FFTs in Graphics and Vision

Representing Rotations

Outline

- Math Review
    - Polynomials
    - Eigenvectors
    - Orthogonal Transformations
    - Classifying the 2D Orthogonal Transformations
- Representing 3D Rotations

Math Review

Polynomials:

Let $P(x)$ be a polynomial of degree d:

$$P(x) = a_0 + a_1 \cdot x + \cdots + a_d \cdot x^d$$

Claim:

If d is odd, the polynomial $P(x)$ must have at least one real root.

Math Review

Polynomials:

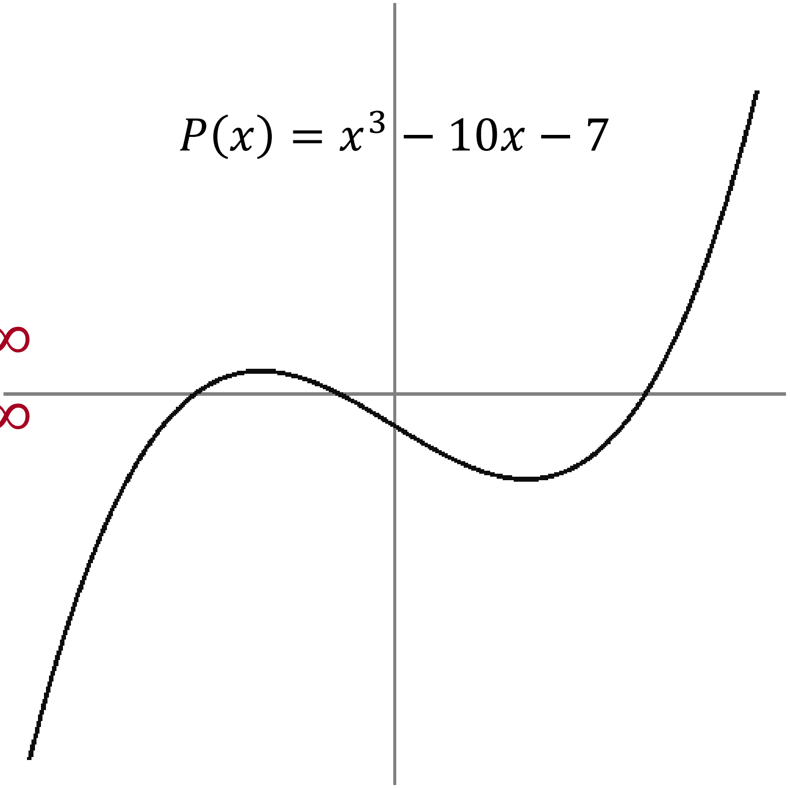
Let $P(x)$ be a polynomial of degree d:

$$P(x) = a_0 + a_1 \cdot x + \cdots + a_d \cdot x^d$$

Proof:

Consider the sign of a_d:

- If a_d is positive:
  - As $x \to -\infty$: $P(x) \to -\infty$
  - As $x \to +\infty$: $P(x) \to +\infty$



Math Review

Polynomials:

Let $P(x)$ be a polynomial of degree d:

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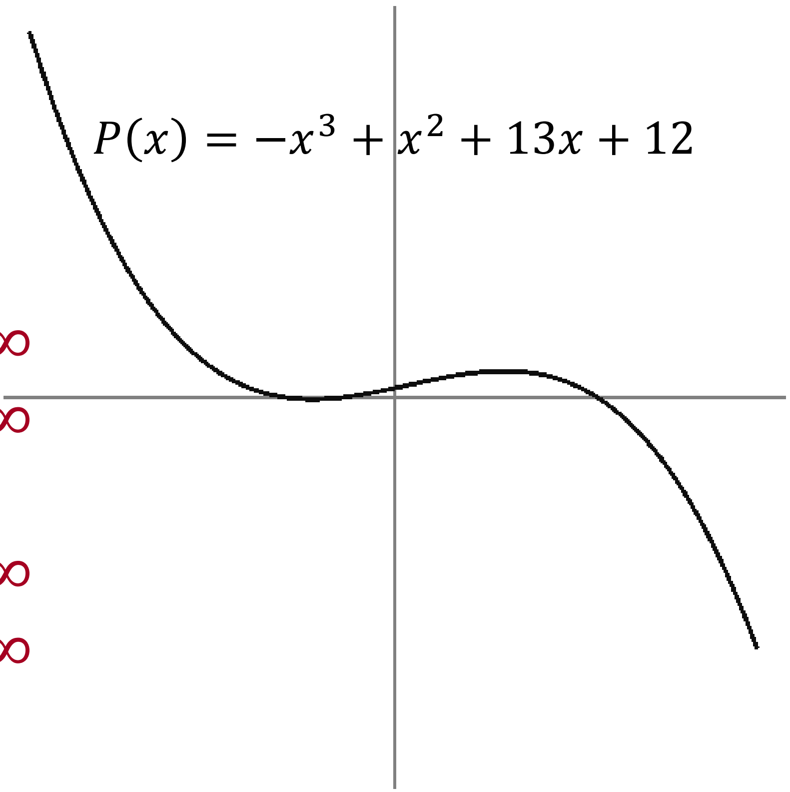
Proof:

Consider the sign of a_d:

- If a_d is positive:
  - As $x \to -\infty$: $P(x) \to -\infty$
  - As $x \to +\infty$: $P(x) \to +\infty$
- If a_d is negative:
  - As $x \to -\infty$: $P(x) \to +\infty$
  - As $x \to +\infty$: $P(x) \to -\infty$



Math Review

Polynomials:

Let $P(x)$ be a polynomial of degree d:

$$P(x) = a_0 + a_1 \cdot x + \cdots + a_d \cdot x^d$$

Proof:

In either case, the value of $P(x)$ changes signs so it must have a zero-crossing somewhere.

Math Review

Eigenvectors:

Given a vector space V and invertible linear op. $A: V \to V$, if v is an e.vector of A with e.value λ then v is an e.vector of A^{-1} with e.value $1/\lambda$.

Since $A^{-1} \cdot A$ is the identity, we have:

$$\begin{aligned} v &= A^{-1}(Av) \\ &= A^{-1}(\lambda v) \\ &= \lambda \cdot A^{-1} v \end{aligned}$$

$$\Downarrow$$

$$\frac{1}{\lambda} \cdot v = A^{-1} v$$

Math Review

Orthogonal Transformations:

For a real inner-product space V, a linear map R is orthogonal if for any $v, w \in V$, we have:

$$\langle v, w \rangle = \langle Rv, Rw \rangle$$

If the determinant of R is 1, the transformation is called a rotation.

Math Review

Orthogonal Transformations (Property 1):

The set of orthogonal transformations is a group.

Math Review

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The set of orthogonal transformations is a group.

To show this we need to show that if R and S are orthogonal transformations than:

- RS is orthogonal
- R^{-1} is orthogonal

Math Review

<u>Orthogonal Transformations (Property 1)</u>:

If R and S are orthogonal transformations, then so is the transformation $R \cdot S$.

Since R is orthogonal:

$$\langle RSv, RSw \rangle = \langle Sv, Sw \rangle$$

Since S is orthogonal:

$$\langle Sv, Sw \rangle = \langle v, w \rangle$$

Thus, as desired, we get:

$$\langle RSv, RSw \rangle = \langle v, w \rangle$$

Math Review

Orthogonal Transformations (Property 1):

If R is an orthogonal transformation, then so is the transformation R^{-1}.

Starting with the identity:

$$\langle v, w \rangle = \langle RR^{-1}v, RR^{-1}w \rangle$$

Since R is orthogonal we get:

$$\langle RR^{-1}v, RR^{-1}w \rangle = \langle R^{-1}v, R^{-1}w \rangle$$

Thus, as desired, we get:

$$\langle v, w \rangle = \langle R^{-1}v, R^{-1}w \rangle$$

Math Review

Orthogonal Transformations (Property 2):

If R is an orthogonal transformation and v is an eigenvector of R with eigenvalue λ, then $\lambda = \pm 1$.

Since R orthogonal, we have:

$$\begin{aligned}\langle v, v\rangle &= \langle Rv, Rv\rangle \\ &= \langle \lambda v, \lambda v\rangle \\ &= \lambda^2 \langle v, v\rangle\end{aligned}$$

$$\Downarrow$$

$$\lambda^2 = 1$$

Equivalently, if λ, is an eigenvalue of (orthogonal) R, then:

$$\frac{1}{\lambda} = \lambda.$$

Math Review

Orthogonal Transformations (Property 3):

If R is an orthogonal transformation and v is an eigenvector of R, then if w is a vector perpendicular to v, Rw is also perpendicular to v.

Since R^{-1} is also an orthogonal transformation:

$$\begin{aligned}\langle v, Rw\rangle &= \langle R^{-1}v, R^{-1}Rw\rangle \\ &= \langle R^{-1}v, w\rangle \\ &= \frac{1}{\lambda}\langle v, w\rangle \\ &= 0\end{aligned}$$

Math Review

Orthogonal Transformations (Property 4):

If R is an orthogonal transformation and v_1 and v_2 are eigenvectors of R with eigenvalues λ_1 and λ_2, then if $\lambda_1 \neq \lambda_2$, v_1 and v_2 must be perpendicular.

Since R^{-1} is orthogonal we have:

$$\begin{aligned}\langle Rv_1, v_2\rangle &= \langle R^{-1}Rv_1, R^{-1}v_2\rangle \\ &= \langle v_1, R^{-1}v_2\rangle\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}\lambda_1\langle v_1, v_2\rangle &= 1/\lambda_2\langle v_1, v_2\rangle \\ &= \lambda_2\langle v_1, v_2\rangle\end{aligned}$$

$$\Downarrow$$

$$\langle v_1, v_2\rangle = 0$$

Math Review

Classifying the 2D Orthogonal Transformations:

Let V be the space of 2D arrays with the standard basis:

$$\{\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\}$$

with the standard inner product:

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$$

Math Review

Classifying the 2D Orthogonal Transformations:

In the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ we can express a linear operator R as a matrix:

$$\mathbf{R} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\mathbf{R}$ is orthogonal if $\mathbf{R}^{\top} \cdot \mathbf{R}$ is the identity:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \end{aligned}$$

Math Review

Classifying the 2D Orthogonal Transformations:

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{pmatrix} = \begin{pmatrix} \boxed{a^2 + c^2} & ab + cd \\ ab + cd & \boxed{b^2 + d^2} \end{pmatrix}$$

The diagonal entries give rise to the equations:

$$1 = a^2 + c^2$$
$$1 = b^2 + d^2$$

For appropriate θ and ϕ, this gives:

$$a = \cos\theta \qquad c = \sin\theta$$
$$b = \cos\phi \qquad d = \sin\phi$$

Math Review

Classifying the 2D Orthogonal Transformations:

$$\begin{pmatrix} 1 & \boxed{0} \\ \boxed{0} & 1 \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & \boxed{ab + cd} \\ \boxed{ab + cd} & b^2 + d^2 \end{pmatrix}$$

The other equations then become:

$$0 = \cos\theta \cdot \cos\phi + \sin\theta \cdot \sin\phi$$

Or equivalently:

$$0 = \cos(\theta - \phi)$$

Which implies that:

$$\phi = \theta + k\pi + \pi/2$$

Math Review

Classifying the 2D Orthogonal Transformations:

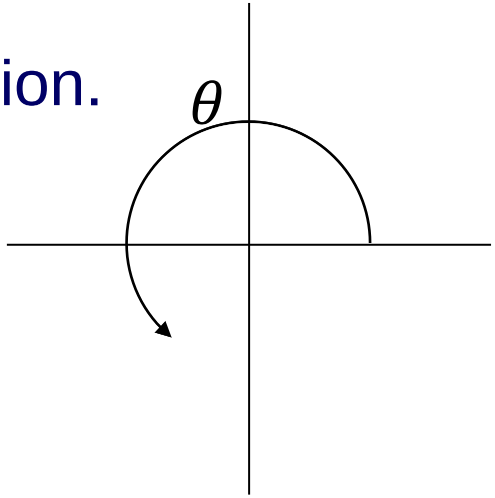
$$\phi = \theta + k\pi + \pi/2$$

If $\mathbf{R}$ is an orthogonal transformation, then in the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ we have one of two cases:

- k is even:

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The determinant is 1 ⇒ this is a rotation.



Math Review

Classifying the 2D Orthogonal Transformations:

$$\phi = \theta + k\pi + \pi/2$$

If $\mathbf{R}$ is an orthogonal transformation, then in the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ we have one of two cases:

- k is even:

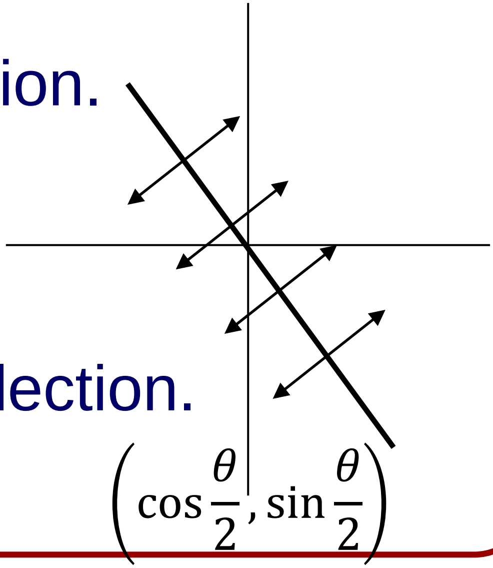
$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

  The determinant is 1 $\Rightarrow$ this is a rotation.

- k is odd:

$$\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

  The determinant is $-1 \Rightarrow$ this is a reflection.

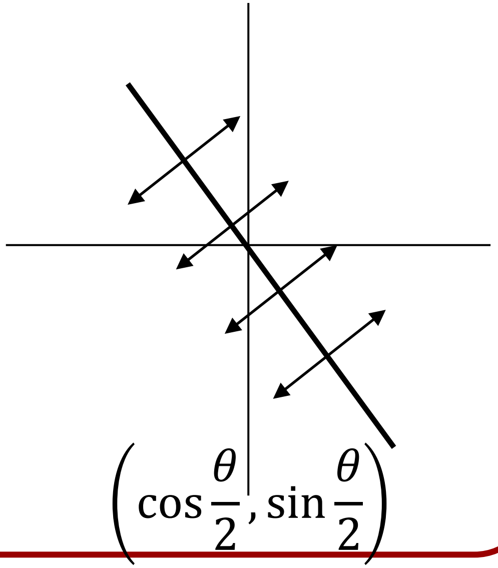


Math Review

Classifying the 2D Orthogonal Transformations:

Claim:

In the case that k is odd, the orthogonal transformation has eigenvalues 1 and -1.



Math Review

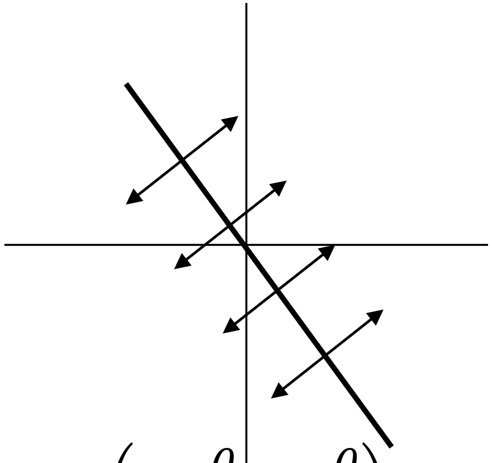
Classifying the 2D Orthogonal Transformations:

Claim:

In the case that k is odd, the orthogonal transformation has eigenvalues 1 and -1.

To compute the eigenvalues, we need to solve for the roots of the polynomial:

$$P_R(\lambda) = \det(R - \lambda \cdot \mathrm{Id})$$



$$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$$

Math Review

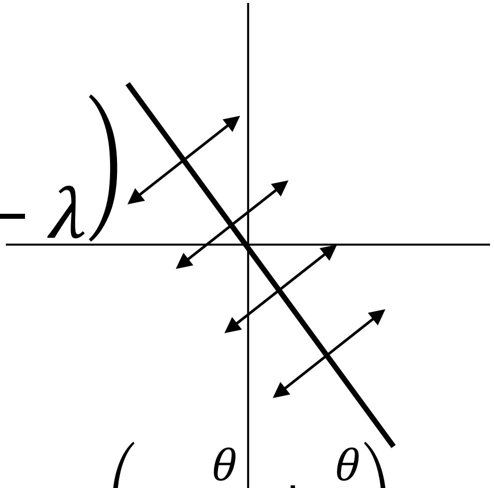
<u>Classifying the 2D Orthogonal Transformations</u>:

<u>Claim</u>:

In the case that k is odd, the orthogonal transformation has eigenvalues 1 and -1.

To compute the eigenvalues, we need to solve for the roots of the polynomial:

$$\begin{aligned} P_R(\lambda) &= \det\begin{pmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{pmatrix} \\ &= \lambda^2 - \cos^2\theta - \sin^2\theta \\ &= \lambda^2 - 1 \end{aligned}$$



$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$

Math Review

<u>Classifying the 2D Orthogonal Transformations</u>:

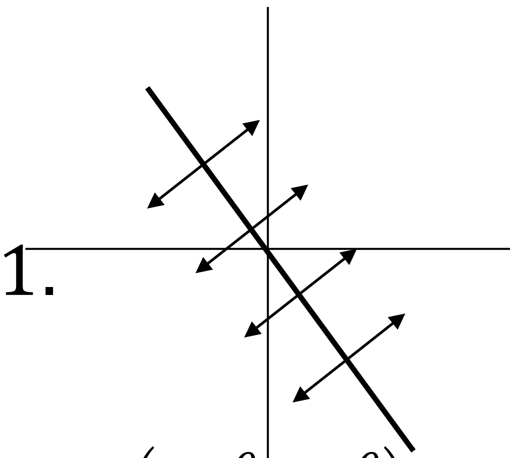
<u>Claim</u>:

In the case that k is odd, the orthogonal transformation has eigenvalues 1 and -1.

To compute the eigenvalues, we need to solve for the roots of the polynomial:

$$P_R(\lambda) = \det(R - \lambda \cdot \mathrm{Id})$$

This polynomial has two roots, $\lambda = \pm 1$.



$$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)$$

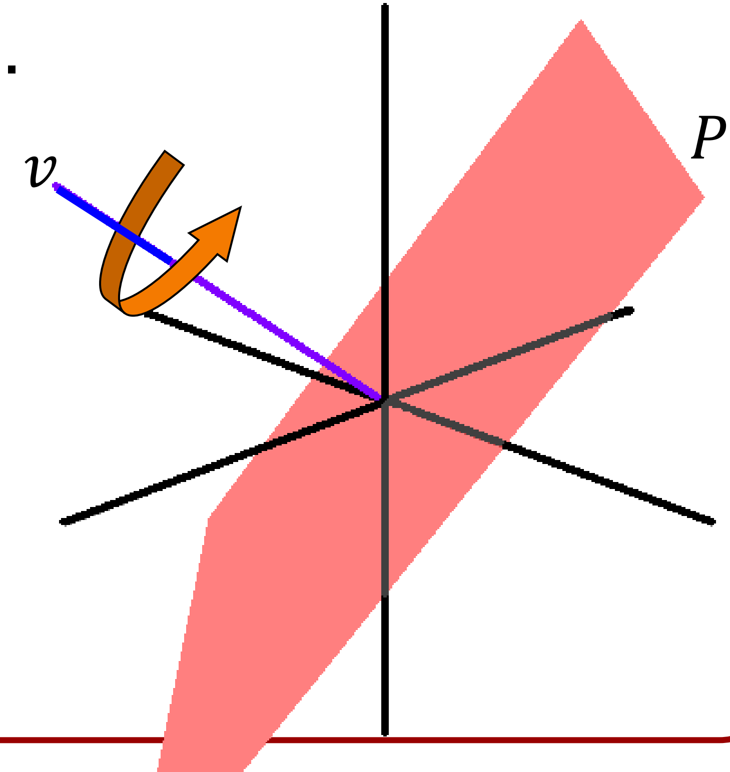
Outline

- Math Review
- Representing 3D Rotations
    - Axis-Angle
    - Euler Angles

Representing 3D Rotations (Axis-Angle)

We will show that any rotation R can be thought of as a rotation about some axis.

In particular, we need to show that every rotation R fixes some vector v and acts as a rotation in the plane P perpendicular to v.



Representing 3D Rotations (Axis-Angle)

Let V be the space of 3D arrays with the standard basis:

$$\{\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)\}$$

with the standard inner product:

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$$

Representing 3D Rotations (Axis-Angle)

In the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can express the linear operator R as a matrix:

$$\mathbf{R} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We can compute the eigenvalues of R by finding the roots of the determinant:

$$P_R(\lambda) = \det\begin{pmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{pmatrix}$$

Representing 3D Rotations (Axis-Angle)

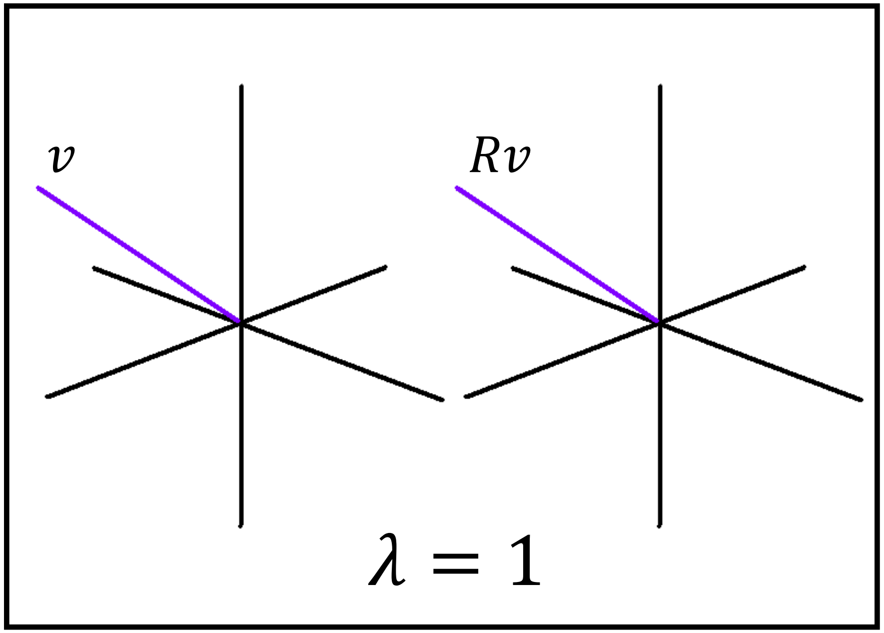
Since $P_R(\lambda)$ has odd degree ($d = 3$), it must have at least one root.

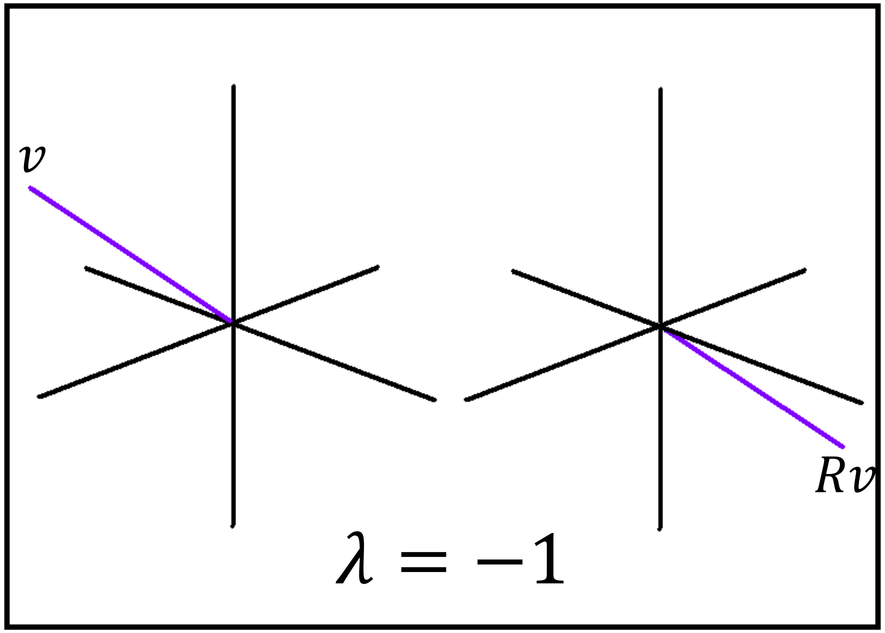
Thus, R has an eigenvector v with eigenvalue λ.

Since R is orthogonal $\lambda = \pm 1$.

Representing 3D Rotations (Axis-Angle)

Thus, for every orthogonal transformation R, there must exist a vector v that is either fixed by R or mapped to its antipode.

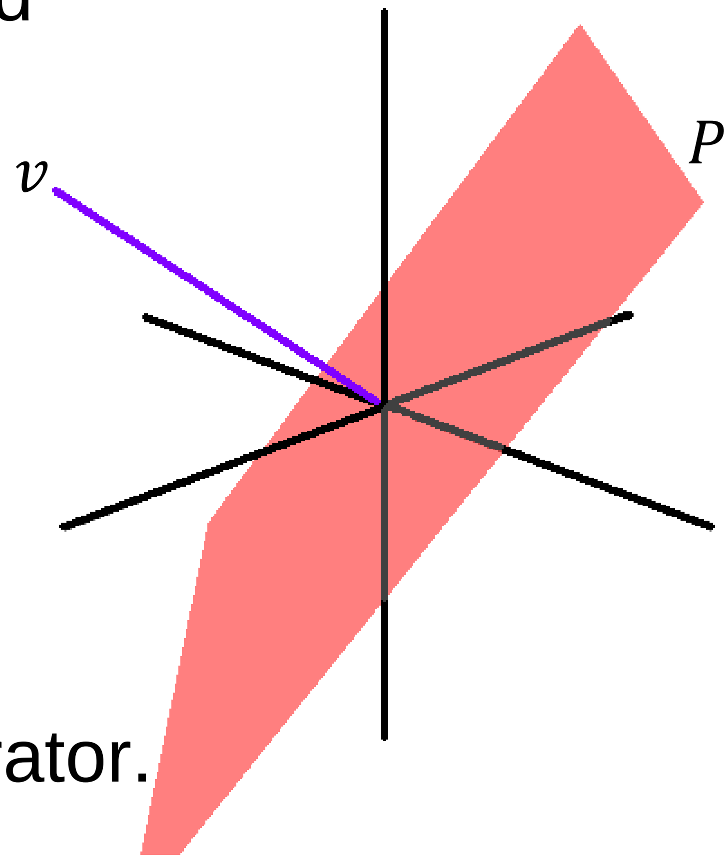




Representing 3D Rotations (Axis-Angle)

What happens to the plane P that is orthogonal to the eigenvector v?

Since R maps the line spanned by v back into itself, and since R is orthogonal, R must map the plane P back into itself.



Since R preserves the inner-product and maps P to itself, the restriction of R to P is a 2D orthogonal operator.

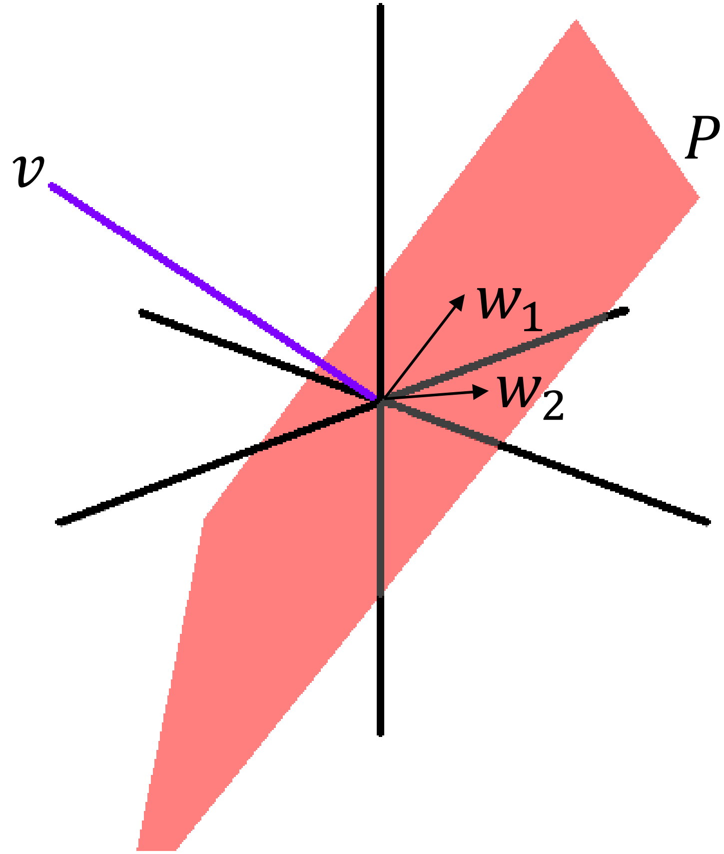
Representing 3D Rotations (Axis-Angle)

Letting $\{w_1, w_2\}$ be an orthonormal basis for the plane P, with respect to the basis $\{v, w_1, w_2\}$, we can express R in matrix form as either:

$$\mathbf{R} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

or:

$$\mathbf{R} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{pmatrix}$$



Representing 3D Rotations (Axis-Angle)

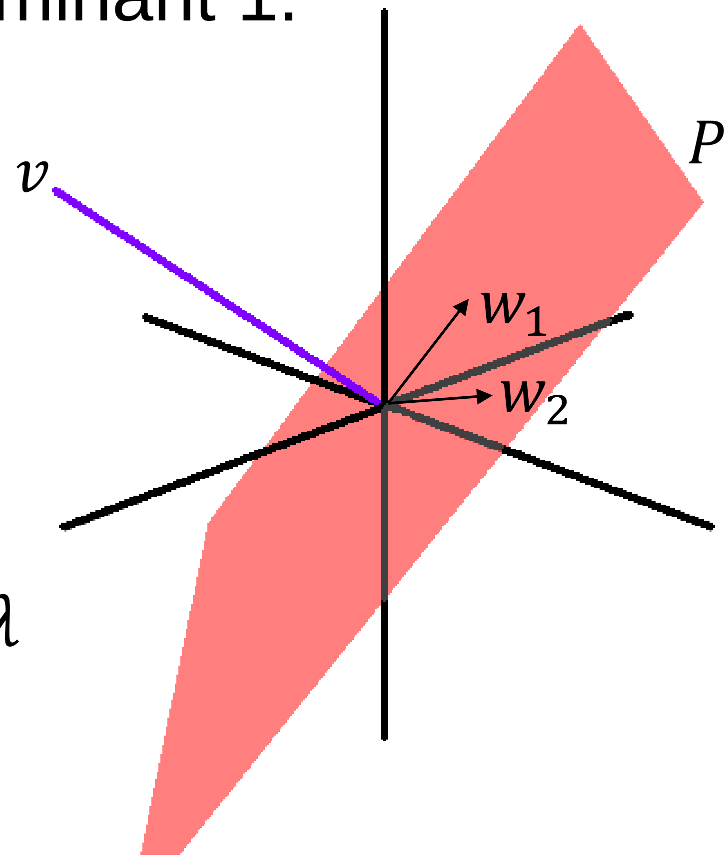
What happens in the case that R is a rotation?

If R is a rotation, then in addition to being orthogonal, it must have determinant 1.

For the two representations of R we get:

$$\det\begin{pmatrix}\lambda & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix} = \lambda$$

$$\det\begin{pmatrix}\lambda & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & \sin\theta & -\cos\theta\end{pmatrix} = -\lambda$$



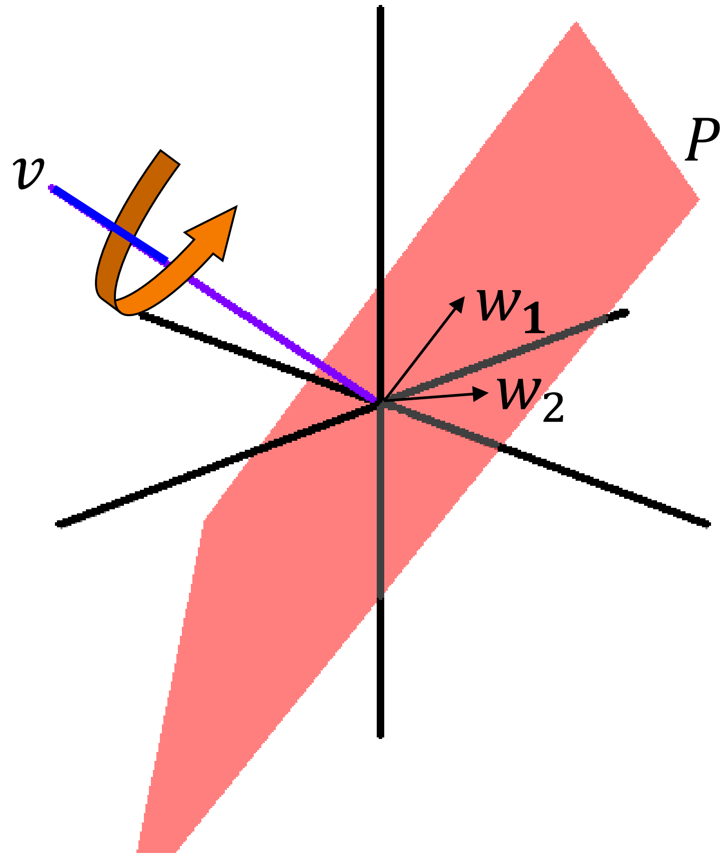
Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

If $\lambda = 1$, we have:

$$\mathbf{R} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

and R is a rotation in the plane P by angle θ.



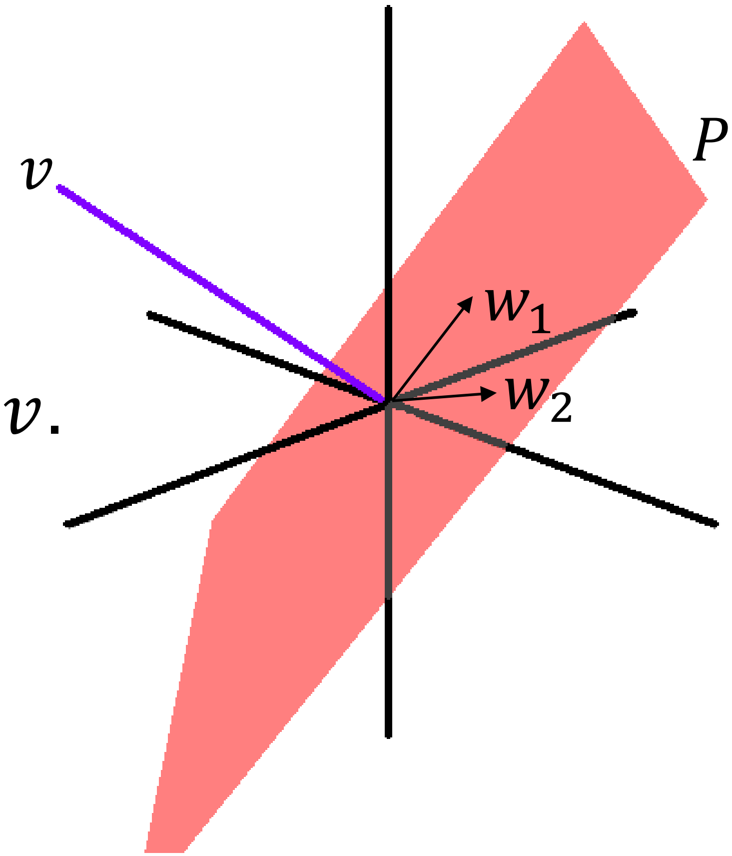
Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

If $\lambda = -1$, we get:

$$\mathbf{R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{pmatrix}$$

and R is the composition of a reflection in the plane P and a flip about the line spanned by v.



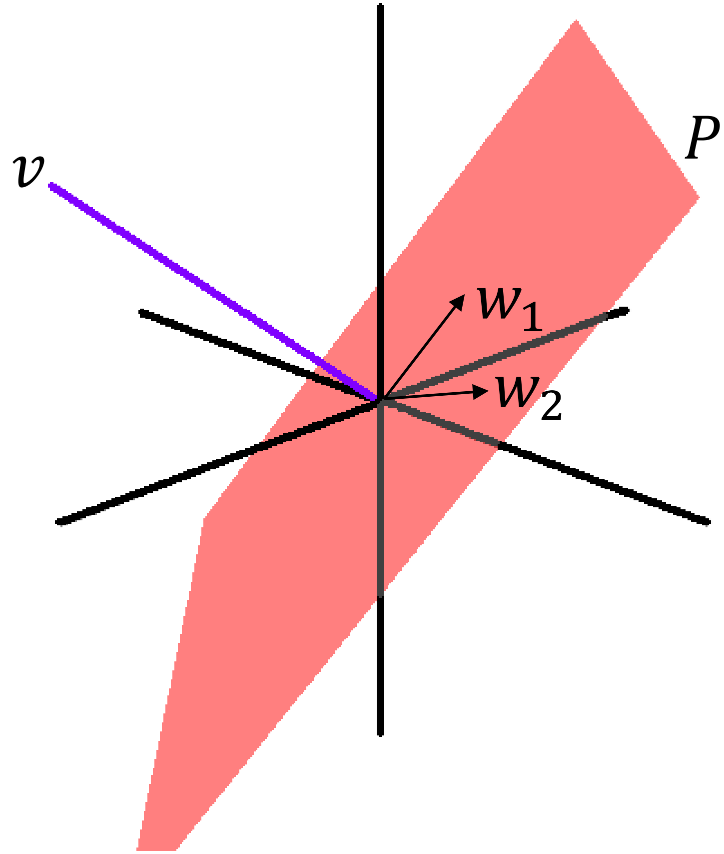
Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

Restricting R to the plane P, in the basis $\{w_1, w_2\}$ we get:

$$\mathbf{R}\Big|_P = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

which has eigenvalues -1 and 1.

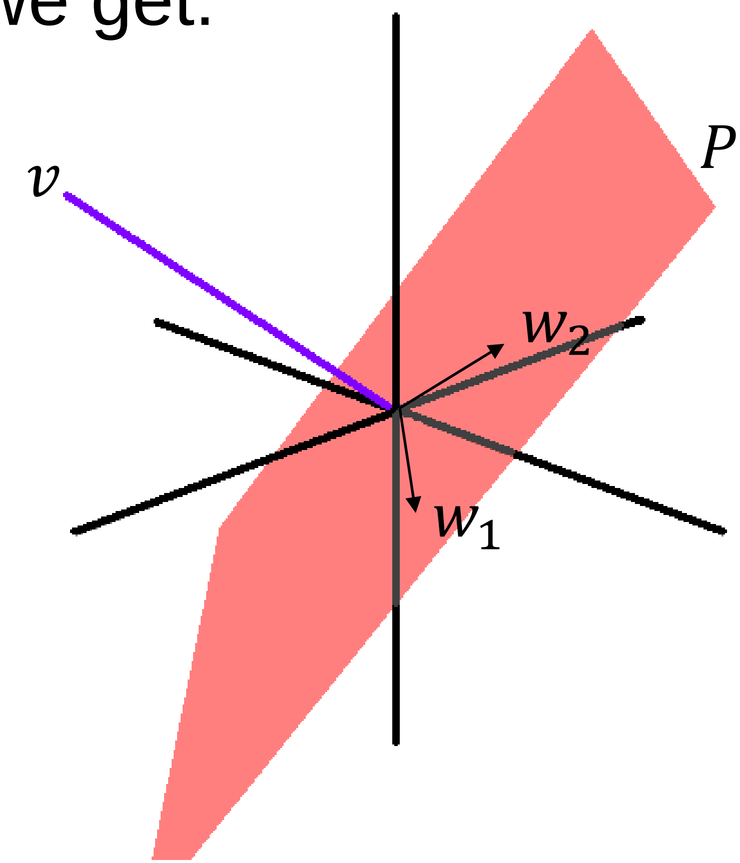


Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

In particular, if we set w_1 and w_2 to be the corresponding eigenvectors, we get:

$$\mathbf{R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

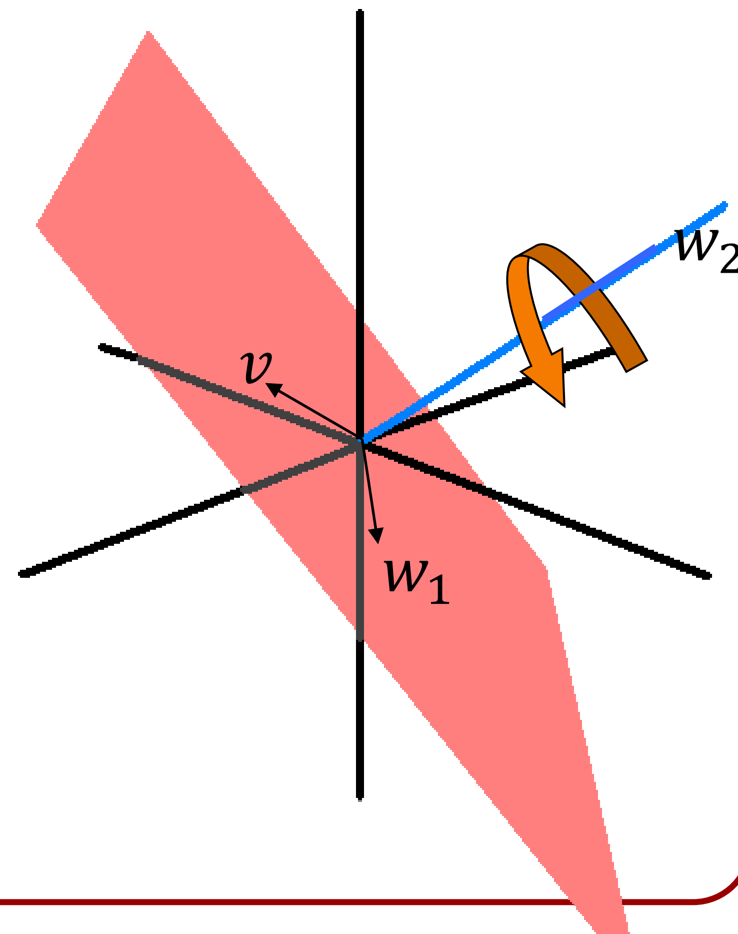


Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

$$\mathbf{R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ R is a rotation by $180°$ in the plane spanned by v and w_1.

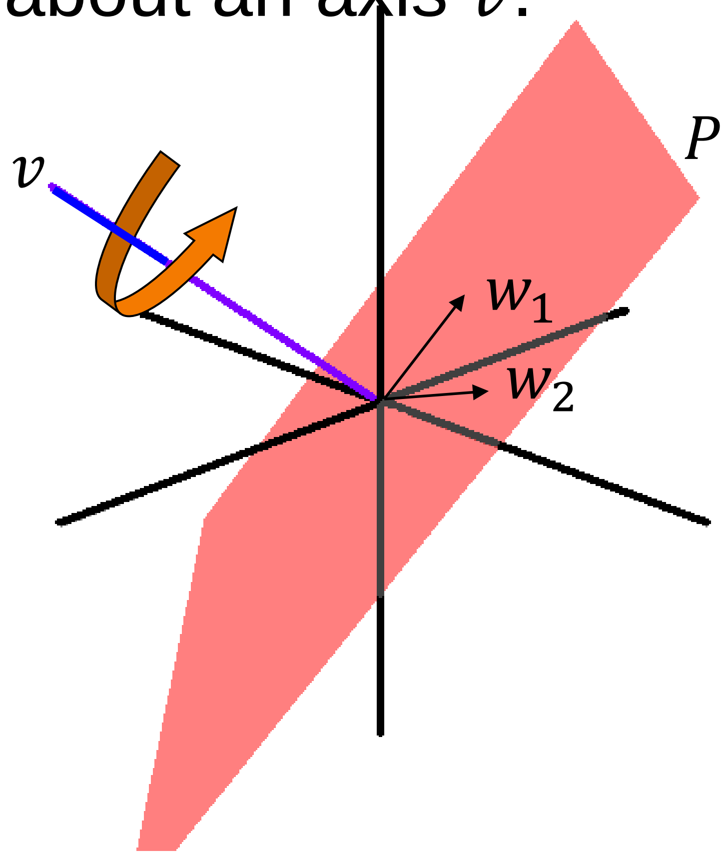


Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

So, in both cases, the rotation R can be realized as a rotation by some angle θ about an axis v.

That is, R sends the vector v back into itself and rotates vectors in the plane that is perpendicular to v by the angle θ.



Representing 3D Rotations (Axis-Angle)

What happens in the case that R is a rotation?

This motivates a representation of rotations by specifying the axis about which the rotation occurs and the angle of the 2D rotation.

Outline

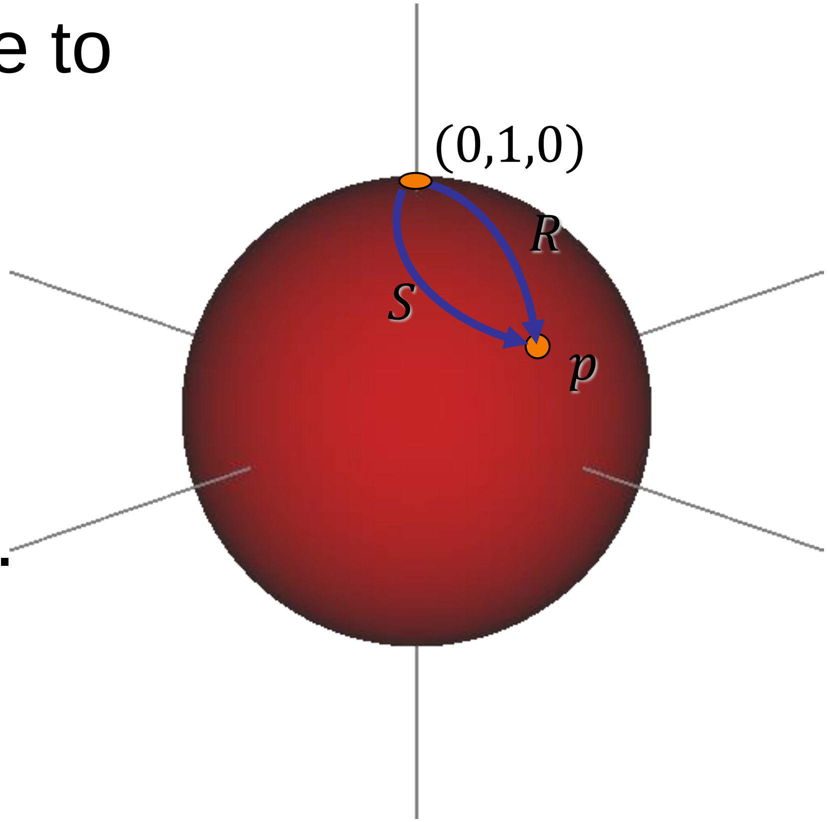
- Math Review
- Representing 3D Rotations
  - Axis-Angle
  - Euler Angles

Representing 3D Rotations (Euler)

We will consider a representation of rotations that describe what the rotation does to $(0,1,0)$.

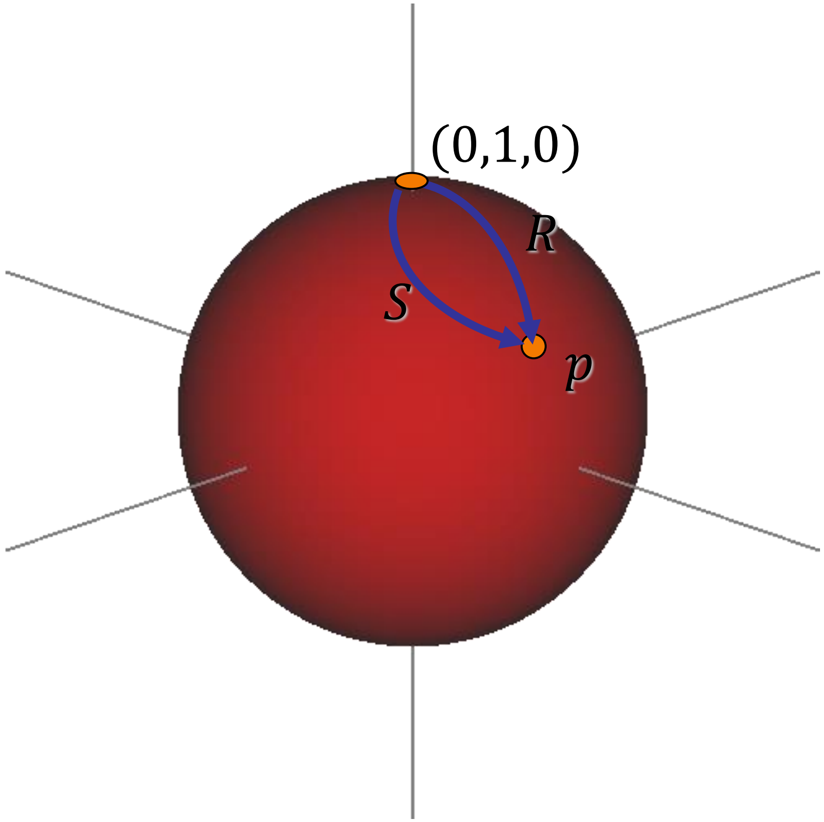
Given a rotation R, if we know that R maps the North pole to the point p, is that enough information to define R?

No. There can be many different rotations that all send $(0,1,0)$ to the point p.



Representing 3D Rotations (Euler)

In particular, if R_p is some (fixed) rotation taking the North pole to p and S is any rotation taking the North pole to p, then $R_p^{-1} \cdot S$ must map the North pole back to itself.



Representing 3D Rotations (Euler)

Denote by $R_y(\psi)$ the rotation about the y-axis (North pole) by ψ degrees:

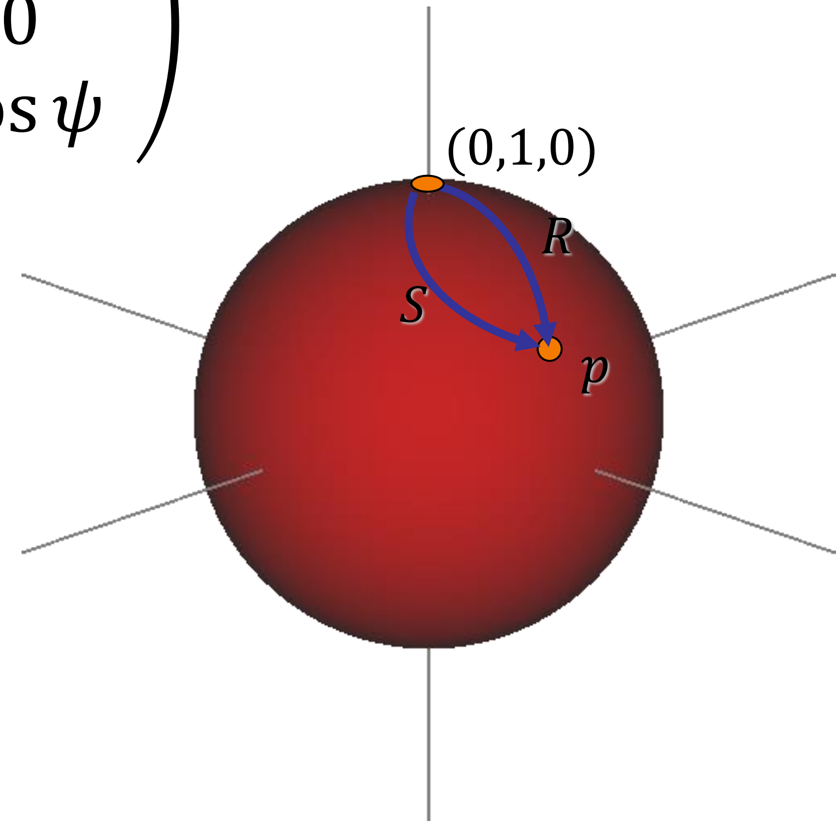
$$\mathbf{R}_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}$$

Then:

$$R_p^{-1} \cdot S = R_y(\psi)$$

$$\Updownarrow$$

$$S = R_p \cdot R_y(\psi)$$



Representing 3D Rotations (Euler)

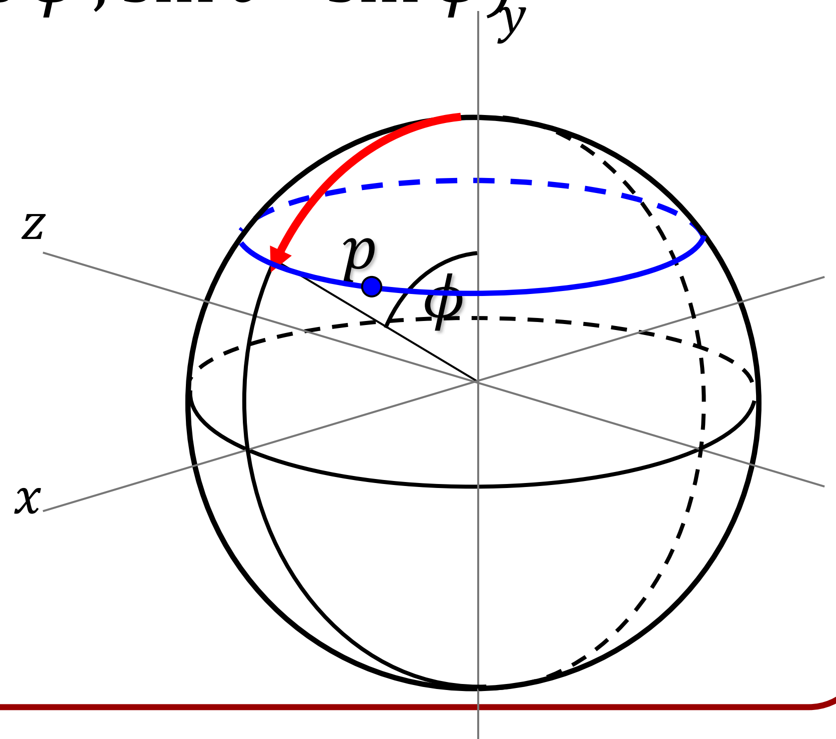
In order to represent all rotations, we need to find an expression for R_p -- some rotation that sends the North pole to the point p.

Let (θ, ϕ) be the spherical coordinates of p:

$$p = (\cos\theta \cdot \sin\phi, \cos\phi, \sin\theta \cdot \sin\phi)$$

The point p must lie on the circle about the y-axis with height $\cos\phi$.

We can get $(0,1,0)$ to this circle with a rotation by an angle of ϕ about the z-axis.



Representing 3D Rotations (Euler)

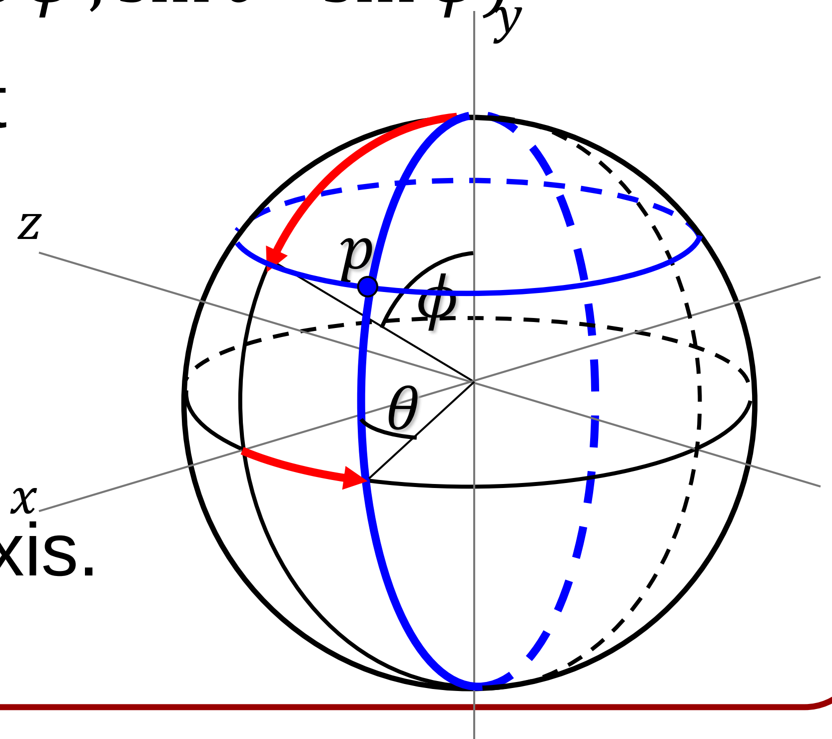
In order to represent all rotations, we need to find an expression for R_p -- some rotation that sends the North pole to the point p.

Let (θ,ϕ) be the spherical coordinates of p:

$$p = (\cos\theta \cdot \sin\phi, \cos\phi, \sin\theta \cdot \sin\phi)$$

We also know that the point p makes an angle of θ with the xz-plane.

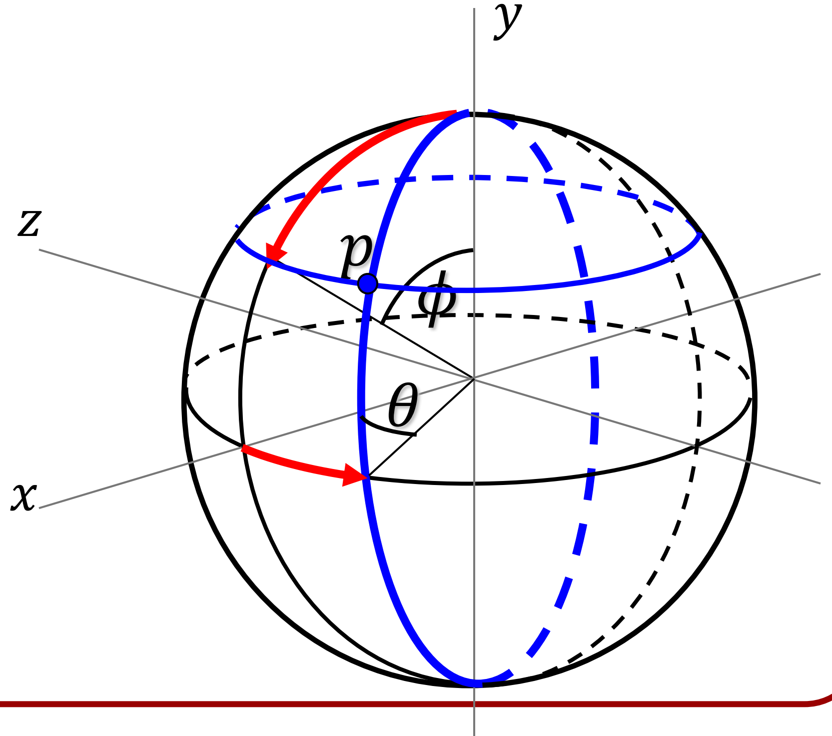
We can get the rotation of $(0,1,0)$ to p by rotating by an angle of θ about the y-axis.



Representing 3D Rotations (Euler)

Thus, when the spherical coordinates of the point p are (θ, ϕ), we can rotate $(0,1,0)$ to p by:

- First rotating by ϕ degrees about the z-axis, and
- Then rotating by θ degrees about the y-axis.



Representing 3D Rotations (Euler)

Since every rotation R can be described by a rotation about the y-axis, followed by a rotation that maps $(0,1,0)$ to $p = \Phi(\theta, \phi)$, we have:

$$R = R_y(\theta) \cdot R_z(\phi) \cdot R_y(\psi)$$

where $R_y(\alpha)$ is the rotation about the y-axis by α, and $R_z(\beta)$ is the rotation about the z-axis by β.

Representing 3D Rotations (Euler)

In matrix form, the triplet of angles (θ, ϕ, ψ) represents the rotation:

$$\mathbf{R}(\theta,\phi,\psi) = \underbrace{\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{\text{Rotation sending} \\ (0,1,0) \to p = \Phi(\theta,\phi)}} \underbrace{\begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}}_{\substack{\text{Rotation about} \\ \text{the } y\text{-axis by } \psi}}$$

This is the <u>Euler Angle</u> parameterization of 3D rotations.