FFTs in Graphics and Vision

Rotational and Reflective
Symmetry Detection

Announcements

Assignment 2 posted:

- Due 3/28

Outline

Representation Theory

Symmetry Detection (1D)

Symmetry Detection (2D)

Representation Theory

Recall:

A group is a set of elements G with a binary operation (often denoted "$\cdot$") such that for all $f, g, h \in G$, the following properties are satisfied:

- Closure:
$$g \cdot h \in G$$
- Associativity:
$$f \cdot (g \cdot h) = (f \cdot g) \cdot h$$
- Identity: $\exists 1 \in G$ s.t.:
$$1 \cdot g = g \cdot 1 = g$$
- Inverse: $\forall g \in G\ \exists g^{-1} \in G$ s.t.:
$$g \cdot g^{-1} = g^{-1} \cdot g = 1$$

Representation Theory

Observation 1:

Given a group $G = \{g_1, \cdots, g_n\}$, for any $g \in G$, the map that multiplies the elements of G on the left by g is invertible.

(The inverse is the map multiplying the elements of G on the left by g^{-1}.)

Representation Theory

Observation 1:

This implies that the set $\{g \cdot g_1, \cdots, g \cdot g_n\}$ is a re-ordering of the set $\{g_1, \cdots, g_n\}$:

$$gG = G$$

Similarly, the set $\{(g_1)^{-1}, \cdots, (g_n)^{-1}\}$ is a re-ordering of the set $\{g_1, \cdots, g_n\}$:

$$G^{-1} = G$$

Representation Theory

Recall:

A Hermitian inner product is a map from $V \times V$ into the complex numbers that is:

1. Linear: $\forall u, v, w \in V$ and $\lambda \in \mathbb{C}$:
$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$
$$\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$$
2. Conjugate Symmetric: $\forall v, w \in V$:
$$\langle v, w\rangle = \overline{\langle w, v\rangle}$$
3. Positive Definite: $\forall v \in V$:
$$\langle v, v\rangle \geq 0 \quad \text{and} \quad \langle v, v\rangle = 0 \Leftrightarrow v = 0$$

Representation Theory

Observation 2:

Given a Hermitian inner-product space V, and given a set of vectors $\{v_1, \cdots, v_n\} \subset V$, the vector minimizing the sum of squared distances is the average of $\{v_1, \cdots, v_n\}$:

$$\frac{1}{n}\sum_{k=1}^{n} v_k = \arg\min_{v \in V}\left(\sum_{k=1}^{n} \|v - v_k\|^2\right)$$

Representation Theory

Recall:

A unitary representation of a group G on a Hermitian inner-product space V is a map ρ that sends every element in G to an orthogonal transformation on V, satisfying:

$$\rho(g \cdot h) = \rho(g) \cdot \rho(h) \quad \forall g, h \in G$$

Representation Theory

Definition:

We say that a vector $v \in V$ is invariant under the action of G if G sends v back to itself:

$$\rho_g(v) = v \quad \forall g \in G$$

Representation Theory

Notation:

We denote by V_G the set of vectors in V that are invariant under the action of G:

$$V_G = \{v \in V \mid \rho_g(v) = v \ \ \forall g \in G\}$$

Representation Theory

Observation 3:

Note that the set V_G is a vector sub-space of V (in fact, a trivial sub-representation).

If $v, w \in V_G$, then:

$$\rho_g(v) = v \quad \text{and} \quad \rho_g(w) = w \qquad \forall g \in G$$

By linearity, we have:

$$\begin{aligned} \rho_g(v + w) &= \rho_g(v) + \rho_g(w) \\ &= v + w \end{aligned}$$

$\Rightarrow v + w \in V_G$ as well.

Representation Theory

Observation 4:

Given a finite group G and given vector $v \in V$, the vector obtained by averaging over G:

$$\text{Average}(v, G) = \frac{1}{|G|} \sum_{g \in G} \rho_g(v)$$

is invariant under the action of G.

Representation Theory

Observation 4:

We would like to show that:

$$\rho_h\big(\text{Average}(v,G)\big) = \text{Average}(v,G) \quad \forall h \in G$$

Expanding the left hand side we get:

$$\begin{aligned}\rho_h\big(\text{Average}(v,G)\big) &= \rho_h\left(\frac{1}{|G|}\sum_{g\in G}\rho_g(v)\right)\\ &= \frac{1}{|G|}\sum_{g\in G}\rho_h\big(\rho_g(v)\big)\\ &= \frac{1}{|G|}\sum_{g\in G}\rho_{h\cdot g}(v)\end{aligned}$$

Representation Theory

Observation 4:

$$\begin{aligned}\rho_h\big(\mathrm{Average}(v,G)\big) &= \frac{1}{|G|}\sum_{g\in G}\rho_{h\cdot g}(v)\\ &= \frac{1}{|G|}\sum_{g\in hG}\rho_g(v)\\ &= \frac{1}{|G|}\sum_{g\in G}\rho_g(v)\\ &= \mathrm{Average}(v,G)\end{aligned}$$

Representation Theory

Observation 5:

Given a finite group G and given a vector $v \in V$, the average of v over G is the closest G-invariant vector to v:

$$\text{Average}(v, G) = \underset{v^* \in V_G}{\arg\min}\left(\|v^* - v\|^2\right)$$

Representation Theory

Observation 5:

$$\text{Average}(v, G) = \underset{v^* \in V_G}{\arg\min}\left(\|v^* - v\|^2\right)$$

Since v^* is invariant under the action of G, we can write out the squared distances as:

$$\begin{aligned}
\|v^* - v\|^2 &= \frac{1}{|G|}\sum_{g\in G}\left\|\rho_g(v^*) - v\right\|^2 \\
&= \frac{1}{|G|}\sum_{g\in G}\left\|v^* - \left(\rho_g\right)^{-1}(v)\right\|^2 \\
&= \frac{1}{|G|}\sum_{g\in G}\left\|v^* - \rho_{g^{-1}}(v)\right\|^2
\end{aligned}$$

Representation Theory

Observation 5:

$$\|v^* - v\|^2 = \frac{1}{|G|}\sum_{g\in G}\left\|v^* - \rho_{g^{-1}}(v)\right\|^2$$

$$= \frac{1}{|G|}\sum_{g\in G^{-1}}\left\|v^* - \rho_g(v)\right\|^2$$

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Representation Theory

Observation 5:

$$\|v^* - v\|^2 = \frac{1}{|G|}\sum_{g\in G}\left\|v^* - \rho_g\,(v)\right\|^2$$

Thus, v^* is the G-invariant vector minimizing the squared distance to v if and only if it minimizes the sum of squared distances to the vectors:

$$\{\rho_{g_1}(v), \cdots, \rho_{g_n}(v)\}$$

So v^* must be the average of these vectors:

$$v^* = \frac{1}{|G|}\sum_{g\in G}\rho_g(v) = \text{Average}(v, G)$$

Representation Theory

Note:

Since the average map:

$$\text{Average}(v,G)=\frac{1}{|G|}\sum_{g\in G}\rho_g(v)$$

is a linear map returning the closest G-invariant vector to v, the average map is the projection map from V to V_G.

Outline

Representation Theory

Symmetry Detection (1D)

Symmetry Detection (2D)

Symmetry Detection (1D)

What kind of reflective/rotational symmetry does the shape have?

Symmetry Detection (1D)

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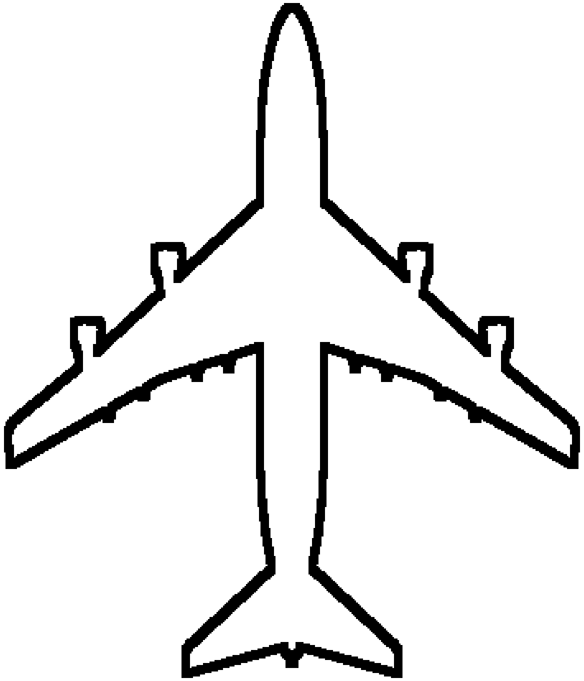

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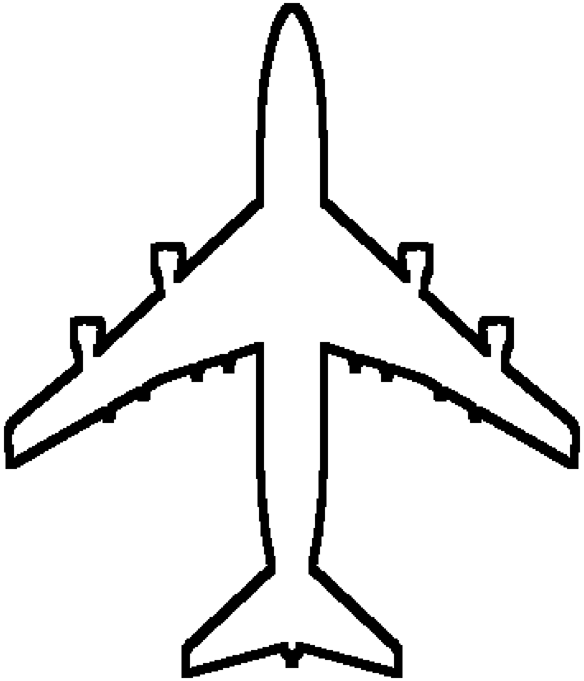

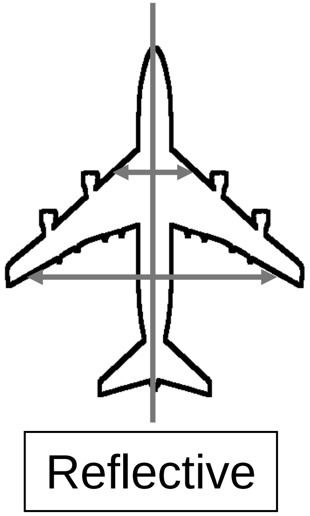

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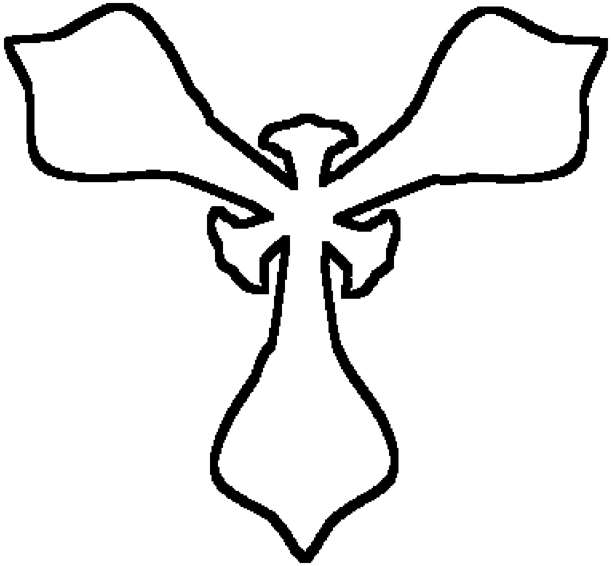

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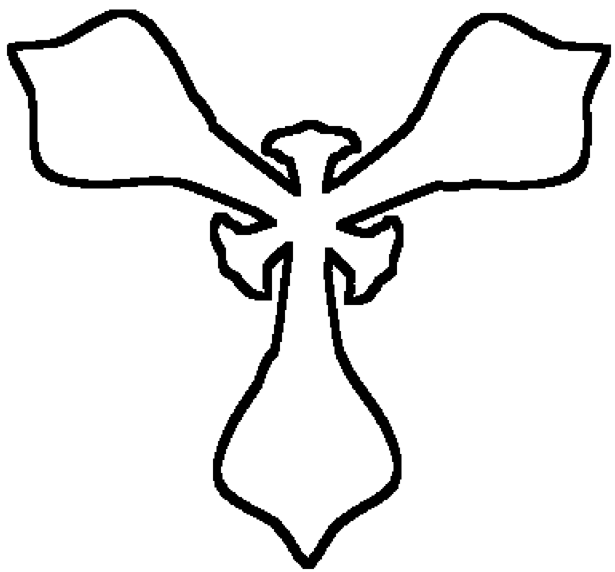

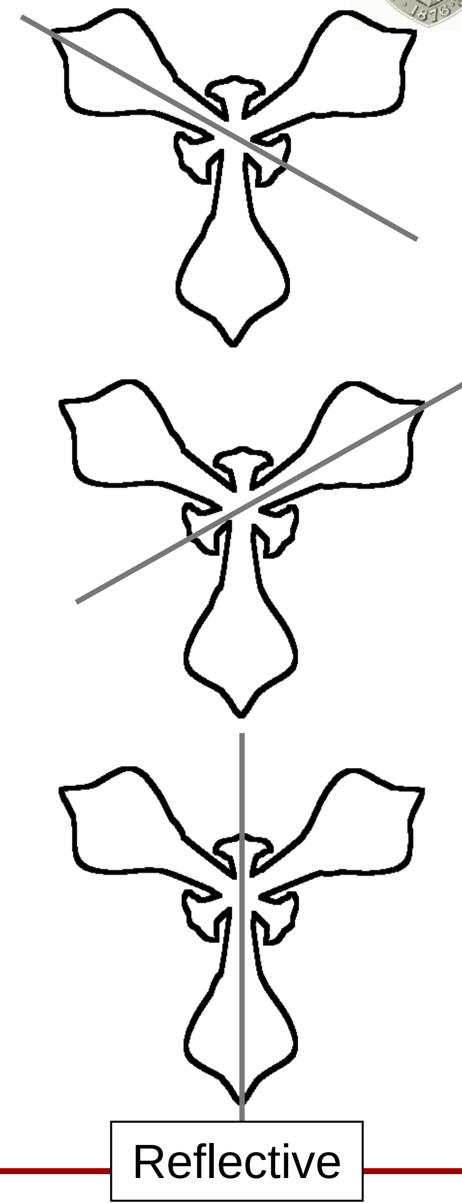

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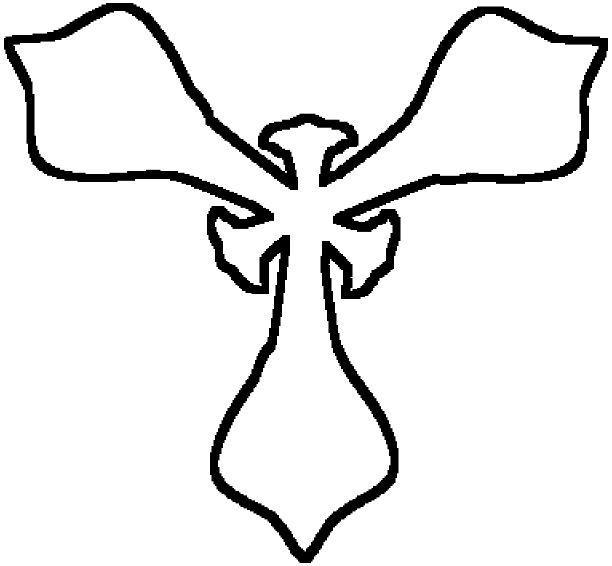

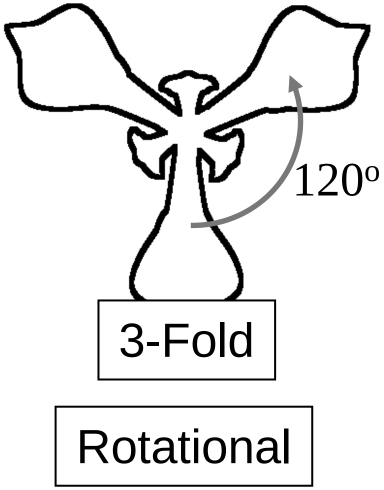

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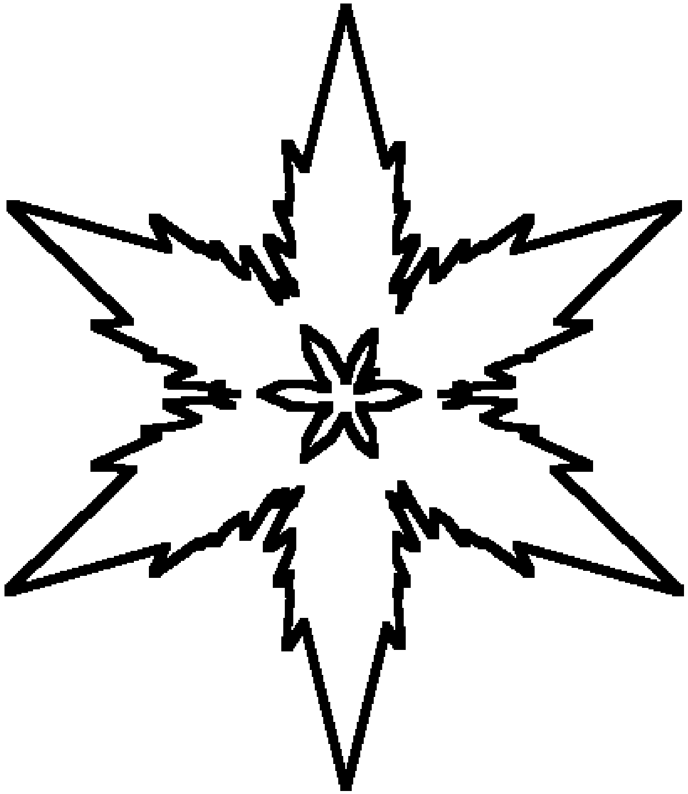

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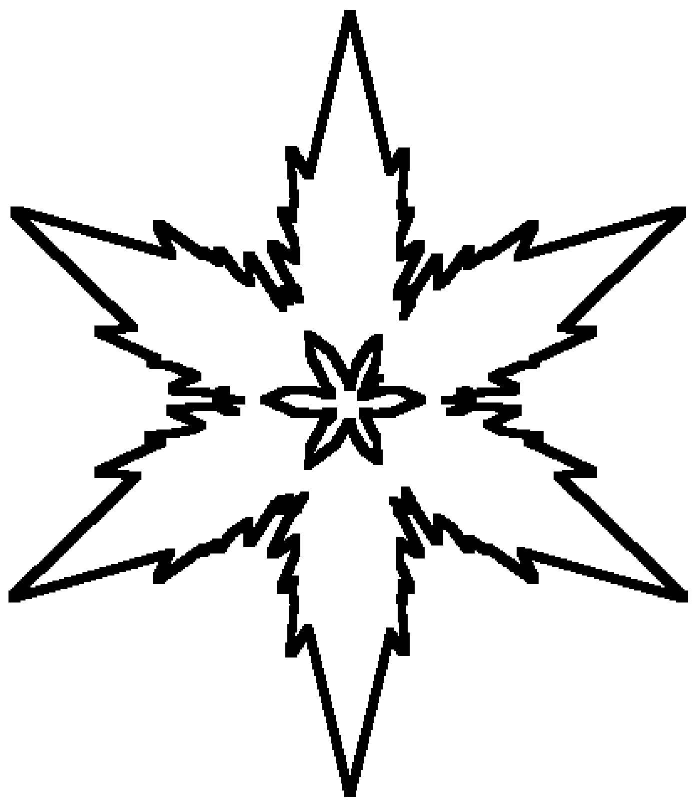

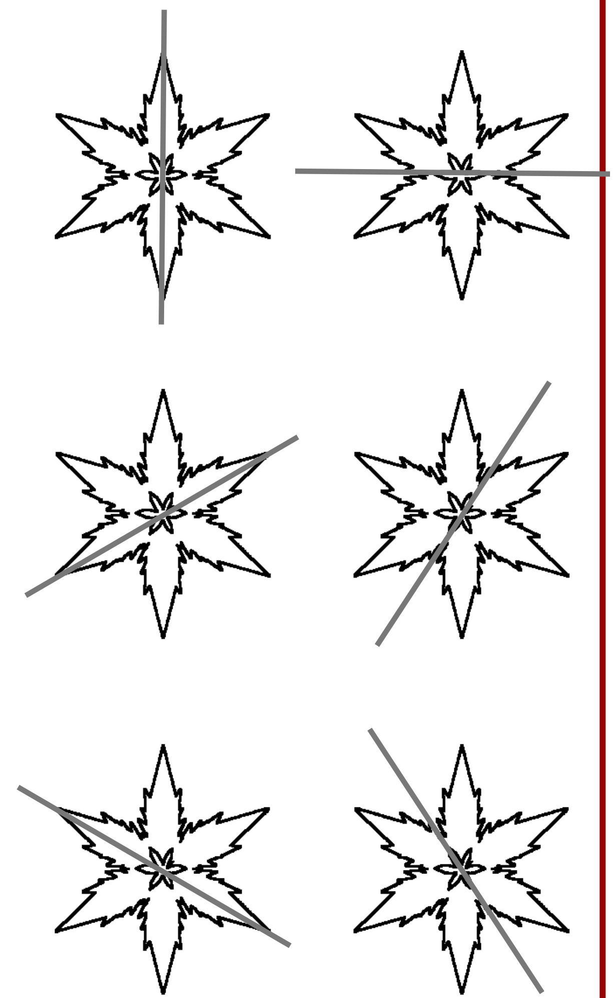

Reflective

Symmetry Detection (1D)

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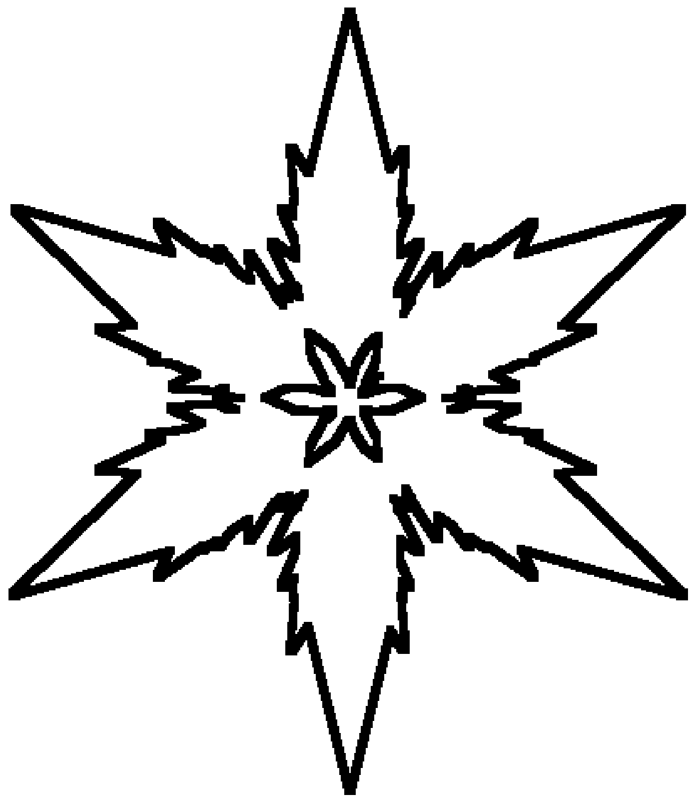

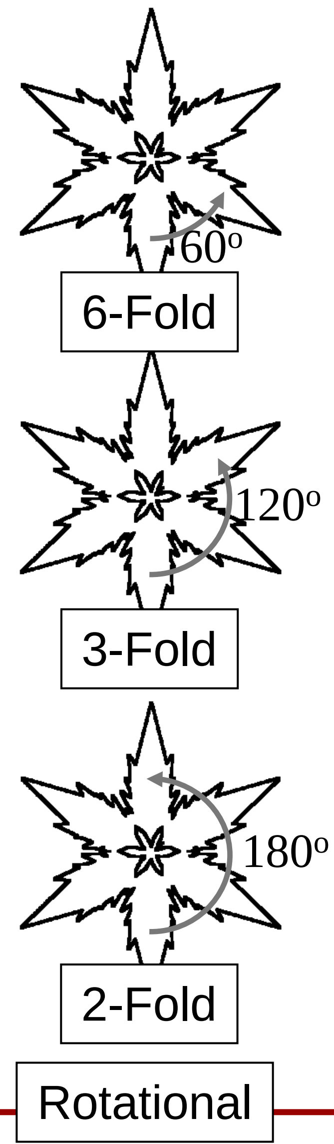

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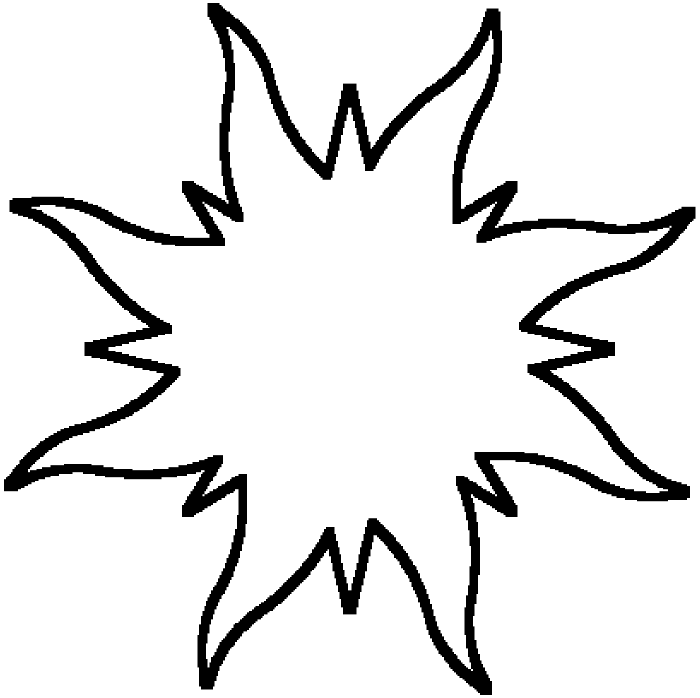

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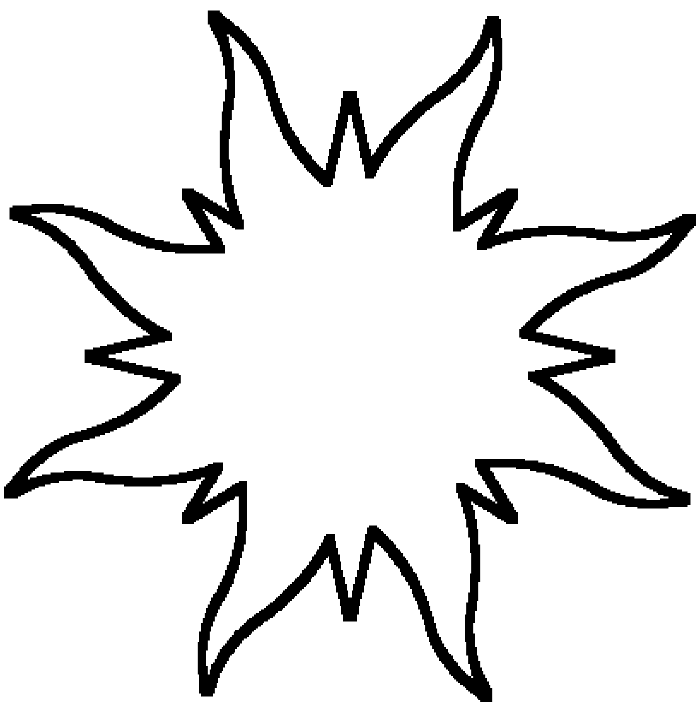
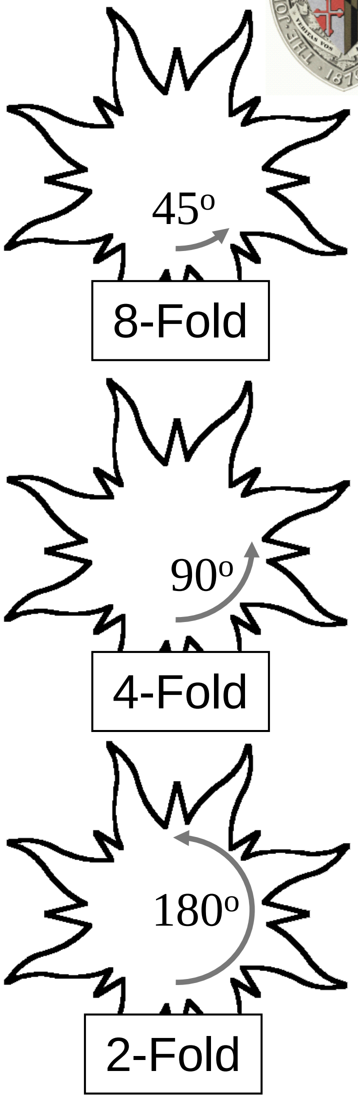
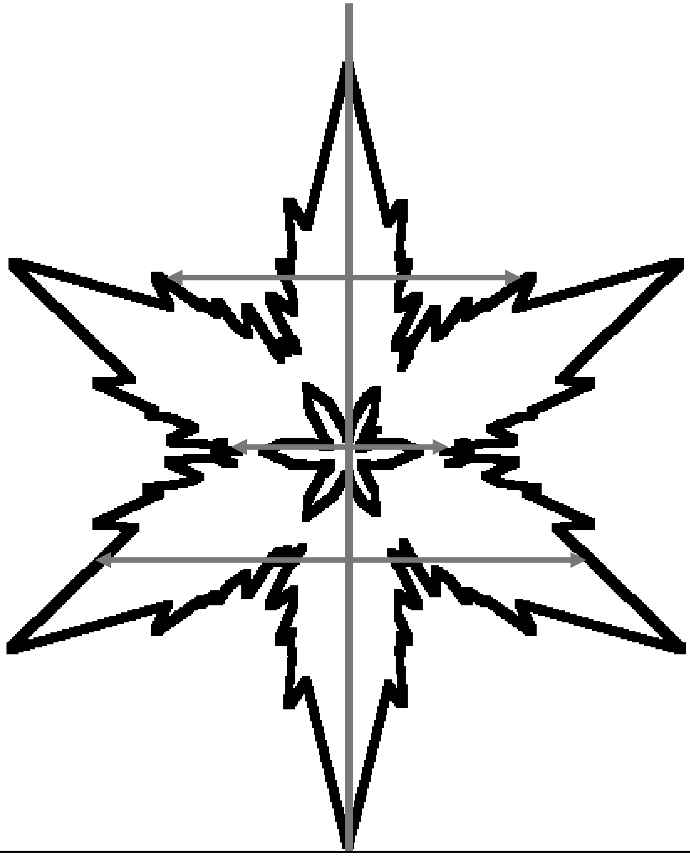




Rotational

Symmetry Detection (1D)

A shape is *symmetric* if there exists a <u>group</u> of transformations that leave the shape unchanged.



Group: {Identity, Reflection about the vertical axis}

Symmetry Detection (1D)

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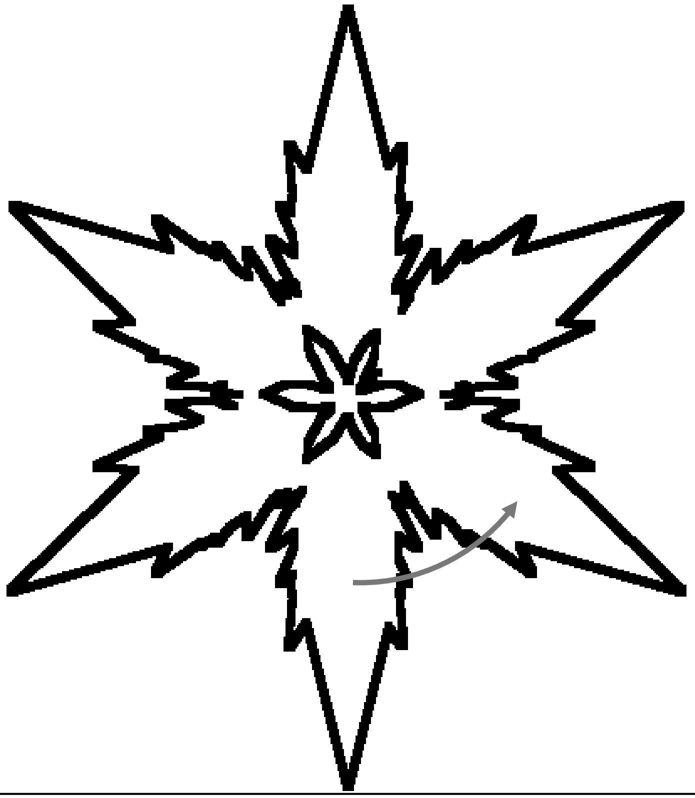


Group: {Identity, 60° Rotation, 120° Rotation, 180° Rotation, 240° Rotation, 300° Rotation}

Symmetry Detection (1D)

A shape is *symmetric* if there exists a group of transformations that leave the shape unchanged.

- Rotational symmetry group:
  Defined by the order of the rotational symmetry:

  k-fold $\Leftrightarrow$ unchanged by $\frac{n360^\circ}{k}$ rotations

- Reflective symmetry group:
  Defined by the axis of reflective symmetry.

Symmetry Detection (1D)

Approach:

1. By considering a representation of a shape by a circular function, we transform the problem:

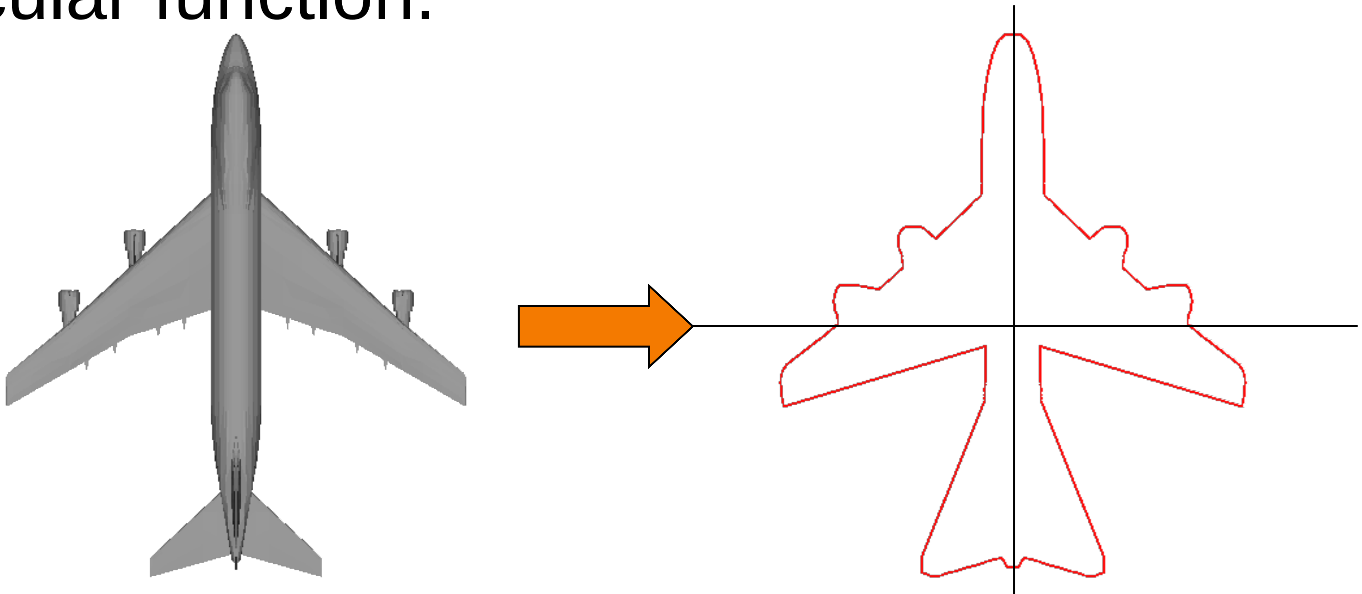
Does the shape have symmetries?

Does the function have symmetries?

Symmetry Detection (1D)

Approach:

1. By considering a representation of a shape by a circular function, we transform the problem to the challenge of detecting the symmetries of a circular function.



Shape

Circular Extent Function

Symmetry Detection (1D)

Approach:

1. By considering a representation of a shape by a circular function, we transform the problem to the challenge of detecting the symmetries of a circular function.
2. To be robust to noise, sampling error, etc., we will focus on answering the question:

How much of each type of symmetry does the shape have?

Symmetry Detection (1D)

Goal:

Given a circular function and a symmetry group, we would like to determine how symmetric the function is.

We have:

- A vector space V (the space of circular functions)
- A group G acting on V (the symmetry group)

We want the size of the projection of a vector on the G-invariant subspace V_G:

$$\mathrm{Sym}^2(v, G) = \|\mathrm{Average}(v, G)\|^2$$

Outline

Representation Theory

Symmetry Detection (1D)

- Rotations
  - Continuous
  - Discrete
- Reflections

Symmetry Detection (2D)

Rotational Symmetry Detection (1D)

Given the group of k-fold rotational symmetries:

$$G_k = \left\{ \text{Identity}, \text{Rotation by } \frac{2\pi}{k}, \cdots, (k-1)\frac{2\pi}{k} \right\}$$

and given a function f, we want to compute:

$$\text{Sym}^2(f, G_k) = \|\text{Average}(f, G_k)\|^2$$

Rotational Symmetry Detection (1D)

We know that rotations map the 1D subspaces spanned by the (unit) complex exponentials:

$$\boldsymbol{\zeta}^l(\theta) = \frac{e^{i2\pi l\theta}}{\sqrt{2\pi}}$$

back to themselves.

So lets look at how averaging acts on each $\boldsymbol{\zeta}^l$.

Rotational Symmetry Detection (1D)

What is the average of $\boldsymbol{\zeta}^l$ under G_k?

Recall that rotating $\boldsymbol{\zeta}^l$ by α is equivalent to multiplying it by $e^{-il\alpha}$, so the j-th element of G_k acts by multiplication by $e^{-il\frac{2\pi j}{k}}$.

$$\Downarrow$$

We can write out the average of $\boldsymbol{\zeta}^l$ under G_k as:

$$\text{Average}\left(\boldsymbol{\zeta}^l, G_k\right) = \frac{1}{k}\sum_{j=0}^{k-1} e^{-il\frac{2\pi j}{k}}\,\boldsymbol{\zeta}^l$$

Rotational Symmetry Detection (1D)

$$\text{Average}(\zeta^l, G_k) = \left(\frac{1}{k}\sum_{j=0}^{k-1} e^{-il\frac{2\pi j}{k}}\right)\boldsymbol{\zeta}^l$$

We can rewrite the sum:

$$\frac{1}{k}\sum_{j=0}^{k-1} e^{-il\frac{2\pi j}{k}} = \frac{1}{k}\sum_{j=0}^{k-1} e^{-ij\frac{2\pi l}{k}}$$

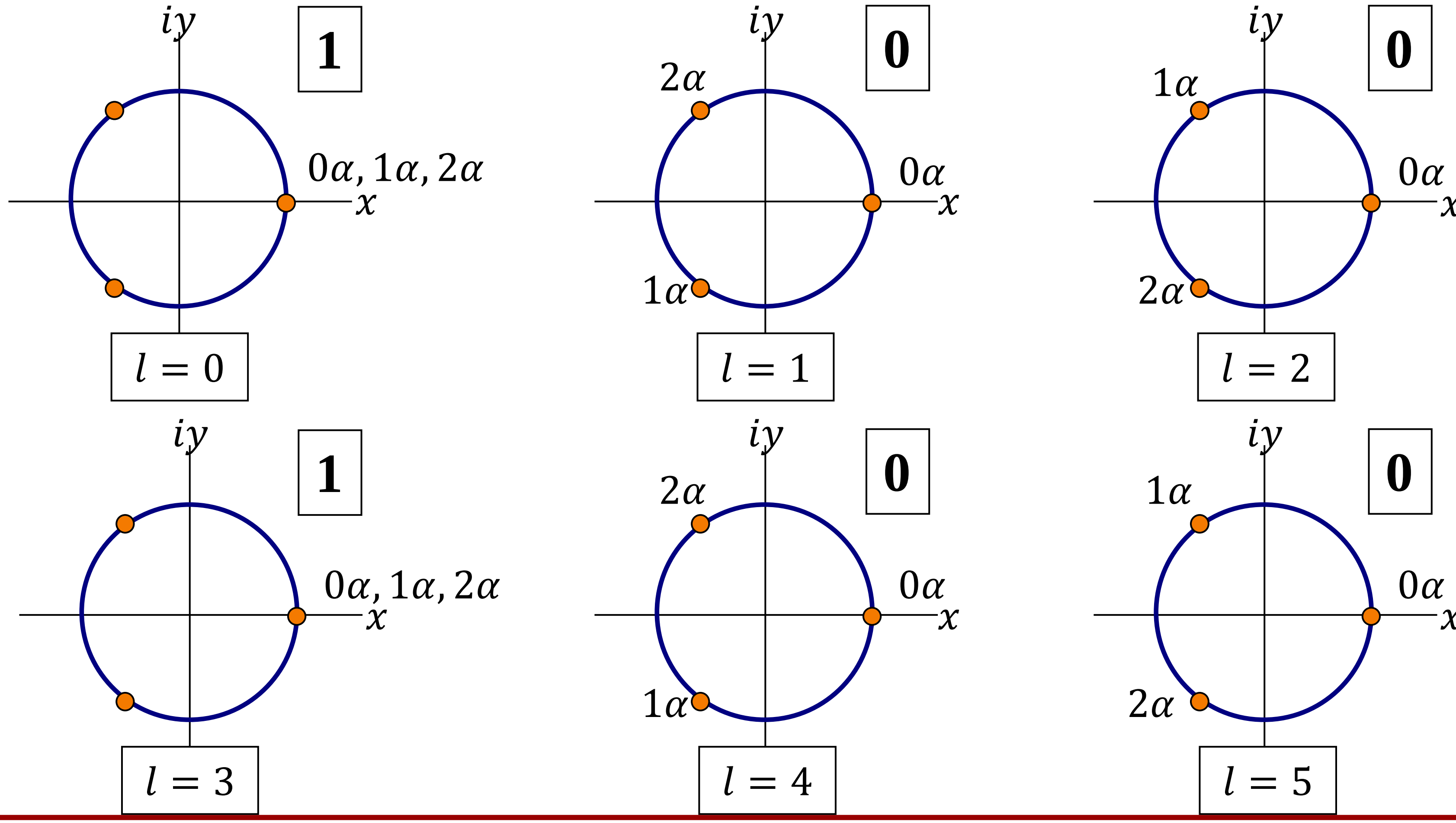
Setting α to be the angle:

$$\alpha = \frac{2\pi l}{k}$$

this is the sum of the unit-norm complex numbers with angles $\{0, \alpha, \cdots, (k-1)\alpha\}$.

Rotational Symmetry Detection (1D)

Example (k=3): $$\frac{1}{3}\sum_{j=0}^{2} e^{-ij\alpha} \quad \text{w/ } \alpha = \frac{2\pi l}{3}$$



Rotational Symmetry Detection (1D)

$$\frac{1}{k}\sum_{j=0}^{k-1} e^{-ij\frac{2\pi l}{k}}$$

When l is a multiple of k, the sum is equal to 1, otherwise, it is equal to zero.

Rotational Symmetry Detection (1D)

$$\mathrm{Average}\left(\boldsymbol{\zeta}^l, G_k\right) = \begin{cases} \boldsymbol{\zeta}^l & l \in k\mathbb{Z} \\ 0 & \text{else} \end{cases}$$

If we take the Fourier decomposition:

$$f = \sum_{l \in \mathbb{Z}} \hat{f}_l \cdot \boldsymbol{\zeta}^l$$

the average of f under G_k can be obtained by zeroing out all the Fourier coefficients of f whose index is not a multiple of k:

$$\mathrm{Average}(f, G_k) = \sum_{l \in k\mathbb{Z}} \hat{f}_l \cdot \boldsymbol{\zeta}^l$$

Rotational Symmetry Detection (1D)

We can compute the measure of k-fold symmetry of f by summing the square norms of the Fourier coefficients of f that are multiples of k:

$$\mathrm{Sym}^2(f, G_k) = \sum_{l \in k\mathbb{Z}} \left\| \hat{f}_l \right\|^2$$

Rotational Symmetry Detection (1D)

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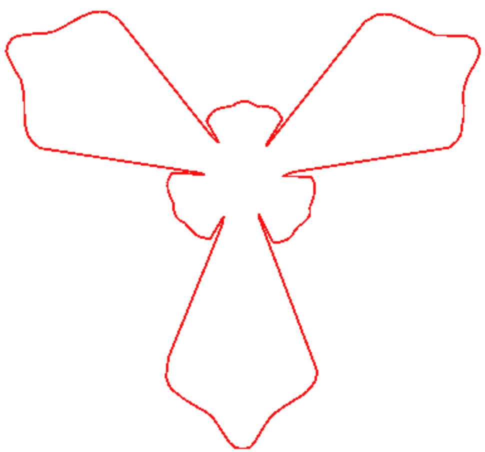

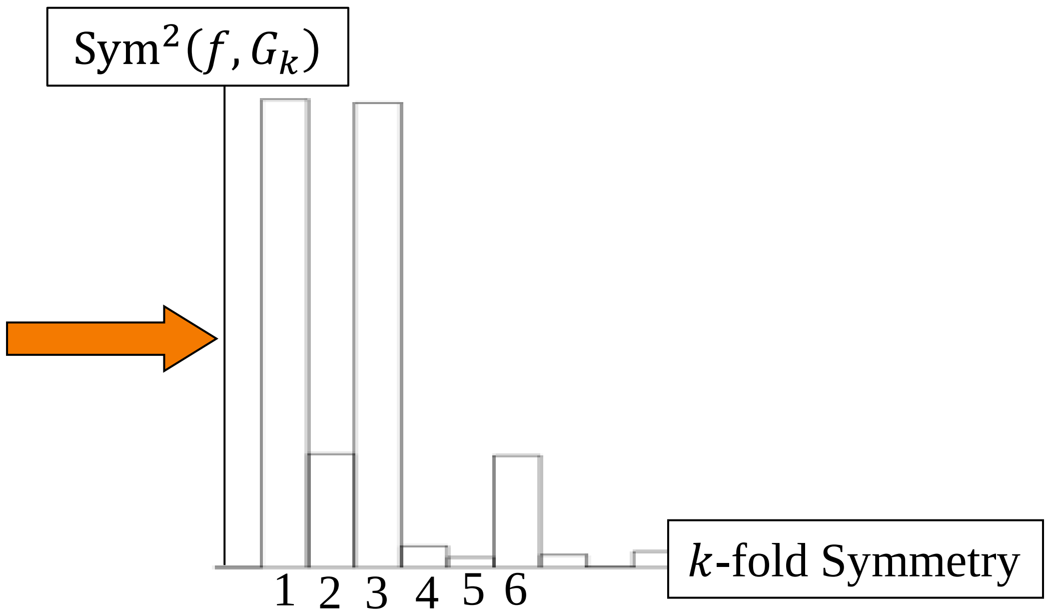

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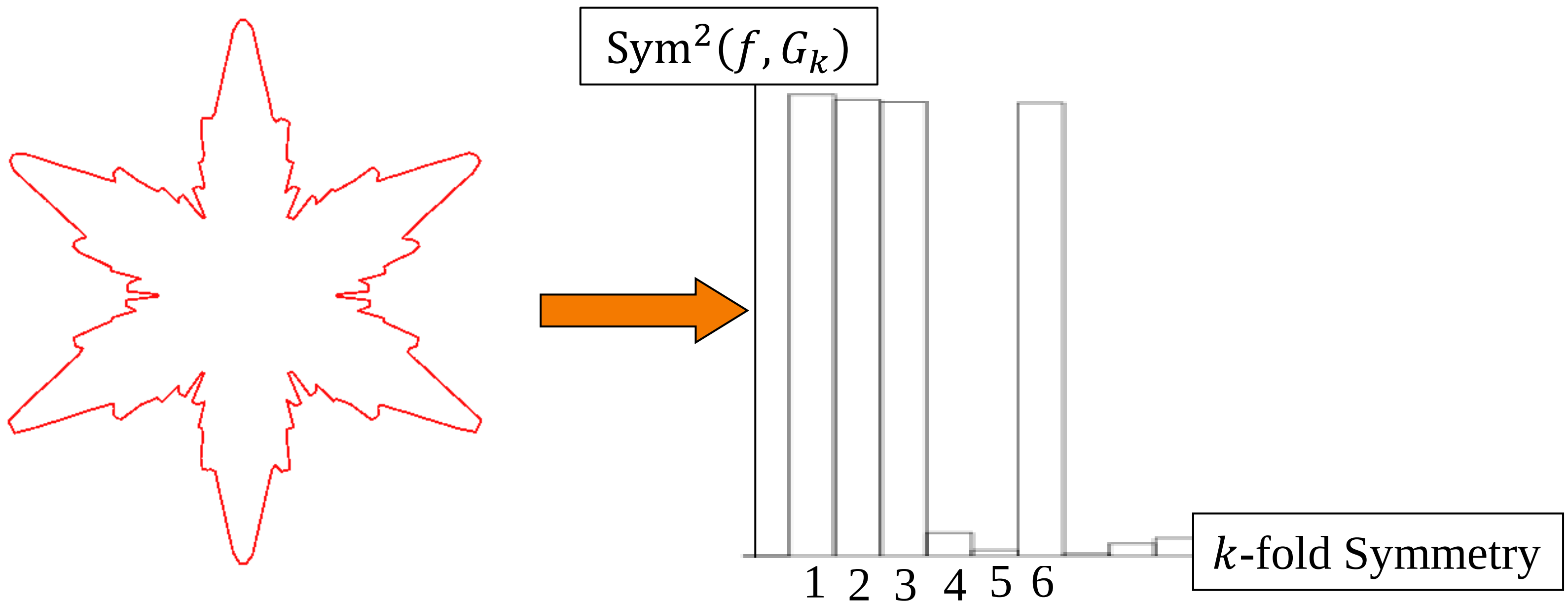

Rotational Symmetry Detection (1D)

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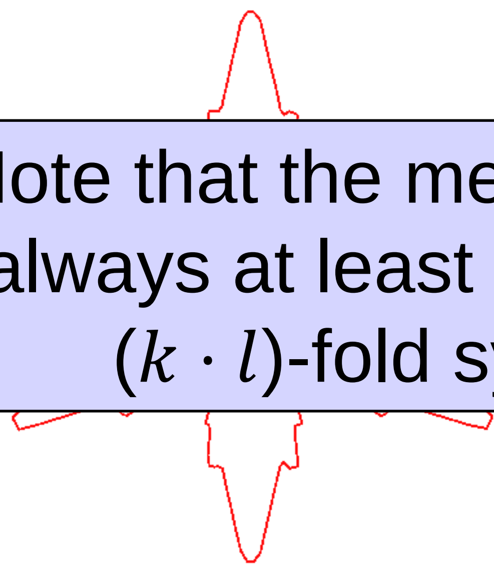
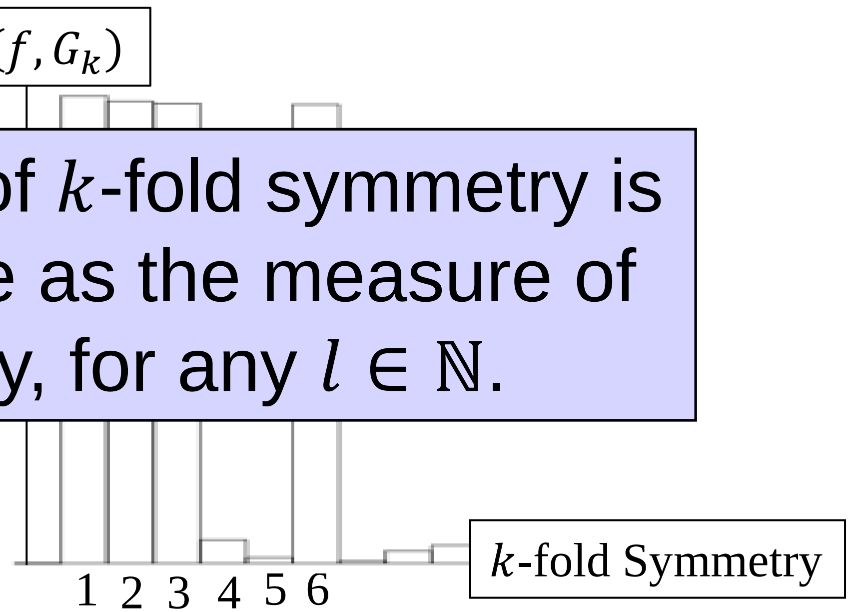




Note that the measure of k-fold symmetry is always at least as large as the measure of $(k \cdot l)$-fold symmetry, for any $l \in \mathbb{N}$.

Outline

Representation Theory

Symmetry Detection (1D)

- Rotations
  - Continuous
  - Discrete
- Reflections

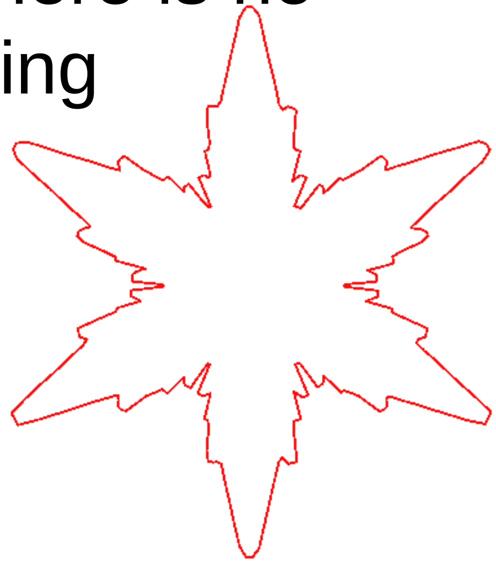
Symmetry Detection (2D)

Rotational Symmetry Detection (1D)

<u>Challenge</u>:

Suppose we represent a signal on a circle by the array $\mathbf{f} \in \mathbb{C}^n$, how do we measure the k-fold rotational symmetry of $\mathbf{f}$?

Specifically, if k does not divide n, there is no integer shift of the array corresponding to a rotation by $360°/k$.



Rotational Symmetry Detection (1D)

Approach:

As before, we can use the Fourier decomposition to think of $\mathbf{f}$ as a continuous function:

$$\mathbf{f} \rightarrow= \sum_j \hat{f}_j \cdot \boldsymbol{\zeta}^j$$

Then the average is the sum using coefficients which are multiples of k:

$$\mathrm{Average}(\mathbf{f}, G_k) = \sum_j \hat{f}_{j \cdot k} \cdot \boldsymbol{\zeta}^{j \cdot k}$$

And the measure of symmetry becomes:

$$\mathrm{Sym}^2(\mathbf{f}, G_k) = \sum_j \left\| \hat{f}_{j \cdot k} \right\|^2$$

Rotational Symmetry Detection (1D)

Danger:

As before, there are multiple ways to interpolate $\mathbf{f}$ with a continuous function:

$$\frac{1}{\sqrt{2\pi}}\sum_{j=-\frac{n}{2}}^{\frac{n}{2}-1}\hat{f}_j\cdot e^{ij\theta}\leftarrow\mathbf{f}\rightarrow\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{n-1}\hat{f}_j\cdot e^{ij\theta}$$

This gives different measures of symmetry:

$$\sum_{j=-\lfloor(n/2)/k\rfloor}^{\lfloor(n/2)/k\rfloor-1}\left\|\hat{f}_{j\cdot k}\right\|^2 \qquad \text{vs.} \qquad \sum_{j=0}^{\lfloor n/k\rfloor-1}\left\|\hat{f}_{j\cdot k}\right\|^2$$

In practice, we use the smoothest interpolant.

Outline

Representation Theory

Symmetry Detection (1D)

- Rotations
- Reflections

Symmetry Detection (2D)

Reflective Symmetry Detection (1D)

Given a circular array $\mathbf{f}$ and given the group of reflections about an axis with angle α:

$$G_\alpha = \{\text{Identity}, \text{Reflection about } \alpha\}$$

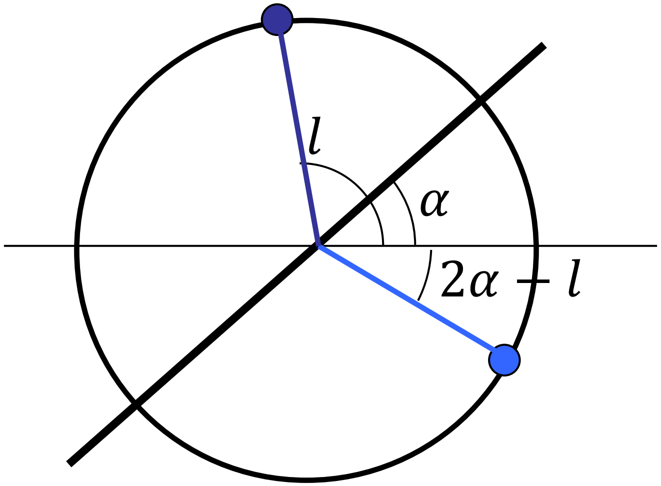
we would like to compute:

$$\text{Sym}^2(\mathbf{f}, G_\alpha) = \|\text{Average}(\mathbf{f}, G_\alpha)\|^2$$

Reflective Symmetry Detection (1D)

To do this we need to know how the group elements act on the circular array $\mathbf{f}$:

- The identity element acts trivially
$$\big(\text{Identity}(\mathbf{f})\big)_l = f_l$$
- Reflection about the line with angle α acts by:
$$\big(\text{Reflection}_\alpha(\mathbf{f})\big)_l = f_{2\alpha-l}$$



Reflective Symmetry Detection (1D)

$$\left(\text{Reflection}_\alpha(\mathbf{f})\right)_l = f_{2\alpha-l}$$

Set $\mathbf{g}$ to the reflection of $\mathbf{f}$ about the origin:

$$g_l = f_{-l}$$

Then we can express the reflection of $\mathbf{f}$ about the line with angle α as:

$$\left(\text{Reflection}_\alpha(\mathbf{f})\right)_l = f_{2\alpha-l} = g_{l-2\alpha} = \left(\rho_{2\alpha}(\mathbf{g})\right)_l$$

Reflective Symmetry Detection (1D)

We can express the average of $\mathbf{f}$ over G_α as:

$$\text{Average}(\mathbf{f}, G_\alpha) = \frac{1}{2}\left(\mathbf{f} + \rho_{2\alpha}(\mathbf{g})\right)$$

And the measure of reflective symmetry becomes:

$$\begin{aligned}\text{Sym}^2(\mathbf{f}, G_\alpha) &= \|\text{Average}(\mathbf{f}, G_\alpha)\|^2 \\ &= \left\|\frac{1}{2}\left(\mathbf{f} + \rho_{2\alpha}(\mathbf{g})\right)\right\|^2\end{aligned}$$

Reflective Symmetry Detection (1D)

$$\mathrm{Sym}^2(\mathbf{f}, G_\alpha) = \left\| \frac{1}{2}\left(\mathbf{f} + \rho_{2\alpha}(\mathbf{g})\right) \right\|^2$$

Expanding this in terms of dot-products, we get:

$$\mathrm{Sym}^2(\mathbf{f}, G_\alpha) = \frac{1}{4}\left(\|\mathbf{f}\|^2 + \|\rho_{2\alpha}(\mathbf{g})\|^2 + 2\langle \mathbf{f}, \rho_{2\alpha}(\mathbf{g})\rangle\right)$$

(where we use the fact that $\mathbf{f}$ is real-valued to lose the complex conjugation).

Reflective Symmetry Detection (1D)

$$\mathrm{Sym}^2(\mathbf{f}, G_\alpha) = \frac{1}{4}\left(\|\mathbf{f}\|^2 + \|\rho_{2\alpha}(\mathbf{g})\|^2 + 2\langle \mathbf{f}, \rho_{2\alpha}(\mathbf{g})\rangle\right)$$

Using the fact that the representation is unitary and that reflecting about the origin does not change the size of $\mathbf{f}$, we get:

$$\begin{aligned}\mathrm{Sym}^2(\mathbf{f}, G_\alpha) &= \frac{1}{4}\left(2\|\mathbf{f}\|^2 + 2\langle \mathbf{f}, \rho_{2\alpha}(\mathbf{g})\rangle\right)\\ &= \frac{1}{2}\left(\|\mathbf{f}\|^2 + (\mathbf{f} \star \mathbf{g})_{2\alpha}\right)\end{aligned}$$

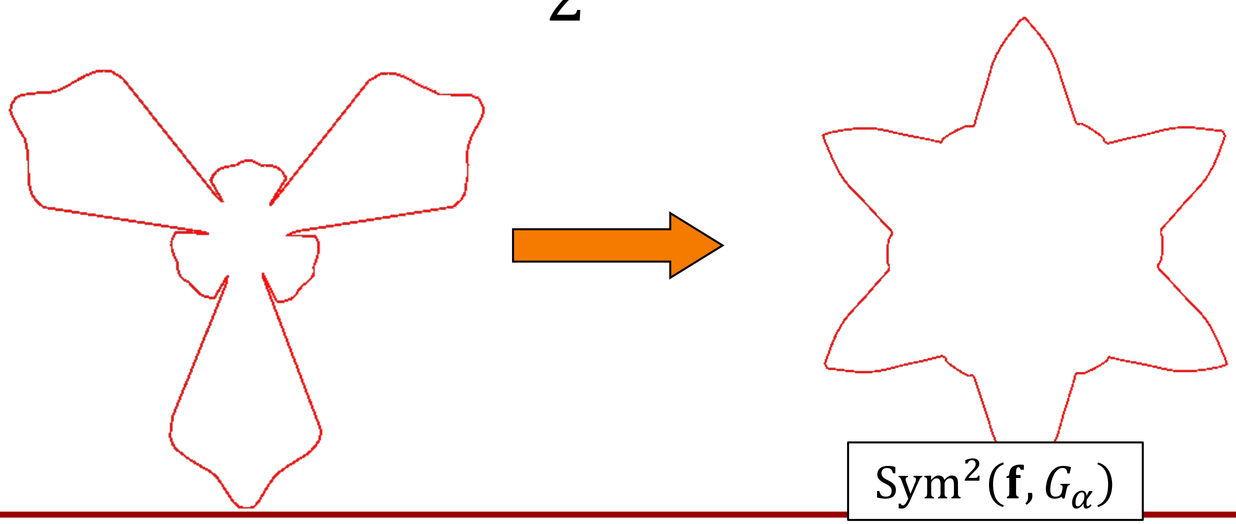
allowing us to express the measure of reflective symmetry in terms of a correlation.

Reflective Symmetry Detection (1D)

$$\mathrm{Sym}^2(\mathbf{f}, G_\alpha) = \frac{1}{4}\left(\|\mathbf{f}\|^2 + \|\rho_{2\alpha}(\mathbf{g})\|^2 + 2\langle \mathbf{f}, \rho_{2\alpha}(\mathbf{g})\rangle\right)$$

Using the fact that the representation is unitary and that reflecting about the origin does not change the size of $\mathbf{f}$, we get:

$$\mathrm{Sym}^2(\mathbf{f}, G_\alpha) = \frac{1}{2}\left(\|\mathbf{f}\|^2 + (\mathbf{f} \star \mathbf{g})_{2\alpha}\right)$$

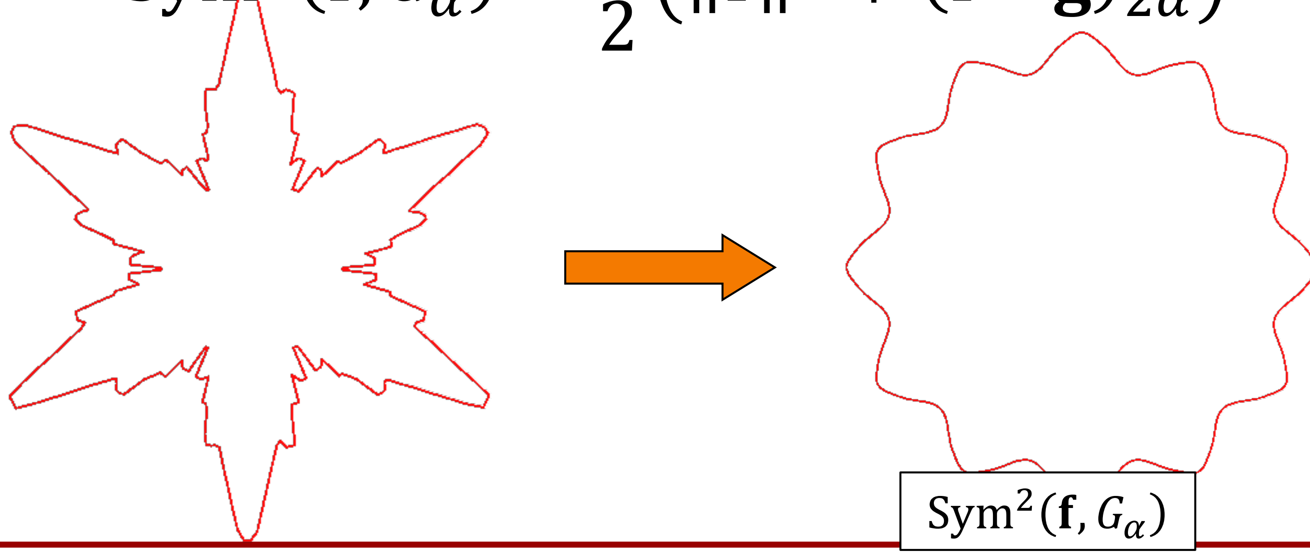


Reflective Symmetry Detection (1D)

$$\mathrm{Sym}^2(\mathbf{f}, G_\alpha) = \frac{1}{4}\left(\|\mathbf{f}\|^2 + \|\rho_{2\alpha}(\mathbf{g})\|^2 + 2\langle \mathbf{f}, \rho_{2\alpha}(\mathbf{g})\rangle\right)$$

Using the fact that the representation is unitary and that reflecting about the origin does not change the size of $\mathbf{f}$, we get:

$$\mathrm{Sym}^2(\mathbf{f}, G_\alpha) = \frac{1}{2}\left(\|\mathbf{f}\|^2 + (\mathbf{f} \star \mathbf{g})_{2\alpha}\right)$$



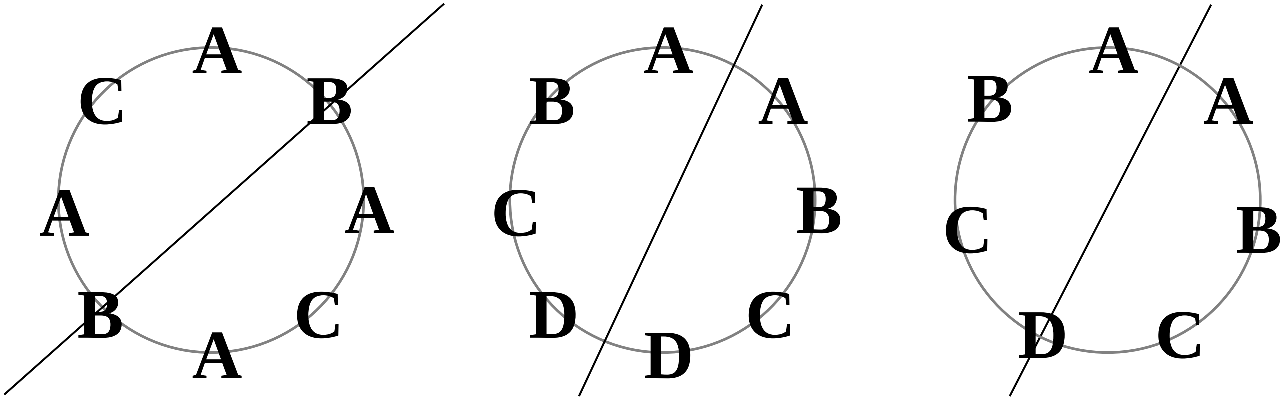
Reflective Symmetry Detection (1D)

$$\mathrm{Sym}^2(\mathbf{f}, G_\alpha) = \frac{1}{4}\left(\|\mathbf{f}\|^2 + \|\rho_{2\alpha}(\mathbf{g})\|^2 + 2\langle \mathbf{f}, \rho_{2\alpha}(\mathbf{g})\rangle\right)$$

<u>Note</u>:

In this case we only require that 2α be an integer.

- 2α "even": reflection line goes through coefficients
- 2α "odd": reflection line goes between coefficients
- 2α "even" and "odd": reflection line goes through and between coefficients



Outline

Representation Theory

Symmetry Detection (1D)

- Rotations
- Reflections

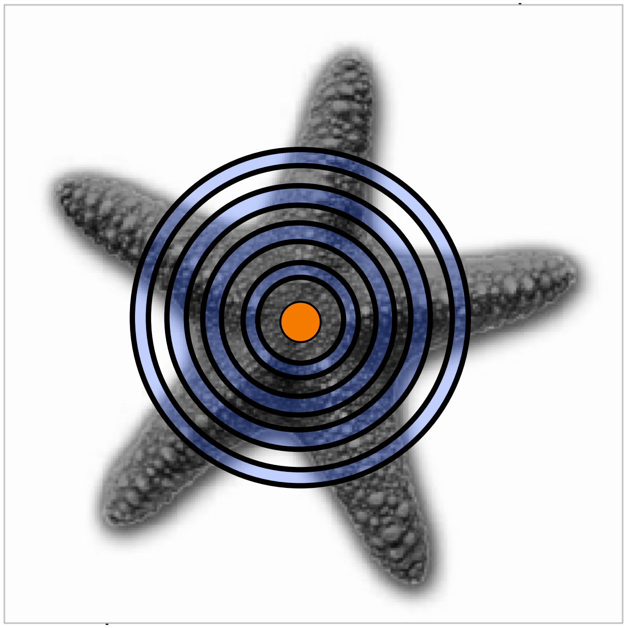
Symmetry Detection (2D)

Symmetry Detection (2D)

What about computing rotational and reflective symmetries of a 2D grid about some point?

We can use the fact that rotations and reflections map concentric circles back into themselves.

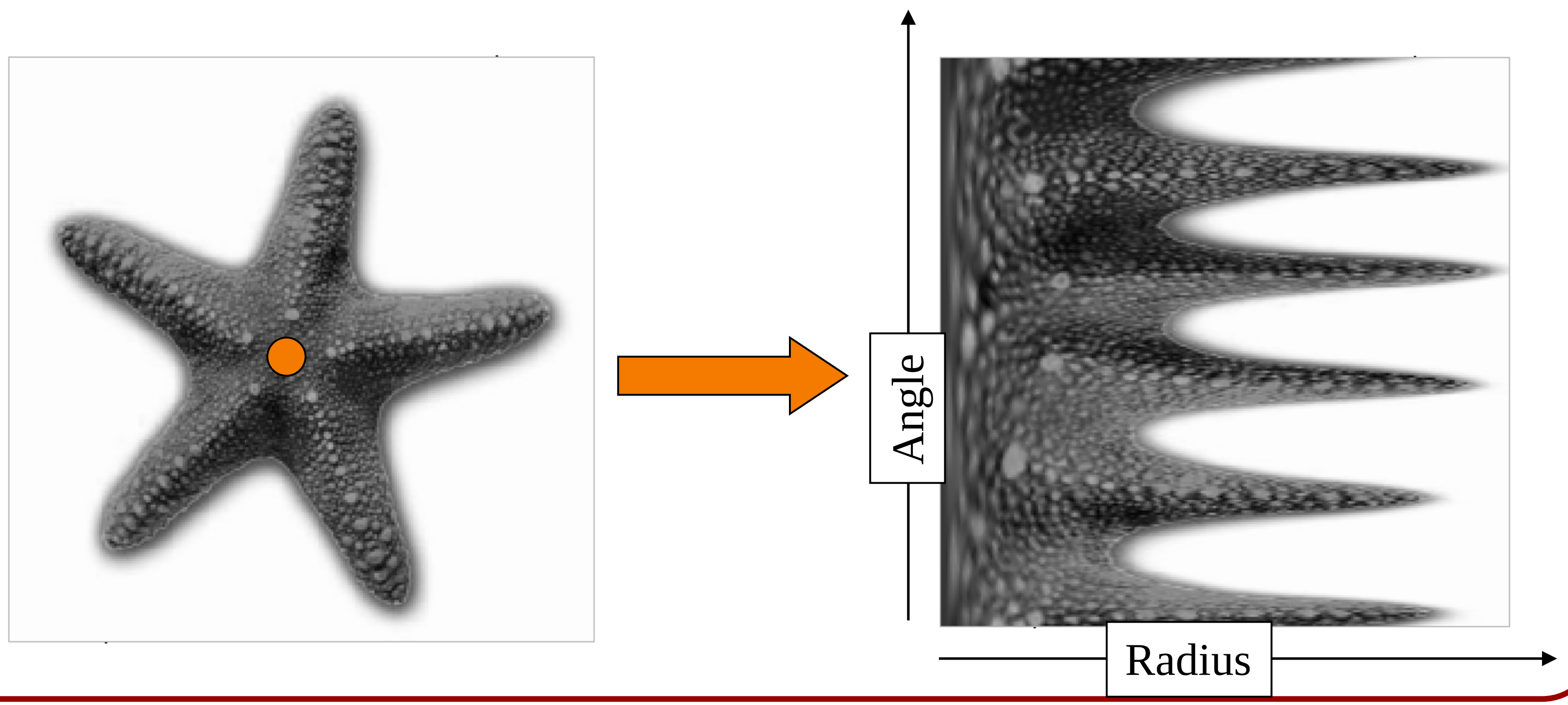
To compute the average over the symmetry group, we can consider the different radii independently.



Symmetry Detection (2D)

To implement this:

- Parameterize the grid in polar coordinates

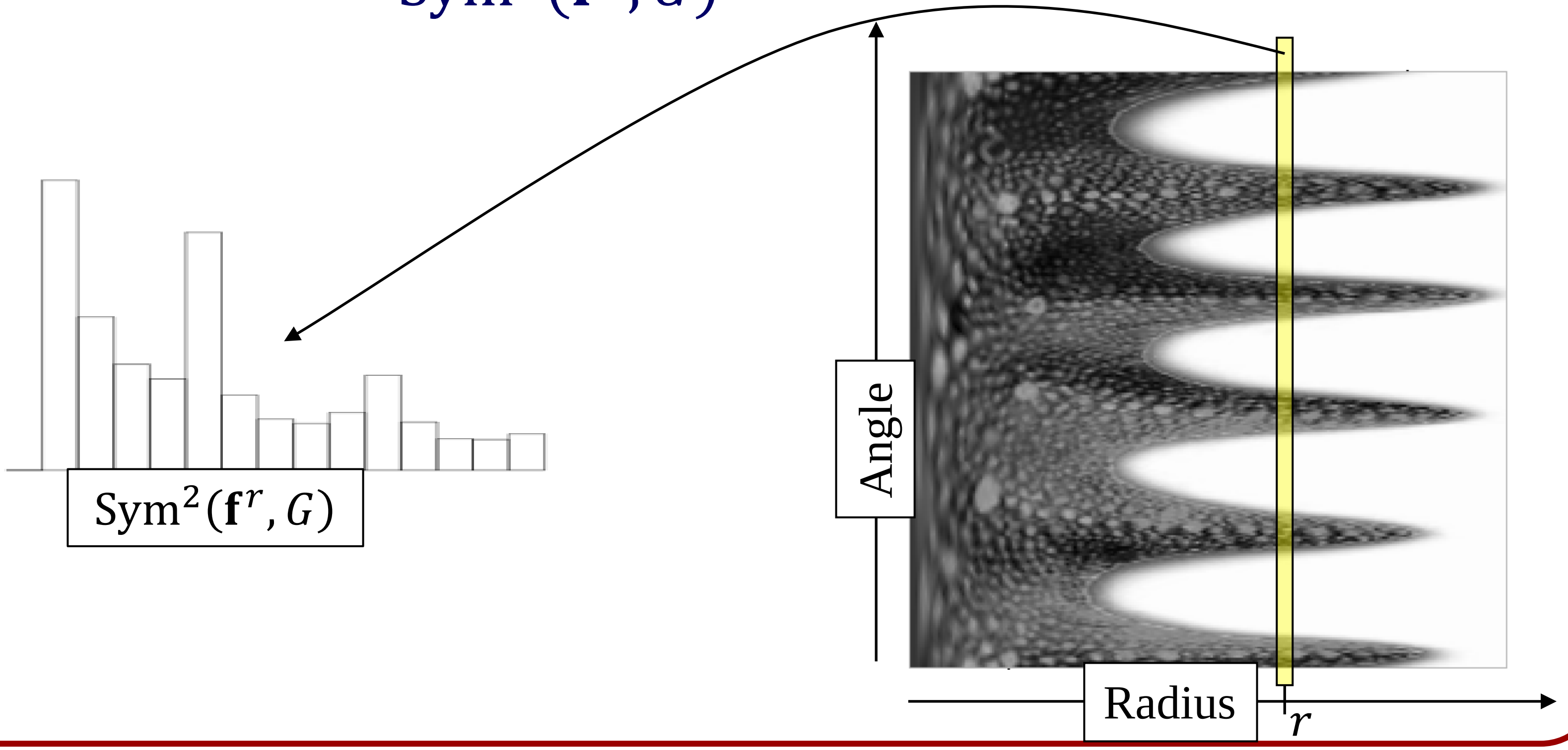


Symmetry Detection (2D)

To implement this:

- Parameterize the grid in polar coordinates
- Compute the (square) measure of symmetry for each radius independently:

$$\mathrm{Sym}^2(\mathbf{f}^r, G)$$



Symmetry Detection (2D)

To implement this:

- Parameterize the grid in polar coordinates
- Compute the (square) measure of symmetry for each radius independently:

$$\mathrm{Sym}^2(\mathbf{f}^r, G)$$

- Sum the symmetry measures over the radii:

$$\mathrm{Sym}^2(\mathbf{f}, G) = \sum_r \mathrm{Sym}^2(\mathbf{f}^r, G) \cdot r$$

Symmetry Detection (2D)

To implement this:

- Parameterize the grid in polar coordinates
- Compute the (square) measure of symmetry for each radius independently:

$$\mathrm{Sym}^2(\mathbf{f}^r, G)$$

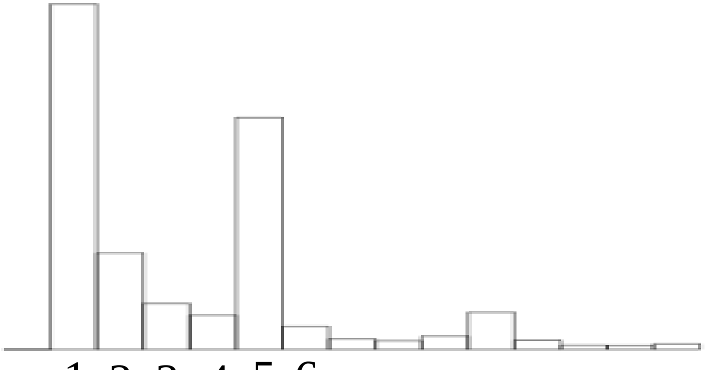
- Sum the symmetry measures over the radii:

$$\mathrm{Sym}^2(\mathbf{f}, G) = \sum_r \mathrm{Sym}^2(\mathbf{f}^r, G) \cdot \boxed{r}$$

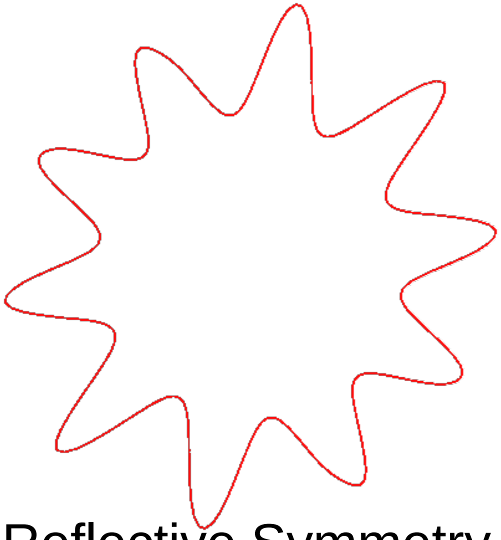
Scaling by r is required to account for the change of variables:

$$\int_{x^2+y^2\leq 1} f(x,y)\,dy\,dy = \int_0^1 \int_0^{2\pi} f(r,\theta)\,d\theta\,\boxed{r}\,dr$$

Symmetry Detection (2D)



Rotational Symmetry



Reflective Symmetry