1, \ldots, g_n\} \), for any \( g \in G \), the (set-theoretic) map that multiplies the elements of \( G \) on the left by \( g \) is invertible.

(The inverse is the map multiplying the elements of \( G \) on the left by \( g^{-1} \).)
Observation 1:

In particular, the set \( \{ g \cdot g_1, \ldots, g \cdot g_n \} \) is a re-ordering of the set \( \{ g_1, \ldots, g_n \} \).

Or more simply, \( g \cdot G = G \).

Similarly, the set \( \{ g_1^{-1}, \ldots, g_n^{-1} \} \) is a re-ordering of the set \( \{ g_1, \ldots, g_n \} \).

Or more simply, \( G^{-1} = G \).
Representation Theory

Recall:

A Hermitian inner product is a map from $V \times V$ into the complex numbers that is:

1. **Linear**: For all $u, v, w \in V$ and any scalar $\lambda \in \mathbb{C}$
   \[
   \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \\
   \langle \lambda v, w \rangle = \lambda \langle v, w \rangle
   \]

2. **Conjugate Symmetric**: For all $v, w \in V$
   \[
   \langle v, w \rangle = \overline{\langle w, v \rangle}
   \]

3. **Positive Definite**: For all $v \in V$
   \[
   \langle v, v \rangle \geq 0 \\
   \langle v, v \rangle = 0 \iff v = 0
   \]
Observation 2:

Given a Hermitian inner-product space $V$, and vectors $\{v_1, \ldots, v_n\} \subset V$, the vector minimizing the sum of squared distances is the average:

$$\frac{1}{n} \sum_{k=1}^{n} v_k = \arg \min_{v \in V} \left( \sum_{k=1}^{n} \|v - v_k\|^2 \right)$$
Recall:

A **unitary representation** of a group $G$ on a Hermitian inner-product space $V$ is a map $\rho$ that sends every element in $G$ to an orthogonal transformation on $V$, satisfying:

$$\rho_{g \cdot h} = \rho_g \cdot \rho_h$$

for all $g, h \in G$. 
Definition:

A vector \( v \in V \) is invariant under the action of \( G \) if:

\[
\rho_g(v) = v
\]

for all \( g \in G \).

We denote by \( V_G \) the set of vectors in \( V \) that are invariant under the action of \( G \):

\[
V_G = \{ v \in V | \rho_g(v) = v, \forall g \in G \}
\]
Observation 3:
The set $V_G$ is a vector sub-space of $V$.

If $v, w \in V_G$, then for any $g \in G$, we have:

$$\rho_g(v) = v \quad \text{and} \quad \rho_g(w) = w$$

And for all scalars $\alpha$ and $\beta$ we have:

$$\rho_g(\alpha \cdot v + \beta \cdot w) = \alpha \cdot \rho_g(v) + \beta \cdot \rho_g(w)$$

$$= \alpha \cdot v + \beta \cdot w$$

So $\alpha \cdot v + \beta \cdot w \in V_G$ as well.
Observation 4:

Given a finite group $G$ and given a vector $v \in V$, the average of $v$ over $G$:

$$\text{Average}(v, G) = \frac{1}{|G|} \sum_{g \in G} \rho_g(v)$$

is invariant under the action of $G$. 
Representation Theory

**Observation 4:**

Let $h$ be any element in $G$.

We show that $h$ maps the average back to itself:

$$\text{Average}(v, G) = \rho_h(\text{Average}(v, G))$$
Observation 4:

\[ \text{Average}(v, G) = \rho_h(\text{Average}(v, G)) \]

\[ = \rho_h \left( \frac{1}{|G|} \sum_{g \in G} \rho_g(v) \right) \]

\[ = \frac{1}{|G|} \sum_{g \in G} \rho_h \cdot \rho_g(v) \]

\[ = \frac{1}{|G|} \sum_{g \in G} \rho_{h \cdot g}(v) \]

\[ = \frac{1}{|G|} \sum_{g \in h \cdot G} \rho_g(v) \]

\[ = \frac{1}{|G|} \sum_{g \in G} \rho_g(v) \]

\[ = \text{Average}(v, G) \]
Representation Theory

Observation 5:

Given a finite group $G$ and given a vector $v \in V$, the average of $v$ over $G$ is the closest $G$-invariant vector to $v$:

$$\text{Average}(v, G) = \arg \min_{v_0 \in V_G} (\|v_0 - v\|^2)$$
Observation 5:

\[ \|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|\rho_g(v_0) - v\|^2 \]

\[ = \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_g^{-1}(v)\|^2 \]

\[ = \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_g^{-1}(v)\|^2 \]

\[ = \frac{1}{|G|} \sum_{g \in G^{-1}} \|v_0 - \rho_g(v)\|^2 \]

\[ = \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_g(v)\|^2 \]
Observation 5:

\[ \|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_g(v)\|^2 \]

Thus, \(v_0\) is the \(G\)-invariant vector minimizing the squared distance to \(v\) if and only if it minimizes the sum of squared distances to the vectors:

\[ \{\rho_{g_1}(v), \ldots, \rho_{g_n}(v)\} \]

So \(v_0\) must be the average of these vectors:

\[ v_0 = \frac{1}{|G|} \sum_{g \in G} \rho_g(v) = \text{Average}(v, G) \]
Since the average map:

$$\text{Average}(v, G) = \frac{1}{|G|} \sum_{g \in G} \rho_g(v)$$

is a linear map returning the closest $G$-invariant vector to $v$, it is the projection map from $V$ to $V_G$. 

Note:
Outline

Representation Theory

Symmetry Detection

- Rotational Symmetry
- Reflective Symmetry
Symmetry Detection

For functions on a circle, we defined measures of:

- **Reflective Symmetry**: for every axis of reflective symmetry.
- **Rotational Symmetry**: for every order of rotational symmetry.

![Reflective Symmetry](image1)

![3-Fold Rotational Symmetry](image2)

![8-Fold Rotational Symmetry](image3)
Symmetry Detection

For functions on a sphere, we would like to define a measure of:

- **Reflective Symmetry**: for every plane of reflective symmetry.
- **Rotational Symmetry**: for every axis through the origin and every order of rotational symmetry.
Symmetry Detection

Goal:

Reflective Symmetry:
- Compute the spherical function giving the measure of reflective symmetry of every plane passing through the origin.

Rotational Symmetry:
- For every order of rotational symmetry $k$:
  - Compute the spherical function giving the measure of $k$-fold symmetry about every axis through the origin.
Symmetry Detection

Goal:

Model

Reflective Symmetries

2-Fold Rotational Symmetries

3-Fold Rotational Symmetries

4-Fold Rotational Symmetries
Symmetry Detection

Approach:
As in the 1D case, we will compute the symmetries of a shape by representing the shape by a spherical function.
Symmetry Detection

Recall:

To measure a function’s symmetry we:

- Associated a group $G$ of transformations to each type of symmetry
- Defined the measure of symmetry as the size of the closest $G$-invariant function:
  \[ \text{Sym}^2(f, G) = \| \pi_G(f) \|^2 \]

Since the nearest symmetric function is the average under the action of the group, we got:

\[ \text{Sym}^2(f, G) = \left\| \frac{1}{|G|} \sum_{g \in G} \rho_g(f) \right\|^2 \]
Outline

Representation Theory

Symmetry Detection
  ◦ Rotational Symmetry
  ◦ Reflective Symmetry
Rotational Symmetry

Given a function on the sphere, and given a fixed order of rotational symmetry $k$, define a function whose value at a point is the measure of $k$-fold rotational symmetry about the associated axis.
Rotational Symmetry

To do this, we need to associate a group to every axis passing through the origin.

We denote by $G_{p,k}$ the group of $k$-fold rotations about the axis through $p$.

The elements of the group are the rotations:

$$g_j = R \left( p, \frac{2j\pi}{k} \right)$$

corresponding to rotations about $p$ by the angle $2j\pi/k$. 
Rotational Symmetry

\[
\text{Sym}^2(f, G_{p,k}) = \left\| \frac{1}{k} \sum_{j=0}^{k-1} \rho g_j(f) \right\|^2 \\
= \frac{1}{k^2} \left( \sum_{i=0}^{k-1} \rho g_i(f), \sum_{j=0}^{k-1} \rho g_j(f) \right) \\
= \frac{1}{k^2} \sum_{i,j=0}^{k-1} \langle \rho g_i(f), \rho g_j(f) \rangle \\
= \frac{1}{k^2} \sum_{i,j=0}^{k-1} \langle \rho g_{i-j}(f), f \rangle \\
= \frac{1}{k} \sum_{j=0}^{k-1} \langle \rho g_j(f), f \rangle 
\]
Rotational Symmetry

\[ \text{Sym}^2(f, G_{p,k}) = \frac{1}{k} \sum_{j=0}^{k-1} \langle \rho_{g_j}(f), f \rangle \]

The measure of \( k \)-fold rotational symmetry about the axis \( p \) can be computed by taking the average of the dot-products of the function \( f \) with its \( k \) rotations about the axis \( p \).
Rotational Symmetry

\[ \text{Sym}^2(f, G_{p,k}) = \frac{1}{k} \sum_{j=0}^{k-1} \langle \rho_{g_j}(f), f \rangle \]

Computing the measures of rotational symmetry reduces to computing the correlation of \( f \) with itself:

\[ D_{f,f}(R) = \langle f, \rho_R(f) \rangle \]

This is something that we can do using the Wigner \( D \)-transform from last lecture.
Rotational Symmetry

\[
\text{Sym}^2(f, G_{p,k}) = \frac{1}{k} \sum_{j=0}^{k-1} \langle \rho_{g_{j}}(f), f \rangle
\]

Algorithm:

Given a function \( f \):

- Compute the correlation of \( f \) with itself (a.k.a. auto-correlation).
- For each order of symmetry \( k \):
  - Compute the spherical function whose value at \( p \) is the average of the correlation values at rotations \( R \left( p, \frac{2\pi j}{k} \right) \), with \( 0 \leq j < k \).
Rotational Symmetry

\[ \text{Sym}^2(f, G_{p,k}) = \frac{1}{k} \sum_{j=0}^{k-1} \langle \rho_{g_j}(f), f \rangle \]

**Complexity:**

- Compute the auto-correlation: \( O(n^3 \log^2 n) \)
- For each order of symmetry \( k \):
  - Compute the spherical function: \( O(n^2 k) \)

Giving a complexity of \( O(n^2 K^2 + n^3 \log^2 n) \) to compute rotational symmetries through order \( K \).
Outline

Representation Theory

Symmetry Detection
  - Rotational Symmetry
  - Reflective Symmetry
Reflective Symmetry

Given a spherical function $f$, we would like to compute a function whose value at a point $p$ is the measure of reflective symmetry with respect to the plane perpendicular to $p$. 
Reflective Symmetry

Reflections through the plane perpendicular to $p$ correspond to a group with two elements:

$$G_p = \{ \text{Id}, \text{Ref}_p \}$$
Reflective Symmetry

Reflections through the plane perpendicular to \( p \) correspond to a group with two elements:

\[
G_p = \{ \text{Id}, \text{Ref}_p \}
\]

So the measure of reflective symmetry becomes:

\[
\text{Sym}^2(f, G_p) = \frac{1}{2} \left( \langle f, f \rangle + \langle \rho_{\text{Ref}_p}(f), f \rangle \right)
\]

\[
= \frac{1}{2} \left( \|f\|^2 + \langle \rho_{\text{Ref}_p}(f), f \rangle \right)
\]
Reflective Symmetry

How do we compute the dot-product of the function \( f \) with the reflection of \( f \) through the plane perpendicular to \( p \)?

Since reflections are not rotations we cannot use the auto-correlation (directly).
Reflective Symmetry

General Approach:

If we have two orthogonal transformations $S$ and $T$, both with determinant $-1$, we can set $R$ to be the transformation:

$$R = T \cdot S$$

Since $S$ and $T$ are both orthogonal, the product $R$ must also be orthogonal.

Since both $S$ and $T$ have determinant $-1$, $R$ must have determinant 1.
Reflective Symmetry

General Approach:

If we have two orthogonal transformations \( S \) and \( T \), both with determinant \(-1\), we can set \( R \) to be the transformation:

\[
R = T \cdot S
\]

Thus, \( R \) must be a rotation and we have:

\[
T = R \cdot S^{-1}
\]

\[\Rightarrow \text{Any orthogonal transformation } T \text{ with det. } -1 \text{ can be expressed as the product of some rotation } R \text{ with (the inverse of) a fixed orthogonal transformation } S \text{ with det. } -1.\]
Reflective Symmetry

General Approach:

Compute the correlation of $f$ with an orthogonal transformation with determinant $-1$, $\rho_S(f)$:

$$D_{\rho_S(f), f}(R) = \langle \rho_R(\rho_S(f)), f \rangle$$

Then we can get the dot-product of $f$ with its reflection through the plane perpendicular to $p$:

$$\langle \rho_{\text{Ref}_p}(f), f \rangle = \langle \rho_{\text{Ref}_p \cdot S^{-1}}(\rho_S(f)), f \rangle$$

$$= D_{\rho_S(f), f}(\text{Ref}_p \cdot S^{-1})$$
Reflective Symmetry

\[ \text{Sym}^2(f, G_p) = \frac{1}{2} \left( \|f\|^2 + D_{\rho_S(f),f}(\text{Ref}_p \cdot S^{-1}) \right) \]

Algorithm:

Given a function \( f \):

- Compute the correlation of \( f \) with \( \rho_S(f) \)
- Compute the spherical function whose value at \( p \) is the average of the size of \( f \) and the dot-product of \( f \) with the rotation of \( \rho_S(f) \) by \( \text{Ref}_p \cdot S^{-1} \).
Reflective Symmetry

\[ \text{Sym}^2(f, G_p) = \frac{1}{2} \left( \|f\|^2 + D_{\rho_S(f),f} (\text{Ref}_p \cdot S^{-1}) \right) \]

Complexity:

- Compute the correlation: \( O(n^3 \log^2 n) \)
- Compute the spherical function: \( O(n^2) \)

Giving a complexity of \( O(n^3 \log^2 n) \) to compute all reflective symmetries.
Reflective Symmetry

There are many different choices for the reflection $S$ we use to compute:

$$D_{\rho_S(f),f}(R)$$

A simple orthogonal transformation with determinant $-1$ is the antipodal map:

$$S = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}$$

Note that $S = S^{-1}$. 
Reflective Symmetry

The advantage of using the antipodal map is that it makes it easy to express $\text{Ref}_p \cdot S$.

Fixing a point $p$, the antipodal map $S$ is the composition of two maps:

- A reflection through the plane perpendicular to $p$, and
- A rotation by $180^\circ$ about the axis through $p$. 
Reflective Symmetry

So a reflection through the plane perpendicular to \( p \) is the product of the antipodal map and a rotation by 180° around the axis through \( p \):

\[
\text{Ref}_p = R(p, \pi) \cdot S
\]
Reflective Symmetry

Setting $S$ to be the antipodal map, we get:

$$\text{Sym}^2(f, G_p) = \frac{1}{2} (\|f\|^2 + \langle \rho_{R(p,\pi)}(\rho_S(f)), f \rangle)$$

Note that evaluating reflective symmetry only requires knowing the correlation values for $180^\circ$ rotations.

For computing reflective symmetries, the computation of the correlation is overkill as we don’t use most of the correlation values.
Reflective Symmetry

Since the spherical harmonics of degree $l$ are homogenous polynomials of degree $l$, we get a simple expression for $\rho_s(f)$:

$$\rho_s(f) = \sum_l (-1)^l \sum_{m=-l}^l \hat{f}_{lm} \cdot Y_l^m$$
Reflective Symmetry
Reflective Symmetry

In particular, if $f$ is antipodally symmetric:

$$\rho_S(f) = f$$

we have:

$$\text{Sym}^2(f, G_p) = \frac{1}{2} (\|f\|^2 + \langle \rho_{R(p,\pi)}(\rho_S(f)), f \rangle)$$

$$= \frac{1}{2} \left( \|f\|^2 + \langle \rho_{R(p,\pi)}(f), f \rangle \right)$$

$$= \text{Sym}^2(f, G_{p,2})$$
Reflective Symmetry

That is, if $f$ is antipodally symmetric, the 2-fold rotational symmetries of $f$ and the reflective symmetries of $f$ are the same.