FFTs in Graphics and Vision

Homogenous Polynomials
and
Irreducible Representations

Outline

Homogenous Polynomials

Representations of Functions on the Unit-Circle

- Sub-Representations for the Circle

- Sub-Representations for the Sphere

Monomials

Definition:

A <u>monomial</u> in variables $\{x_1, \cdots, x_n\}$ is a product of non-negative integer powers of the variables.

The <u>degree</u> of a monomial is the sum of the powers.

Monomials

Examples:

- Degree 0: 1

- Degree 1: x, y, z

- Degree 2: $x^2, y^2, z^2, xy, xz, yz$

- Degree 3: $x^3, x^2y, x^2z, xy^2, xz^2, xyz, y^3, y^2z, yz^2, z^3$

Polynomials

Definition:

A polynomial of degree d in variables $\{x_1, \cdots, x_n\}$ is a linear sum of monomials in $\{x_1, \cdots, x_n\}$, each of whose degree is no greater than d.

Notation:

Denote by $P^d(x_1, \cdots, x_n)$ the set of polynomials in $\{x_1, \cdots, x_n\}$ of degree d (with complex coefficients).

Polynomials

Examples:

- $d = 0$:
  - $P^0(x) = \{a | a \in \mathbb{C}\}$
  - $P^0(x, y) = \{a | a \in \mathbb{C}\}$
  - $P^0(x, y, z) = \{a | a \in \mathbb{C}\}$

- $d = 1$:
  - $P^1(x) = \{ax + b | a, b \in \mathbb{C}\}$
  - $P^1(x, y) = \{ax + by + c | a, b, c \in \mathbb{C}\}$
  - $P^1(x, y, z) = \{ax + by + cz + d | a, b, c, d \in \mathbb{C}\}$

- $d = 2$:
  - $P^2(x) = \{ax^2 + bx + c | a, b, c \in \mathbb{C}\}$
  - $P^2(x, y) = \{ax^2 + by^2 + cxy + dx + ey + f | a, b, c, d, e, f \in \mathbb{C}\}$
  - …

Polynomials

Properties:

- The weighted sum of polynomials $p, q \in P^d(x_1, \cdots, x_n)$ is a polynomial:

$$a \cdot p(x_1, \cdots, x_n) + b \cdot q(x_1, \cdots, x_n) \in P^d(x_1, \cdots, x_n)$$

- The product of polynomials $p \in P^{d_1}(x_1, \cdots, x_n)$ and $q \in P^{d_2}(x_1, \cdots, x_n)$ is a polynomial:

$$p(x_1, \cdots, x_n) \cdot q(x_1, \cdots, x_n) \in P^{d_1+d_2}(x_1, \cdots, x_n)$$

- The k-th power of a polynomial $p \in P^d(x_1, \cdots, x_n)$ is a polynomial:

$$p^k(x_1, \cdots, x_n) \in P^{d \cdot k}(x_1, \cdots, x_n)$$

Homogenous Polynomials

Definition:
A degree d polynomial is said to be <u>homogenous</u> if the individual monomials all have degree d.

Notation:
Denote by $HP^d(x_1, \ldots, x_n)$ the set of homogenous polynomials in $\{x_1, \ldots, x_n\}$ of degree d.

Homogenous Polynomials

Examples:

- $d = 0$:
  - $HP^0(x) = \{a | a \in \mathbb{C}\}$
  - $HP^0(x, y) = \{a | a \in \mathbb{C}\}$
  - $HP^0(x, y, z) = \{a | a \in \mathbb{C}\}$

- $d = 1$:
  - $HP^1(x) = \{ax | a \in \mathbb{C}\}$
  - $HP^1(x, y) = \{ax + by | a, b \in \mathbb{C}\}$
  - $HP^1(x, y, z) = \{ax + by + cz | a, b, c \in \mathbb{C}\}$

- $d = 2$:
  - $HP^2(x) = \{ax^2 | a \in \mathbb{C}\}$
  - $HP^2(x, y) = \{ax^2 + by^2 + cxy | a, b, c \in \mathbb{C}\}$
  - $HP^2(x, y, z) = \{ax^2 + by^2 + cz^2 + dxy + exz + fyz | a, b, c, d, e, f \in \mathbb{C}\}$

Homogenous Polynomials

Properties:

- The weighted sum of homogenous polynomials $p, q \in HP^d(x_1, \cdots, x_n)$ is a homogenous polynomial:

  $a \cdot p(x_1, \cdots, x_n) + b \cdot p(x_1, \cdots, x_n) \in HP^d(x_1, \cdots, x_n)$

- The product of homogenous polynomials $p \in HP^{d_1}(x_1, \cdots, x_n)$ and $q \in HP^{d_2}(x_1, \cdots, x_n)$ is a homogenous polynomial:

  $p(x_1, \cdots, x_n) \cdot q(x_1, \cdots, x_n) \in HP^{d_1 + d_2}(x_1, \cdots, x_n)$

- The k-th power of a homogenous polynomial $p \in HP^d(x_1, \cdots, x_n)$ is a homogenous polynomial:

  $p^k(x_1, \cdots, x_n) \in HP^{d \cdot k}(x_1, \cdots, x_n)$

Homogenous Polynomials

Note 1:

Any degree d polynomial in $\{x_1, \ldots, x_n\}$ can be uniquely expressed as the sum of homogenous polynomials in $\{x_1, \ldots, x_n\}$ of degrees 0 through d:

$$P^d(x_1, \cdots, x_n) = HP^0(x_1, \cdots, x_n) \oplus \cdots \oplus HP^d(x_1, \cdots, x_n)$$

Homogenous Polynomials

Note 1:
$$P^d(x_1, \cdots, x_n) = HP^0(x_1, \cdots, x_n) \oplus \cdots \oplus HP^d(x_1, \cdots, x_n)$$

Example:

- $\underbrace{p(x,y)}_{\in P^2(x,y)} = \underbrace{2x^2 + 3y^2 - xy}_{\in HP^2(x,y)} + \underbrace{5x - 7y}_{\in HP^1(x,y)} + \underbrace{2}_{\in HP^0(x,y)}$

Homogenous Polynomials

Note 2:

Any homogenous polynomial in $\{x_1, \ldots, x_n\}$ of degree d can be uniquely expressed as:

- x_1 times a degree $d-1$ homogenous polynomial in $\{x_1, \ldots, x_n\}$, plus
- a degree d homogenous polynomial in $\{x_2, \ldots, x_n\}$.

$$HP^d(x_1, \cdots, x_n) = x_1 \cdot HP^{d-1}(x_1, \cdots x_n) \oplus HP^d(x_2, \cdots, x_n)$$

Homogenous Polynomials

Note 2:

$$HP^d(x_1, \cdots, x_n) = x_1 \cdot HP^{d-1}(x_1, \cdots x_n) \oplus HP^d(x_2, \cdots, x_n)$$

Example:

- $\underbrace{p(x, y)}_{\in HP^2(x, y)} = 2x^2 + 3y^2 - xy$

    $= x \cdot \underbrace{(2x - y)}_{\in HP^1(x, y)} + \underbrace{3y^2}_{\in HP^2(y)}$

Dimensions

What is the dimension of $P^d(x_1, \ldots, x_n)$?

Since every polynomial of degree d can be uniquely expressed as the sum of homogenous polynomials of degrees 0 through d:

$$\dim\left(P^d(x_1, \cdots, x_n)\right) = \dim\left(HP^0(x_1, \cdots, x_n)\right) + \cdots + \dim\left(HP^d(x_1, \cdots, x_n)\right)$$

Dimensions

What is the dimension of $HP^d(x_1, \ldots, x_n)$?

Dimensions

Three properties give us a recursive definition:

1. The space of homogenous polynomials in $\{x_1, \cdots, x_n\}$ of degree 0 is:
$$HP^0(x_1, \cdots, x_n) = \{a | a \in \mathbb{C}\}$$

2. The space of homogenous polynomials in $\{x\}$ of degree d is:
$$HP^d(x) = \{ax^d | a \in \mathbb{C}\}$$

3. A homogenous polynomial of degree d factors as:
$$HP^d(x_1, \cdots, x_n) = x_1 \cdot HP^{d-1}(x_1, \cdots, x_n) \oplus HP^d(x_2, \cdots, x_n)$$

Dimensions

<u>Homogenous Polynomials of Degree Zero</u>:
$$HP^0(x_1, \cdots, x_n) = \{a \,|\, a \in \mathbb{C}\}$$

The dimension of the space of homogenous polynomials of degree 0 in any number of variables is one:
$$\dim[HP^0(x_1, \cdots, x_n)] = 1$$

Dimensions

Homogenous Polynomials in One Variable:
$$HP^d(x) = \{ax^d | a \in \mathbb{C}\}$$

The dimension of the space of homogenous polynomials of degree d in one variable is one, for all degrees d:
$$\dim\left[HP^d(x)\right] = 1$$

Dimensions

Homogenous Polynomials in n Variables:
$$HP^d(x_1, \cdots, x_n) = x_1 \cdot HP^{d-1}(x_1, \cdots, x_n) \oplus HP^d(x_2, \cdots, x_n)$$

The dimension of the space of homogenous polynomials of degree d in n variables is:
$$\dim[HP^d(x_1, \cdots, x_n)] = \dim[HP^d(x_2, \cdots x_n)]$$
$$+ \dim[HP^{d-1}(x_1, \cdots, x_n)]$$

Dimensions

Homogenous Polynomials in n Variables:

$$HP^d(x_1, \cdots, x_n) = x_1 \cdot HP^{d-1}(x_1, \cdots, x_n) \oplus HP^d(x_2, \cdots, x_n)$$

The dimension of the space of homogenous polynomials of degree d in n variables is:

$$\dim[HP^d(x_1, \cdots, x_n)] = \dim[HP^d(x_2, \cdots x_n)]$$
$$+ \dim\left[HP^{d-1}(x_2, \cdots, x_n)\right]$$
$$+ \dim\left[HP^{d-2}(x_1, \cdots, x_n)\right]$$

Dimensions

<u>Homogenous Polynomials in n Variables</u>:

$$HP^d(x_1, \cdots, x_n) = x_1 \cdot HP^{d-1}(x_1, \cdots, x_n) \oplus HP^d(x_2, \cdots, x_n)$$

The dimension of the space of homogenous polynomials of degree d in n variables is:

$$\dim[HP^d(x_1, \cdots, x_n)] = \sum_{i=1}^{d} \dim[HP^i(x_2, \cdots x_n)] + \dim[HP^0(x_1, \cdots, x_n)]$$

Dimensions

Homogenous Polynomials in n Variables:

$$HP^d(x_1, \cdots, x_n) = x_1 \cdot HP^{d-1}(x_1, \cdots, x_n) \oplus HP^d(x_2, \cdots, x_n)$$

The dimension of the space of homogenous polynomials of degree d in n variables is:

$$\dim[HP^d(x_1, \cdots, x_n)] = \sum_{i=0}^{d} \dim[HP^i(x_2, \cdots x_n)]$$

Dimensions

$$\dim\left[HP^d(x_1, \cdots, x_n)\right] = \sum_{i=0}^{d} \dim\left[HP^i(x_2, \cdots x_n)\right]$$

Homogenous Polynomials in n Variables:

One Variable:

$$\dim\left[HP^d(x)\right] = 1$$

Dimensions

$$\dim\left[HP^d(x_1, \cdots, x_n)\right] = \sum_{i=0}^{d} \dim\left[HP^i(x_2, \cdots x_n)\right]$$

$$\dim\left[HP^d(x)\right] = 1$$

<u>Homogenous Polynomials in n Variables</u>:

Two Variables:

$$\dim\left[HP^d(x, y)\right] = \sum_{i=0}^{d} \dim\left[HP^i(x)\right]$$

$$= \sum_{i=0}^{d} 1$$

$$= d + 1$$

Dimensions

$$\dim\left[HP^d(x_1, \cdots, x_n)\right] = \sum_{i=0}^{d} \dim\left[HP^i(x_2, \cdots x_n)\right]$$

$$\dim\left[HP^d(x, y)\right] = d + 1$$

<u>Homogenous Polynomials in n Variables</u>:

Three Variables:

$$\dim\left[HP^d(x, y, z)\right] = \sum_{i=0}^{d} \dim\left[HP^i(x, y)\right]$$

$$= \sum_{i=0}^{d} (i + 1)$$

$$= \frac{(d + 2) \cdot (d + 1)}{2}$$

Dimensions

Homogenous Polynomials in n Variables:

One Variable: $\dim\big[HP^d(x)\big] = 1$

Two Variables: $\dim\big[HP^d(x, y)\big] = d + 1$

Three Variables: $\dim\big[HP^d(x, y, z)\big] = \dfrac{(d+2)\cdot(d+1)}{2}$

Outline

Homogenous Polynomials
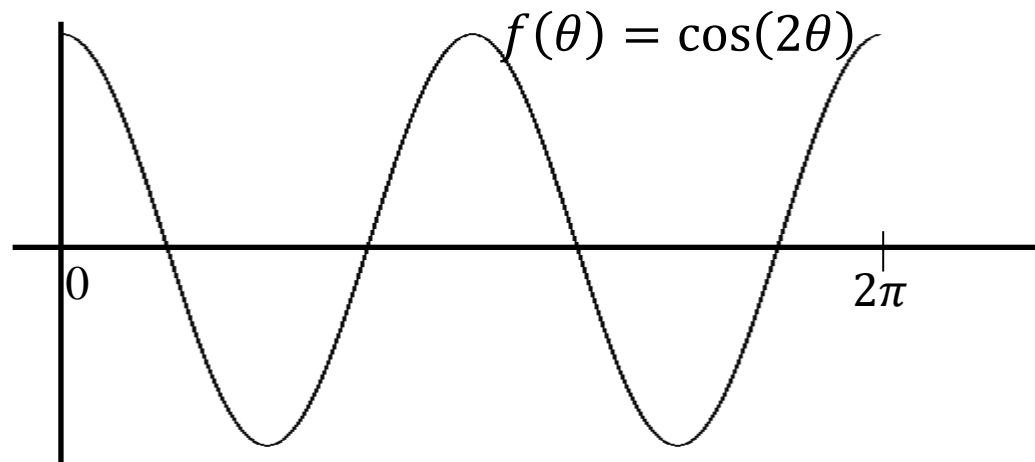
Representations of Functions on the Unit-Circle

- Sub-Representations for the Circle

- Sub-Representations for the Sphere

Representing Functions on the Unit-Circle

There are two ways we can represent a function on the unit-circle:

    1. <u>By Parameter</u>: Every point on the circle can be represented by an angle in the range $[0, 2\pi)$.

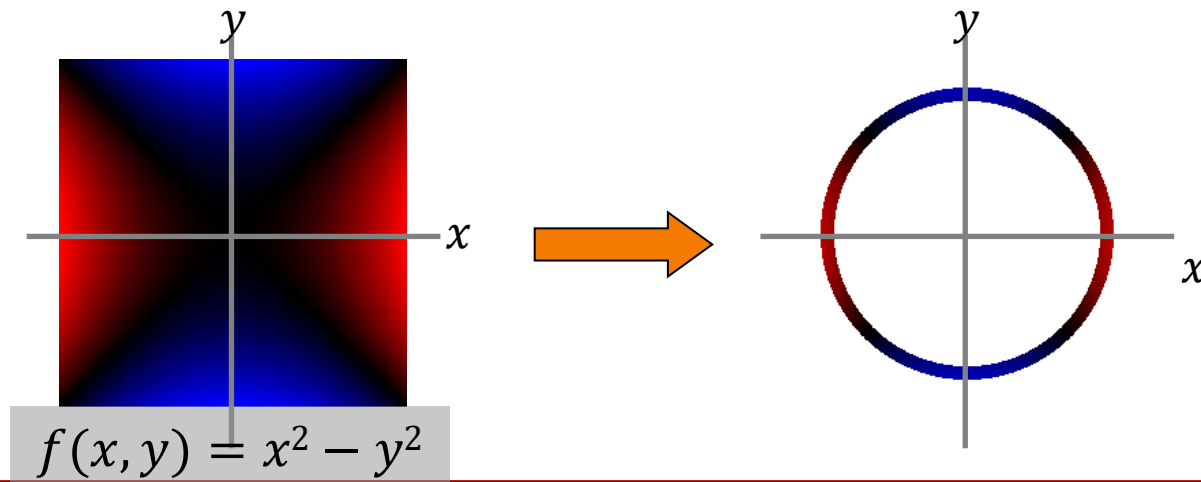 $\Rightarrow$ We can represent circular functions as (periodic) 1D functions on the domain $[0, 2\pi)$.

$$f(\theta) = \cos(2\theta)$$

0   2π

Representing Functions on the Unit-Circle

There are two ways we can represent a function on the unit-circle:

2. <u>By Restriction</u>: We know that the unit-circle "lives" in 2D, i.e. it is the set of points (x, y) satisfying:
$$x^2 + y^2 = 1$$

$\Rightarrow$ We can represent circular functions by looking at the restriction of 2D functions to the unit-circle.

$f(x, y) = x^2 - y^2$

Representing By Restriction

Observation 1:

On a circle, a point with angle θ has x- and y-coordinates:

$$x = \cos(\theta) \qquad y = \sin(\theta)$$

This lets us transform a (circular) function represented by the restriction of a 2D function $f(x, y)$ to a function represented by parameter:

$$f(x, y) \rightarrow g(\theta) \equiv f(\cos\theta, \sin\theta)$$
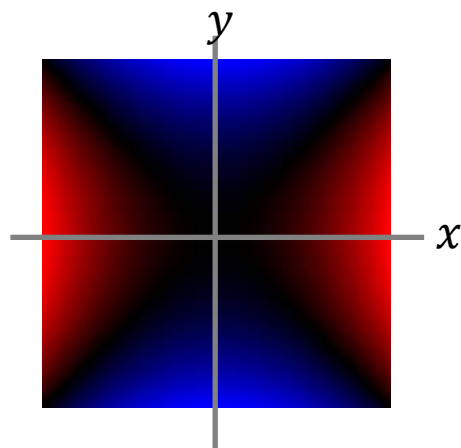
Representing By Restriction

Example:

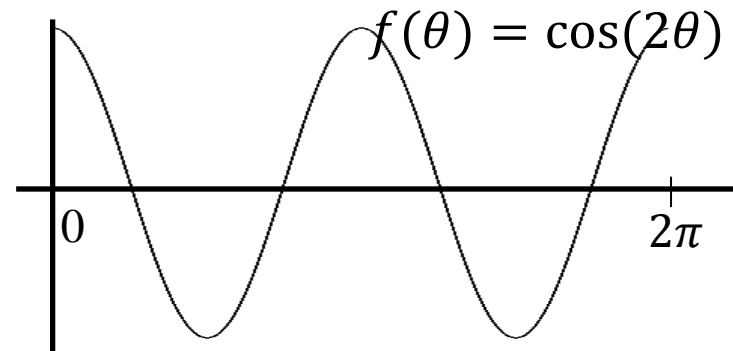If the circular function is defined as the restriction of the 2D function:

$$f(x, y) = x^2 - y^2$$

Then the representation in terms of angle is:

$$g(\theta) = \cos^2 \theta - \sin^2 \theta$$
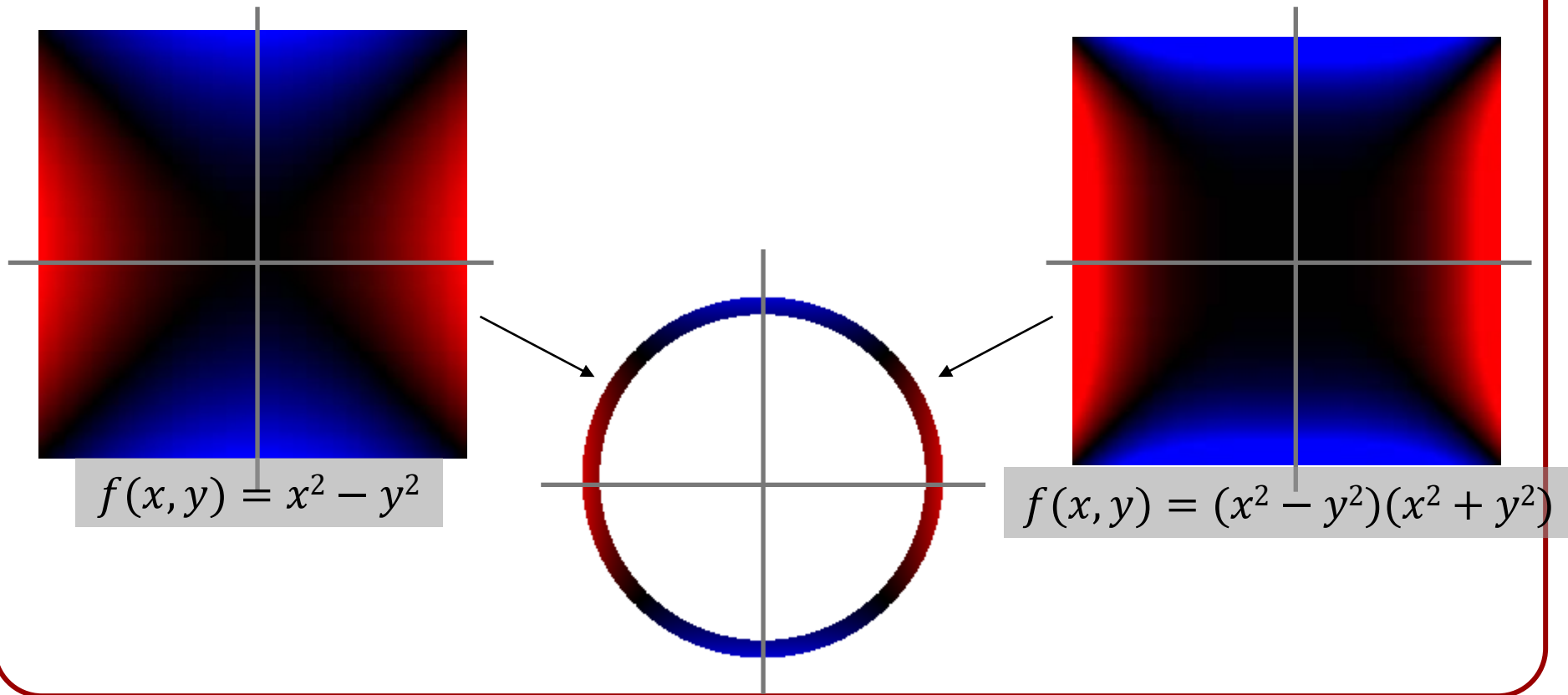$$= \cos 2\theta$$

$f(x, y) = x^2 - y^2$

$f(\theta) = \cos(2\theta)$

Representing By Restriction

Observation 2:

Two different functions in 2D, can have the same restriction to the unit-circle.

$$f(x, y) = x^2 - y^2$$

$$f(x, y) = (x^2 - y^2)(x^2 + y^2)$$

Outline

Homogenous Polynomials

Representations of Functions on the Unit-Circle

- **Sub-Representations for the Circle**

- Sub-Representations for the Sphere

Irreducible Representations

Recall:

In various shape/image analysis tasks:
- Rotation invariant representation
- Image filtering
- Symmetry detection
- (2D) Rotational alignment

we needed to consider the representation of the group of 2D rotations on the space of circular functions.

Irreducible Representations

<u>Recall</u>:

To perform these tasks efficiently and/or effectively, we depended on Schur's Lemma.

$\Rightarrow$ Since the group was commutative, the irreducible representations were all one (complex) dimensional

Irreducible Representations

Challenge:

We know that the irreducible representations exist. How do we find them?

Sub-Representations

How do we find a sub-space of functions that is also a sub-representation?

⇔ How do we find a space of functions such that a rotation of a function from this space gives some other function in the space?

Fourier Basis

For the circles, we know that these spaces are one-dimensional, spanned by:

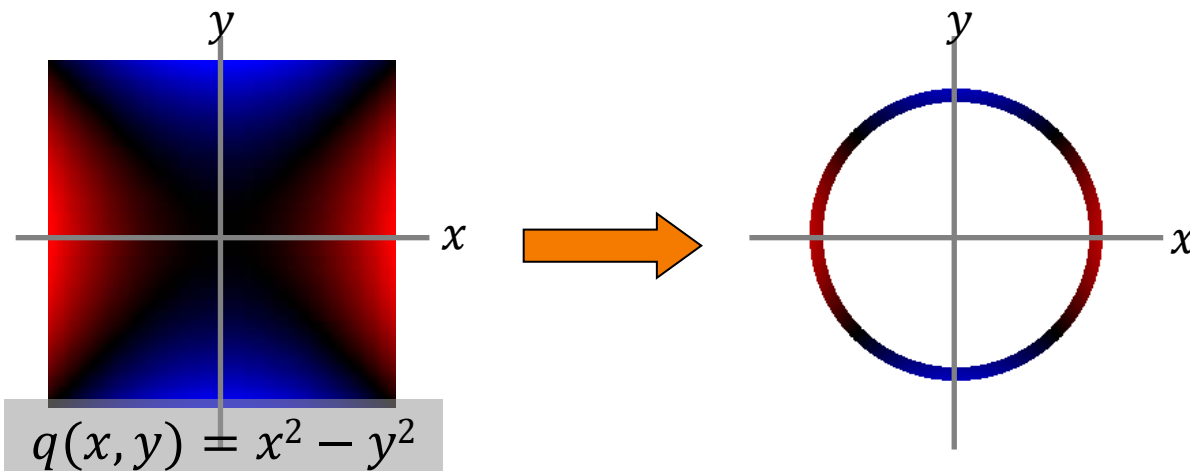$$\boldsymbol{\zeta}^k(\theta) = \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

But how would we go about finding them if we didn't know?

Polynomials

Consider the circular functions that are obtained by restricting degree d polynomials to the circle:

$$q(x, y) = \sum_{j+k \leq d} a_{jk} \cdot x^j \cdot y^k$$

$q(x, y) = x^2 - y^2$

Polynomials

Consider the circular functions that are obtained by restricting degree d polynomials to the circle:

$$q(x, y) = \sum_{j+k \leq d} a_{jk} \cdot x^j \cdot y^k$$

How does a rotation act on this function?

$$\mathbf{R} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Polynomials

$$\mathbf{R} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Rotations act on the space of functions by rotating the domain of evaluation:
$$\big(\rho_{\mathbf{R}}(q)\big)(x, y) = q\big(\mathbf{R}^{-1}(x, y)\big)$$

Since the inverse of a rotation is its transpose, the rotation $\mathbf{R}^{-1}$, acts on the 2D space by:
$$\mathbf{R}^{-1}(x, y) = (ax + cy, bx + dy)$$

Polynomials

$$(\rho_{\mathbf{R}}(q))(x, y) = q(ax + cy, bx + dy)$$

$\Rightarrow$ The rotation acts on the polynomial:

$$q(x, y) = \sum_{j+k \leq d} a_{jk} \cdot x^j \cdot y^k$$

by sending it to:

$$(\rho_{\mathbf{R}}(q))(x, y) = \sum_{j+k \leq d} a_{jk} \cdot \underbrace{(ax + cy)^j}_{\text{Degree 1}} \cdot \underbrace{(bx + dy)^k}_{\text{Degree 1}}$$

Polynomials

$$\left(\rho_{\mathbf{R}}(q)\right)(x, y) = q(ax + cy, bx + dy)$$

$\Rightarrow$ The rotation acts on the polynomial:

$$q(x, y) = \sum_{j+k \leq d} a_{jk} \cdot x^j \cdot y^k$$

by sending it to:

$$\left(\rho_{\mathbf{R}}(q)\right)(x, y) = \sum_{j+k \leq d} a_{jk} \cdot \underbrace{(ax + cy)^j}_{\text{Degree } j} \cdot \underbrace{(bx + dy)^k}_{\text{Degree } k}$$

Polynomials

$$\left(\rho_{\mathbf{R}}(q)\right)(x, y) = q(ax + cy, bx + dy)$$

$\Rightarrow$ The rotation acts on the polynomial:

$$q(x, y) = \sum_{j+k \leq d} a_{jk} \cdot x^j \cdot y^k$$

by sending it to:

$$\left(\rho_{\mathbf{R}}(q)\right)(x, y) = \sum_{j+k \leq d} a_{jk} \cdot \underbrace{(ax + cy)^j \cdot (bx + dy)^k}_{\text{Degree } j + k}$$

$\Rightarrow$ Since $j + k \leq d$, the rotation of $q(x, y)$ by $\mathbf{R}$ is also a polynomial of degree d.

Polynomials

If we start with a polynomial of degree d:
$$q(x, y) \in P^d(x, y)$$

and we apply any rotation $\mathbf{R}$ to it, the rotated polynomial will also be a polynomial of degree d:
$$\rho_{\mathbf{R}}(q) \in P^d(x, y)$$

$\Rightarrow$ The space of functions obtained by restricting polynomials of degree d to the unit circle is a sub-representation.

Polynomials

We can repeat the argument for restrictions of <u>homogenous</u> polynomials:

$$q(x,y) = \sum_{j+k=d} a_{jk} \cdot x^j \cdot y^k$$

$$\Updownarrow$$

$$\left(\rho_{\mathbf{R}}(q)\right)(x,y) = \sum_{j+k=d} a_{jk} \cdot (ax+cy)^j \cdot (bx+dy)^k$$

Polynomials

We can repeat the argument for restrictions of <u>homogenous</u> polynomials:

$$q(x, y) = \sum_{j+k=d} a_{jk} \cdot x^j \cdot y^k$$

$$\Updownarrow$$

$$\left(\rho_{\mathbf{R}}(q)\right)(x, y) = \sum_{j+k=d} a_{jk} \cdot \underbrace{(ax + cy)^j}_{\text{Degree 1}} \cdot \underbrace{(bx + dy)^k}_{\text{Degree 1}}$$

Polynomials

We can repeat the argument for restrictions of <u>homogenous</u> polynomials:

$$q(x,y) = \sum_{j+k=d} a_{jk} \cdot x^j \cdot y^k$$

$$\Updownarrow$$

$$\big(\rho_{\mathbf{R}}(q)\big)(x,y) = \sum_{j+k=d} a_{jk} \cdot \underbrace{(ax+cy)^j}_{\text{Degree } j} \cdot \underbrace{(bx+dy)^k}_{\text{Degree } k}$$

Polynomials

We can repeat the argument for restrictions of <u>homogenous</u> polynomials:

$$q(x,y) = \sum_{j+k=d} a_{jk} \cdot x^j \cdot y^k$$

$$\Updownarrow$$

$$\big(\rho_{\mathbf{R}}(q)\big)(x,y) = \sum_{j+k=d} a_{jk} \cdot \underbrace{(ax+cy)^j \cdot (bx+dy)^k}_{\text{Degree } j+k}$$

$\Rightarrow$ The space of functions obtained by restricting homogenous polynomials of degree d to the unit circle is a sub-representation.

Homogenous Polynomials

How small are these sub-representations?

The space of homogenous polynomials of degree d in two variables is $(d+1)$-dimensional.
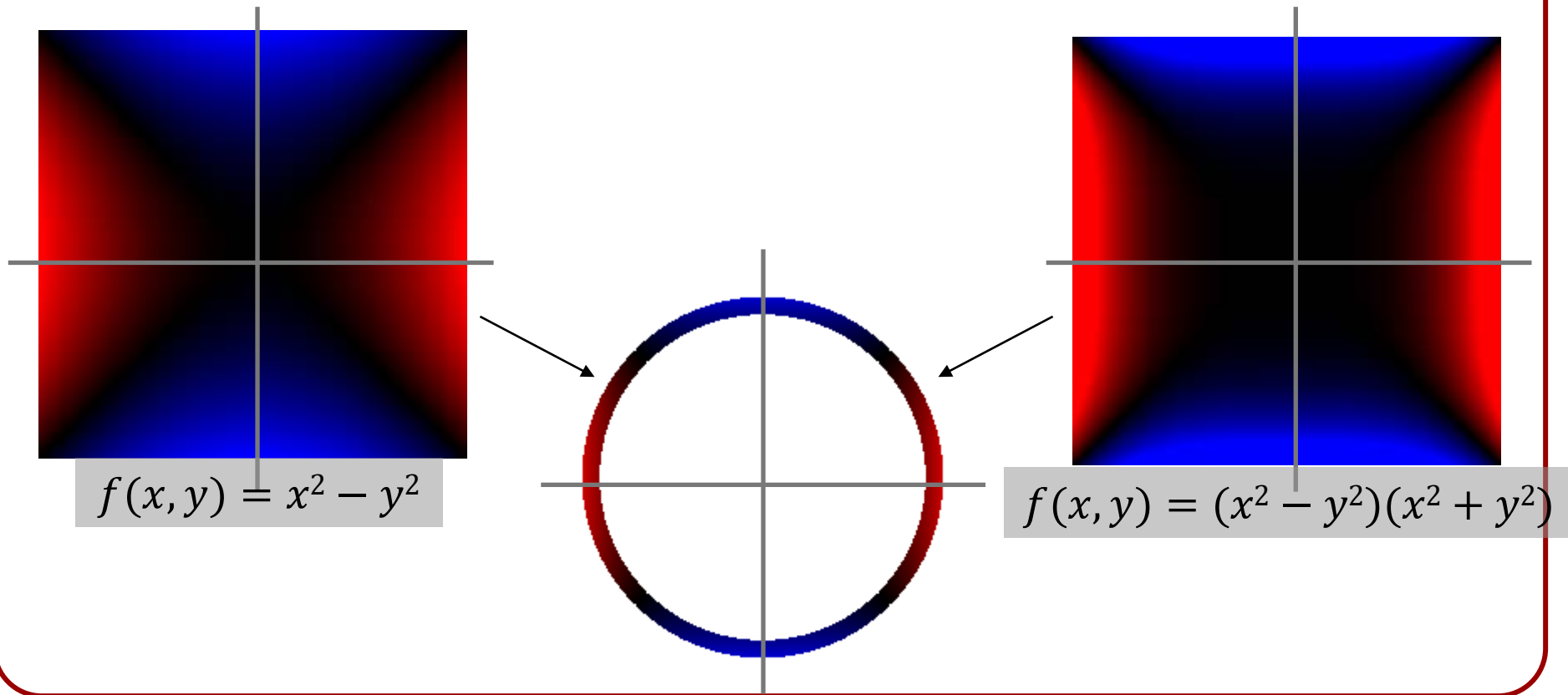
We know that the irreducible representations all have to be one-dimensional – what's going on?

Homogenous Polynomials

Recall:

Two different functions in 2D, can have the same restriction to the unit-circle.

$$f(x, y) = x^2 - y^2$$

$$f(x, y) = (x^2 - y^2)(x^2 + y^2)$$

Homogenous Polynomials

Note that every point (x, y) on the circle satisfies:
$$x^2 + y^2 + 1$$

$\Rightarrow$ For any $q(x, y) \in HP^d(x, y)$, the homogenous polynomial $q(x, y) \cdot (x^2 + y^2) \in HP^{d+2}(x, y)$ will have the same restriction to the unit circle.

Homogenous Polynomials

When considering the restriction of homogenous polynomials to the circle, degree d polynomials are "contained" in the restriction of the degree $(d + 2)$ polynomials.

Since the restrictions of degree d polynomials to the circle form a sub-representation, we want the polynomials of degree $(d + 2)$ whose restrictions are orthogonal to those of degree d polynomials.

Homogenous Polynomials

Example:

- $d = 0$:

  $HP^d(x, y)$ is spanned by $\{1\}$ so the restriction is the space of constant functions.

Homogenous Polynomials

Example:

- $d = 1$:

  $HP^d(x, y)$ is spanned by $\{x, y\}$ so the restriction is the space of functions of the form $ax + by$.

  Since we can write out the x and y coordinates in terms of the angle θ:

  $$x = \cos\theta \qquad y = \sin\theta$$

  this gives the space of functions of the form:

  $$f(\theta) = a \cdot \cos\theta + b \cdot \sin\theta$$

Homogenous Polynomials

Example:

- $d = 2$:

  $HP^d(x, y)$ is spanned by $\{x^2, xy, y^2\}$ so the restriction is the space of functions of the form $ax^2 + bxy + cy^2$. In terms of the angle, this gives the space of functions of the form:

  $$f(\theta) = a \cdot \cos^2\theta + b \cdot \cos\theta \cdot \sin\theta + c \cdot \sin^2\theta$$

Homogenous Polynomials

<u>Example</u>:

- $d = 2$:
    $$f(\theta) = a \cdot \cos^2 \theta + b \cdot \cos \theta \cdot \sin \theta + c \cdot \sin^2 \theta$$

    Since we know that:
    $$\cos^2 \theta + \sin^2 \theta = 1$$

    is a constant function accounted for by the $d = 0$ case, we want the space of homogenous polynomial restrictions that are perpendicular to those accounted for by the $d = 0$ case.

Homogenous Polynomials

Example:

- $d = 2$:

    A function of the form:
    $$f(\theta) = a \cdot \cos^2\theta + b \cdot \cos\theta \cdot \sin\theta + c \cdot \sin^2\theta$$
    is perpendicular to the function:
    $$\cos^2\theta + \sin^2\theta = 1$$
    if and only if:
    $$0 = \langle 1, a \cdot \cos^2\theta + b \cdot \cos\theta \cdot + c \cdot \sin^2\theta \rangle$$

Homogenous Polynomials

Example:

- $d = 2$:

$$0 = \langle 1, a \cdot \cos^2 \theta + b \cdot \cos \theta \cdot \sin \theta + c \cdot \sin^2 \theta \rangle$$

$$\Updownarrow$$

$$0 = \int_0^{2\pi} \left(a \cdot \cos^2 \theta + b \cdot \cos \theta \cdot \sin \theta + c \cdot \sin^2 \theta \right) d\theta$$

$$\Updownarrow$$

$$0 = a \cdot \pi + c \cdot \pi$$

$$\Updownarrow$$

$$a = -c$$

Homogenous Polynomials

Example:

- $d = 2$:

    Homogenous polynomials of degree two can be expressed as:
    $$f(\theta) = a \cdot \cos^2 \theta + b \cdot \cos \theta \cdot \sin \theta + c \cdot \sin^2 \theta$$
    and orthogonality implies that:
    $$c = -a$$

    $\Rightarrow$ A basis for the sub-representation is:
    $$\{\cos^2 \theta - \sin^2 \theta , \cos \theta \cdot \sin \theta\}$$
    $$\Updownarrow$$
    $$\{\cos 2\theta , \sin 2\theta\}$$

Homogenous Polynomials

Example:

- $d \geq 2$:

    As in the $d = 2$ case, we start with the space of homogenous polynomials of degree d.

    Since the space of homogenous polynomials of degree $d - 2$ is contained in this space, we "remove" the degree $d - 2$ polynomials.

    Thus, the final dimension of the sub-representation is:
    $$\dim\left[HP^d(x, y)\right] - \dim[HP^{d-2}(x, y) = (d + 1) - (d - 1)$$
    $$= 2$$

Homogenous Polynomials

Example:

- $d \geq 2$:

    As in the $d = 2$ case, that the two functions:
    $$\{\cos d\theta \,, \sin d\theta\}$$
    are a basis for the sub-representation.

Homogenous Polynomials

Note:

These sub-representations are reducible.

By Schur's lemma, the irreducible representations are all one-dimensional and for $d > 0$, we are getting two-dimensional sub-representations.

To get the irreducible representations, we need to further break apart these sub-representations.

$$\{\cos d\theta\,,\sin d\theta\} = \begin{Bmatrix}\cos d\theta + i\sin d\theta \\ \cos d\theta - i\sin d\theta\end{Bmatrix} = \begin{Bmatrix}e^{i\theta} \\ e^{-i\theta}\end{Bmatrix}$$

These two-dimensional representations are irreducible representations for the group of orthogonal transformations (i.e. rotations <u>and reflections</u>).

Outline

Homogenous Polynomials

Representations of Functions on the Unit-Circle

- Sub-Representations for the Circle

- Sub-Representations for the Sphere

Spherical Functions

As in the case of circular functions, we would like to find the sub-representations of the spherical functions – sub-spaces of spherical functions which get rotated back into themselves.

In this case, the group does not commute, so we do not expect the sub-representations to be one-dimensional.

Homogenous Polynomials

As with circular functions, we consider spherical functions obtained by restricting homogenous polynomials of degree d to the unit sphere:

$$q(x, y, z) = \sum_{j+k+l=d} a_{jkl} \cdot x^j \cdot y^k \cdot z^l$$

Homogenous Polynomials

If $\mathbf{R}$ is a rotation:

$$\mathbf{R} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

then $\mathbf{R}$ will rotate the polynomial q by:

$$\big(\rho_{\mathbf{R}}(q)\big)(x, y, z) =$$

$$= \sum_{j+k+l=d} a_{jkl} \cdot (ax + dy + gz)^j \cdot (bx + ey + hz)^k \cdot (cx + fy + iz)^l$$

Again, rotations fix the space of homogenous polynomials – mapping homogenous polynomials of degree d back into homogenous polynomials of degree d.

Homogenous Polynomials

As in the 2D case, we know that the restrictions of homogenous polynomials of degree d to the unit sphere contain the restrictions of homogenous polynomials of degree $d - 2$ to the unit sphere.

So for any $q(x, y, z) \in HP^d(x, y, z)$, the polynomial $q(x, y, z) \cdot (x^2 + y^2 + z^2) \in HP^{d+2}(x, y, z)$ will have the same restriction to the unit sphere.

Homogenous Polynomials

⇒ Sub-representations can be obtained by considering the restrictions of homogenous polynomials of degree d to the unit sphere, and removing those that were already accounted for at degree $d - 2$.

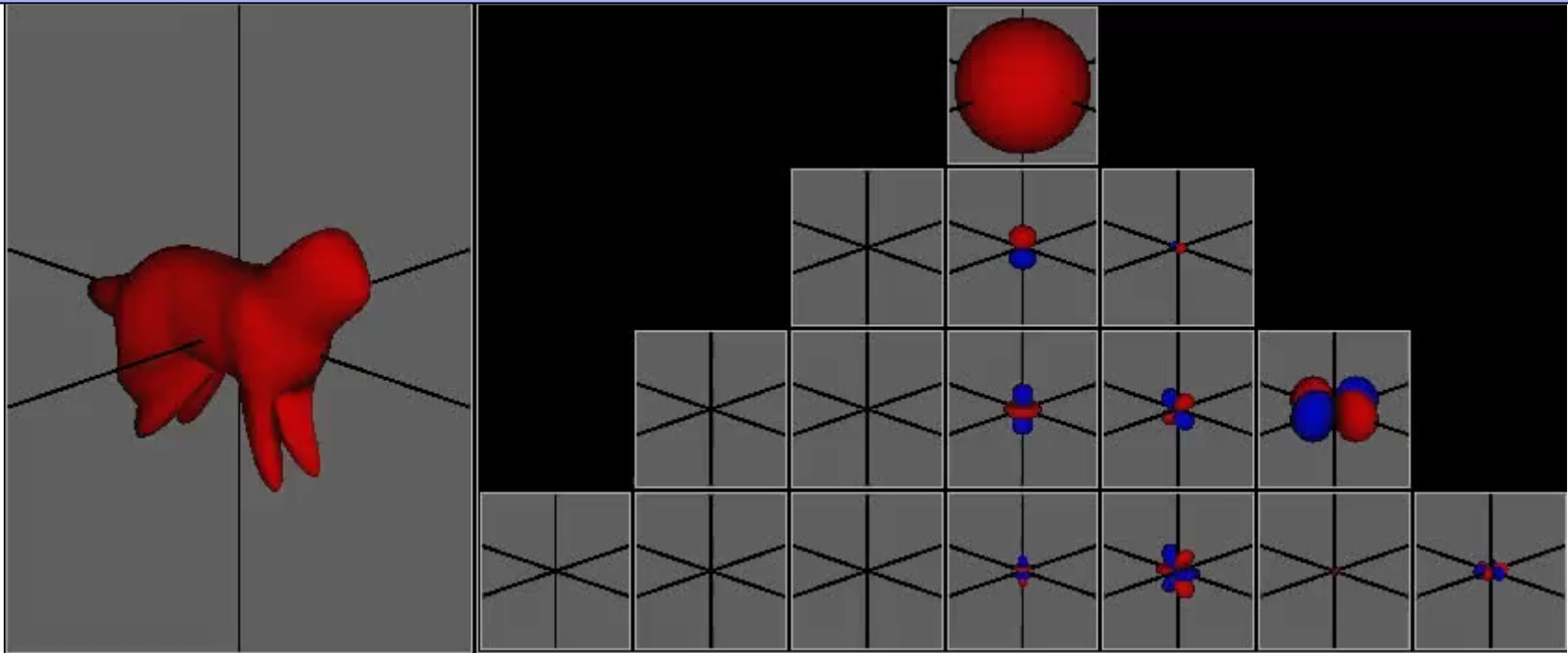⇒ The dimension of the space obtained from the degree d homogenous polynomials will be:

$$\dim\big[HP^d(x, y, z)\big] - \dim\big[HP^{d-2}(x, y, z)\big]$$

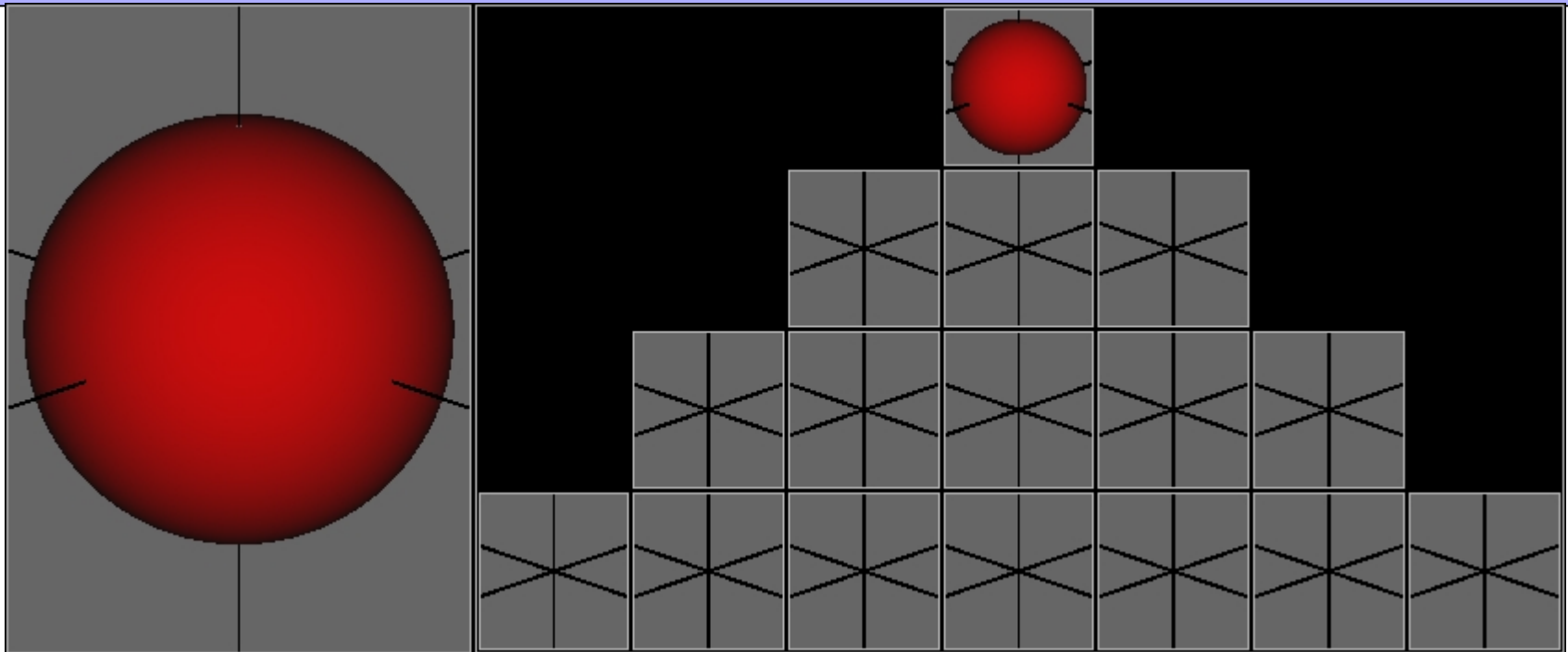$$= \frac{(d + 2) \cdot (d + 1)}{2} - \frac{d \cdot (d - 1)}{2}$$
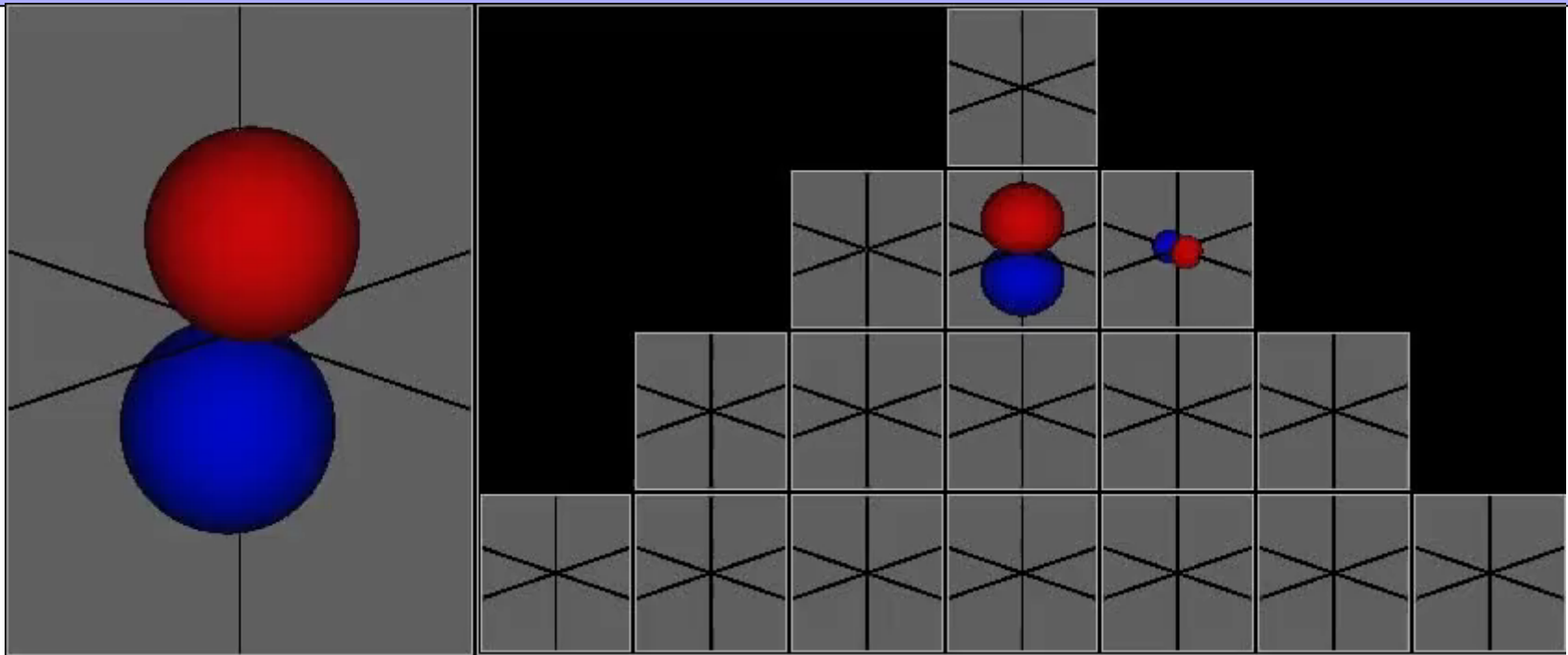
$$= 2d + 1$$

Homogenous Polynomials

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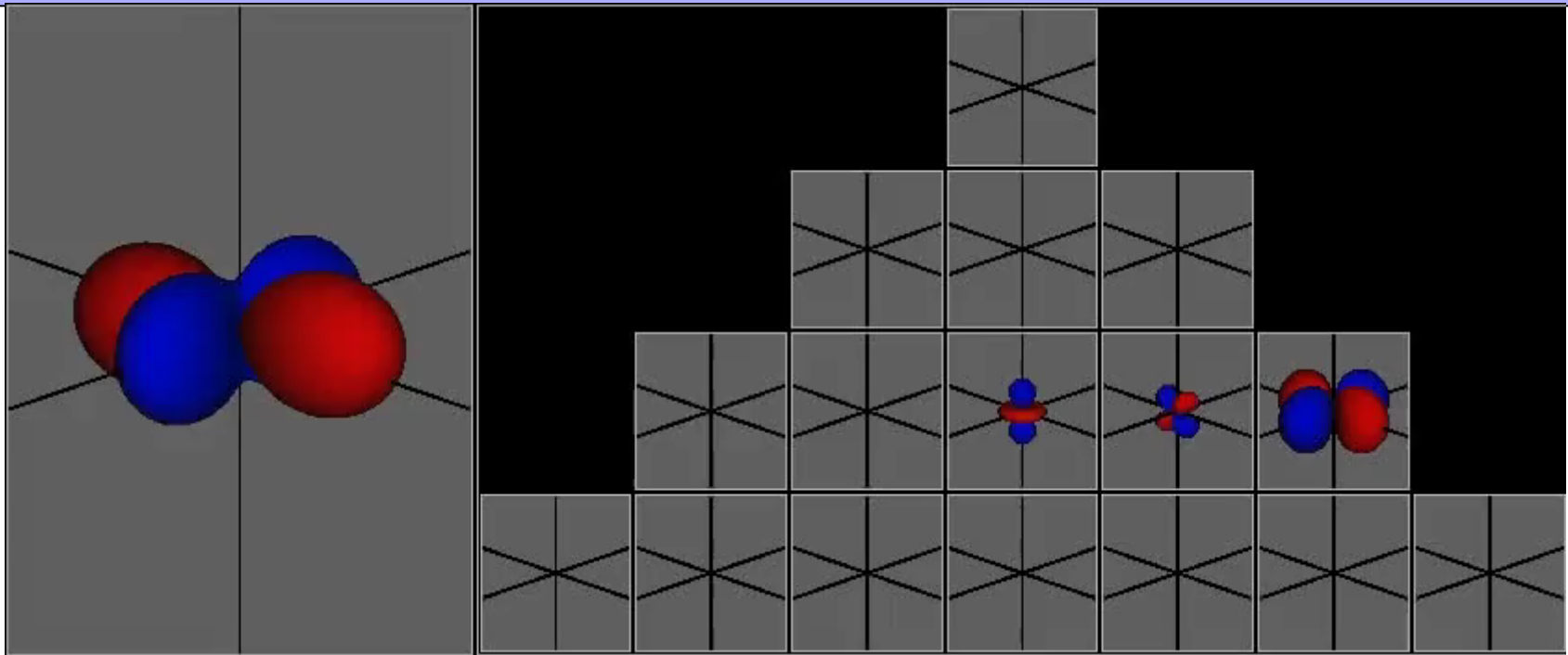
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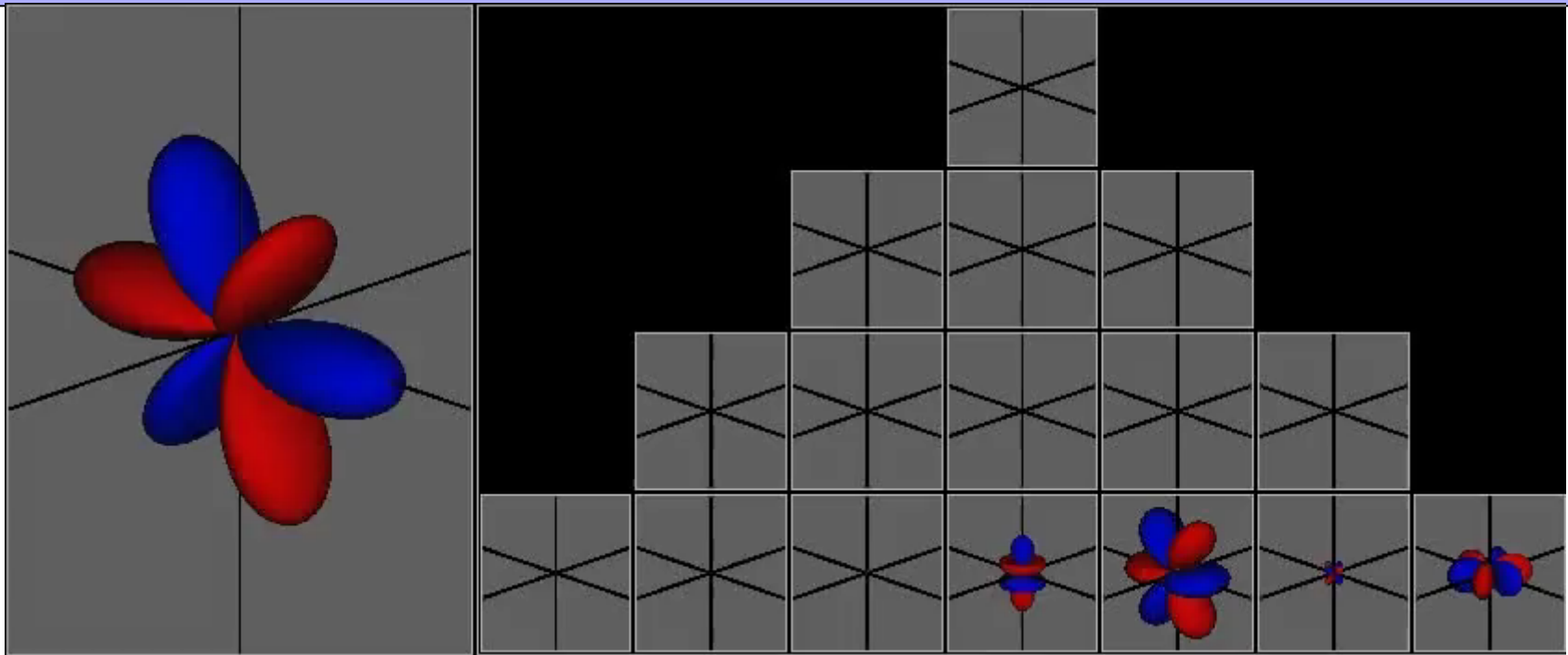

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