FFTs in Graphics and Vision

The Spherical Laplacian
Outline

• Stokes’ Theorem
• Tangent Spaces
• Gradients
• The Spherical Laplacian
• Applications
Stokes’ Theorem

Stokes’ Theorem equates the integral of a vector field over the boundary of a region to the integral of the divergence of the vector field over the region itself:

$$
\int_{\partial V} \langle \vec{F}, \vec{n} \rangle \cdot dA = \int_{V} (\nabla \cdot \vec{F}) \, dV
$$

where $\vec{n}$ is the normal at the boundary.
Stokes’ Theorem

Stokes’ Theorem equates the integral of a vector field over the boundary of a region to the integral of the divergence of the vector field over the region itself:

\[
\int_{\partial V} \langle \vec{F}, \vec{n} \rangle \cdot dA = \int_{V} \left( \nabla \cdot \vec{F} \right) dV
\]
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Tangent Spaces

Given a curve $C(t) = (x(t), y(t))$, the tangent line to the curve at a point $p_0 = C(t_0)$ is the line passing through $p_0$ with direction $C'(t_0) = (x'(t_0), y'(t_0))$.

This is the line that most closely approximates the curve $C(t)$ at the point $p_0$. 
Tangent Spaces

Often, we want a unit vector.

In this case, we normalize:

\[ T_C(t) = \frac{C'(t)}{|C'(t)|} \]
Tangent Spaces

Given a surface $S(u, v)$ the tangent plane to the curve at a point $p_0 = S(u_0, v_0)$ is the plane passing through $p_0$, parallel to the plane spanned by:

$$\frac{\partial S(u, v)}{\partial u} \bigg|_{(u_0, v_0)}$$

and

$$\frac{\partial S(u, v)}{\partial v} \bigg|_{(u_0, v_0)}$$

This is the plane that most closely approximates $S(u, v)$ at the point $p_0$. 

$$\Phi(\theta, \phi) = (\cos \theta \cdot \sin \phi, \cos \phi, \sin \theta \cdot \sin \phi)$$
Tangent Spaces

In the case of the sphere, points are parameterized by the equation:

$$\Phi(\theta, \phi) = (\cos \theta \cdot \sin \phi, \cos \phi, \sin \theta \cdot \sin \phi)$$

⇒ The two tangent directions are:

$$\frac{\partial \Phi}{\partial \theta} = (-\sin \theta \cdot \sin \phi, 0, \cos \theta \cdot \sin \phi)$$

$$\frac{\partial \Phi}{\partial \phi} = (\cos \theta \cdot \cos \phi, -\sin \phi, \sin \theta \cdot \cos \phi)$$
Tangent Spaces

\[
\frac{\partial \Phi}{\partial \theta} = (- \sin \theta \cdot \sin \phi, 0, \cos \theta \cdot \sin \phi)
\]

\[
\frac{\partial \Phi}{\partial \phi} = (\cos \theta \cdot \cos \phi, -\sin \phi, \sin \theta \cdot \cos \phi)
\]

Taking the dot-product of the tangent vectors gives:

\[
\left< \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \theta} \right> = \sin^2 \theta \cdot \sin^2 \phi + \cos^2 \theta \cdot \sin^2 \phi
\]

\[
= \sin^2 \phi \cdot (\sin^2 \theta + \cos^2 \theta)
\]

\[
= \sin^2 \phi
\]
Tangent Spaces

\[
\frac{\partial \Phi}{\partial \theta} = (- \sin \theta \cdot \sin \phi, 0, \cos \theta \cdot \sin \phi)
\]

\[
\frac{\partial \Phi}{\partial \phi} = (\cos \theta \cdot \cos \phi, -\sin \phi, \sin \theta \cdot \cos \phi)
\]

Taking the dot-product of the tangent vectors gives:

\[
\begin{bmatrix} \frac{\partial \Phi}{\partial \theta} & \frac{\partial \Phi}{\partial \phi} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \Phi}{\partial \theta}^\top \end{bmatrix} = \sin^2 \theta
\]

\[
\begin{bmatrix} \frac{\partial \Phi}{\partial \theta} & \frac{\partial \Phi}{\partial \phi} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \Phi}{\partial \phi}^\top \end{bmatrix} = \cos^2 \theta \cdot \cos^2 \phi + \sin^2 \phi + \sin^2 \theta \cdot \cos^2 \phi
\]

\[
= (\cos^2 \theta + \sin^2 \theta) \cdot \cos^2 \phi + \sin^2 \phi
\]

\[
= \cos^2 \phi + \sin^2 \phi
\]

\[
= 1
\]
Tangent Spaces

\[
\frac{\partial \Phi}{\partial \theta} = (- \sin \theta \cdot \sin \phi, 0, \cos \theta \cdot \sin \phi)
\]

\[
\frac{\partial \Phi}{\partial \phi} = (\cos \theta \cdot \cos \phi, -\sin \phi, \sin \theta \cdot \cos \phi)
\]

Taking the dot-product of the tangent vectors gives:

\[
\begin{bmatrix}
\frac{\partial \Phi}{\partial \theta} & \frac{\partial \Phi}{\partial \theta} \\
\frac{\partial \Phi}{\partial \phi} & \frac{\partial \Phi}{\partial \phi}
\end{bmatrix} = \sin^2 \theta
\]

\[
\begin{bmatrix}
\frac{\partial \Phi}{\partial \phi} & \frac{\partial \Phi}{\partial \phi} \\
\frac{\partial \Phi}{\partial \phi} & \frac{\partial \Phi}{\partial \phi}
\end{bmatrix} = 1
\]

\[
\begin{bmatrix}
\frac{\partial \Phi}{\partial \theta} & \frac{\partial \Phi}{\partial \theta} \\
\frac{\partial \Phi}{\partial \phi} & \frac{\partial \Phi}{\partial \phi}
\end{bmatrix} = - \sin \theta \cdot \cos \theta \cdot \sin \phi \cdot \cos \phi + \sin \theta \cdot \cos \theta \cdot \sin \phi \cdot \cos \phi
\]

\[
= 0
\]
Tangent Spaces

\[
\begin{bmatrix}
\frac{\partial \Phi}{\partial \theta} & \frac{\partial \Phi}{\partial \theta} \\
\frac{\partial \Phi}{\partial \phi} & \frac{\partial \Phi}{\partial \phi}
\end{bmatrix} = \sin^2 \theta
\]

\[
\begin{bmatrix}
\frac{\partial \Phi}{\partial \theta} \\
\frac{\partial \Phi}{\partial \phi}
\end{bmatrix} = 1
\]

\[
\begin{bmatrix}
\frac{\partial \Phi}{\partial \theta} \\
\frac{\partial \Phi}{\partial \phi}
\end{bmatrix} = 0
\]

So, the vectors:

\[
\Phi_\theta(\theta, \phi) = \frac{1}{\sin \theta} \frac{\partial \Phi}{\partial \theta} \quad \text{and} \quad \Phi_\phi(\theta, \phi) = \frac{\partial \Phi}{\partial \phi}
\]

form an orthonormal basis for the tangent plane to the sphere at the point \(\Phi(\theta, \phi)\).
Outline

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Function Gradients

Given a function, $f$, the gradient of $f$ at $p$, $\nabla f|_p$, is a vector in the tangent plane at $p$ which tells us how the function changes as we move in different directions.

Given a function $f$ and given a direction $v$:

$$\frac{d}{dt} \bigg|_{t=0} f(p + tv) = \left\langle \nabla f|_p , v \right\rangle$$
Function Gradients

To compute the gradient, we can choose two orthonormal unit vectors $u$ and $v$, and set:

$$\nabla f \bigg|_p = \left. \frac{d}{dt} \right|_{t=0} f(p + tu) \cdot u + \left. \frac{d}{dt} \right|_{t=0} f(p + tv) \cdot v$$
Curve Gradients

Given a curve $C(t)$, and given a function $f(t)$ the gradient of the function is a vector field on the curve telling us how the function changes as we move along the curve.
Curve Gradients

Example:

Let $C$ be the curve defined by:

$$C(t) = (t, t^2)$$

and let $f(t)$ be the function on the curve defined by:

$$f(t) = t$$

What is the gradient $\nabla_C f$ of $f$ along the curve?
Curvature Gradients

Example:

Note that:

\[ \nabla_c f \neq 1 \cdot T_c(t) \]

This would imply that at any point on the curve moving a unit distance forward would change the value by a constant amount.

\[ C(t) = (t, t^2) \]
\[ f(t) = t \]
Curve Gradients

Example:

Note that:
\[ \nabla_c f \neq 1 \cdot T_c(t) \]

As we move from \( t = 1 \) to \( t = 2 \), the function changes by a value of 1.
Similarly, as we move from \( t = 10 \) to \( t = 11 \), the function changes by a value of 1.

But in the first case, we moved a distance of:
\[ d_1 \approx \| C(2) - C(1) \| = \sqrt{1^2 + 3^2} \]
Curve Gradients

Example:

Note that:

\[ \nabla_c f \neq 1 \cdot T_c(t) \]

As we move from \( t = 1 \) to \( t = 2 \), the function changes by a value of 1.

Similarly, as we move from \( t = 10 \) to \( t = 11 \), the function changes by a value of 1.

And in the second case, we moved a distance of:

\[ d_2 \approx \| C(10) - C(11) \| = \sqrt{1^2 + 21^2} \]
Curve Gradients

Example:

We need to measure the ratio of the change in $f$ over the distance traveled:

\[
\frac{f(t + \varepsilon) - f(t)}{|C(t + \varepsilon) - C(t)|} \cdot T_C(t)
\]

\[
\downarrow
\]

\[
\frac{f'(t)}{|C'(t)|} \cdot T_C(t)
\]

\[
= \frac{1}{\sqrt{1 + 2t}} \cdot T_C(t)
\]

\[
C(t) = (t, t^2)
\]

\[
f(t) = t
\]
Spherical Gradients

Given a function on the sphere, \( f(\theta, \phi) \), we would like to compute the gradient:

\[
\nabla f \bigg|_{(\theta, \phi)}
\]

We need to pick two directions in parameter space whose images on the surface of the sphere are orthogonal.

The lines of longitude and latitude, defined by the derivatives w.r.t. \( \theta \) and \( \phi \) are two such directions:
Spherical Gradients

We could try taking the partial derivatives in the $\theta$ and $\phi$ directions:

$$\nabla f \bigg|_{(\theta, \phi)} = \left( \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right)$$

But this introduces bias!

Shifting by a constant $\varepsilon$ will move us different distances depending on where we are on the sphere.
Spherical Gradients

How does the scale change as we change $\theta$ or $\phi$ by a value of $\varepsilon$?

At the point $p = \Phi(\theta, \phi)$, changing the value of $\theta$ by $\varepsilon$, moves us a distance of $\varepsilon r$ along the circle about the $y$-axis, where $r$ is the radius of the circle.

On the sphere, the radius is:

$$r(\phi) = \sin \phi$$
Spherical Gradients

How does the scale change as we change $\theta$ or $\phi$ by a value of $\varepsilon$?

At the point $p = \Phi(\theta, \phi)$, changing the value of $\phi$ by $\varepsilon$, moves us a distance of $\varepsilon$ along a great circle regardless of where on the sphere we are:
Spherical Gradients

Taking the scaling into account, we get:

\[ \nabla f \bigg|_{(\theta, \phi)} = \left( \frac{f(\theta + \varepsilon, \phi) - f(\theta, \phi)}{\varepsilon \sin \phi}, \frac{f(\theta, \phi + \varepsilon) - f(\theta, \phi)}{\varepsilon} \right) \]

\[ \Downarrow \]

\[ \nabla f \bigg|_{(\theta, \phi)} = \left( \frac{1}{\sin \phi} \frac{\partial f}{\partial \theta'}, \frac{\partial f}{\partial \phi} \right) \]
Outline

- Stokes’ Theorem
- Tangent Spaces
- Gradients
- The Spherical Laplacian
- Applications
The Spherical Laplacian

Recall:

The Laplacian operator is self-adjoint (symmetric)

⇒ There is an orthogonal basis of eigenvectors.

The Laplacian operator commutes with rotations

⇒ If $V^\lambda$ are the eigenfunctions of the Laplacian with eigenvalue $\lambda$, rotations fix $V^\lambda$.

⇒ The irreducible representations are subspaces of the $V^\lambda$. 
The Spherical Laplacian

This implies that for a fixed degree $l$, the spherical harmonics of degree $l$:

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^{|m|}(\cos \phi) \quad |m| \leq l$$

must be eigenvectors of the Laplacian with the same eigenvalue.

1. What is the Laplacian?
2. What are the eigenvalues?
The Spherical Laplacian

How do we compute the Laplacian of a spherical function $f(\theta, \phi)$?

Recall:
The Laplacian of a function is the divergence of its gradient:

$$\Delta f = \nabla \cdot (\nabla f)$$
The Spherical Laplacian

By Stokes’ Theorem, we can compute the integral of the Laplacian (the divergence of the gradient) over a region by computing the surface integral of the gradient field over the boundary.
The Spherical Laplacian

Consider the “square” on the sphere with vertices $(\theta, \phi)$, $(\theta + \varepsilon, \phi)$, $(\theta + \varepsilon, \phi + \varepsilon)$, and $(\theta, \phi + \varepsilon)$.

The integral of the Laplacian is approximately:

$$\int_{\mathcal{R}} \Delta f \, dR \approx \text{Area}(\mathcal{R}) \cdot \Delta f (\theta, \phi)$$

$$= \varepsilon^2 \cdot \sin \phi \cdot \Delta f (\theta, \phi)$$
The Spherical Laplacian

Consider the “square” on the sphere with vertices $(\theta, \phi)$, $(\theta + \varepsilon, \phi)$, $(\theta + \varepsilon, \phi + \varepsilon)$, and $(\theta, \phi + \varepsilon)$.

On the curve $c_1$ the boundary integral of the Laplacian is approximately:

$$\int_{c_1} \langle \nabla f, \vec{n} \rangle \cdot dA \approx \text{Length}(c_1) \cdot \langle \nabla f, (0,1) \rangle$$

$$= \varepsilon \cdot \sin(\phi + \varepsilon) \cdot \left( \frac{1}{\sin(\phi + \varepsilon)} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right) (0,1)$$

$$= \varepsilon \cdot \sin(\phi + \varepsilon) \cdot \frac{\partial f}{\partial \phi} (\theta, \phi + \varepsilon)$$
The Spherical Laplacian

Consider the “square” on the sphere with vertices 
\((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon), \) and \((\theta, \phi + \varepsilon)\).

On the curve \(c_2\) the boundary integral of the Laplacian is approximately:

\[
\int_{c_2} \langle \nabla f, \vec{n} \rangle \cdot dA \approx -\varepsilon \cdot \sin \phi \cdot \frac{\partial f}{\partial \phi} (\theta, \phi)
\]
The Spherical Laplacian

Consider the “square” on the sphere with vertices \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon),\) and \((\theta, \phi + \varepsilon).\)

On the curve \(c_3\) the boundary integral of the Laplacian is approximately:

\[
\int_{c_3} \langle \nabla f, \vec{n} \rangle \cdot dA \approx \text{Length}(c_3) \cdot \langle \nabla f, (1,0) \rangle
\]

\[
= \varepsilon \left( \frac{1}{\sin \phi} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi} \right), (1,0)
\]

\[
= \varepsilon \frac{1}{\sin \phi} \frac{\partial f}{\partial \theta} (\theta + \varepsilon, \phi)
\]
The Spherical Laplacian

Consider the “square” on the sphere with vertices \((\theta, \phi), (\theta + \varepsilon, \phi), (\theta + \varepsilon, \phi + \varepsilon),\) and \((\theta, \phi + \varepsilon)\).

On the curve \(c_4\) the boundary integral of the Laplacian is approximately:

\[
\int_{c_4} \langle \nabla f, \vec{n} \rangle \cdot dA \approx -\varepsilon \frac{1}{\sin \phi} \frac{\partial f}{\partial \theta} (\theta, \phi)
\]
Consider the “square” on the sphere with vertices $(\theta, \phi)$, $(\theta + \varepsilon, \phi)$, $(\theta + \varepsilon, \phi + \varepsilon)$, and $(\theta, \phi + \varepsilon)$.

Summing these together, we approximate the boundary integral by:

$$\int_{\partial R} \langle \nabla f, \hat{n} \rangle \cdot dA \approx \varepsilon \left( \frac{1}{\sin \phi} \left[ \frac{\partial f}{\partial \theta} (\theta + \varepsilon, \phi) - \frac{\partial f}{\partial \theta} (\theta, \phi) \right] \right) +$$

$$\varepsilon \left( \sin(\phi + \varepsilon) \frac{\partial f}{\partial \phi} (\theta, \phi + \varepsilon) - \sin \phi \frac{\partial f}{\partial \phi} (\theta, \phi) \right)$$
The Spherical Laplacian

Consider the “square” on the sphere with vertices $(\theta, \phi)$, $(\theta + \varepsilon, \phi)$, $(\theta + \varepsilon, \phi + \varepsilon)$, and $(\theta, \phi + \varepsilon)$.

Summing these together, we approximate the boundary integral by:

\[
\int_{\partial R} \langle \nabla f, \vec{n} \rangle \cdot dA \approx \varepsilon \left( \frac{1}{\sin \phi} \varepsilon \frac{\partial}{\partial \theta} \left[ \frac{\partial f}{\partial \theta} (\theta, \phi) \right] \right) + \varepsilon \left( \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} (\theta, \phi) \right] \right)
\]

\[
= \varepsilon^2 \left( \frac{1}{\sin \phi} \frac{\partial}{\partial \theta} \left[ \frac{\partial f}{\partial \theta} \right] \right) + \varepsilon^2 \left( \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} \right] \right)
\]
The Spherical Laplacian

The boundary integral can be approximated by:

\[
\int_{\partial R} \langle \nabla f, \vec{n} \rangle \cdot dA \approx \varepsilon^2 \left( \frac{1}{\sin \phi} \frac{\partial}{\partial \theta} \left[ \frac{\partial f}{\partial \theta} \right] \right) + \varepsilon^2 \left( \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} \right] \right)
\]

The surface integral can be approximated by:

\[
\int_{R} \Delta f \, dR \approx \varepsilon^2 \cdot \sin \phi \cdot \Delta f (\theta, \phi)
\]

⇒ Applying Stokes’ Theorem we get:

\[
\Delta f = \frac{1}{\sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} \right]
\]
The Spherical Laplacian

\[ \Delta f = \frac{1}{\sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} \right] \]

To compute the eigenvalues of the Laplacian associated to the spherical harmonics, we need to compute:

\[ \Delta Y^m_l(\theta, \phi) = \Delta \left( e^{im\theta} \cdot P^{|m|}_l(\cos \phi) \right) \]
The Spherical Laplacian

\[
\Delta f = \frac{1}{\sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} \right]
\]

Taking the derivative with respect to \( \theta \) is reasonably easy:

\[
\frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2} Y_l^m(\theta, \phi) = \frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2} \left( e^{im\theta} \cdot P_l^m(\cos \phi) \right)
\]

\[
= -\frac{m^2}{\sin^2 \phi} e^{im\theta} \cdot P_l^m(\cos \phi)
\]

\[
= -\frac{m^2}{\sin^2 \phi} Y_l^m(\theta, \phi)
\]
The Spherical Laplacian

\[ \Delta f = \frac{1}{\sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial f}{\partial \phi} \right] \]

Taking the derivative with respect to \( \phi \) is more complicated:

\[
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial}{\partial \phi} Y_l^m(\theta, \phi) \right] = \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \cdot \frac{\partial}{\partial \phi} \left( e^{im\theta} \cdot P_l^{|m|}(\theta, \phi) \right) \right] \\
= e^{im\theta} \frac{\partial}{\partial \phi} \left[ \sin \phi \cdot \frac{\partial P_l^{|m|}}{\partial \phi} (\cos \phi) \right]
\]

This requires taking the derivatives of the associated Legendre polynomials.
Associated Legendre Polynomials

Recall:

The associated Legendre polynomials are defined by the (somewhat hairy) formula:

\[ P_l^m(x) = \frac{(-1)^m}{2^l l!} \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} \left(x^2 - 1\right)^l \]
Associated Legendre Polynomials

One can show, (but we won’t) that the associated Legendre polynomials satisfy the identities:

\[
\frac{d\mathbf{P}_l^m(\cos \phi)}{d\phi} = \frac{l \cdot \cos \phi \cdot \mathbf{P}_l^m(\cos \phi) - (l + m) \cdot \mathbf{P}_{l-1}^m(\cos \phi)}{\sin \phi}
\]

\[
0 = (l - m) \cdot \mathbf{P}_l^m(\cos \phi) - \cos \phi \cdot (2l - 1) \cdot \mathbf{P}_{l-1}^m(\cos \phi) + (l + m - 1) \cdot \mathbf{P}_{l-2}^m(\cos \phi)
\]
The Spherical Laplacian

\[
\frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2} Y_l^m(\theta, \phi) = -\frac{m^2}{\sin^2 \phi} Y_l^m(\theta, \phi)
\]

Plugging these identities into the equation for the Laplacian, we get (see appendix):

\[
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial Y_l^m}{\partial \phi} \right] = (-l^2 - l) \cdot Y_l^m + m^2 \cdot \frac{Y_l^m}{\sin^2 \phi}
\]

\[
\downarrow
\]

\[
\frac{1}{\sin^2 \phi} \frac{\partial^2 Y_l^m}{\partial \theta^2} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial Y_l^m}{\partial \phi} \right] = -l \cdot (l + 1) \cdot Y_l^m
\]

\[
\downarrow
\]

\[
\Delta Y_l^m(\theta, \phi) = -l \cdot (l + 1) \cdot Y_l^m(\theta, \phi)
\]
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Smoothing

In the case of a functions on a plane, we had Newton’s Law of Cooling:

“The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.”
Smoothing

In the case of a functions on a plane, we had Newton’s Law of Cooling:

“The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.”

This is expressed by the PDE:

\[
\frac{\partial F}{\partial t} = \eta \cdot \Delta F
\]
Smoothing

Using the fact that the spherical harmonics are eigenvectors of the Laplacian, we get solutions of the form:

\[ \mathbf{F}_l^m(\theta, \phi, t) = e^{-\eta \cdot l \cdot (l+1) \cdot t} \cdot \mathbf{Y}_l^m(\theta, \phi) \]

Since the linear sum of solutions is also a solution, solutions to the PDE will have the form:

\[ \mathbf{F}(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} \cdot e^{-\eta \cdot l \cdot (l+1) \cdot t} \cdot \mathbf{Y}_l^m(\theta, \phi) \]

and we have freedom in choosing the linear coefficients.
Smoothing

To satisfy the initial condition:

\[ F(\theta, \phi, 0) = f(\theta, \phi) \]

we need to compute the spherical harmonic decomposition of \( f \):

\[ f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{f}_{lm} \cdot Y_{lm}(\theta, \phi) \]

Then we set the solution to be:

\[ F(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{f}_{lm} \cdot e^{-\eta \cdot l \cdot (l+1) \cdot t} \cdot Y_{lm}(\theta, \phi) \]
Smoothing

\[ F(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{f}_{lm} \cdot e^{-\eta \cdot l \cdot (l+1) \cdot t} \cdot Y_l^m(\theta, \phi) \]
The Spherical Wave Equation

We can repeat the same type of argument for the wave equation, where the acceleration is proportional to the difference in values:

\[ \frac{\partial^2 F}{\partial t^2} = \eta \cdot \Delta F \]
The Spherical Wave Equation

Again, using the fact that the spherical harmonics $Y_l^m$ are eigenvectors of the Laplacian with eigenvalues $l \cdot (l + 1)$ we get solutions of the form:

$$F_l^{m^+}(\theta, \phi, t) = e^{i\sqrt{\eta \cdot l \cdot (l+1)} \cdot t} \cdot Y_l^m(\theta, \phi)$$

$$F_l^{m^-}(\theta, \phi, t) = e^{-i\sqrt{\eta \cdot l \cdot (l+1)} \cdot t} \cdot Y_l^m(\theta, \phi)$$
The Spherical Wave Equation

Thus, given the initial conditions:

\[ F(\theta, \phi, 0) = f(\theta, \phi) \]
\[ \frac{\partial}{\partial t} F(\theta, \phi, 0) = 0 \]

we get the solution:

\[ F(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{f}_{lm} \cdot \cos \left( \sqrt{\eta \cdot l \cdot (l+1)} \cdot t \right) \cdot Y_{lm}(\theta, \phi) \]
The Spherical Wave Equation

\[ F(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{f}_{lm} \cdot \cos \left( \sqrt{\eta \cdot l(l + 1)} \cdot t \right) \cdot Y_l^m(\theta, \phi) \]
\[
\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial}{\partial \phi} Y_l^m(\theta, \phi) \right] = e^{im\theta} \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left[ \sin \phi \frac{\partial P_l^m}{\partial \phi} (\cos \phi) \right]
\]
\[
= e^{im\theta} \left( \frac{\cos \phi \frac{\partial P_l^m(\cos \phi)}{\partial \phi}}{\sin \phi} + \frac{\partial^2 P_l^m(\cos \phi)}{\partial \phi^2} \right)
\]
Appendix

\[
\frac{dP_l^m(\cos \phi)}{d\phi} = \frac{l \cdot \cos \phi \cdot P_l^m(\cos \phi) - (l + m) \cdot P_{l-1}^m(\cos \phi)}{\sin \phi}
\]

\[
0 = (l - m) \cdot P_l^m(\cos \phi) - \cos \phi \cdot (2l - 1) \cdot P_{l-1}^m(\cos \phi) + (l + m - 1) \cdot P_{l-2}^m(\cos \phi)
\]
Appendix

\[
\frac{dP_l^m(\cos \phi)}{d\phi} = \frac{l \cdot \cos \phi \cdot P_l^m(\cos \phi) - (l + m) \cdot P_{l-1}^m(\cos \phi)}{\sin \phi}
\]

\[
0 = (l - m) \cdot P_l^m(\cos \phi) - \cos \phi \cdot (2l - 1) \cdot P_{l-1}^m(\cos \phi) + (l + m - 1) \cdot P_{l-2}^m(\cos \phi)
\]
Appendix

\[
\frac{d \mathbf{P}^m_l (\cos \phi)}{d \phi} = \frac{l \cdot \cos \phi \cdot \mathbf{P}^m_l (\cos \phi) - (l + m) \cdot \mathbf{P}^m_{l-1} (\cos \phi)}{\sin \phi}
\]

0 = (l - m) \cdot \mathbf{P}^m_l (\cos \phi) - \cos \phi \cdot (2l - 1) \cdot \mathbf{P}^m_{l-1} (\cos \phi) + (l + m - 1) \cdot \mathbf{P}^m_{l-2} (\cos \phi)

\[
\frac{(-l \cdot \sin^2 \phi + l^2 \cdot \cos^2 \phi) \cdot \mathbf{P}^m_l (\cos \phi) - (l + m) \cdot (2l - 1) \cdot \cos \phi \cdot \mathbf{P}^m_{l-1} (\cos \phi) + (l + m) \cdot (l - 1 + m) \cdot \mathbf{P}^m_{l-2} (\cos \phi)}{\sin^2 \phi}
\]

\[
= \frac{(-l^2 - l) \cdot \sin^2 \phi + l^2 \cdot \mathbf{P}^m_l (\cos \phi) - (l + m) \cdot (2l - 1) \cdot \cos \phi \cdot \mathbf{P}^m_{l-1} (\cos \phi) + (l + m) \cdot (l - 1 + m) \cdot \mathbf{P}^m_{l-2} (\cos \phi)}{\sin^2 \phi}
\]

\[
= (-l^2 - l) \cdot \mathbf{P}^m_l (\cos \phi) + \frac{l^2 \cdot \mathbf{P}^m_l (\cos \phi) - (l + m) \cdot (2l - 1) \cdot \cos \phi \cdot \mathbf{P}^m_{l-1} (\cos \phi) + (l + m) \cdot (l - 1 + m) \cdot \mathbf{P}^m_{l-2} (\cos \phi)}{\sin^2 \phi}
\]
Appendix

\[
\frac{dP_l^m(\cos \phi)}{d\phi} = \frac{l \cdot \cos \phi \cdot P_l^m(\cos \phi) - (l + m) \cdot P_{l-1}^m(\cos \phi)}{\sin \phi}
\]

\[
0 = (l - m) \cdot P_l^m(\cos \phi) - \cos \phi \cdot (2l - 1) \cdot P_{l-1}^m(\cos \phi) + (l + m - 1) \cdot P_{l-2}^m(\cos \phi)
\]

\[
(-l^2 - l) \cdot P_l^m(\cos \phi) + \frac{l^2 \cdot P_l^m(\cos \phi) - (l + m) \cdot (2l - 1) \cdot \cos \phi \cdot P_{l-1}^m(\cos \phi) + (l + m) \cdot (l - 1 + m) \cdot P_{l-2}^m(\cos \phi)}{\sin^2 \phi}
\]

\[
= (-l^2 - l) \cdot P_l^m(\cos \phi) + \frac{m^2 \cdot P_l^m(\cos \phi) + (l - m) \cdot (l + m) \cdot P_l^m(\cos \phi) - (l + m) \cdot (2l - 1) \cdot \cos \phi \cdot P_{l-1}^m(\cos \phi) + (l + m) \cdot (l - 1 + m) \cdot P_{l-2}^m(\cos \phi)}{\sin^2 \phi}
\]

\[
= (-l^2 - l) \cdot P_l^m(\cos \phi) + \frac{m^2 \cdot P_l^m(\cos \phi) + (l + m) \cdot (l - m) \cdot P_l^m(\cos \phi) - (2l - 1) \cdot \cos \phi \cdot P_{l-1}^m(\cos \phi) + (l - 1 + m) \cdot P_{l-2}^m(\cos \phi)}{\sin^2 \phi}
\]

\[
= (-l^2 - l) \cdot P_l^m(\cos \phi) + \frac{m^2 \cdot P_l^m(\cos \phi) + (l + m) \cdot (l - m) \cdot P_l^m(\cos \phi) - (2l - 1) \cdot \cos \phi \cdot P_{l-1}^m(\cos \phi) + (l - 1 + m) \cdot P_{l-2}^m(\cos \phi)}{\sin^2 \phi}
\]