FFTs in Graphics and Vision

Spherical Harmonics
and
Legendre Polynomials

Outline

- Math Stuff
  - Gram-Schmidt Orthogonalization
  - Completing Homogenous Polynomials
- Review
- Defining the Harmonics

Recall

Given a monomial x^d, we have:

$$\int_{-1}^{1} x^d dx = \frac{1}{d+1} x^{d+1} \Big|_{-1}^{1} = \frac{1-(-1)^{d+1}}{d+1}$$

$\Rightarrow \int_{-1}^{1} x^d \, dx = 0$ if d is odd.

$\Rightarrow$ If $p(x)$ is a polynomial whose even-degree coefficients are all zero:

$$\int_{-1}^{1} p(x) dx = 0$$

Gram–Schmidt Orthogonalization

Given an inner product space V, and given a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ we can define an orthonormal basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ for V such that:

$$\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \delta_{ij}$$

Algorithm:

Start by making $\mathbf{v}_1$ a unit vector:

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

Gram–Schmidt Orthogonalization

Given an inner product space V, and given a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ we can define an orthonormal basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ for V such that:

$$\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \delta_{ij}$$

Algorithm:

To get the 2nd basis element, subtract from $\mathbf{v}_2$ the $\mathbf{w}_1$ component and normalize:

$$\mathbf{w}_2 = \frac{\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{w}_1 \rangle \mathbf{w}_1}{\|\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{w}_1 \rangle \mathbf{w}_1\|}$$

Gram–Schmidt Orthogonalization

Given an inner product space V, and given a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ we can define an orthonormal basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ for V such that:

$$\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \delta_{ij}$$

Algorithm:

To get the i-th basis element, subtract off all the earlier components from $\mathbf{v_i}$ and normalize:

$$\mathbf{w}_i = \frac{\mathbf{v}_i - \langle \mathbf{v}_i, \mathbf{w}_{i-1} \rangle \mathbf{w}_{i-1} - \cdots - \langle \mathbf{v}_i, \mathbf{w}_1 \rangle \mathbf{w}_1}{\|\mathbf{v}_i - \langle \mathbf{v}_i, \mathbf{w}_{i-1} \rangle \mathbf{w}_{i-1} - \cdots - \langle \mathbf{v}_i, \mathbf{w}_1 \rangle \mathbf{w}_1\|}$$

Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree N on the interval $[-1,1]$, with the inner-product:

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)\cdot g(x)\,dx$$

We would like to obtain an orthonormal basis:

$$\left\{\mathbf{p}_0(x),\cdots,\mathbf{p}_{N(x)}\right\}$$

Start with the basis of monomials:

$$\{1, x, x^2, \cdots, x^N\}$$

and perform Gram-Schmidt orthogonalization.

Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree N on the interval $[-1,1]$, with the inner-product:

By induction, $\mathbf{p}_k(x)$ is a polynomial of degree k since G.S. orthogonalization only subtracts off lower-degree basis functions.

We would like to obtain an orthonormal basis:

$$\left\{\mathbf{p}_0(x),\cdots,\mathbf{p}_{N(x)}\right\}$$

Start with the basis of monomials:

$$\{1,x,x^2,\cdots,x^N\}$$

and perform Gram-Schmidt orthogonalization.

Gram–Schmidt Orthogonalization

Example:

Starting with the constant term, we get:

$$\mathbf{p}_0(x) = \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_{-1}^{1} dx}} = \frac{1}{\sqrt{2}}$$

Gram–Schmidt Orthogonalization

Example:

For the linear term, we get:

$$
\begin{aligned}
\mathbf{p}_1(x) &= \frac{x - \langle x, \mathbf{p}_0(x)\rangle \mathbf{p}_0(x)}{\|x - \langle x, \mathbf{p}_0(x)\rangle \mathbf{p}_0(x)\|} \\
&= \frac{x - \left(\int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}}\, dx\right) \frac{1}{\sqrt{2}}}{\left\| x - \left(\int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}}\, dx\right) \frac{1}{\sqrt{2}} \right\|} \\
&= \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} \\
&= \sqrt{\frac{3}{2}}x
\end{aligned}
$$

Gram–Schmidt Orthogonalization

Example:

And for the quadratic term:

$$\mathbf{p}_2(x) = \frac{x^2 - \langle x^2, \mathbf{p}_1(x)\rangle \mathbf{p}_1(x) - \langle x^2, \mathbf{p}_0(x)\rangle \mathbf{p}_0(x)}{\| x^2 - \langle x^2, \mathbf{p}_1(x)\rangle \mathbf{p}_1(x) - \langle x^2, \mathbf{p}_0(x)\rangle \mathbf{p}_0(x) \|}$$

These are the *Legendre Polynomials*.

Legendre Polynomials

Claim:

The monomials comprising the k-th Legendre polynomials have degree with the parity of k:

$$\mathbf{p}_k(x) = a_k \cdot x^k + a_{k-2} \cdot x^{k-2} + \cdots$$

Proof by Induction:

Legendre Polynomials

Claim:

The monomials comprising the k-th Legendre polynomials have degree with the parity of k:

$$\mathbf{p}_k(x) = a_k \cdot x^k + a_{k-2} \cdot x^{k-2} + \cdots$$

Proof by Induction ($k = 0$):

$$\mathbf{p}_0(x) = \frac{1}{\sqrt{2}}$$

Legendre Polynomials

Claim:

The monomials comprising the k-th Legendre polynomials have degree with the parity of k:

$$\mathbf{p}_k(x) = a_k \cdot x^k + a_{k-2} \cdot x^{k-2} + \cdots$$

Proof by Induction (assume true for $k = n$):

$$\mathbf{p}_{n+1}(x) = \frac{x^{n+1} - \langle x^{n+1}, \mathbf{p}_n(x)\rangle \mathbf{p}_n(x) - \cdots - \langle x^{n+1}, \mathbf{p}_0(x)\rangle \mathbf{p}_0(x)}{\|x^{n+1} - \langle x^{n+1}, \mathbf{p}_n(x)\rangle \mathbf{p}_n(x) - \cdots - \langle x^{n+1}, \mathbf{p}_0(x)\rangle \mathbf{p}_0(x)\|}$$

Recall that:

$$\langle x^{n+1}, \mathbf{p}_m(x)\rangle = \int_{-1}^{1} x^{n+1} \cdot \mathbf{p}_m(x)\, dx$$

Legendre Polynomials

Claim:

The monomials comprising the k-th Legendre polynomials have degree with the parity of k:

$$\mathbf{p}_k(x) = a_k \cdot x^k + a_{k-2} \cdot x^{k-2} + \cdots$$

Proof by Induction (assume true for $k = n$):

$$\langle x^{n+1}, \mathbf{p}_m(x) \rangle = \int_{-1}^{1} x^{n+1} \cdot \mathbf{p}_m(x)\, dx$$

Since $m \leq n$ we can assume that the monomials comprising $\mathbf{p}_m(x)$ have the parity of m.

Legendre Polynomials

Claim:

The monomials comprising the k-th Legendre polynomials have degree with the parity of k:

$$\mathbf{p}_k(x) = a_k \cdot x^k + a_{k-2} \cdot x^{k-2} + \cdots$$

Proof by Induction (assume true for $k = n$):

So if n and m are both even/odd, then the polynomial $x^{n+1} \cdot \mathbf{p}_m(x)$ is comprised of strictly odd-powered monomials:

$$\langle x^{n+1}, \mathbf{p}_m(x) \rangle = \int_{-1}^{1} x^{n+1} \cdot \mathbf{p}_m(x)\, dx = 0$$

Legendre Polynomials

Claim:

The monomials comprising the k-th Legendre polynomials have degree with the parity of k:

$$\mathbf{p}_k(x) = a_k \cdot x^k + a_{k-2} \cdot x^{k-2} + \cdots$$

Proof by Induction (assume true for $k = n$):

$$\mathbf{p}_{n+1}(x) = \frac{x^{n+1} \cancel{- \langle x^{n+1}, \mathbf{p}_n(x) \rangle \mathbf{p}_n(x)} - \langle x^{n+1}, \mathbf{p}_{n-1}(x) \rangle \mathbf{p}_{n-1}(x) - \cdots}{\| x^{n+1} \cancel{- \langle x^{n+1}, \mathbf{p}_n(x) \rangle \mathbf{p}_n(x)} - \langle x^{n+1}, \mathbf{p}_{n-1}(x) \rangle \mathbf{p}_{n-1}(x) - \cdots \|}$$

$\Rightarrow$ $\mathbf{p}_{n+1}(x)$ is obtained by starting with the monomial x^{n+1} and subtracting off monomials with the same parity.

Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree N on the interval $[-1,1]$, with an inner-product giving more weight to points near the origin:

$$\langle f(x), g(x)\rangle_m = \int_{-1}^{1} \left(1 - x^2\right)^m \cdot f(x) \cdot g(x)\, dx$$

We would like to obtain an orthonormal basis:

$$\{\mathbf{p}_0^m(x), \cdots, \mathbf{p}_N^m(x)\}$$

Gram–Schmidt Orthogonalization

Example:

$$\langle f(x), g(x)\rangle_m = \int_{-1}^{1}\left(1-x^2\right)^m \cdot f(x)\cdot g(x)\,dx$$

We proceed as before with the new inner-product.

Since the weighting function is even, if f is an even function and g is an odd function (or vice-versa), the inner product must be zero:

$$\langle f(x), g(x)\rangle_m = 0$$

As before, the degree of the monomials comprising $\mathbf{p}_l^m(x)$ must all have the parity of l.

Completing Homogenous Polynomials

Consider a polynomial $p(x, y, z)$ of degree d, consisting of monomials of degree d, $d - 2$, …:

$$p(x,y,z) = \sum_{k=0}^{\lfloor d/2 \rfloor} \left(\sum_{l+m+n=d-2k} a_{lmn} \cdot x^l \cdot y^m \cdot z^n \right)$$

This is *not* a homogenous polynomial.

However, if we restrict it to the sphere, we can think of it as homogenous:

$$p(x,y,z) = \sum_{k=0}^{\lfloor d/2 \rfloor} \left(x^2 + y^2 + z^2\right)^k \left(\sum_{l+m+n=d-2k} a_{lmn} \cdot x^l \cdot y^m \cdot z^n \right)$$

Completing Homogenous Polynomials

Example:

$$p(x, y, z) = x^2y + y + z$$

is not a homogenous polynomial.

But on the unit-sphere, the polynomial:

$$q(x, y, z) = x^2y + (y + z)(x^2 + y^2 + z^2)$$

has identical values and is homogenous of degree 3.

Outline

Math Stuff

Review

- Spherical Harmonics

Defining the Harmonics

Spherical Harmonics

For each non-negative integer l, there are $2l+1$ spherical harmonics of degree l.

1. Each spherical harmonic of degree l can be expressed as the restriction of a homogenous polynomial of degree l to the unit-sphere.
2. The different spherical harmonics are orthogonal to each other.

Spherical Harmonics

We saw that by considering just rotations about the y-axis, we could factor the spherical harmonics as:

$$\mathbf{Y}_l^m(\theta,\phi) = e^{im\theta}\cdot \mathbf{g}_l^m(\phi)$$
$$= (\cos\theta + i\sin\theta)^m \cdot \mathbf{P}_l^{|m|}(\cos\phi)$$

where $|m| \leq l$.

The functions $\mathbf{P}_l^m(x)$ are the associated Legendre polynomials:

$$\mathbf{P}_l^k(x) = \frac{(-1)^k}{2^l l!}\left(1-x^2\right)^{k/2}\frac{d^{l+k}}{dx^{l+k}}\left(x^2-1\right)^l$$

Outline

Math Stuff

Review

Defining the Harmonics

Defining the Harmonics ($m \geq 0$)

To define the spherical harmonics, we would like to express the function:

$$\mathbf{Y}_l^m(\theta,\phi) = (\cos\theta + i\sin\theta)^m \cdot \mathbf{P}_l^m(\cos\phi)$$

as the restriction of a homogenous polynomial of degree l to the unit sphere.

Defining the Harmonics ($m \geq 0$)

Using the parameterization of the unit-sphere

$$\Phi(\theta,\phi) = (\cos\theta\cdot\sin\phi\,, \cos\phi\,, \sin\theta\cdot\sin\phi)$$

we get:

$$\begin{aligned}
\mathbf{Y}_l^m(\theta,\phi) &= (\cos\theta + i\sin\theta)^m\cdot\mathbf{P}_l^m(\cos\phi)\\
&= \left(\frac{x}{\sin\phi} + i\,\frac{z}{\sin\phi}\right)^m\cdot\mathbf{P}_l^m(y)\\
&= (x+iz)^m\cdot\frac{\mathbf{P}_l^m(y)}{\sin^m\phi}\\
&= (x+iz)^m\cdot\frac{\mathbf{P}_l^m(y)}{\left(\sqrt{1-y^2}\right)^m}
\end{aligned}$$

Defining the Harmonics ($m \geq 0$)

$$\boxed{\mathbf{Y}_l^m(\theta,\phi)} = \boxed{(x+iz)^m} \cdot \boxed{\frac{\mathbf{P}_l^m(y)}{(1-y^2)^{m/2}}}$$

This • is a homogenous polynomial of degree l.

This • is a homogenous polynomial of degree m.

So we want:

1. This • to complete to a homogenous polynomial of degree $l - m$.
2. The different $\mathbf{Y}_l^m$ to be orthogonal

Defining the Harmonics ($m \geq 0$)

$$\mathbf{Y}_l^m(\theta,\phi) = (x+iz)^m \cdot \frac{\mathbf{P}_l^m(y)}{(1-y^2)^{m/2}}$$

<u>Homogeneous Completion</u>:

To satisfy the homogeneity constraint, we need:

$$\frac{\mathbf{P}_l^m(y)}{(1-y^2)^{m/2}} = a_{l-m} \cdot y^{l-m} + a_{l-m-2} \cdot y^{l-m-2} + \cdots$$

$$\Downarrow$$

$$\mathbf{P}_l^m(y) = \mathbf{q}_l^m(y) \cdot \left(1-y^2\right)^{m/2}$$

for some polynomial:

$$\mathbf{q}_l^m(y) = a_{l-m} \cdot y^{l-m} + a_{l-m-2} \cdot y^{l-m-2} + \cdots$$

Defining the Harmonics ($m \geq 0$)

$$\mathbf{Y}_l^m(\theta,\phi) \; = e^{im\theta} \cdot \mathbf{P}_l^m(\cos\phi)$$

<u>Orthogonality</u>:

To satisfy the orthogonality constraint, we need:

$$\langle \mathbf{Y}_l^m(\theta,\phi), \mathbf{Y}_{l'}^{m'}(\theta,\phi) \rangle = 0 \qquad \forall\; l \neq l' \text{ or } \; m \neq m'$$

Defining the Harmonics ($m \geq 0$)

$$\mathbf{Y}_l^m(\theta,\phi) \ = e^{im\theta} \cdot \mathbf{P}_l^m(\cos\phi)$$

Orthogonality ($m \neq m'$):

Since we have:

$$\mathbf{Y}_l^m(\theta,\phi) = e^{im\theta} \cdot \mathbf{P}_l^m(\cos\phi)$$

we know that:

$$\begin{aligned}\langle \mathbf{Y}_l^m(\theta,\phi), \mathbf{Y}_{l'}^{m'}(\theta,\phi)\rangle &= \int_0^{\pi}\int_0^{2\pi} e^{im\theta} \cdot \mathbf{P}_l^m(\cos\phi) \cdot \overline{e^{im'\theta} \cdot \mathbf{P}_{l'}^{m'}(\cos\phi)}\, d\theta \sin\phi\, d\phi \\ &= \left(\int_0^{\pi} \mathbf{P}_l^m(\cos\phi) \cdot \overline{\mathbf{P}_{l'}^{m'}(\cos\phi)} \cdot \sin\phi\, d\phi\right) \cdot \left(\int_0^{2\pi} e^{i(m-m')\theta}\, d\theta\right)\end{aligned}$$

Defining the Harmonics ($m \geq 0$)

$$\mathbf{Y}_l^m(\theta,\phi) \;= e^{im\theta} \cdot \mathbf{P}_l^m(\cos\phi)$$

<u>Orthogonality ($m \neq m'$)</u>:

$$\langle \mathbf{Y}_l^m(\theta,\phi), \mathbf{Y}_{l'}^{m'}(\theta,\phi)\rangle = \left(\int_0^\pi \mathbf{P}_l^m(\cos\phi)\cdot\overline{\mathbf{P}_{l'}^{m'}(\cos\phi)}\cdot\sin\phi\, d\phi\right)\cdot\boxed{\left(\int_0^{2\pi} e^{i(m-m')\theta}\,d\theta\right)}$$

But this is zero whenever $m \neq m'$:

$$\int_0^{2\pi} e^{i(m-m')\theta}\,d\theta = \frac{1}{i(m-m')}\cdot e^{i(m-m')\theta}\Big|_0^{2\pi} = 0$$

Defining the Harmonics ($m \geq 0$)

$$\mathbf{Y}_l^m(\theta,\phi) \; = e^{im\theta} \cdot \mathbf{P}_l^m(\cos\phi)$$

Orthogonality ($m = m'$ and $l \neq l'$):

We have to choose the function:

$$\mathbf{P}_l^m(y) = \mathbf{q}_l^m(y) \cdot \left(1 - y^2\right)^{m/2}$$

so that:

$$0 = \int_0^{\pi} \int_0^{2\pi} e^{im\theta} \cdot \mathbf{P}_l^m(\cos\phi) \cdot \overline{e^{im\theta} \cdot \mathbf{P}_{l'}^m(\cos\phi)} \, d\theta \sin\phi \, d\phi$$

$$\Downarrow$$

$$0 = 2\pi \int_0^{\pi} \mathbf{P}_l^m(\cos\phi) \cdot \overline{\mathbf{P}_{l'}^m(\cos\phi)} \cdot \sin\phi \, d\phi$$

Defining the Harmonics ($m \geq 0$)

$$\mathbf{Y}_l^m(\theta,\phi) = e^{im\theta} \cdot \mathbf{P}_l^m(\cos\phi)$$

<u>Orthogonality ($m = m'$ and $l \neq l'$)</u>:

We have to choose the function:

$$\mathbf{P}_l^m(y) = \mathbf{q}_l^m(y) \cdot \left(1 - y^2\right)^{m/2}$$

Changing variables:

$$\begin{aligned} 0 &= \int_0^\pi \mathbf{P}_l^m(\cos\phi) \cdot \overline{\mathbf{P}_{l'}^m(\cos\phi)} \cdot \sin\phi \, d\phi \\ &= \int_{-1}^1 \mathbf{P}_l^m(y) \cdot \overline{\mathbf{P}_{l'}^m(y)} \, dy \\ &= \int_{-1}^1 \mathbf{q}_l^m(y) \cdot \overline{\mathbf{q}_{l'}^m(y)} \cdot \left(1 - y^2\right)^m dy \end{aligned}$$

Defining the Harmonics ($m \geq 0$)

$$\mathbf{Y}_l^m(\theta,\phi) = e^{im\theta} \cdot \mathbf{P}_l^m(\cos\phi)$$

$$\mathbf{P}_l^m(y) = \mathbf{q}_l^m(y) \cdot \left(1 - y^2\right)^{m/2}$$

Thus, the polynomials $\mathbf{q}_l^m(y)$ should:

1. Complete to homogeneous polynomials of degree $l - m$:

   $$\mathbf{q}_l^m(y) = a_{l-m} \cdot y^{l-m} + a_{l-m-2} \cdot y^{l-m-2} + \cdots$$

2. Satisfy the orthogonality condition:

   $$0 = \int_{-1}^{1} \mathbf{q}_l^m(y) \cdot \overline{\mathbf{q}_{l'}^m(y)} \cdot \left(1 - y^2\right)^m dy$$

Defining the Harmonics ($m \geq 0$)

$$\mathbf{Y}_l^m(\theta,\phi) = e^{im\theta} \cdot \mathbf{P}_l^m(\cos\phi)$$
$$\mathbf{P}_l^m(y) = \mathbf{q}_l^m(y) \cdot \left(1-y^2\right)^{m/2}$$

This is what we get with G.S. orthogonalization $\{1, y, y^2, \cdots\} \rightarrow \{\mathbf{p}_0^m(y), \mathbf{p}_1^m(y), \mathbf{p}_2^m(y), \cdots\}$ relative to the inner-product:

$$\langle f(y), g(y)\rangle_m = \int_{-1}^{1} f(y) \cdot g(y) \cdot \left(1-y^2\right)^m dy$$

and set:

$$\mathbf{q}_l^m(y) = \mathbf{p}_{l-m}^m(y)$$

Defining the Harmonics ($m \geq 0$)

$$\mathbf{Y}_l^m(\theta,\phi) = e^{im\theta}\cdot \mathbf{P}_l^m(\cos\phi)$$
$$\mathbf{P}_l^m(y) = \mathbf{q}_l^m(y)\cdot\left(1-y^2\right)^{m/2}$$

In sum, we get an expression for the spherical harmonics as:

$$\begin{aligned}\mathbf{Y}_l^m(\theta,\phi) &= e^{im\theta}\cdot \mathbf{p}_{l-m}^m(\cos\phi)\cdot\left(\sqrt{1-\cos^2\phi}\right)^m \\ &= e^{im\theta}\cdot \mathbf{p}_{l-m}^m(\cos\phi)\cdot \sin^m\phi\end{aligned}$$

where $\mathbf{p}_{l-m}^m(y)$ is a polynomial of degree $l-m$ whose monomials have degree with the same parity as $l-m$.

Defining the Harmonics

$$\mathbf{Y}_l^m(\theta,\phi) = \sin^{|m|}\phi \cdot \mathbf{p}_{l-|m|}^{|m|}(\cos\phi) \cdot e^{im\theta}$$

Examples ($l = 0$):

$$\mathbf{Y}_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}} = \sin^0\phi \cdot \left(\frac{1}{\sqrt{4\pi}}\right) \cdot e^{i(0)\theta}$$

Defining the Harmonics

$$\mathbf{Y}_l^m(\theta,\phi) = \sin^{|m|}\phi \cdot \mathbf{p}_{l-|m|}^{|m|}(\cos\phi) \cdot e^{im\theta}$$

Examples ($l = 1$):

$$\mathbf{Y}_1^{-1}(\theta,\phi) = \sin^1\phi \cdot \left(\sqrt{\frac{3}{8\pi}}\right) \cdot e^{i(-1)\theta}$$

$$\mathbf{Y}_1^{0}(\theta,\phi) = \sin^0\phi \cdot \left(\sqrt{\frac{3}{4\pi}}\cos\phi\right) \cdot e^{i(0)\theta}$$

$$\mathbf{Y}_1^{1}(\theta,\phi) = \sin^1\phi \cdot \left(\sqrt{\frac{3}{8\pi}}\right) \cdot e^{i(1)\theta}$$

Defining the Harmonics

$$\mathbf{Y}_l^m(\theta,\phi) = \sin^{|m|}\phi \cdot \mathbf{p}_{l-|m|}^{|m|}(\cos\phi) \cdot e^{im\theta}$$

Examples ($l = 2$):

$$\mathbf{Y}_2^{-2}(\theta,\phi) = \sin^2(\phi) \cdot \left(\sqrt{\frac{15}{32\pi}}\right) \cdot e^{i(-2)\theta}$$

$$\mathbf{Y}_{\mathbf{2}}^{-\mathbf{1}}(\theta,\phi) = \sin^1\phi \cdot \left(\sqrt{\frac{15}{8\pi}}\cos\phi\right) \cdot e^{i(-1)\theta}$$

$$\mathbf{Y}_2^0(\theta,\phi) = \sin^0\phi \cdot \left(\sqrt{\frac{5}{16\pi}}\left(3\cos^2\phi - 1\right)\right) \cdot e^{i(0)\theta}$$

$$\mathbf{Y}_2^1(\theta,\phi) = \sin^1\phi \cdot \left(\sqrt{\frac{15}{8\pi}}\cos\phi\right) \cdot e^{i(1)\theta}$$

$$\mathbf{Y}_2^2(\theta,\phi) = \sin^2\phi \cdot \left(\sqrt{\frac{15}{32\pi}}\right) \cdot e^{i2\theta}$$