FFTs in Graphics and Vision

Differential Equations

Outline

A Simple PDE

Solving the PDE

Relationship to the Fourier Transform

Generalizations

Examples

Evolving Systems

In many physical systems, the way the system changes over time only depends on the system's current state.

Examples:

- Population growth
- Radioactive decay
- Vibrations of a plucked string
- Heat dissipation
- Advection of particles in a vector field

Evolving Systems

In many physical systems, the way the system changes over time only depends on the system's current state.

What we would like to be able to answer is:

- Given the dependency of the change in the system on its current state, and
- Given the initial state of the system,

How does the system evolve over time?

Evolving Systems

A Simple Case:

Consider a 1D system represented by the function $F(x,t)$, where x represents the point in space and t the point in time.

Evolving Systems

A Simple Case:

If the change in the system can be described by:

$$\frac{\partial F(x,t)}{\partial t} = a_0 \cdot F(x,t) + \cdots + a_n \cdot \frac{\partial^n F(x,t)}{\partial x^n}$$
$$= \sum_{p=0}^{n} a_p \cdot \frac{\partial^p F(x,t)}{\partial x^p}$$

and the initial state is defined by:

$$F(x,0) = f^0(x)$$

How do we compute the state at time t:

$$F(x,t) = ?$$

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Solving the PDE

General Approach:

1. Solve the time PDE $h'(t) = \lambda \cdot h(t)$
2. Get a set of solutions to the space/time PDE
3. Find the linear combination of solutions that satisfies the initial condition

Solve the time PDE $h'(t) = \lambda \cdot h(t)$

This we know how to do:

$$h'(t) = \lambda \cdot h(t)$$

$$\Updownarrow$$

$$h(t) = C \cdot e^{\lambda t}$$

Get Solutions to the space/time PDE

$$\frac{\partial F(x,t)}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p F(x,t)}{\partial x^p}$$

Approach (separation of variables):

Consider the case when $F(x,t)$ is the product:

$$F(x,t) = g(x) \cdot h(t)$$

Our goal is to solve for $g(x)$ and $h(t)$ s.t.:

$$\frac{\partial (g(x) \cdot h(t))}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p (g(x) \cdot h(t))}{\partial x^p}$$

Get Solutions to the space/time PDE

Observation 1:

Taking the p-th derivative is a linear operation:

$$\frac{\partial^p\big(\alpha \cdot f(x) + \beta \cdot g(x)\big)}{\partial x^p} = \alpha \cdot \frac{\partial^p f(x)}{\partial x^p} + \beta \cdot \frac{\partial^p g(x)}{\partial x^p}$$

$\Rightarrow$ The map:

$$f(x) \to \sum_{p=0}^{n} a_p \cdot \frac{\partial^p f(x)}{\partial x^p}$$

is linear.

Get Solutions to the space/time PDE

Observation 1:

The map:

$$f(x) \to \sum_{p=0}^{n} a_p \cdot \frac{\partial^p f(x)}{\partial x^p}$$

is linear.

We will write out this linear operator as:

$$D_{\mathbf{a}} = \sum_{p=0}^{n} a_k \cdot \frac{\partial^p}{\partial x^p}$$

Get Solutions to the space/time PDE

Observation 2:

If we can find an eigenvalue/eigenvector of $D_{\mathbf{a}}$, we can find a solution to the space/time PDE.

Get Solutions to the space/time PDE

Observation 2:

Suppose that $\mathbf{g}^{\lambda}(x)$ is an eigenvector of $D_{\mathbf{a}}$ with eigenvalue λ:

$$D_{\mathbf{a}}\left(\mathbf{g}^{\lambda}(x)\right) = \lambda \cdot \mathbf{g}^{\lambda}(x)$$

We want to find $\mathbf{h}^{\lambda}(t)$ such that:

$$\frac{\partial\left(\mathbf{g}^{\lambda}(x) \cdot \mathbf{h}^{\lambda}(t)\right)}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p\left(\mathbf{g}^{\lambda}(x) \cdot \mathbf{h}^{\lambda}(t)\right)}{\partial x^p}$$

Get Solutions to the space/time PDE

Observation 2:

$$\frac{\partial\left(\mathbf{g}^{\lambda}(x)\cdot\mathbf{h}^{\lambda}(t)\right)}{\partial t}=\sum_{p=0}^{n}a_p\cdot\frac{\partial^p\left(\mathbf{g}^{\lambda}(x)\cdot\mathbf{h}^{\lambda}(t)\right)}{\partial x^p}$$

Since $\mathbf{g}^{\lambda}(x)$ does not depend on t we can re-write the left-hand side as:

$$\mathrm{LHS}=\mathbf{g}^{\lambda}(x)\cdot\frac{\partial\mathbf{h}^{\lambda}(t)}{\partial t}$$

Get Solutions to the space/time PDE

Observation 2:

$$\mathbf{g}^{\lambda}(x)\cdot\frac{\partial\mathbf{h}^{\lambda}(t)}{\partial t}=\sum_{p=0}^{n}a_p\cdot\frac{\partial^p\left(\mathbf{g}^{\lambda}(x)\cdot\mathbf{h}^{\lambda}(t)\right)}{\partial x^p}$$

Since $\mathbf{h}^{\lambda}(t)$ does not depend on x we can re-write the right-hand side as:

$$RHS=\mathbf{h}^{\lambda}(t)\sum_{p=0}^{n}a_p\cdot\frac{\partial^p\mathbf{g}^{\lambda}(x)}{\partial x^p}$$

$$=\mathbf{h}^{\lambda}(t)\cdot D_{\mathbf{a}}\left(\mathbf{g}^{\lambda}(x)\right)$$

$$=\lambda\cdot\mathbf{h}^{\lambda}(t)\cdot\mathbf{g}^{\lambda}(x)$$

Get Solutions to the space/time PDE

Observation 2:

$$\mathbf{g}^{\lambda}(x) \cdot \frac{\partial \mathbf{h}^{\lambda}(t)}{\partial t} = \lambda \cdot \mathbf{h}^{\lambda}(t) \cdot \mathbf{g}^{\lambda}(x)$$

$$\Uparrow$$

$$\frac{\partial \mathbf{h}^{\lambda}(t)}{\partial t} = \lambda \cdot \mathbf{h}^{\lambda}(t)$$

$$\Updownarrow$$

$$\mathbf{h}^{\lambda}(t) = C \cdot e^{\lambda t}$$

Get Solutions to the space/time PDE

Observation 2:

⇒ If $\mathbf{g}^{\lambda}(x)$ is an eigenvector of the linear operator $D_{\mathbf{a}}$ with eigenvalue λ, then any multiple of:

$$\mathbf{F}^{\lambda}(x,t) = e^{\lambda \cdot t} \cdot \mathbf{g}^{\lambda}(x)$$

is a solution to the differential equation:

$$\frac{\partial F(x,t)}{\partial t} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p F(x,t)}{\partial x^p}$$

Satisfying the Initial Condition

Observation 3:

If $\mathbf{F}^1(x,t)$ and $\mathbf{F}^2(x,t)$ are solutions to the (partial) differential equation then their linear combination is also a solution:

$$\frac{\partial \mathbf{F}^1}{\partial t} = D_{\mathbf{a}}(\mathbf{F}^1) \quad \text{and} \quad \frac{\partial \mathbf{F}^2}{\partial t} = D_{\mathbf{a}}(\mathbf{F}^2)$$

$$\Downarrow$$

$$\frac{\partial\left(\alpha \cdot \mathbf{F}^1 + \beta \cdot \mathbf{F}^2\right)}{\partial t} = D_{\mathbf{a}}(\alpha \cdot \mathbf{F}^1 + \beta \cdot \mathbf{F}^2)$$

Satisfying the Initial Condition

- For every $\mathbf{g}^{\lambda}(x)$ that is an eigenvector of $D_{\mathbf{a}}$ with eigenvalue λ, the function:
$$\mathbf{F}^{\lambda}(x,t) = e^{\lambda\cdot t}\cdot\mathbf{g}^{\lambda}(x)$$
is a solution.

- A linear combination of solutions is a solution.

$\Rightarrow$ Any function $F(x,t)$ that is expressible as:
$$F(x,t) = \sum_{\lambda} c_{\lambda}\cdot e^{\lambda\cdot t}\cdot\mathbf{g}^{\lambda}(x)$$
is a solution to the PDE.

Satisfying the Initial Condition

To satisfy the initial value conditions, we need to choose c_λ so that the function:

$$F(x,t) = \sum_\lambda c_\lambda \cdot e^{\lambda \cdot t} \cdot \mathbf{g}^\lambda(x)$$

satisfies the initial value conditions:

$$F(x,0) = f^0(x)$$

But this implies that:

$$f^0(x) = \sum_\lambda c_\lambda \cdot \mathbf{g}^\lambda(x)$$

Satisfying the Initial Condition

To satisfy the initial value conditions, we need to choose c_λ so that the function:

$$F(x,t) = \sum_\lambda c_\lambda \cdot e^{\lambda \cdot t} \cdot \mathbf{g}^\lambda(x)$$

sat

Satisfying the initial value conditions is equivalent to finding the coefficients of $f^0(x)$ with respect to the functions $\{\mathbf{g}^\lambda(x)\}$

But

$$f^0(x) = \sum_\lambda c_\lambda \cdot \mathbf{g}^\lambda(x)$$

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Relationship to the Fourier Transform

Recall that the Fourier decomposition expresses a circular function $f(\theta)$ as a sum of complex exponentials of different frequencies:

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_k \cdot \boldsymbol{\zeta}^k(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_k \cdot \frac{e^{ik\theta}}{\sqrt{2\pi}}$$

Relationship to the Fourier Transform

The complex exponentials are the eigenvectors of (all) the derivative operator:

$$\frac{\partial}{\partial\theta} e^{i\cdot k\cdot\theta} = i\cdot k\cdot e^{i\cdot k\cdot\theta}$$

$$\frac{\partial^2}{\partial\theta^2} e^{i\cdot k\cdot\theta} = -k^2\cdot e^{i\cdot k\cdot\theta}$$

$$\vdots$$

$$\frac{\partial^n}{\partial\theta^n} e^{i\cdot k\cdot\theta} = (i\cdot k)^n\cdot e^{i\cdot k\cdot\theta}$$

Relationship to the Fourier Transform

So, if we are given the linear map:

$$D_{\mathbf{a}} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p}{\partial \theta^p}$$

it acts on $e^{i \cdot k \cdot \theta}$ as:

$$D_{\mathbf{a}}\left(e^{i \cdot k \cdot \theta}\right) = \left(\sum_{p=0}^{n} a_p \cdot (i \cdot k)^p\right) e^{i \cdot k \cdot \theta}$$

$\Rightarrow \boldsymbol{\zeta}^k(\theta)$ is an eigenvector with eigenvalue:

$$\lambda_k = \sum_{p=0}^{n} a_p \cdot (i \cdot k)^p$$

Relationship to the Fourier Transform

In particular, this implies that:

$$\mathbf{F}^k(\theta,t)=e^{\lambda_k\cdot t}\cdot\boldsymbol{\zeta}^k(\theta)\quad \text{w/ } \lambda_k=\sum_{p=0}^{n}a_p\cdot(i\cdot k)^p$$

are solutions to the partial differential equation:

$$\frac{\partial F(\theta,t)}{\partial t}=\sum_{p=0}^{n}a_p\cdot\frac{\partial^p F(\theta,t)}{\partial\theta^p}$$

⇒ All functions of the form:

$$F(\theta,t)=\sum_{k=-\infty}^{\infty}c_k\cdot e^{\lambda_k\cdot t}\cdot\boldsymbol{\zeta}^k(\theta)$$

are solutions to the PDE

Relationship to the Fourier Transform

$$F(\theta,t)=\sum_{k=-\infty}^{\infty}c_k\cdot e^{\lambda_k t}\cdot\boldsymbol{\zeta}^k(\theta)$$

To satisfy the initial condition:

$$F(\theta,0)=f^0(\theta)$$

we need to solve for the values of c_k such that:

$$f^0(\theta)=\sum_{k=-\infty}^{\infty}c_k\cdot\boldsymbol{\zeta}^k(\theta)$$

$\Rightarrow$ c_k is the k-th Fourier coefficients of $f^0(\theta)$:

$$c_k=\hat{f}_k^0$$

Relationship to the Fourier Transform

The solution to the PDE is the function $F(\theta, t)$ whose k-th Fourier coefficient at time t is the modulation of the k-th Fourier coefficient of $f^0(\theta)$ by a function of t:

$$\hat{f}_k(t) = \hat{f}_k^0 \cdot e^{\lambda_k \cdot t}$$

Relationship to the Fourier Transform

To implement this, we start off by:

- Computing the Fourier coefficients of $f^0(\theta)$

Then, at each time t, we:

- Compute the modulated Fourier coefficients:
$$\hat{f}_k(t) = \hat{f}_k^0 \cdot e^{\lambda_k \cdot t}$$
- And compute the inverse Fourier transform.

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- Higher dimensions
- Second order time derivatives

Examples

2D Systems

In the case that the system is 2D, we want to consider functions of the form $F(\theta, \phi, t)$.

The linear partial differential equation becomes:

$$\frac{\partial F(\theta, \phi, t)}{\partial t} = \sum_{p=0}^{m} \sum_{q=0}^{n} a_{pq} \cdot \frac{\partial^p \partial^q F(\theta, \phi, t)}{\partial \theta^p \partial \phi^q}$$

The initial state becomes:

$$F(\theta, \phi, 0) = f^0(\theta, \phi)$$

And the challenge is to compute the state of the system at an arbitrary point in time:

$$F(\theta, \phi, t) = ?$$

2D Systems

As in the 1D case, we can use the fact that the a periodic 2D function $f(\theta,\phi)$ can be expressed in terms of its Fourier decomposition:

$$f(\theta,\phi) = \sum_{k,l=-\infty}^{\infty} \hat{f}_{kl} \cdot \boldsymbol{\zeta}^{kl}(\theta,\phi) = \sum_{k,l=-\infty}^{\infty} \hat{f}_{kl} \cdot \frac{e^{i \cdot k \cdot \theta}}{\sqrt{2\pi}} \cdot \frac{e^{i \cdot l \cdot \phi}}{\sqrt{2\pi}}$$

2D Systems

And again, we use the fact that the complex exponentials are eigenvectors of the partial derivative operator:

$$\frac{\partial}{\partial\theta} e^{i\cdot k\cdot\theta}\cdot e^{i\cdot l\cdot\phi} = (i\cdot k)\cdot e^{i\cdot k\cdot\theta}\cdot e^{i\cdot l\cdot\phi}$$

$$\frac{\partial}{\partial\phi} e^{i\cdot k\cdot\theta}\cdot e^{i\cdot l\cdot\phi} = (i\cdot l)\cdot e^{i\cdot k\cdot\theta}\cdot e^{i\cdot l\cdot\phi}$$

$$\Downarrow$$

$$\frac{\partial^m\partial^n}{\partial\theta^m\partial\phi^n} e^{i\cdot k\cdot\theta}\cdot e^{i\cdot l\cdot\phi} = (i\cdot k)^m\cdot(i\cdot l)^n e^{i\cdot k\cdot\theta}\cdot e^{i\cdot l\cdot\phi}$$

2D Systems

Thus, given the linear map:

$$D_{\mathbf{a}} = \sum_{p=0}^{m} \sum_{q=0}^{n} a_{pq} \cdot \frac{\partial^p \partial^q}{\partial \theta^p \partial \phi^q}$$

the complex exponential $\boldsymbol{\zeta}^{kl}(\theta, \phi)$ is an eigenvector with eigenvalue:

$$\lambda_{kl} = \sum_{p=0}^{m} \sum_{q=0}^{n} a_{pq} \cdot (i \cdot k)^p \, (i \cdot l)^q$$

2D Systems

$$\lambda_{kl} = \sum_{p=0}^{m} \sum_{q=0}^{n} a_{pq} \cdot (i \cdot k)^p \, (i \cdot l)^q$$

Given these eigenvectors, we can proceed as before, obtaining a solution to the differential equation for each eigenvector:

$$\mathbf{F}^{kl}(\theta, \phi, t) = e^{\lambda_{kl} \cdot t} \cdot \boldsymbol{\zeta}^{kl}(\theta, \phi)$$

And we can satisfy the initial condition:

$$F(\theta, \phi, 0) = f^0(\theta, \phi)$$

by setting the Fourier coefficients of $F(\theta, \phi, t)$ at time t equal to:

$$\hat{f}_{kl}(t) = \hat{f}_{kl}^0 \cdot e^{\lambda_{kl} \cdot t}$$

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- Higher dimensions
- Second order time derivatives

Second Order Time Derivatives

What if the change in the system is characterized by the second derivative with respect to time:

$$\frac{\partial^2 F(\theta,t)}{\partial t^2} = \sum_{p=0}^{n} a_p \cdot \frac{\partial^p F(\theta,t)}{\partial \theta^p}$$

Second Order Time Derivatives

Recall that the solution to the original PDE was derived by solving the (first-order time) derivative:

$$h'(t) = \lambda \cdot h(t)$$

In this case, we need to solve the second-order time derivative:

$$h''(t) = \lambda \cdot h(t)$$

$$\Downarrow$$

$$h(t) = e^{\sqrt{\lambda} t} \quad \text{and} \quad h(t) = e^{-\sqrt{\lambda} t}$$

Second Order Time Derivatives

As before, if $\mathbf{g}^{\lambda}(\theta)$ is an eigenvector of the linear operator $D_{\mathbf{a}}$ with eigenvalue λ, then the functions:

$$\mathbf{F}^{\lambda,+}(\theta,t) = e^{\sqrt{\lambda}t} \cdot \mathbf{g}^{\lambda}(\theta)$$

$$\mathbf{F}^{\lambda,-}(\theta,t) = e^{-\sqrt{\lambda}t} \cdot \mathbf{g}^{\lambda}(\theta)$$

will both be solutions to the PDE.

Second Order Time Derivatives

Note that in this case, one eigenvector of $D_{\mathbf{a}}$ gives us two solutions to the differential equation.

↓

We have more functions with which we can satisfy the initial boundary conditions.

↓

We can specify more boundary conditions.

Second Order Time Derivatives

In practice, this amounts to specifying two boundary conditions:

- Initial value conditions:

$$F(\theta,0) = f^0(\theta)$$

- Initial derivative conditions:

$$\frac{\partial}{\partial t}F(\theta,0) = v^0(\theta)$$

Second Order Time Derivatives

Intuitively:

In the first-order case the PDE gives the velocity at every point.

⇒ If we know the initial positions, this is enough to know where things end up.

In the second-order case the PDE gives the acceleration at every point.

⇒ To know where things end up, we need to know the initial position and the initial velocity.

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- The Laplacian
- The 2D heat equation
- The 2D wave equation

The Laplacian

Given a function f in 1D, how do we interpret its second derivative:

$$\Delta f = \frac{\partial f}{\partial x^2}$$

The Laplacian

The first derivative of $f(x)$ is approximated by looking at the difference between the value of f at x and the value of f at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

The Laplacian of $f(x)$ is approximated by applying the process to the derivative of $f(x)$:

$$\begin{aligned} f''(x) &\approx f'(x) - f'(x-1) \\ &\approx \big(f(x+1) - f(x)\big) - \big(f(x) - f(x-1)\big) \\ &= f(x+1) + f(x-1) - 2f(x) \end{aligned}$$

The Laplacian

The first derivative of $f(x)$ is approximated by looking at the difference between the value of f at x and the value of f at a neighboring point:

$$f'(x) \approx f(x+1) - f(x)$$

The Laplacian of $f(x)$ is approximated by applying the process to the derivative of $f(x)$:

$$f''(x) \approx 2\left(\frac{f(x+1)+f(x-1)}{2} - f(x)\right)$$

i.e. it is a measure of the difference between the value of f at x and the average value of f at the neighbors of x.

The Laplacian

The same interpretation holds for the Laplacian of a 2D function f:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

i.e. The Laplacian of a function is a measure of how the value of f at a point (x, y) differs from the average of the values of its neighbors.

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Newton's Law of Cooling

"*The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.*"

Translating this into the PDE, if $F(\theta, \phi, t)$ is the heat at position (x, y) at time t, then:

$$\frac{\partial F}{\partial t} = \eta \cdot \Delta F$$

Newton's Law of Cooling

$$\frac{\partial F}{\partial t} = \eta \cdot \Delta F$$

In this case, the linear operator $D_{\mathbf{a}}$ is defined by:

$$D_{\mathbf{a}} = \eta \cdot \Delta$$

and the complex exponential $\boldsymbol{\zeta}^{kl}$ is an eigenvector with eigenvalue:

$$\lambda_{kl} = -\eta \cdot \left(k^2 + l^2\right)$$

Newton's Law of Cooling

$$\frac{\partial F}{\partial t} = \eta \cdot \Delta F$$

Thus, the solution to this equation:

$$F(\theta, \phi, 0) = f^0(\theta, \phi)$$

is the function whose (k, l)-th Fourier coefficient at time t is:

$$\hat{f}_{kl}(t) = \hat{f}_{kl}^0 \cdot e^{-\eta \cdot (k^2 + l^2) \cdot t}$$

Newton's Law of Cooling

$$\hat{f}_{kl}(t) = \hat{f}_{kl}^0 \cdot e^{-\eta \cdot (k^2 + l^2) \cdot t}$$

This is equivalent to Gaussian smoothing for the plane, but extends the definition to other domains.

This looks remarkably like what we get when implement Gaussian smoothing…

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The 2D Wave Equation

Consider a square (periodic) rubber sheet.

If we displace the points on the sheet, then at any point (θ,ϕ), the neighbors of (θ,ϕ) will exert a force to pull the point towards them.

The force exerted on (θ,ϕ) is proportional to the distance of (θ,ϕ) from its neighbors.

The 2D Wave Equation

If the height of the point (θ, ϕ) is given by the function $h(\theta, \phi)$ then the force at (θ, ϕ) is:

$$f(\theta, \phi) = \eta \cdot \Delta h(\theta, \phi) \quad \text{w/ } \eta > 0$$

Using the fact that Force = Mass · Acceleration, the PDE for the height at time t is:

$$\frac{\partial^2 H}{\partial t^2} = \eta \cdot \Delta H$$

The 2D Wave Equation

$$\frac{\partial^2 H}{\partial t^2} = \eta \cdot \Delta H$$

We would like to solve this equation, subject to the constraints that at the initial time-step:

- The height at each point is given by:

$$H(\theta, \phi, 0) = h^0(\theta, \phi)$$

- The sheet is not moving:

$$\frac{\partial}{\partial t} H(\theta, \phi, 0) = 0$$

The 2D Wave Equation

$$\frac{\partial^2 H}{\partial t^2} = \eta \cdot \Delta H$$

Again, the linear operator $D_{\mathbf{a}}$ is defined by:

$$D_{\mathbf{a}} = \eta \cdot \Delta$$

and the complex exponential $\boldsymbol{\zeta}^{kl}$ is an eigenvector with eigenvalue:

$$\lambda_{kl} = -\eta \cdot (k^2 + l^2)$$

The 2D Wave Equation

$$\frac{\partial^2 H}{\partial t^2} = \eta \cdot \Delta H$$

Thus, the solutions to this equation are:

$$\mathbf{H}^{kl,+}(\theta,\phi,t) = e^{i\sqrt{\eta\cdot(k^2+l^2)}\cdot t} \cdot \boldsymbol{\zeta}^{kl}(\theta,\phi)$$

$$\mathbf{H}^{kl,-}(\theta,\phi,t) = e^{-i\sqrt{\eta\cdot(k^2+l^2)}\cdot t} \cdot \boldsymbol{\zeta}^{kl}(\theta,\phi)$$

and a general solution takes the form:

$$H(\theta,\phi,t) = \sum_{k,l}\left(a_{kl}\cdot e^{i\sqrt{\eta\cdot(k^2+l^2)}\cdot t} + b_{kl}\cdot e^{-i\sqrt{\eta\cdot(k^2+l^2)}\cdot t}\right)\boldsymbol{\zeta}^{kl}(\theta,\phi)$$

The 2D Wave Equation

$$H(\theta,\phi,t) = \sum_{k,l} \left(a_{kl} \cdot e^{i\sqrt{\eta\cdot(k^2+l^2)}\cdot t} + b_{kl} \cdot e^{-i\sqrt{\eta\cdot(k^2+l^2)}\cdot t} \right) \boldsymbol{\zeta}^{kl}(\theta,\phi)$$

To satisfy the initial value condition:

$$H(\theta,\phi,0) = h^0(\theta,\phi)$$

we need to have:

$$h^0(\theta,\phi) = \sum_{k,l} (a_{kl} + b_{kl}) \cdot \boldsymbol{\zeta}^{kl}(\theta,\phi)$$

$$\Downarrow$$

$$\hat{h}^0_{kl} = a_{kl} + b_{kl}$$

The 2D Wave Equation

$$H(\theta,\phi,t)=\sum_{k,l}\left(a_{kl}\cdot e^{i\sqrt{\eta\cdot(k^2+l^2)}\cdot t}+b_{kl}\cdot e^{-i\sqrt{\eta\cdot(k^2+l^2)}\cdot t}\right)\boldsymbol{\zeta}^{kl}(\theta,\phi)$$

To satisfy the initial derivative condition:

$$\frac{\partial}{\partial t}H(\theta,\phi,0)=0$$

we need to have:

$$0=\sum_{k,l}(a_{kl}-b_{kl})\left(i\sqrt{\eta\cdot(k^2+l^2)}\right)\cdot\boldsymbol{\zeta}^{kl}(\theta,\phi)$$

$$\Downarrow$$

$$a_{kl}=b_{kl}$$

The 2D Wave Equation

$$H(\theta,\phi,t)=\sum_{k,l}\left(a_{kl}\cdot e^{i\sqrt{\eta\cdot(k^2+l^2)}\cdot t}+b_{kl}\cdot e^{-i\sqrt{\eta(k^2+l^2)}\cdot t}\right)\boldsymbol{\zeta}^{kl}(\theta,\phi)$$

$$\hat{h}_{kl}^{0}=a_{kl}+b_{kl}\qquad\text{and}\qquad a_{kl}=b_{kl}$$

⇒ The solution to the 2D wave equation, with initial position $h^0(\theta,\phi)$ and zero initial derivative is the function $H(\theta,\phi,t)$ whose Fourier coefficients at time t are equal to:

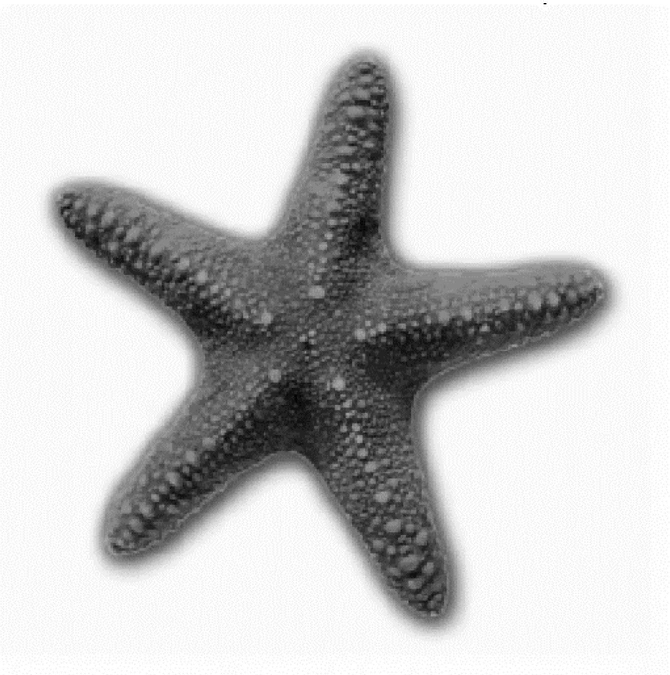
$$\hat{h}_{kl}(t)=\hat{h}_{kl}^{0}\cdot\frac{e^{i\sqrt{\eta\cdot(k^2+l^2)}\cdot t}+e^{-i\sqrt{\eta\cdot(k^2+l^2)}\cdot t}}{2}$$

$$=\hat{h}_{kl}^{0}\cdot\cos\left(\sqrt{\eta\cdot(k^2+l^2)}\cdot t\right)$$

The 2D Wave Equation

$$\hat{h}_{kl}(t) = \hat{h}^0_{kl} \cdot \cos\left(\sqrt{\eta \cdot (k^2 + l^2)} \cdot t\right)$$

This looks <u>nothing</u> like what we get when implement Gaussian smoothing…



The 2D Wave Equation

Note:

The discussion didn't need the full force of the Fourier transform. As long as we can compute the eigenvalues/eigenvectors $\{\lambda, \mathbf{g}^\lambda\}$ of the differential operator $D_{\mathbf{a}}$, we can generate solutions as the linear combinations of:

$$\mathbf{H}^{\lambda,+}(p,t) = e^{\sqrt{\lambda}t} \cdot \mathbf{g}^\lambda(p)$$
$$\mathbf{H}^{\lambda,-}(p,t) = e^{-\sqrt{\lambda}t} \cdot \mathbf{g}^\lambda(p)$$