FFTs in Graphics and Vision

Spherical Convolution
and
Axial Symmetry Detection

Announcements

Assignment 3 has been posted!

Outline

- Math Review
  - Symmetry
  - General Convolution
- Spherical Convolution
- Axial Symmetry Detection

Math Review

Symmetry:

Given a unitary representation of a group G on a vector space V, we say that a vector $v \in V$ is invariant under the action of G if for all $g \in G$:

$$\rho_g(v) = v$$

The set of G-invariant vector V_G is a vector space.

Math Review

Symmetry:

The linear map π_G is a projection onto V_G, if:

- $\pi_G(v) \in V_G$ for all $v \in V$
- $\pi_G(v) = v$ for all $v \in V_G$
- $\langle v, w - \pi_G(w) \rangle = 0$ for all $v \in V_G, w \in V$.

The map π_G is the map sending a vector v to the closest G-invariant vector.

Math Review

<u>Symmetry</u>:

The measure of symmetry of a vector v with respect to the group G is the size of its projection onto the space of G-invariant vectors:

$$\mathrm{Sym}^2(v, G) = \|\pi_G(v)\|^2$$

Math Review

Convolution:

Given functions $f(p)$ and $g(p)$ on the circle/torus, the convolution of the two functions is defined as:

$$(f * g)(q) = \int f(q-p) \cdot g(p) dp$$

Math Review

<u>Convolution</u>:

If we hold the function f fixed we get a map from the space of functions back into itself:

$$C_f(g) = f * g$$

<u>Claim</u>:

The map C_f is a linear operator.

Math Review

Convolution:

If we hold the function f fixed we get a map from the space of functions back into itself:

$$C_f(g) = f * g$$

Claim:

Given functions f and h and scalars α and β:

$$C_f(\alpha g + \beta h)(q) = \int f(q-p) \cdot \big(\alpha \cdot g(p) + \beta \cdot h(p)\big) dp$$

$$= \alpha \int f(q-p) \cdot g(p) dp + \beta \int f(q-p) \cdot h(p) dp$$

$$= \alpha \cdot C_f(g) + \beta \cdot C_f(h)$$

Math Review

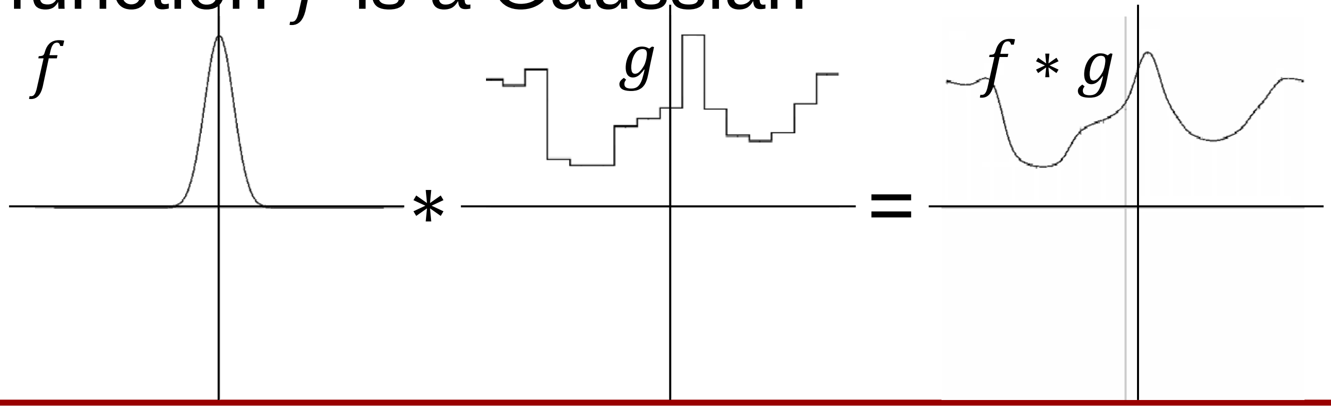
Convolution:

Assume that the function f is real-valued and radial, i.e. the value of f at p is determined by the distance of p from the origin:

$$f(p) = \tilde{f}(|p|)$$

Example:

The function f is a Gaussian



Math Review

Convolution:

Assume that the function f is real-valued and radial, i.e. the value of f at p is determined by the distance of p from the origin:

$$f(p) = \tilde{f}(|p|)$$

Claim:

In this case, C_f is self-adjoint (i.e. symmetric).

Math Review

$$C_f(g) = f * g$$

$$(f * g)(q) = \int f(q-p) \cdot g(p) dp$$

Proof:

We need to show that for any functions g and h:

$$\langle C_f(g), h \rangle = \langle g, C_f(h) \rangle$$

Expanding the left side, we get:

$$\langle C_f(g), h \rangle = \int \big(C_f(g)\big)(p) \cdot \overline{h(p)}\, dp$$

Math Review

$$C_f(g) = f * g$$

$$(f * g)(q) = \int f(q-p) \cdot g(p) dp$$

Proof:

$$\langle C_f(g), h \rangle = \int \Big(C_f(g)\Big)(p) \cdot \overline{h(p)}\, dp$$

Writing out the operator C_f, we get:

$$\langle C_f(g), h \rangle = \int (f * g)(p) \cdot \overline{h(p)}\, dp$$

Math Review

$$C_f(g) = f * g$$

$$(f * g)(q) = \int f(q-p) \cdot g(p) dp$$

<u>Proof</u>:

$$\langle C_f(g), h \rangle = \int (f * g)(p) \cdot \overline{h(p)}\, dp$$

Expressing the convolution as an integral gives:

$$\langle C_f(g), h \rangle = \int \left(\int f(p-q) \cdot g(q) dq \right) \cdot \overline{h(p)}\, dp$$

Math Review

$$C_f(g) = f * g$$

$$(f * g)(q) = \int f(q-p) \cdot g(p) dp$$

<u>Proof</u>:

$$\langle C_f(g), h \rangle = \int \left(\int f(p-q) \cdot g(q) dq \right) \cdot \overline{h(p)}\, dp$$

Changing the order of integration, we get:

$$\langle C_f(g), h \rangle = \int \int f(p-q) \cdot g(q) \cdot \overline{h(p)}\, dp\, dq$$

Math Review

$$C_f(g) = f * g$$

$$(f * g)(q) = \int f(q-p) \cdot g(p) dp$$

Proof:

$$\langle C_f(g), h \rangle = \int\int f(p-q) \cdot g(q) \cdot \overline{h(p)}\, dp\, dq$$

Using the fact that f is real-valued and radial:

$$\langle C_f(g), h \rangle = \int\int g(q) \cdot \overline{f(q-p) \cdot h(p)}\, dp\, dq$$

Math Review

$$C_f(g) = f * g$$

$$(f * g)(q) = \int f(q-p) \cdot g(p) dp$$

Proof:

$$\langle C_f(g), h \rangle = \int \int g(q) \cdot \overline{f(q-p) \cdot h(p)}\, dp\, dq$$

Moving the integration inside:

$$\langle C_f(g), h \rangle = \int g(q) \left(\int \overline{f(q-p) \cdot h(p)}\, dp \right) dq$$

Math Review

$$C_f(g) = f * g$$

$$(f * g)(q) = \int f(q-p) \cdot g(p)dp$$

Proof:

$$\langle C_f(g), h \rangle = \int g(q) \left(\int \overline{f(q-p) \cdot h(p)}\, dp \right) dq$$

Using the equation for convolution, we get:

$$\langle C_f(g), h \rangle = \int g(q) \cdot \overline{(f * h)(q)}\, dq$$

Math Review

$$C_f(g) = f * g$$

$$(f * g)(q) = \int f(q-p) \cdot g(p) dp$$

<u>Proof</u>:

$$\langle C_f(g), h \rangle = \int g(q) \cdot \overline{(f * h)(q)}\, dq$$

Using the equation for C_f, we get:

$$\langle C_f(g), h \rangle = \int f(q) \cdot \overline{\left(C_f(h)\right)(q)}\, dq$$

Math Review

$$C_f(g) = f * g$$

$$(f * g)(q) = \int f(q-p) \cdot g(p) dp$$

Proof:

$$\langle C_f(g), h \rangle = \int g(q) \cdot \overline{\Big(C_f(h)\Big)(q)}\, dq$$

And finally, using the equation for the dot-product:

$$\langle C_f(g), h \rangle = \langle f, C_f(h) \rangle$$

Outline

- Math Review
- Spherical Convolution
- Axial Symmetry Detection

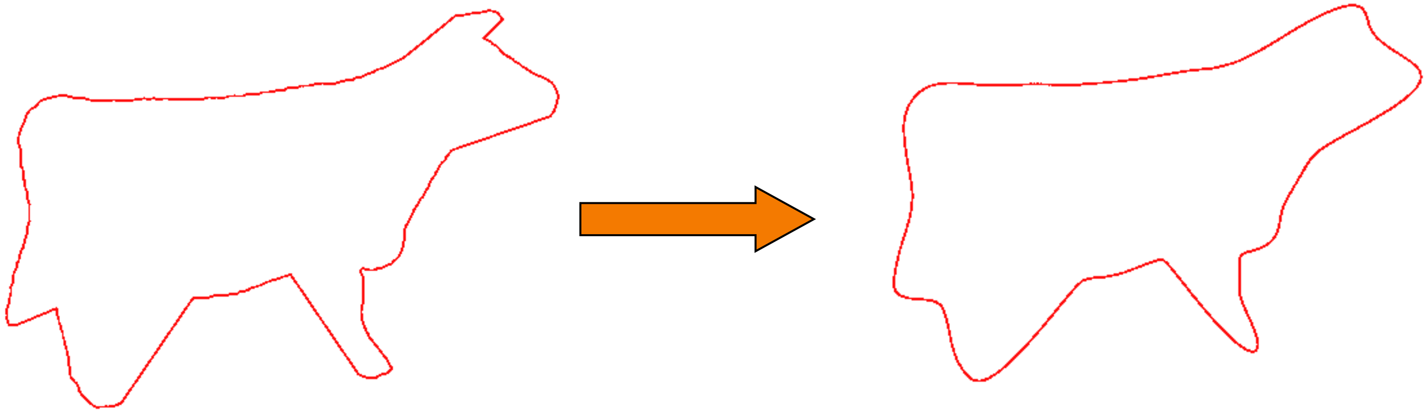
Spherical Convolution/Correlation

In the case of the circle we used convolution / correlation for two different tasks:

Spherical Convolution/Correlation

In the case of the circle we used convolution / correlation for two different tasks:

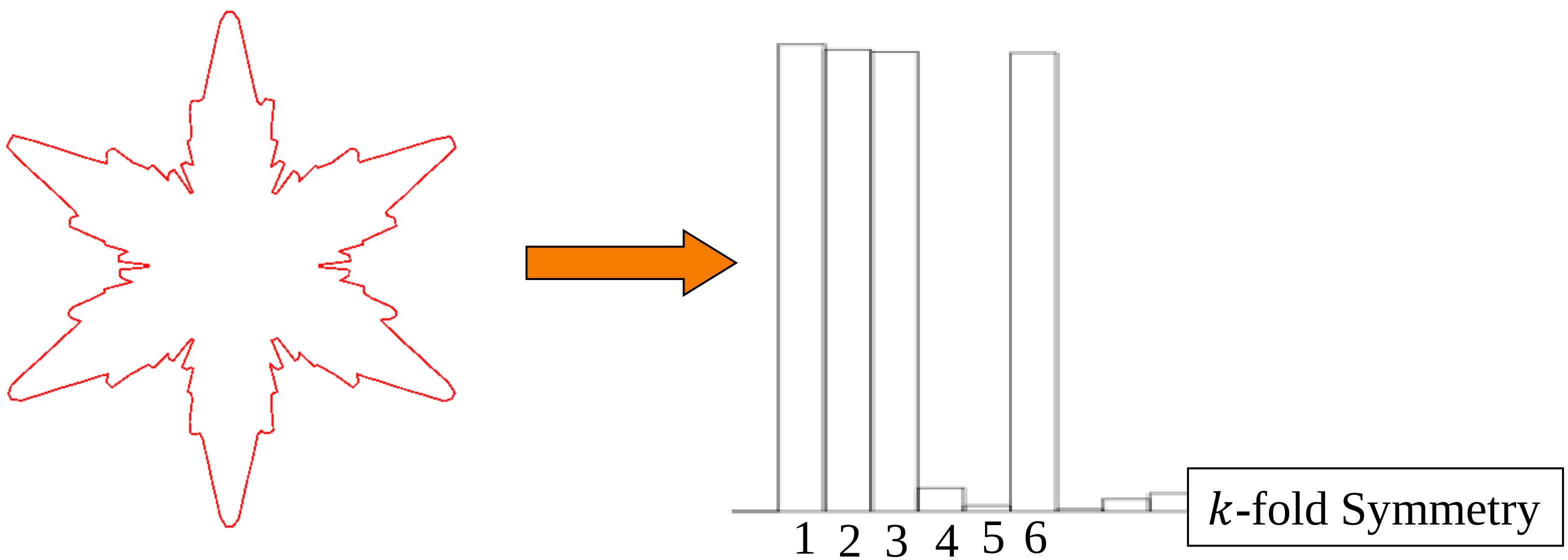
1. We used convolution for operations like smoothing



Spherical Convolution/Correlation

In the case of the circle we used convolution / correlation for two different tasks:

1. We used convolution for operations like smoothing
2. We used correlation for operations like alignment and symmetry detection



Spherical Convolution/Correlation

Up to now, we thought of these two operations as essentially the same.

The situation changes when we consider functions on the sphere.

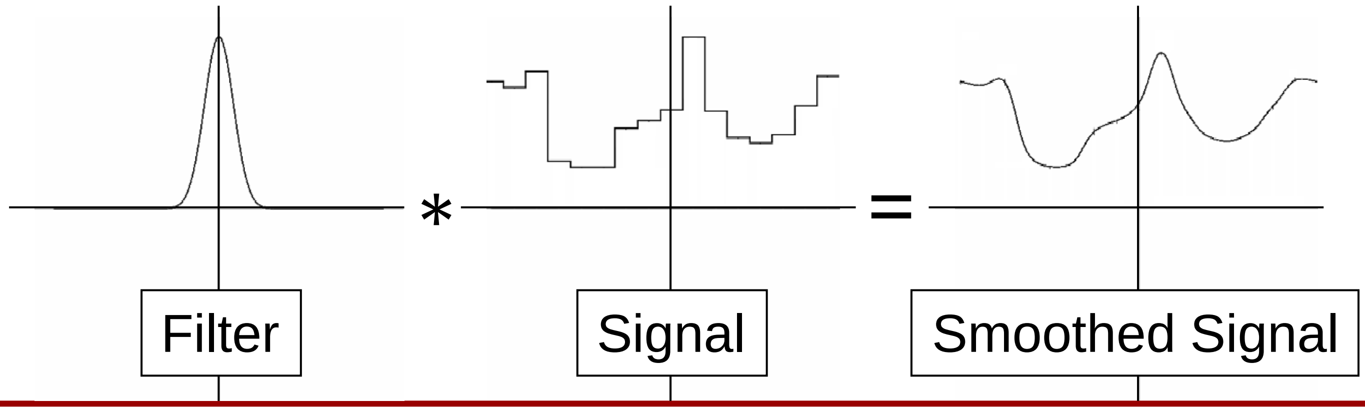
Spherical Convolution/Correlation

When we perform an operation like smoothing, the input consists of:

- Two functions on a circle, the signal and the filter

The output of the operation is:

- A function on the circle



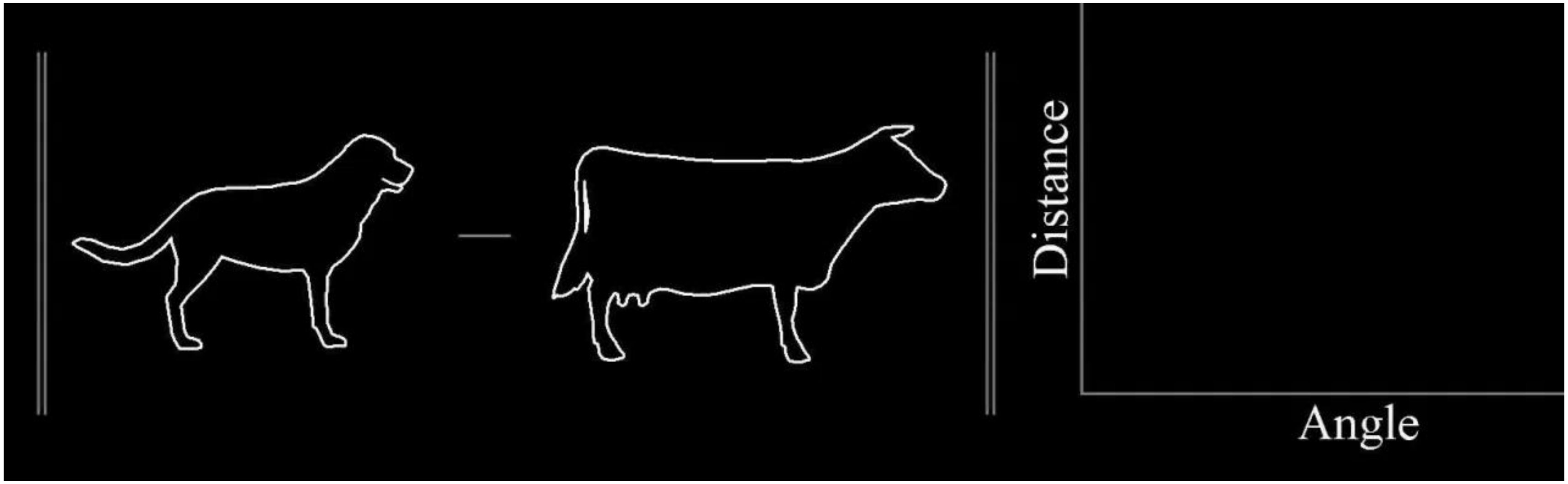
Spherical Convolution/Correlation

When we perform an operation like alignment, the input consists of:

- Two functions on a circle, the source and target

The output is:

- A function on the space of 2D rotations



Spherical Convolution/Correlation

In the case of a circle, the situation is simpler because the space of rotations is itself a circle:

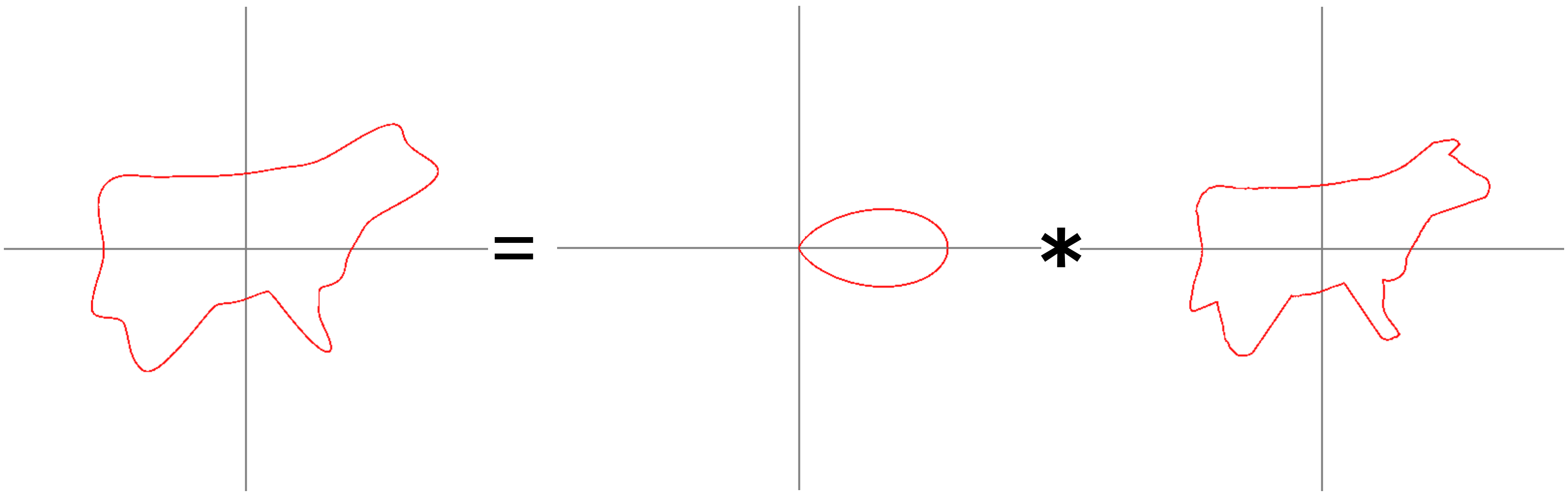
- There is a one-to-one mapping from points on a circle to rotations, with a point on a circle with angle θ corresponding to a rotation by an angle of θ.

In the case of the sphere, the situation is more complicated:

- The sphere is a 2D space while the space of rotations is a 3D space, so they cannot be equivalent.

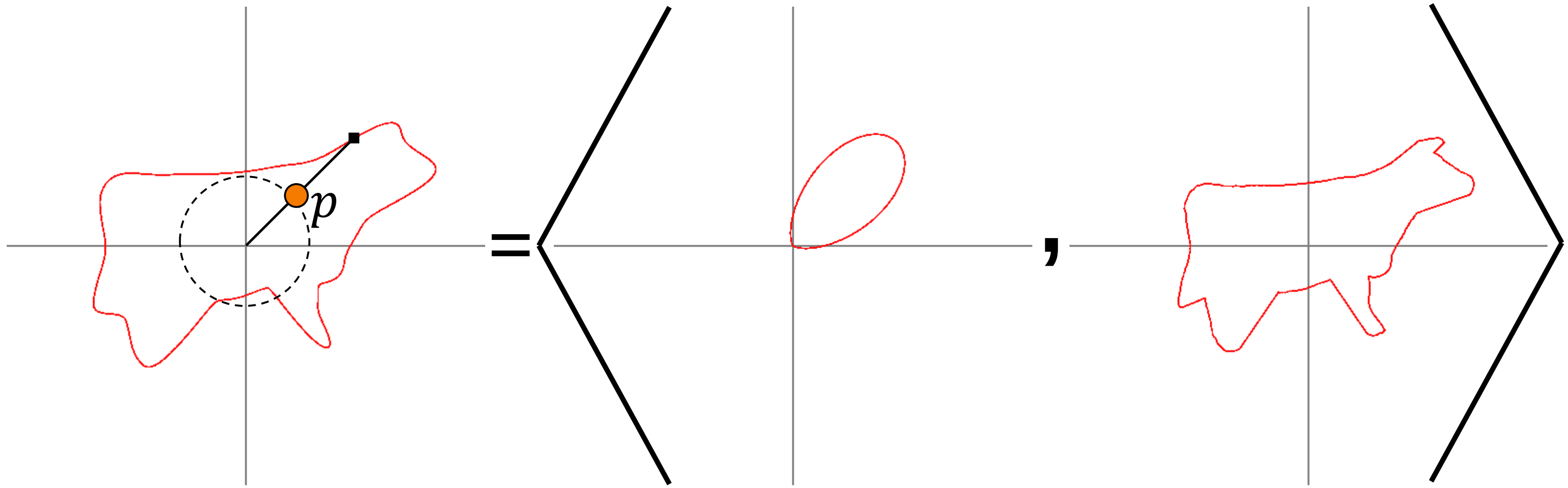
Spherical Convolution

In the case of a circle, we compute the value of the smoothed function at p by rotating the filter so that $(1,0)$ maps to p and then we compute the inner product of the signal with the rotated filter.



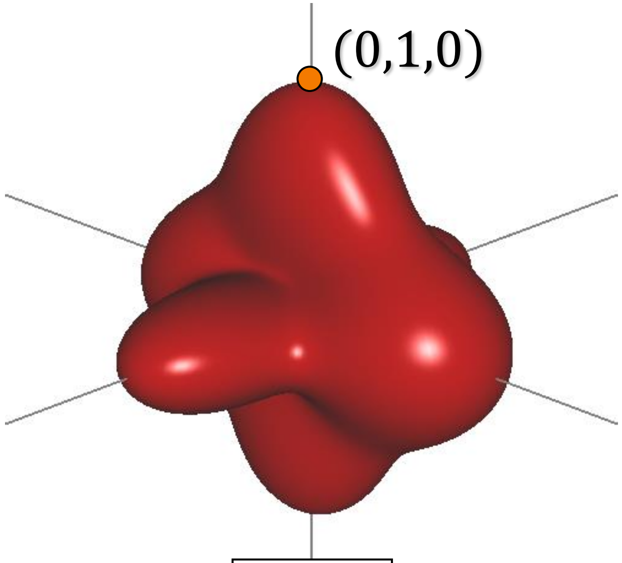
Spherical Convolution

In the case of a circle, we compute the value of the smoothed function at p by rotating the filter so that $(1,0)$ maps to p and then we compute the inner product of the signal with the rotated filter.

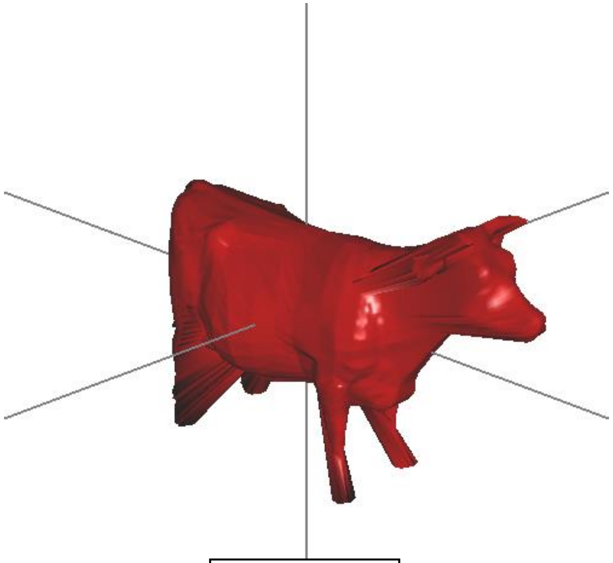


Spherical Convolution

We can try an apply the same type of approach to the case of spherical functions.



Filter

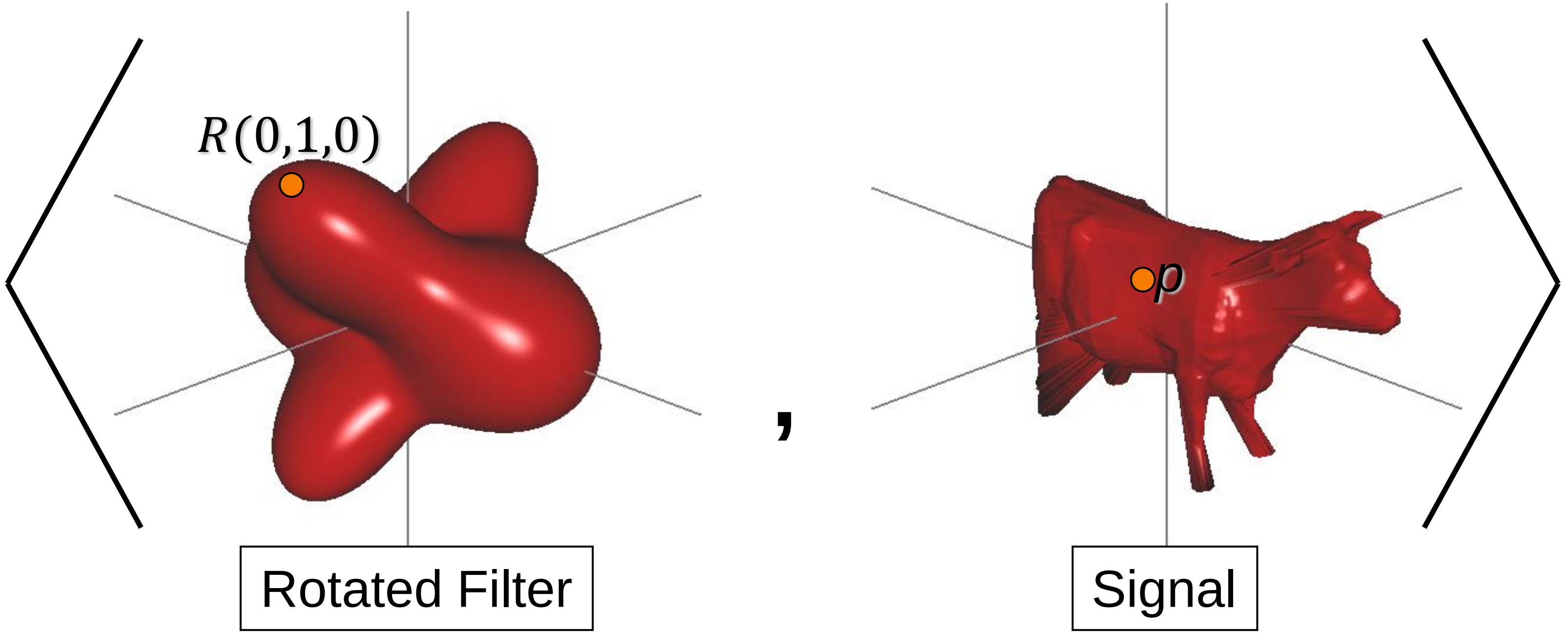


Signal

Spherical Convolution

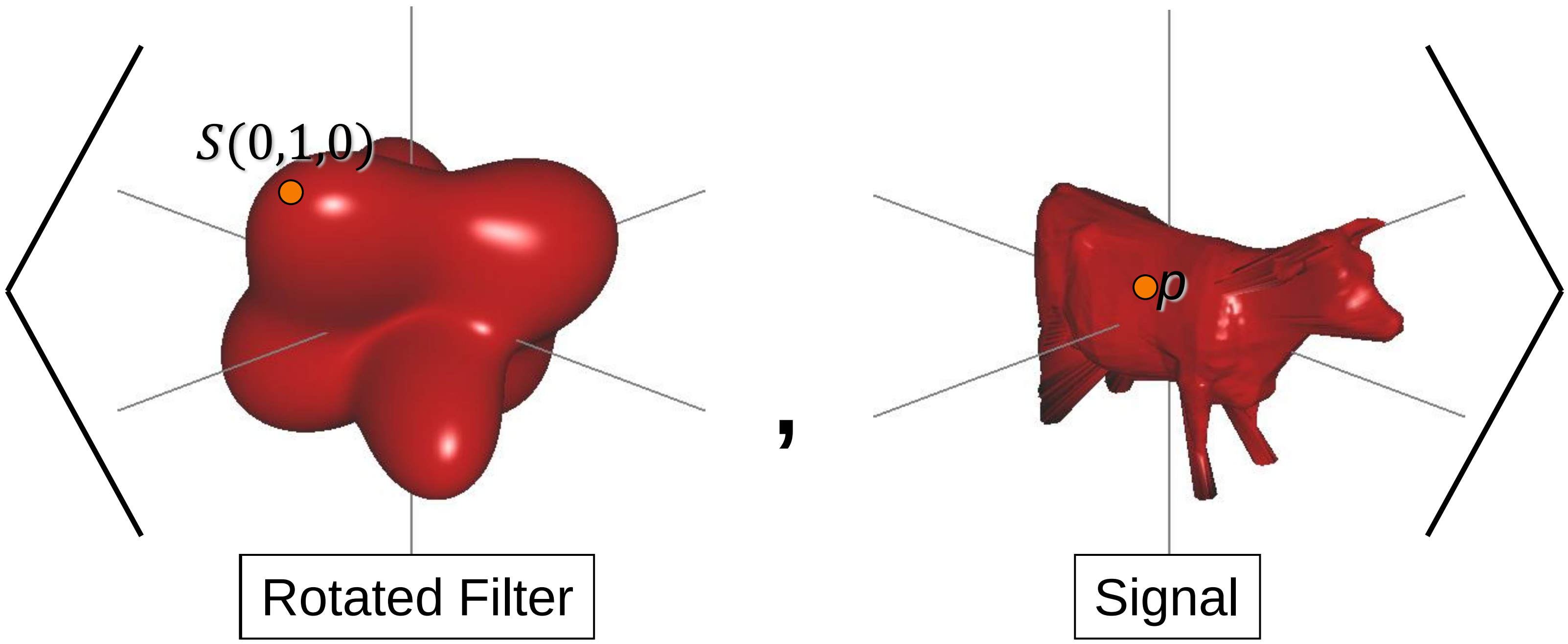
We would like to define a new function on the sphere whose value at the point p is obtained by:

Finding a rotation R that maps the North pole to p and then compute the inner product of the signal with the rotated filter.



Spherical Convolution

But there are many different rotations that send the North pole to the point p, so this does not lead to a well-defined notion of smoothing!

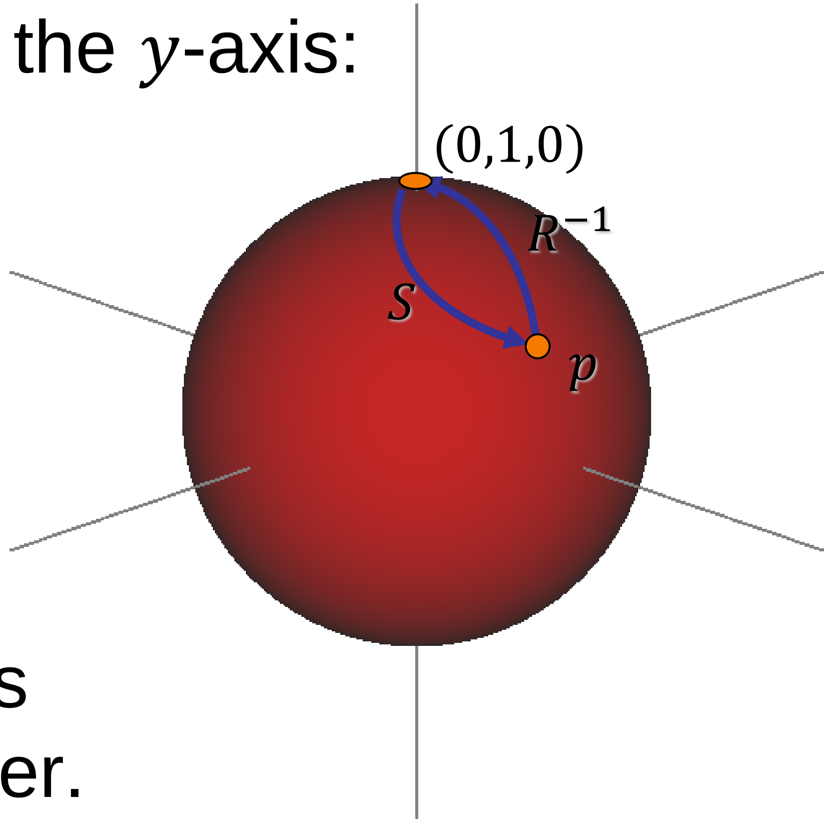


Spherical Convolution

<u>Recall</u>:

If we have two rotations R and S mapping the North pole to the point p, the rotations must differ by an initial rotation about the y-axis:

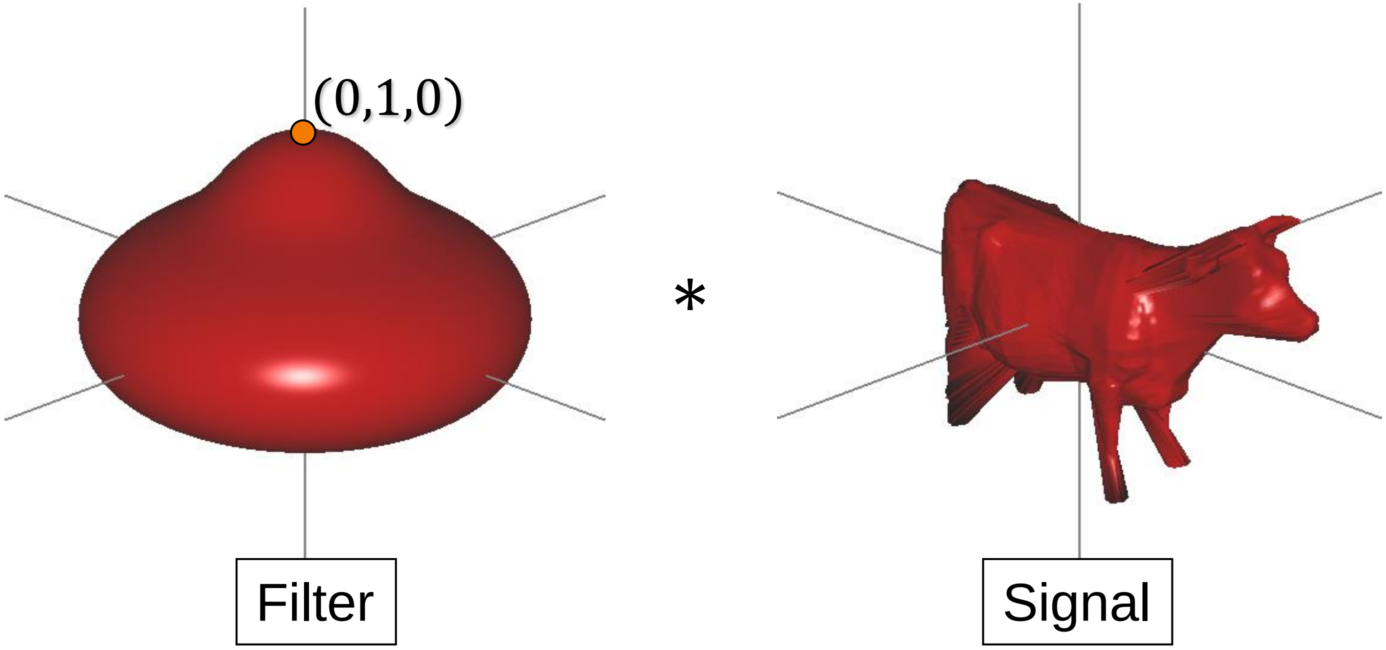
$$S = R \cdot R_y(\psi)$$



⇒ Convolution becomes well-defined if we can ensure that the initial rotation about the y-axis does not change the filter.

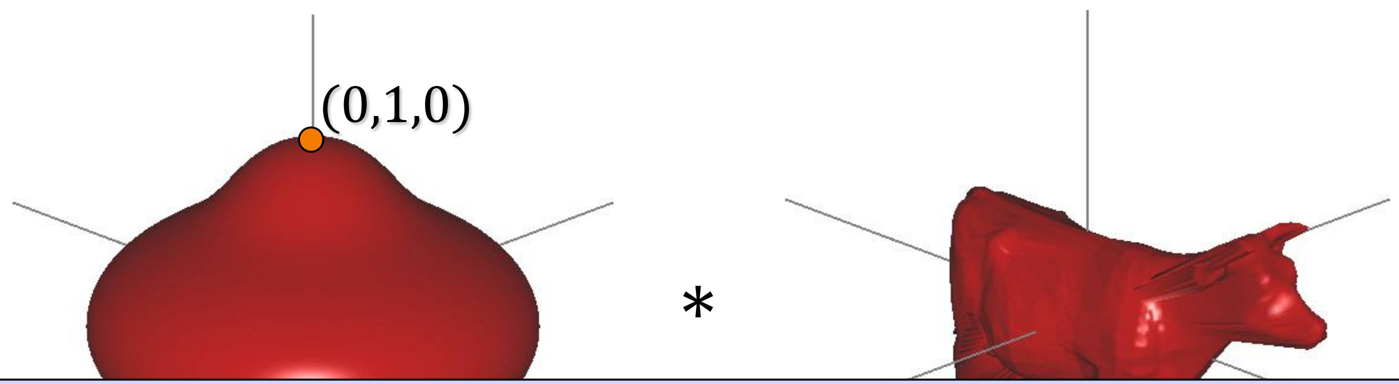
Spherical Convolution

⇒ We can extend the circular notion of smoothing to the sphere if we ensure that the filter is symmetric about the y-axis:



Spherical Convolution

⇒ We can extend the circular notion of smoothing to the sphere if we ensure that the filter is symmetric about the y-axis:



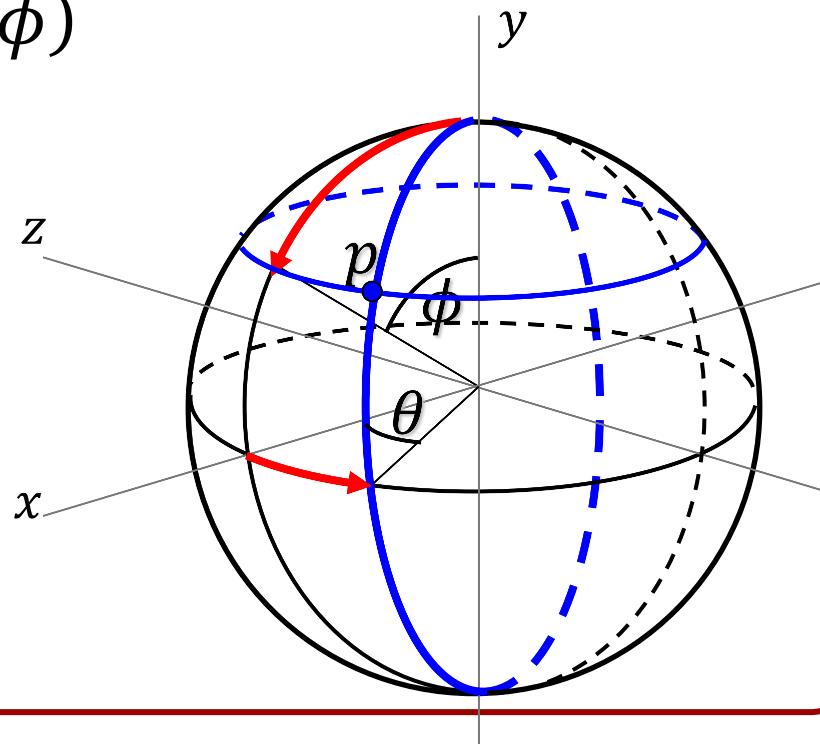
If R and S are rotations mapping the North pole to p, then the rotation of the filter by either R or S will give the same spherical function!

Spherical Convolution

Convolution:

Using the Euler angle representation, we know that the rotation taking the North pole to the point $p = \Phi(\theta, \phi)$ is the rotation:

$$R(\theta, \phi) = R_y(\theta) \cdot R_z(\phi)$$



Spherical Convolution

Convolution:

Thus, given

- A spherical function $g(\theta,\phi)$
- A spherical filter $f(\theta,\phi)$, rotationally-symmetric about the y-axis

The convolution of g with f at $p=\Phi(\theta,\phi)$ can be expressed by rotating f so the North pole gets mapped to p and computing the inner product:

$$(f*g)(\theta,\phi)=\langle\rho_{R(\theta,\phi)}(f),\bar{g}\rangle$$

Spherical Convolution

<u>Convolution</u>:

Expressing the spherical functions f and g in terms of the spherical harmonic basis, we get:

$$f(\theta,\phi) = \sum_{l} \sum_{m=-l}^{l} \hat{f}_{lm} \cdot \mathbf{Y}_l^m(\theta,\phi)$$

$$g(\theta,\phi) = \sum_{l} \sum_{m=-l}^{l} \hat{g}_{lm} \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Spherical Convolution

Recall:

A spherical harmonic can be expressed as a complex exponential in θ times a "polynomial" in $\cos\phi$:

$$\mathbf{Y}_l^m(\theta,\phi) = \mathbf{P}_l^m(\cos\phi)\cdot e^{im\theta}$$

So a rotation by α degrees about the y-axis acts on the (l,m)-th spherical harmonic by:

$$\rho_{R_y(\alpha)}(\mathbf{Y}_l^m) = e^{-im\alpha}\cdot\mathbf{Y}_l^m$$

Spherical Convolution

Convolution:

If the filter f is rotationally symmetric about the y-axis, any rotation about the y-axis must not change f. That is, for all α we must have:

$$\rho_{R_y(\alpha)}(f) = f$$

Or in terms of the spherical harmonics:

$$\sum_l \sum_{m=-l}^{l} \hat{f}_{lm} \cdot \mathbf{Y}_l^m(\theta, \phi) = \sum_l \sum_{m=-l}^{l} \hat{f}_{lm} \cdot e^{-im\alpha} \cdot \mathbf{Y}_l^m(\theta, \phi)$$

$$\Downarrow$$

$$\hat{f}_{lm} = \hat{f}_{lm} \cdot e^{-im\alpha}$$

Spherical Convolution

Convolution:

$$\hat{f}_{lm} = \hat{f}_{lm} \cdot e^{-im\alpha}$$

For this to be true, either:

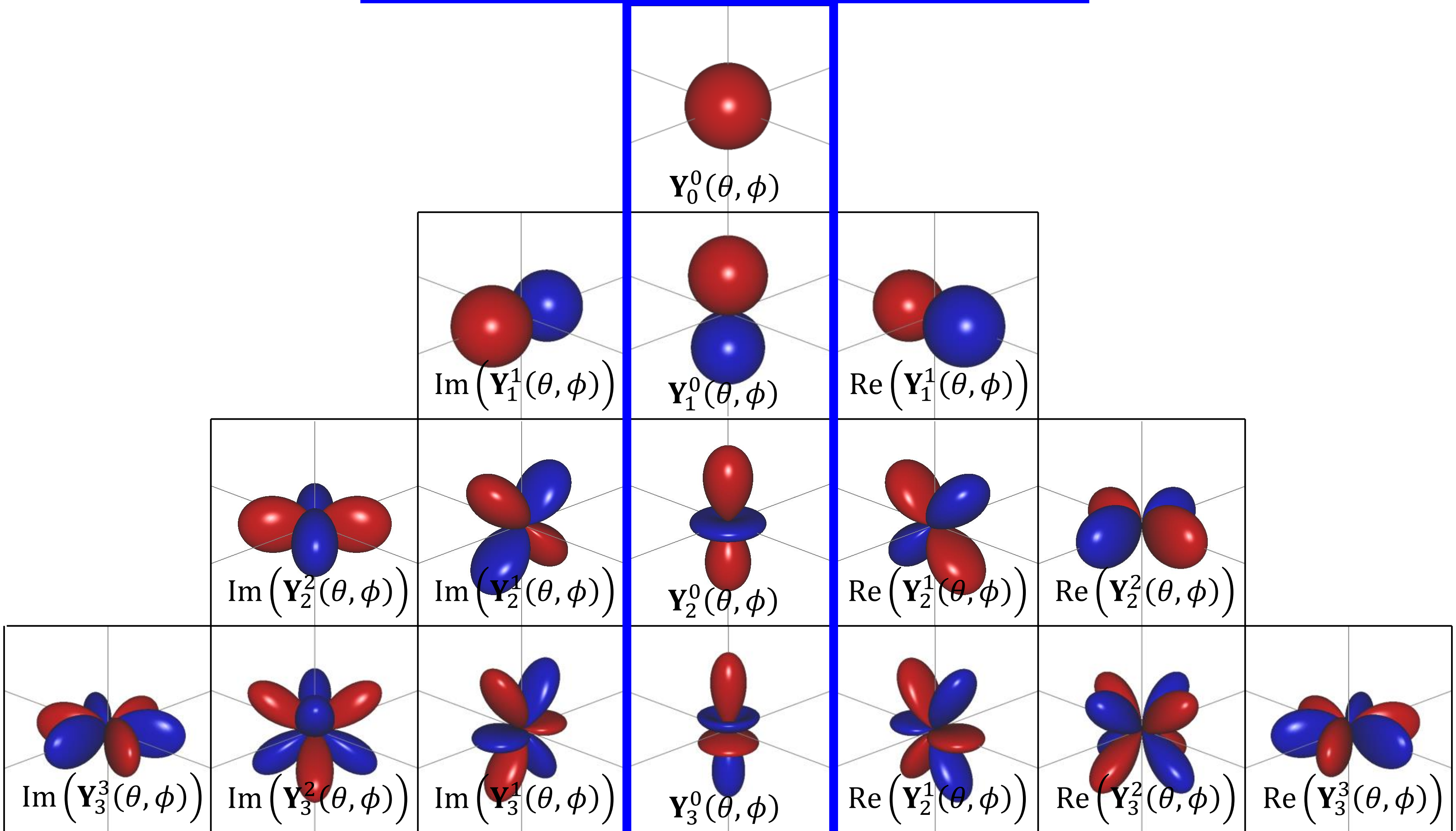
- $e^{-im\alpha} = 1$ for all $\alpha \quad \Rightarrow \quad m = 0$, or
- $\hat{f}_{lm} = 0$

Thus, in terms of the spherical harmonics, we get:

$$f(\theta, \phi) = \sum_{l} \hat{f}_{l0} \cdot \mathbf{Y}_l^0(\theta, \phi)$$

Spherical Convolution

$$f(\theta,\phi) = \sum_l \hat{f}_{l0} \cdot \mathbf{Y}_l^0(\theta,\phi)$$



Spherical Convolution

Convolution:

Thus, the expression for the functions in terms of their spherical harmonic decomposition becomes:

$$f(\theta,\phi) = \sum_{l'} \hat{f}_{l'0} \cdot \mathbf{Y}_{l'}^{0}(\theta,\phi)$$

$$g(\theta,\phi) = \sum_{l} \sum_{m=-l}^{l} \hat{g}_{lm} \cdot \mathbf{Y}_{l}^{m}(\theta,\phi)$$

and we get an expression for the convolution:

$$(f * g)(\theta,\phi) = \left\langle \rho_{R(\theta,\phi)} \left(\sum_{l'} \hat{f}_{l'0} \cdot \mathbf{Y}_{l'}^{0} \right), \sum_{l} \sum_{m=-l}^{l} \overline{\hat{g}_{lm} \cdot \mathbf{Y}_{l}^{m}} \right\rangle$$

Spherical Convolution

Convolution:

$$(f * g)(\theta, \phi) = \left\langle \rho_{R(\theta,\phi)}\left(\sum_{l'} \hat{f}_{l'0} \cdot \mathbf{Y}_{l'}^{0}\right), \sum_{l} \sum_{m=-l}^{l} \overline{\hat{g}_{lm} \cdot \mathbf{Y}_{l}^{m}} \right\rangle$$

By leveraging the conjugate-linearity of the inner product and using the fact that the transformation ρ_R is linear, we get:

$$(f * g)(\theta, \phi) = \sum_{l,l'} \sum_{m=-l}^{l} \hat{f}_{l'0} \cdot \hat{g}_{lm} \langle \rho_{R(\theta,\phi)}(\mathbf{Y}_{l'}^{0}), \overline{\mathbf{Y}_{l}^{m}} \rangle$$

Spherical Convolution

<u>Recall</u>:

- A rotation of an l-th frequency function will still be an l-th frequency function
- The space of l-th frequency functions is orthogonal to the space of l'-th frequency functions (if $l \neq l'$)
- The spherical harmonics satisfy $\mathbf{Y}_l^{-m} = \overline{\mathbf{Y}_l^m}$

$\Rightarrow$ For all $l \neq l'$, we have:

$$\left\langle \rho_R\left(\mathbf{Y}_{l'}^{m'}\right), \overline{\mathbf{Y}_l^m} \right\rangle = 0$$

Spherical Convolution

Convolution:

This lets us simplify the expression for the convolution:

$$(f * g)(\theta,\phi) = \sum_{l,l'} \sum_{m=-l}^{l} \hat{f}_{l'0} \cdot \hat{g}_{lm} \left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_{l'}^{0}\right), \overline{\mathbf{Y}_{l}^{m}} \right\rangle$$

$$\Downarrow$$

$$(f * g)(\theta,\phi) = \sum_{l} \sum_{m=-l}^{l} \hat{f}_{l0} \cdot \hat{g}_{lm} \left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_{l}^{0}\right), \overline{\mathbf{Y}_{l}^{m}} \right\rangle$$

Spherical Convolution

Convolution:

$$(f * g)(\theta,\phi) = \sum_{l} \sum_{m=-l}^{l} \hat{f}_{l0} \cdot \hat{g}_{lm} \left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_l^m} \right\rangle$$

To compute the convolution, we need to be able to evaluate the inner product:

$$\left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_l^m} \right\rangle$$

Spherical Convolution

Convolution:

What is the meaning of the function:

$$\left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_l^m} \right\rangle$$

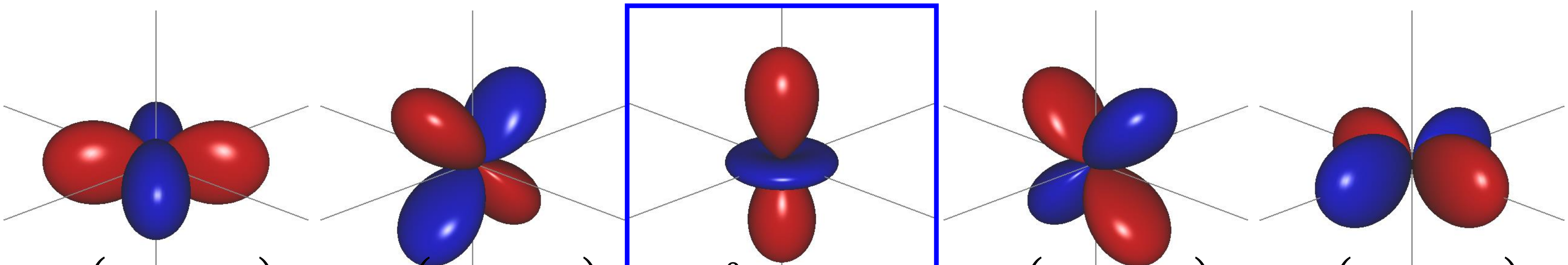
This is a function on the sphere whose value at the point $p = \Phi(\theta, \phi)$ is the dot-product of (the conjugate of) $\mathbf{Y}_l^m$ with the rotation of $\mathbf{Y}_l^0$, where the rotation takes the North pole to p.

Spherical Convolution

Convolution:

We would like to show that this function acts very simply on the spherical harmonics:

$$\left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_l^m}\right\rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$



$\mathrm{Im}\left(\mathbf{Y}_2^2(\theta,\phi)\right)$ $\mathrm{Im}\left(\mathbf{Y}_2^1(\theta,\phi)\right)$ $\boxed{\mathbf{Y}_2^0(\theta,\phi)}$ $\mathrm{Re}\left(\mathbf{Y}_2^1(\theta,\phi)\right)$ $\mathrm{Re}\left(\mathbf{Y}_2^2(\theta,\phi)\right)$

Spherical Convolution

Convolution:

For a given frequency l, consider the operator C_l taking spherical functions to spherical functions:

$$\big(C_l(g)\big)(\theta,\phi) = \left\langle \rho_{R(\theta,\phi)}\big(\mathbf{Y}_l^0\big), \bar{g} \right\rangle$$

As before, it turns out this map is a symmetric linear operator on the space of functions.

Thus, there exists an orthonormal basis with respect to which C_l is diagonal.

Spherical Convolution

Convolution:

For a given frequency l, consider the operator C_l taking spherical functions to spherical functions:

$$\big(C_l(g)\big)(\theta,\phi) = \left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \bar{g}\right\rangle$$

This operator also has the property that it commutes with rotations:

- Rotating a spherical function and then convolving with $\mathbf{Y}_l^0$ is the same as first convolving with $\mathbf{Y}_l^0$ and then rotating.

Spherical Convolution

<u>Convolution</u>:

As with the Laplacian, we have a symmetric operator that is G-linear:

⇒ Functions within an irreducible representation are all eigenvectors of C_l with the same eigenvalue.

⇒ The spherical harmonics of frequency l' are eigenvectors of C_l with eigenvalue λ_{ll}':

$$C_l\left(\mathbf{Y}_{l'}^m\right) = \lambda_{ll'} \cdot \mathbf{Y}_{l'}^m$$

Spherical Convolution

<u>Convolution</u>:

$$C_l\left(\mathbf{Y}_{l'}^m\right) = \lambda_{ll'} \cdot \mathbf{Y}_{l'}^m$$

Since $\left(C_l\left(\mathbf{Y}_{l'}^m\right)\right)(\theta,\phi) = \left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_{l'}^m} \right\rangle$, we have:

$$C_l\left(\mathbf{Y}_{l'}^m\right) = \mathbf{Y}_{l'}^m \cdot \begin{cases} 0 & \text{if} \quad l \neq l' \\ \lambda_l & \text{otherwise} \end{cases}$$

Spherical Convolution

Convolution:

Putting this together, we get:

$$\left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_l^m} \right\rangle = \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Thus, the equation for the convolution becomes:

$$(f * g)(\theta,\phi) = \sum_l \sum_{m=-l}^{l} \hat{f}_{l0} \cdot \hat{g}_{lm} \cdot \left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_l^m} \right\rangle$$

$$\Downarrow$$

$$(f * g)(\theta,\phi) = \sum_l \sum_{m=-l}^{l} \hat{f}_{l0} \cdot \hat{g}_{lm} \cdot \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

Spherical Convolution

Convolution:

$$(f * g)(\theta,\phi) = \sum_{l}\sum_{m=-l}^{l} \hat{f}_{l0}\cdot \hat{g}_{lm}\cdot \lambda_l \cdot \mathbf{Y}_l^m(\theta,\phi)$$

$\Rightarrow$ The convolution of f with g can be obtained by multiplying the (l,m)-th spherical harmonic coefficients of g by $\lambda_l \cdot \hat{f}_{l0}$.

As with functions on a circle, convolution in the spatial domain is multiplication in the frequency domain.

Spherical Convolution

Convolution:

To use the convolution theorem for spherical functions, we need to know what the λ_l are.

It turns out that these are:

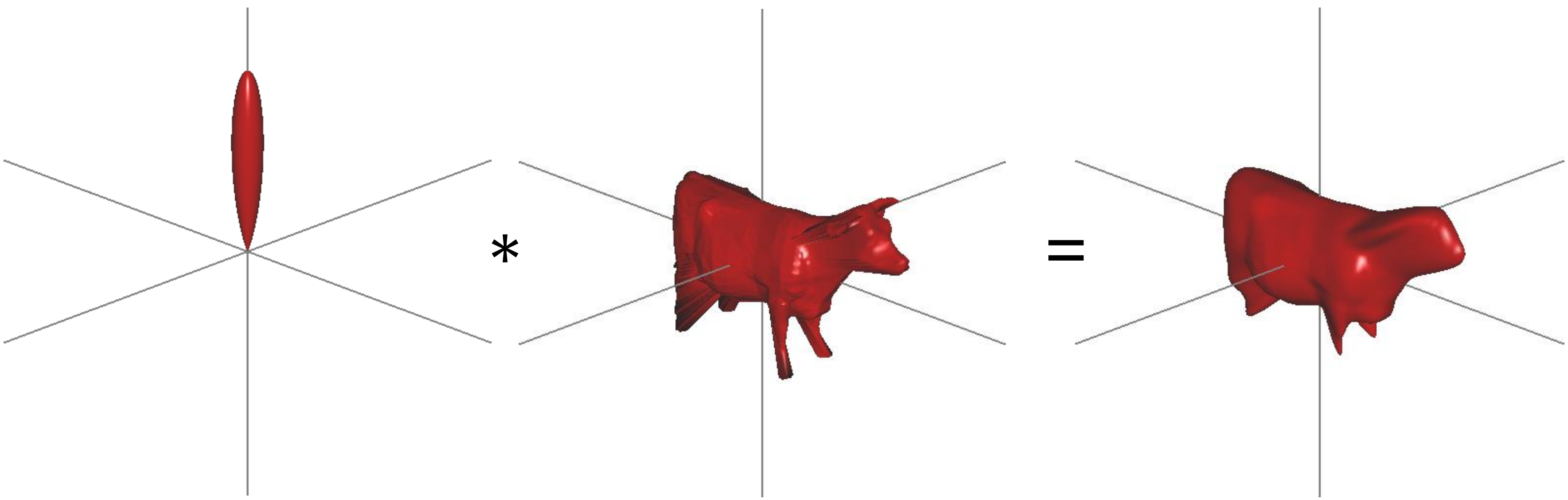
$$\lambda_l = \sqrt{\frac{4\pi}{2l+1}}$$

Spherical Convolution

Convolution:

Which gives us the equation:

$$\langle f * g, \mathbf{Y}_l^m \rangle = \sqrt{\frac{4\pi}{2l+1}} \cdot \hat{f}_{l0} \cdot \hat{g}_{lm}$$

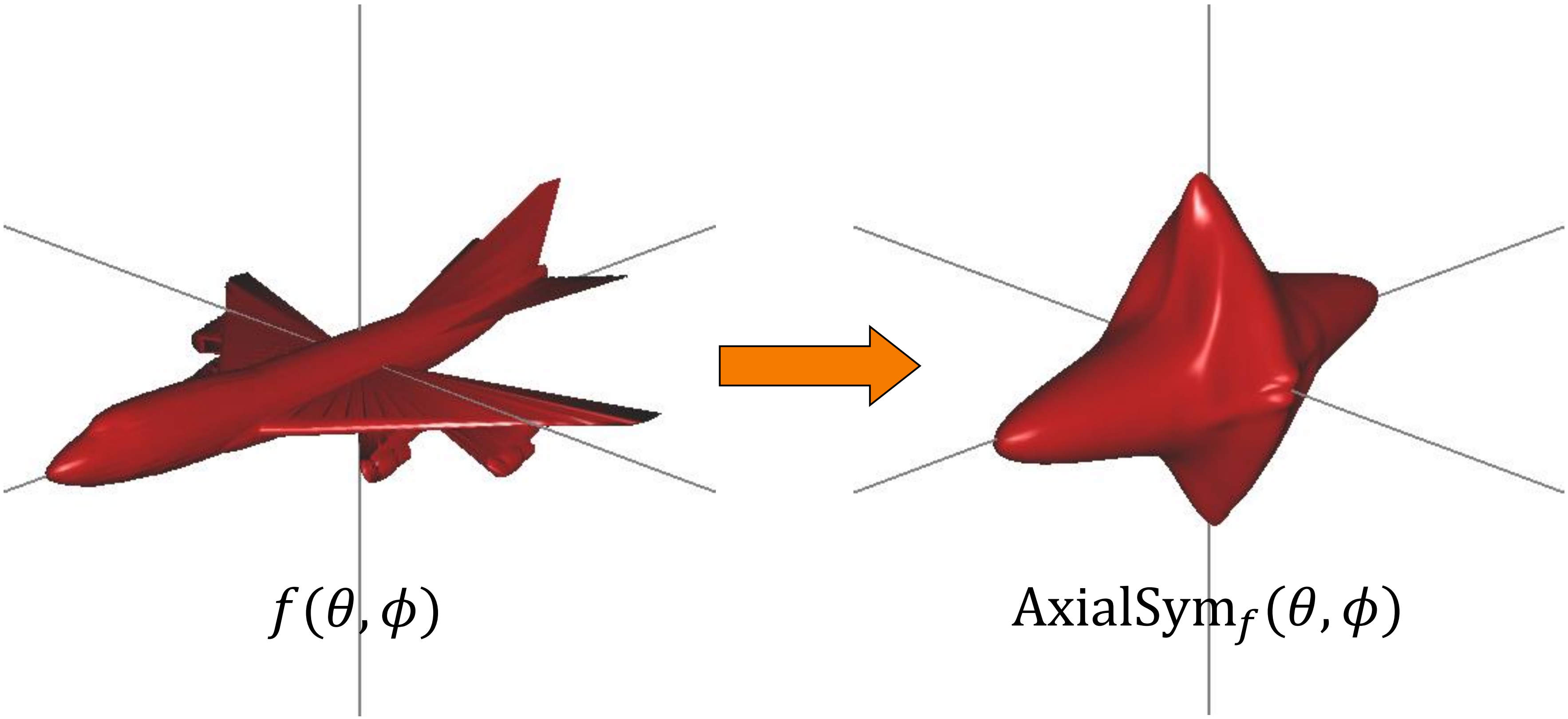


Outline

- Math Review
- Spherical Convolution
- Axial Symmetry Detection

Axial Symmetry Detection

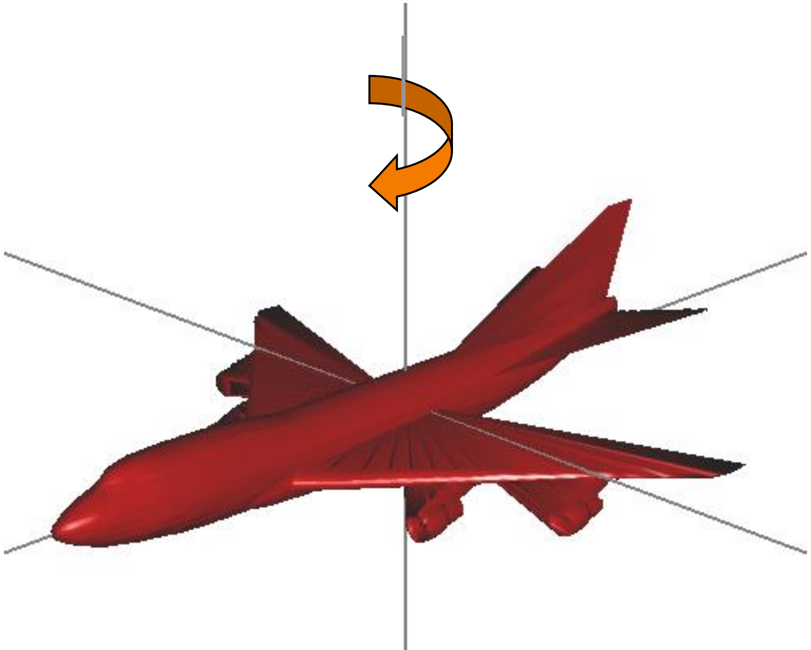
Given a spherical function f, we would like to compute the measure of the axial symmetry of f with respect to every axis through the origin.



$f(\theta, \phi)$ $\quad$ $\mathrm{AxialSym}_f(\theta, \phi)$

Axial Symmetry Detection

What is the measure of the axial symmetry of f about the y-axis?

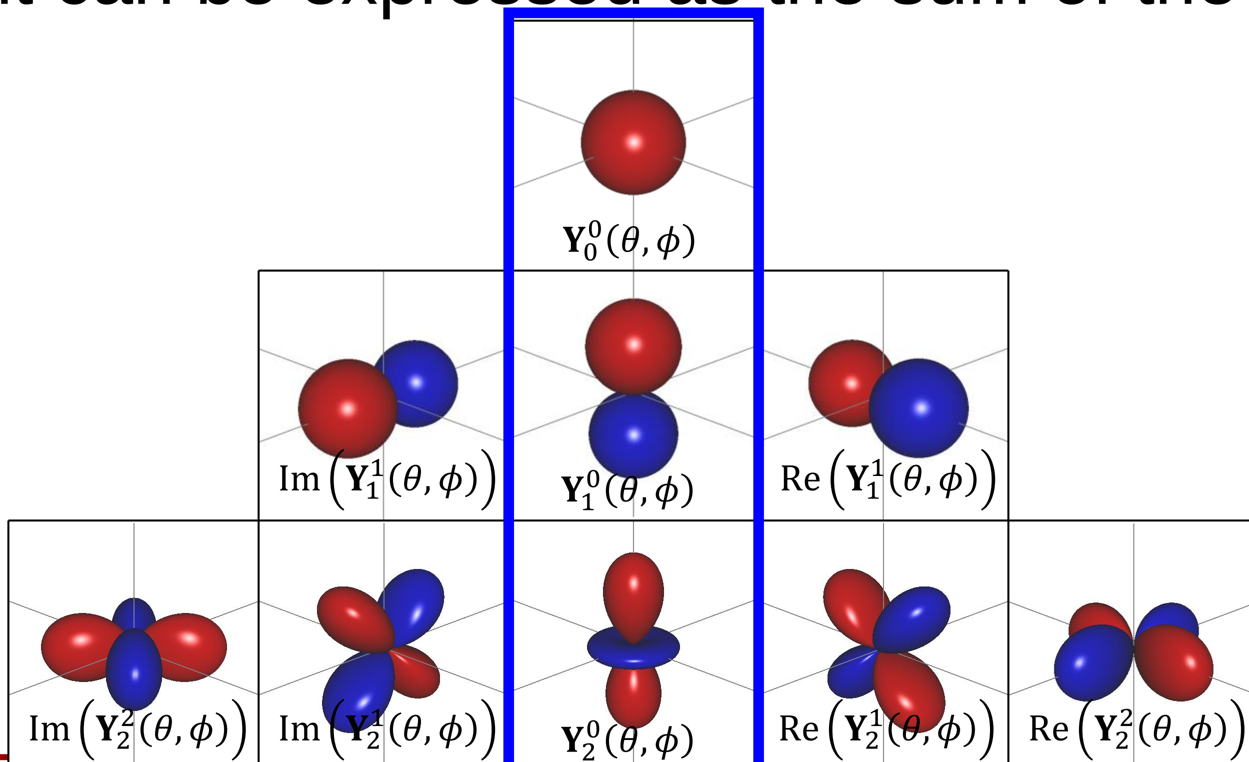


$f(\theta, \phi)$

Axial Symmetry Detection

What is the measure of the axial symmetry of f about the y-axis?

We know that f is axially symmetric about the y-axis if it can be expressed as the sum of the $\mathbf{Y}_l^0$:



Axial Symmetry Detection

What is the measure of the axial symmetry of f about the y-axis?

We know that f is axially symmetric about the y-axis if it can be expressed as the sum of the $\mathbf{Y}_l^0$.

We also know that for $m \neq 0$:

$$\langle \mathbf{Y}_l^0, \mathbf{Y}_l^m \rangle = 0$$

So the projection onto the space of functions that are axially symmetric about the y-axis is:

$$\pi_y\left(\sum_l \sum_{m=-l}^{k} \hat{f}_{lm} \cdot \mathbf{Y}_l^m\right) = \sum_l \hat{f}_{l0} \cdot \mathbf{Y}_l^0$$

Axial Symmetry Detection

What is the measure of the axial symmetry of f about the y-axis?

Thus, the measure of the axial symmetry of f about the y-axis is defined as:

$$\mathrm{YAxialSym}^2(f) = \left\| \sum_l \hat{f}_{l0} \cdot \mathbf{Y}_l^0 \right\|^2$$

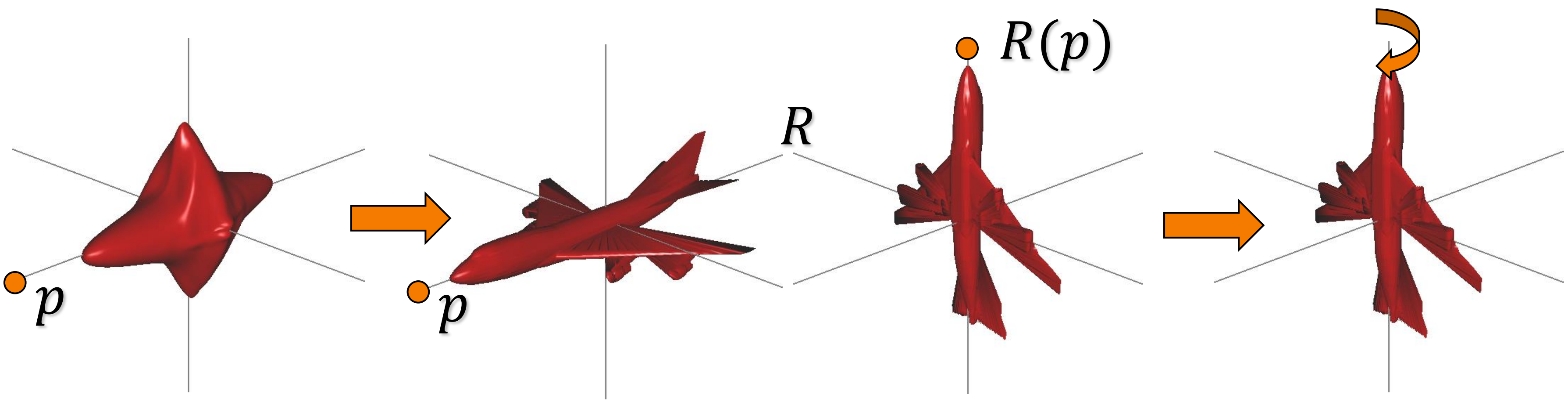
$$= \sum_l \left\| \hat{f}_{l0} \right\|^2$$

Axial Symmetry Detection

More generally, we would like to compute the measure of the axial symmetry of f with respect to any axis.

To compute the symmetry measure about the line through $p = \Phi(\theta, \phi)$ we:

- Rotate so that p goes to the North pole, and
- Compute the symmetry measure about the y-axis.



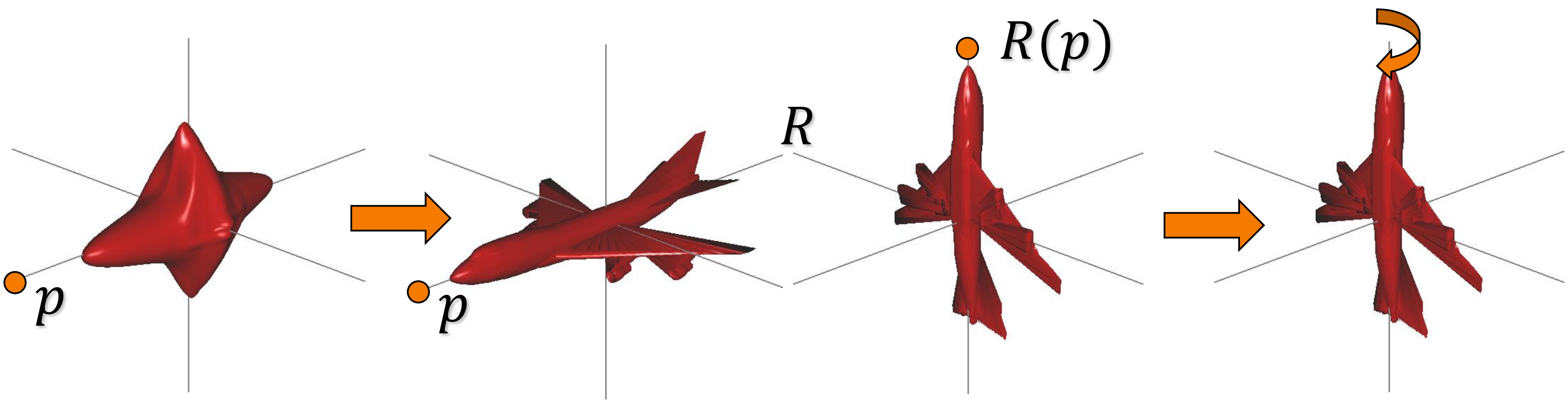
Axial Symmetry Detection

More generally, we would like to compute the measure of the axial symmetry of f with respect to a…

Since the rotation $R(\theta,\phi)$ maps the North pole to p, the rotation we are interested in is the inverse, $R^{-1}(\theta,\phi)$.

To c… line thro…

- Rotate so that p goes to the North pole, and
- Compute the symmetry measure about the y-axis.



Axial Symmetry Detection

Using the fact that the spherical harmonics form an orthonormal basis, we know that the (l, m)-th harmonic coefficient of f is defined by:

$$\hat{f}_{lm} = \langle f, \mathbf{Y}_l^m \rangle$$

Thus, to compute the measure of axial symmetry about the axis through p we need to compute:

$$\text{AxialSym}_f^2(\theta, \phi) = \sum_l \left\| \left\langle \rho_{R^{-1}(\theta,\phi)}(f), \mathbf{Y}_l^0 \right\rangle \right\|^2$$

Axial Symmetry Detection

$$\mathrm{AxialSym}_f^2(\theta,\phi) = \sum_l \left\| \left\langle \rho_{R^{-1}(\theta,\phi)}(f), \mathbf{Y}_l^0 \right\rangle \right\|^2$$

Using the facts that ρ is unitary and the zonal harmonics are real-valued, we can re-write this equation as:

$$\begin{aligned}\mathrm{AxialSym}_f^2(\theta,\phi) &= \sum_l \left\| \left\langle f, \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right) \right\rangle \right\|^2 \\ &= \sum_l \left\| \left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \bar{f} \right\rangle \right\|^2\end{aligned}$$

Axial Symmetry Detection

$$\mathrm{AxialSym}_f^2(\theta,\phi) = \sum_l \left\| \left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \bar{f} \right\rangle \right\|^2$$

Expressing f in terms of its spherical harmonic decomposition, we get:

$$\mathrm{AxialSym}_f^2(\theta,\phi) = \sum_l \left\| \sum_{m=-l}^{l} \hat{f}_{lm} \left\langle \rho_{R(\theta,\phi)}\left(\mathbf{Y}_l^0\right), \overline{\mathbf{Y}_l^m} \right\rangle \right\|^2$$

Axial Symmetry Detection

$$\mathrm{AxialSym}_f^2(\theta,\phi) = \sum_l \left\| \sum_{m=-l}^{l} \hat{f}_{lm} \langle \rho_{R(\theta,\phi)}(\mathbf{Y}_l^0), \overline{\mathbf{Y}_l^m} \rangle \right\|^2$$

Applying the identity:

$$\langle \rho_{R(\theta,\phi)}(\mathbf{Y}_l^0), \overline{\mathbf{Y}_l^m} \rangle = \sqrt{\frac{4\pi}{2l+1}} \mathbf{Y}_l^m(\theta,\phi)$$

we get an expression for the symmetry measure:

$$\mathrm{AxialSym}_f^2(\theta,\phi) = \sum_l \frac{4\pi}{2l+1} \left(\left\| \sum_{m=-l}^{l} \hat{f}_{lm} \cdot \mathbf{Y}_l^m(\theta,\phi) \right\|^2 \right)$$

Axial Symmetry Detection

$$\mathrm{AxialSym}_f^2(\theta,\phi)=\sum_l \frac{4\pi}{2l+1}\left(\left\|\sum_{m=-l}^{l}\hat{f}_{lm}\cdot \mathbf{Y}_l^m(\theta,\phi)\right\|^2\right)$$

Thus, the measure of axial symmetry can be computed by taking the weighted sum of the squares of the frequency components of f.

Axial Symmetry Detection

$$\mathrm{AxialSym}_f^2(\theta,\phi) = \sum_l \frac{4\pi}{2l+1}\left(\left\|\sum_{m=-l}^{l} \hat{f}_{lm}\cdot \mathbf{Y}_l^m(\theta,\phi)\right\|^2\right)$$