FFTs in Graphics and Vision

More Math Review

Outline

Inner Product Spaces

- Real Inner Products
- Hermitian Inner Products
- Orthogonal Transforms
- Unitary Transforms
- Function Spaces

Inner Product Spaces

Given a real vector space V, a *real inner product* is a function $\langle\cdot,\cdot\rangle: V \times V \to \mathbb{R}$ that takes a pair of vectors and returns a real value.

Inner Product Spaces

An inner product is a map $\langle\cdot,\cdot\rangle: V \times V \rightarrow \mathbb{R}$ that is:

1. Linear in the first term: For all $u, v, w \in V$ and all $\alpha \in \mathbb{R}$:
$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$
$$\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$$

2. Symmetric: For all $v, w \in V$:
$$\langle v, w\rangle = \langle w, v\rangle$$

3. Positive Definite: For all $v \in V$:
$$\langle v, v\rangle \geq 0$$
$$\langle v, v\rangle = 0 \Leftrightarrow v = 0$$

Inner Product Spaces

Note:

Combining linearity in the first term with symmetry, gives:

$$\begin{aligned}\langle u, v+w\rangle &= \langle v+w, u\rangle \\ &= \langle v,u\rangle + \langle w,u\rangle \\ &= \langle u,v\rangle + \langle u,w\rangle\end{aligned}$$

$$\begin{aligned}\langle u, \alpha v\rangle &= \langle \alpha v, u\rangle \\ &= \alpha\langle v,u\rangle \\ &= \alpha\langle u,v\rangle\end{aligned}$$

$\Rightarrow$ Linearity in the second term

Inner Product Spaces

An inner product defines a notion of distance on a vector space by setting:

$$\text{Dist}(v, w) = \sqrt{\langle v - w, v - w \rangle} = \|v - w\|$$

Inner Product Spaces

<u>Examples</u>:

1. For n-dimensional, $\mathbb{R}$-valued, arrays, the standard inner product is:

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b} \rangle &= a_1 b_1 + \cdots + a_{\mathbf{n}} b_n \\ &= \mathbf{a}^\top \mathbf{b}\end{aligned}$$

   for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$

2. For continuous, $\mathbb{R}$-valued functions, defined on a circle, the standard inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot g(\theta) d\theta$$

Inner Product Spaces

Examples:

3. We will think of an $\mathbb{R}$-valued, n-dimensional array, $\mathbf{a} = (a_1, \ldots, a_n)^\top \in \mathbb{R}^n$, as a sampling of a continuous $\mathbb{R}$-valued function on the circle, $f: [0,2\pi) \to \mathbb{R}$:

$$a_j = f\left(\frac{2\pi j}{n}\right)$$

$$\Downarrow$$

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b} \rangle_{[0,2\pi)} &= \frac{2\pi}{n}(a_1 b_1 + \cdots + a_n b_n) \\ &= \frac{2\pi}{n}\mathbf{a}^\top \mathbf{b}\end{aligned}$$

Inner Product Spaces

Examples:

4. For n-dimensional, $\mathbb{R}$-valued arrays, suppose we have a matrix:

$$\mathbf{M} = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix}$$

Does the map:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^{\top}\mathbf{M}\mathbf{b}$$

define an inner product?

Inner Product Spaces

Examples:

4. Does the map:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^\top \mathbf{M} \mathbf{b}$$

define an inner product?

- Is it linear?
- Is it symmetric?
- Is it positive definite?

Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^\top \mathbf{M} \mathbf{b}$$

Is it linear?

$$\begin{aligned}\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}} &= (\mathbf{a} + \mathbf{b})^\top \mathbf{M} \mathbf{c} \\ &= (\mathbf{a}^\top + \mathbf{b}^\top) \mathbf{M} \mathbf{c} \\ &= \mathbf{a}^\top \mathbf{M} \mathbf{c} + \mathbf{b}^\top \mathbf{M} \mathbf{c} \\ &= \langle \mathbf{a}, \mathbf{c} \rangle_{\mathbf{M}} + \langle \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}}\end{aligned}$$

Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^\top \mathbf{M} \mathbf{b}$$

Is it linear?

$$\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}} = \langle \mathbf{a}, \mathbf{c} \rangle_{\mathbf{M}} + \langle \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}}$$

$$\begin{aligned} \langle \alpha\mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} &= (\alpha\mathbf{a})^\top \mathbf{M} \mathbf{b} \\ &= \alpha\mathbf{a}^\top \mathbf{M} \mathbf{b} \\ &= \alpha \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \end{aligned}$$

Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^{\top} \mathbf{M} \mathbf{b}$$

Is it linear? Yes

$$\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}} = \langle \mathbf{a}, \mathbf{c} \rangle_{\mathbf{M}} + \langle \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}}$$

$$\langle \alpha \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} = \alpha \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}}$$

Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^\top \mathbf{M} \mathbf{b}$$

Is it symmetric?

$$\begin{aligned} \langle \mathbf{b}, \mathbf{a} \rangle_{\mathbf{M}} &= \mathbf{b}^\top \mathbf{M} \mathbf{a} \\ &= (\mathbf{b}^\top \mathbf{M} \mathbf{a})^\top \\ &= \mathbf{a}^\top \mathbf{M}^\top \mathbf{b} \\ &= \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}^\top} \end{aligned}$$

Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^\top \mathbf{M} \mathbf{b}$$

Is it symmetric? Only if $\mathbf{M}$ is $(\mathbf{M} = \mathbf{M}^\top)$

$$\langle \mathbf{b}, \mathbf{a} \rangle_{\mathbf{M}} = \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}^\top}$$

Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^\top \mathbf{M} \mathbf{b}$$

Assuming $\mathbf{M}$ is symmetric, is the map $\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}}$ positive definite?

If $\mathbf{M}$ is symmetric, there exists an orthogonal basis $\{v^1, \ldots, v^n\} \subset \mathbb{R}^n$ with respect to which $\mathbf{M}$ is diagonal:

$$\mathbf{M} = \mathbf{B}^\top \mathbf{\Lambda} \mathbf{B} = \mathbf{B}^\top \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & & 0 & \lambda_n \end{pmatrix} \mathbf{B}$$

Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^\top \mathbf{M} \mathbf{b}$$

Assuming $\mathbf{M}$ is symmetric, is the map $\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}}$ positive definite?

Setting $\mathbf{M} = \mathbf{B}^\top \mathbf{\Lambda} \mathbf{B}$, we have:

$$\begin{aligned} \langle \mathbf{a}, \mathbf{a} \rangle_{\mathbf{M}} &= \mathbf{a}^\top \mathbf{M} \mathbf{a} \\ &= \mathbf{a}^\top \mathbf{B}^\top \mathbf{\Lambda} \mathbf{B} \mathbf{a} \\ &= \langle \mathbf{B}\mathbf{a}, \mathbf{B}\mathbf{a} \rangle_{\mathbf{\Lambda}} \end{aligned}$$

Writing $\mathbf{b} = \mathbf{B}\mathbf{a}$, we get:

$$\begin{aligned} \langle \mathbf{a}, \mathbf{a} \rangle_{\mathbf{M}} = \langle \mathbf{B}\mathbf{a}, \mathbf{B}\mathbf{a} \rangle_{\mathbf{\Lambda}} &= \langle \mathbf{b}, \mathbf{b} \rangle_{\mathbf{\Lambda}} \\ &= \lambda_1 b_1^2 + \cdots + \lambda_n b_n^2 \end{aligned}$$

Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^\top \mathbf{M} \mathbf{b}$$

Assuming $\mathbf{M}$ is symmetric, is the map $\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}}$ positive definite? Only if $\lambda_i > 0$ for all i.

Writing $\mathbf{b} = \mathbf{B}\mathbf{a}$, we get:

$$\langle \mathbf{a}, \mathbf{a} \rangle_{\mathbf{M}} = \lambda_1 b_1^2 + \cdots + \lambda_n b_n^2$$

Inner Product Spaces

<u>Examples</u>:

5. For continuous, $\mathbb{R}$-valued functions, defined on a circle, given $\omega\colon [0,2\pi) \to \mathbb{R}$, does the map:
$$\langle f, g \rangle_\omega = \int_0^{2\pi} f(\theta) \cdot g(\theta) \cdot \omega(\theta)\, d\theta$$
define an inner product?

Inner Product Spaces

Examples:

5. For continuous, $\mathbb{R}$-valued functions, defined on a circle, given $\omega: [0,2\pi) \to \mathbb{R}$, does the map:
$$\langle f, g \rangle_\omega = \int_0^{2\pi} f(\theta) \cdot g(\theta) \cdot \omega(\theta)\, d\theta$$
define an inner product? No

Inner Product Spaces

Examples:

5. For continuous, $\mathbb{R}$-valued functions, defined on a circle, given $\omega: [0,2\pi) \to \mathbb{R}$, does the map:
$$\langle f,g\rangle_\omega = \int_0^{2\pi} f(\theta)\cdot g(\theta)\cdot \omega(\theta)\,d\theta$$
define an inner product? No
What if $\omega(\theta) > 0$?

Inner Product Spaces

Examples:

5. For continuous, $\mathbb{R}$-valued functions, defined on a circle, given $\omega: [0,2\pi) \to \mathbb{R}$, does the map:
$$\langle f, g \rangle_\omega = \int_0^{2\pi} f(\theta) \cdot g(\theta) \cdot \omega(\theta)\, d\theta$$
define an inner product? No
What if $\omega(\theta) > 0$? Yes

Hermitian Inner Product Spaces

Given a complex vector space V, a *Hermitian inner product* is a function $\langle\cdot,\cdot\rangle: V \times V \to \mathbb{C}$ that takes a pair of vectors and returns a complex value.

Hermitian Inner Product Spaces

A Hermitian inner product is a map $\langle\cdot,\cdot\rangle: V \times V \to \mathbb{C}$ that is:

1. Linear in the first term: For all $u, v, w \in V$ and any $\alpha \in \mathbb{C}$:
$$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$
$$\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$$

2. Conjugate Symmetric: For all $u, v \in V$:
$$\langle v, w\rangle = \overline{\langle w, v\rangle}$$

3. Positive Definite: For all $v \in V$:
$$\langle v, v\rangle \geq 0$$
$$\langle v, v\rangle = 0 \Leftrightarrow v = 0$$

Hermitian Inner Product Spaces

Note:

Combining linearity in the first term with conjugate symmetry, gives:

$$\begin{aligned}\langle u, v+w\rangle &= \overline{\langle v+w, u\rangle}\\ &= \overline{\langle v,u\rangle} + \overline{\langle w,u\rangle}\\ &= \langle u,v\rangle + \langle u,w\rangle\end{aligned}$$

$$\begin{aligned}\langle u, \alpha v\rangle &= \overline{\langle \alpha v, u\rangle}\\ &= \bar{\alpha}\overline{\langle v,u\rangle}\\ &= \bar{\alpha}\langle u,v\rangle\end{aligned}$$

$\Rightarrow$ Conjugate linearity in the second term

Hermitian Inner Product Spaces

As in the real case, a Hermitian inner product defines a notion of distance on a complex vector space by setting:

$$\mathrm{Dist}(v, w) = \sqrt{\langle v - w, v - w \rangle} = \|v - w\|$$

Hermitian Inner Product Spaces

Examples:

1. For n-dimensional, $\mathbb{C}$-valued arrays, the standard Hermitian inner product is:

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b} \rangle &= a_1 \bar{b}_1 + \cdots + a_n \bar{b}_n \\ &= \mathbf{a}^\top \bar{\mathbf{b}}\end{aligned}$$

   for all $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$

2. For continuous, $\mathbb{C}$-valued functions, defined on a circle, the standard Hermitian inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot \overline{g(\theta)}\, d\theta$$

Hermitian Inner Product Spaces

Examples:

3. We will think of a $\mathbb{C}$-valued n-dimensional arrays $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)^\top \in \mathbb{C}^n$ as a sampling of a continuous, $\mathbb{C}$-valued functions on the circle, $f: [0,2\pi) \to \mathbb{C}$:

$$a_j = f\left(\frac{2\pi j}{n}\right)$$

$$\Downarrow$$

$$\langle \mathbf{a}, \bar{\mathbf{b}} \rangle_{[0,2\pi)} = \frac{2\pi}{n}\left(a_1\bar{b}_1 + \cdots + a_n\bar{b}_n\right)$$

$$= \frac{2\pi}{n}\mathbf{a}^\top\bar{\mathbf{b}}$$

Structure Preservation

Recall:

Given vector spaces V and W, a linear map L is a function $L: V \to W$ that preserves the linear structure:

$$L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)$$

for all $v, w \in V$ and all scalars α and β.

Structure Preservation

Orthogonal Transformations:

Given real vector spaces V and W, with inner-products $\langle \cdot,\cdot \rangle_V$ and $\langle \cdot,\cdot \rangle_W$, we would also like to consider those functions from V to W that preserve the underlying structure.

If R is such a function, then:

- R must be a linear operator, in order to preserve the underlying vector space structure.
- R must also preserve the underlying inner product.

Structure Preservation

Orthogonal Transformations:

Given real inner-product spaces $\{V, \langle\cdot,\cdot\rangle_V\}$ and $\{W, \langle\cdot,\cdot\rangle_W\}$, a linear operator $L: V \rightarrow W$ is called orthogonal if it preserves the inner product:

$$\langle L(v), L(w)\rangle_W = \langle v, w\rangle_V$$

for all $v, w \in V$.

Structure Preservation

Orthogonal Transformations:

In particular, if $L: V \rightarrow W$ is an orthogonal transformation, we have $\|L(v)\|_W = \|V\|_V$ so the kernel of R is the zero vector.

$\Rightarrow$ If the vector spaces V and W are equal (and finite-dimensional), the map is invertible.

Structure Preservation

<u>Example</u>:

On n-dimensional, $\mathbb{R}$-valued, arrays the matrix $\mathbf{L} \in \mathbb{R}^{n \times n}$ is orthogonal if for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$:

$$\langle \mathbf{La}, \mathbf{Lb} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle$$

$$\Updownarrow$$

$$(\mathbf{La})^\top (\mathbf{Lb}) = \mathbf{a}^\top \mathbf{b}$$

$$\Updownarrow$$

$$\mathbf{a}^\top \mathbf{L}^\top \mathbf{Lb} = \mathbf{a}^\top \mathbf{b}$$

$$\Updownarrow$$

$$\mathbf{L}^\top = \mathbf{L}^{-1}$$

Structure Preservation

Example:

On the space of $\mathbb{R}$-valued, n-dimensional arrays, a matrix is orthogonal if and only if:

$$\mathbf{L}^{\top} = \mathbf{L}^{-1}$$

Note:

The determinant of an orthogonal matrix always has absolute value 1:

$$\begin{aligned}\det(\mathbf{L})^2 &= \det(\mathbf{L}) \cdot \det(\mathbf{L}^{\top}) \\ &= \det(\mathbf{L}) \cdot \det(\mathbf{L}^{-1}) \\ &= \det(\mathbf{L}\mathbf{L}^{-1}) \\ &= 1\end{aligned}$$

Structure Preservation

Example:

On the space of $\mathbb{R}$-valued, n-dimensional arrays, a matrix is orthogonal if and only if:

$$\mathbf{L}^{\top} = \mathbf{L}^{-1}$$

Note: The determinant of an orthogonal matrix always has absolute value 1.

If the determinant of an orthogonal matrix is equal to 1, the matrix is called a rotation.

Orthogonal Matrices and Eigenvalues

If $L: V \to V$ is an orthogonal transformation and L has an eigenvalue λ, then $|\lambda| = 1$.

To see this, let v be an eigenvector of L corresponding to the eigenvalue λ.

Since L is orthogonal, we have:

$$\begin{aligned}\langle v, v\rangle &= \langle Lv, Lv\rangle \\ &= \langle \lambda v, \lambda v\rangle \\ &= \lambda^2 \langle v, v\rangle\end{aligned}$$

$\Rightarrow |\lambda| = 1.$

Structure Preservation

Unitary Transformations:

For a complex vector space V and W with Hermitian inner-products, a linear operator L is called unitary if it preserves the Hermitian inner product:

$$\langle Lv, Lw\rangle_W = \langle v, w\rangle_V$$

for all $v, w \in V$.

Structure Preservation

Example:

On n-dimensional, $\mathbb{C}$-valued, arrays the matrix $\mathbf{L} \in \mathbb{C}^{n\times n}$ is unitary if for all $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$:

$$\langle \mathbf{La}, \mathbf{Lb} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle$$

$$\Updownarrow$$

$$(\mathbf{La})^\top \left(\overline{\mathbf{Lb}}\right) = \mathbf{a}^\top \bar{\mathbf{b}}$$

$$\Updownarrow$$

$$\mathbf{a}^\top \mathbf{L}^\top \overline{\mathbf{Lb}} = \mathbf{a}^\top \bar{\mathbf{b}}$$

$$\Updownarrow$$

$$\bar{\mathbf{L}}^\top = \mathbf{L}^{-1}$$

Structure Preservation

Example:

On n-dimensional, $\mathbb{C}$-valued, arrays the matrix $\mathbf{L} \in \mathbb{C}^{n\times n}$ is unitary if:

$$\bar{\mathbf{L}}^\top = \mathbf{L}^{-1}$$

Note:

The determinant of a unitary matrix has norm 1:

$$\begin{aligned} |\det(\mathbf{L})|^2 &= \det(\mathbf{L}) \cdot \overline{\det(\mathbf{L})} \\ &= \det(\mathbf{L}) \cdot \det(\bar{\mathbf{L}}^\top) \\ &= \det(\mathbf{L}) \cdot \det(\mathbf{L}^{-1}) \\ &= \det(\mathbf{L}\mathbf{L}^{-1}) \\ &= 1 \end{aligned}$$

Unitary Matrices and Eigenvalues

If $L: V \to V$ is a unitary transformation and L has an eigenvalue λ, then $|\lambda| = 1$.

To see this, let v be an eigenvector of L corresponding to the eigenvalue λ.

Since L is unitary, we have:

$$\begin{aligned}\langle v, v\rangle &= \langle Lv, Lv\rangle \\ &= \langle \lambda v, \lambda v\rangle \\ &= \lambda\bar{\lambda}\langle v, v\rangle \\ &= |\lambda|^2\langle v, v\rangle\end{aligned}$$

$\Rightarrow |\lambda| = 1.$

Function Spaces

In this course, the vector spaces we will be looking at most often are the infinite-dimensional vector spaces of continuous (complex-valued) functions defined over some domain:

- S^1: The unit circle
- D^2: The unit disk
- T^2: The (flat) torus
- S^2: The unit sphere
- B^3: The unit ball

Function Spaces: Unit Circle

$S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$:

If we have functions $f, g: S^1 \to \mathbb{C}$ then:

$$\langle f, g \rangle = \int_{p \in S^1} f(p) \cdot \overline{g(p)}\, dp$$

We can represent points on the circle in terms of angle $\theta \in [0, 2\pi)$:

$$\theta \to (\cos\theta, \sin\theta)$$

For $f, g: [0, 2\pi) \to \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot \overline{g(\theta)}\, d\theta$$

Function Spaces: Unit Circle

$S^1 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$:

The space of functions on $[0,2\pi)$ is equivalent to the space of periodic functions on the line:

$$\{f: \mathbb{R} \to \mathbb{C} | f(x) = f(x+2\pi)\}$$

We can represent points on the circle in terms of angle $\theta \in [0,2\pi)$:

$$\theta \to (\cos\theta, \sin\theta)$$

For $f, g: [0,2\pi) \to \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot \overline{g(\theta)}\, d\theta$$

Function Spaces: Unit Disk

$\underline{D^2 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}}$:

If we have functions $f, g: D^2 \to \mathbb{C}$ then:

$$\langle f, g \rangle = \int_{p \in D^2} f(p) \cdot \overline{g(p)} dp$$

We can represent points on the disk in terms of radius $r \in [0,1]$ and angle $\theta \in [0,2\pi)$:

$$(r, \theta) \to (r \cdot \cos\theta, r \cdot \sin\theta)$$

For $f, g: [0,1] \times [0,2\pi) \to \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^1 f(r, \theta) \cdot \overline{g(r, \theta)} \cdot r \, dr \, d\theta$$

Function Spaces: (Flat) Torus

$\underline{T^2 = S^1 \times S^1}$:

If we have functions $f, g: T^2 \to \mathbb{C}$ then:

$$\langle f, g \rangle = \int_{p \in T^2} f(p) \cdot \overline{g(p)}\, dp$$

We can represent points on the torus in terms of angles $(\theta, \phi) \in [0, 2\pi) \times [0, 2\pi)$:

$$(\theta, \phi) \to (\cos\theta, \sin\theta, \cos\phi, \sin\phi)$$

For $f, g: [0, 2\pi) \times [0, 2\pi) \to \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^{2\pi} f(\theta, \phi) \cdot \overline{g(\theta, \phi)}\, d\theta\, d\phi$$

Function Spaces: (Flat) Torus

$\underline{T^2 = S^1 \times S^1}$:

The space of functions on $[0,2\pi) \times [0,2\pi)$ is equivalent to the space of periodic functions in the plane:

$$\{f: \mathbb{R}^2 \to \mathbb{C} \big| f(x,y) = f(x+2\pi, y) = f(x, y+2\pi)\}$$

We can represent points on the torus in terms of angles $(\theta,\phi) \in [0,2\pi) \times [0,2\pi)$:

$$(\theta,\phi) \to (\cos\theta, \sin\theta, \cos\phi, \sin\phi)$$

For $f, g: [0,2\pi) \times [0,2\pi) \to \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^{2\pi} f(\theta,\phi) \cdot \overline{g(\theta,\phi)}\, d\theta\ d\phi$$

Function Spaces: Unit Sphere

$S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$

If we have functions $f, g: S^2 \to \mathbb{C}$ then:

$$\langle f, g \rangle \int_{p \in S^2} f(p) \cdot \overline{g(p)}\, dp$$

We can represent points on the sphere in terms of spherical angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$:

$$(\theta, \phi) \to (\sin\theta \cdot \cos\phi, \cos\theta, \sin\theta \cdot \sin\phi)$$

For $f, g: [0, \pi] \times [0, 2\pi) \to \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \cdot \overline{g(\theta, \phi)} \cdot \sin(\theta)\ d\theta\ d\phi$$

Function Spaces: Unit Ball

$$\underline{B^3 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}}$$

If we have functions $f, g: B^3 \to \mathbb{C}$ then:

$$\langle f, g \rangle \int_{p \in B^3} f(p) \cdot \overline{g(p)}\, dp$$

We can represent points in the ball in terms of radius $r \in [0,1]$ and spherical angles $\theta \in [0, \pi]$, $\phi \in [0,2\pi)$:

$$(r, \theta, \phi) \to (r \cdot \sin\theta \cdot \cos\phi, r \cdot \cos\theta, r \cdot \sin\theta \cdot \sin\phi)$$

For $f, g: [0,1] \times [0,1] \times [0,2\pi) \to \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^{\pi} \int_0^1 f(r, \theta, \phi) \cdot \overline{g(r, \theta, \phi)} \cdot r^2 \cdot \sin(\theta)\ dr\ d\theta\ d\phi$$

Function Spaces

<u>Examples</u>

If we consider the space of continuous, $\mathbb{C}$-valued functions on the unit circle:

- Is the map:

$$f(p) \to f(p) + 1$$

  a linear transformation?

Function Spaces

Examples

If we consider the space of continuous, $\mathbb{C}$-valued functions on the unit circle:

- Is the map:

$$f(p) \to f(p) + 1$$

  a linear transformation? **No**

Function Spaces

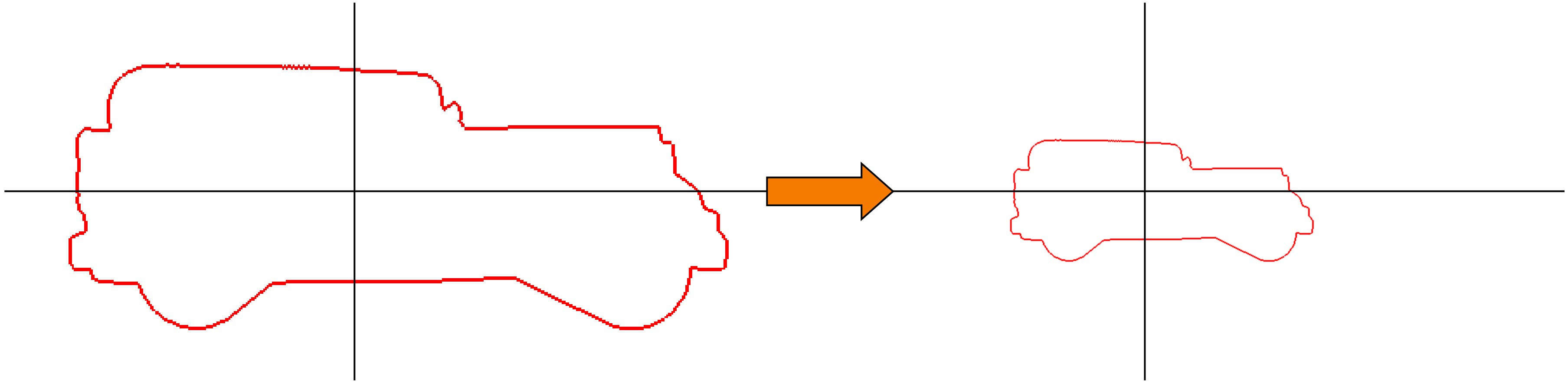
Examples

If we consider the space of continuous, $\mathbb{C}$-valued functions on the unit circle:

- For any scalar value λ, is:

$$f(p) \to \lambda \cdot f(p)$$

  a linear transformation?



Function Spaces

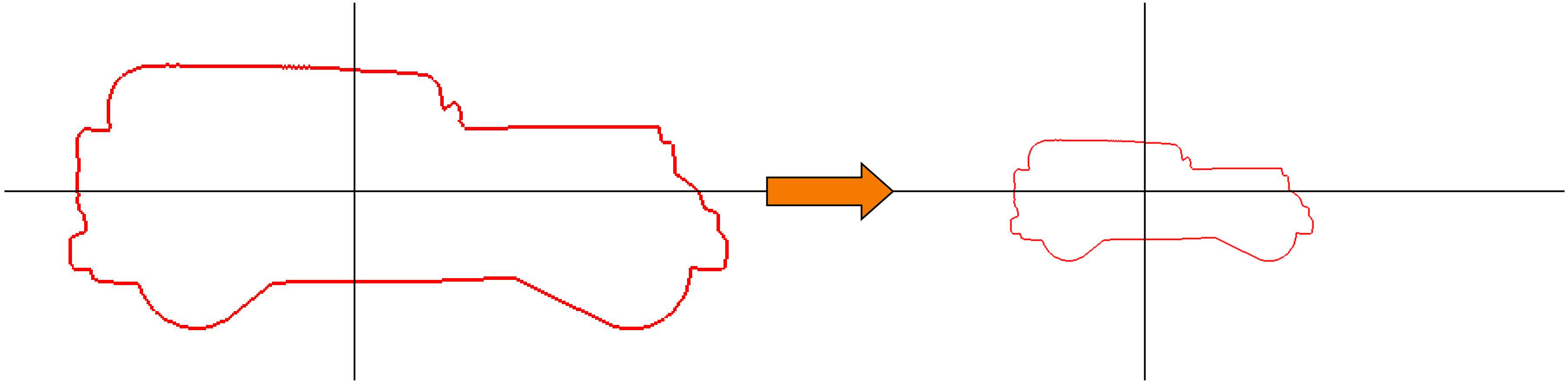
Examples

If we consider the space of continuous, ℂ-valued functions on the unit circle:

- For any scalar value λ, is:

  $$f(p) \to \lambda \cdot f(p)$$

  a linear transformation? **Yes**



Function Spaces

Examples

If we consider the space of continuous, $\mathbb{C}$-valued functions on the unit circle:

- For any scalar value λ, is:
  $$f(p) \to \lambda \cdot f(p)$$
  a linear transformation? **Yes**

- Is it unitary?

Function Spaces

Examples

If we consider the space of continuous, $\mathbb{C}$-valued functions on the unit circle:

- For any scalar value λ, is:
  $$f(p) \to \lambda \cdot f(p)$$
  a linear transformation? **Yes**

- Is it unitary? **No**

Function Spaces

Examples

If we consider the space of continuous, $\mathbb{C}$-valued functions on the unit circle:

- For any scalar value λ, is:
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If we consider the space of continuous, $\mathbb{C}$-valued functions on the periodic line:*

- Is the integration operator:
$$f(x) \to \int_0^x f(t)\,dt - \frac{1}{2\pi}\int_0^{2\pi}\int_0^x f(t)\,dt\;ds$$
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*Technically, we are looking at functions $f: \mathbb{R} \to \mathbb{R}$ with $\int_0^{2\pi} f = 0$.

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Note:
The image evaluates to zero at $x = 0$ so the map isn't invertible.

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If we consider the space of continuous, $\mathbb{C}$-valued functions on the unit circle:

- For any 2D rotation R is the transformation:
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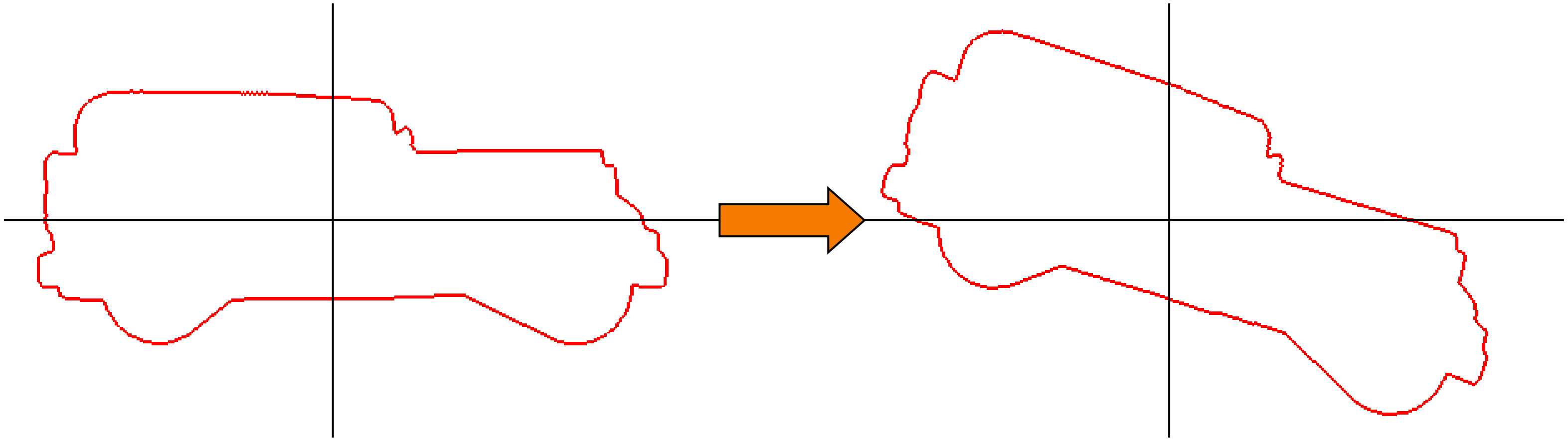

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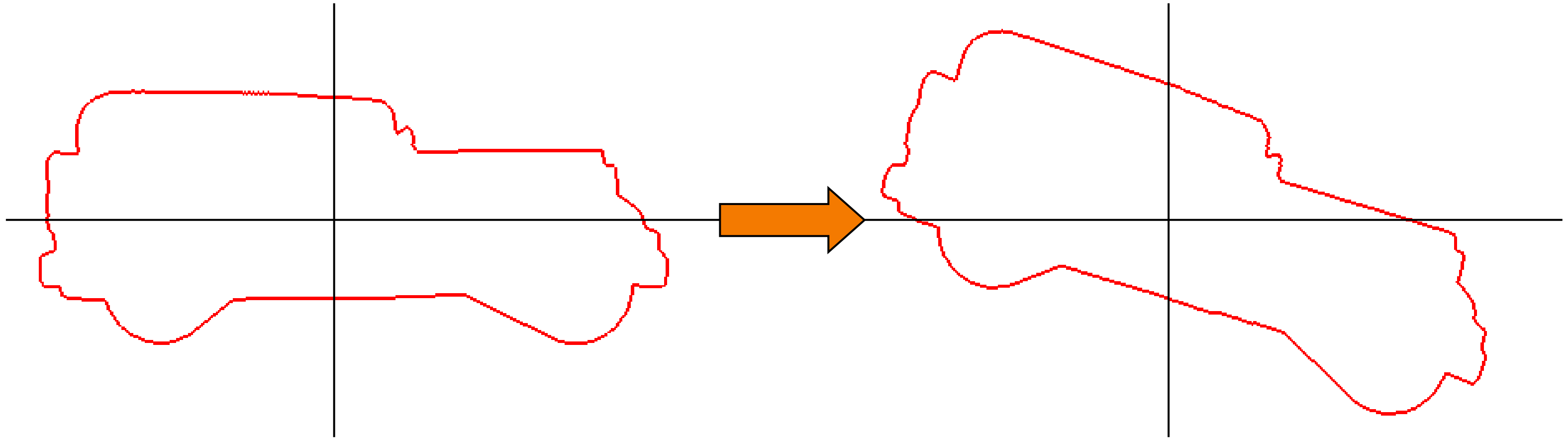

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If we consider the space of continuous, infinitely-differentiable, periodic, $\mathbb{C}$-valued functions:

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Note:
The map takes functions that are constant in x to zero, so it's not invertible.

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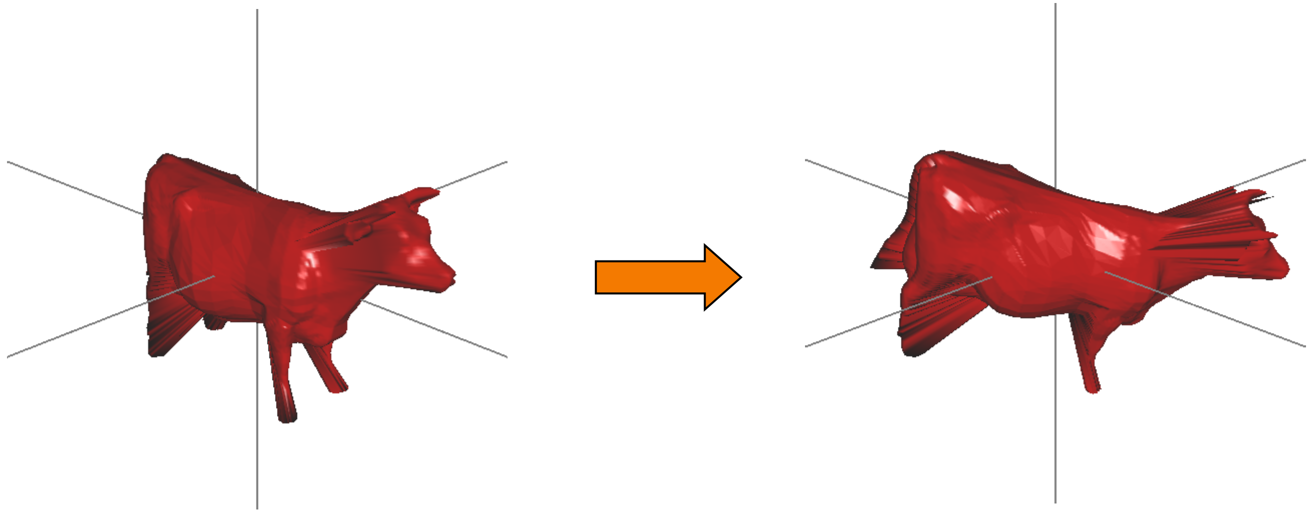

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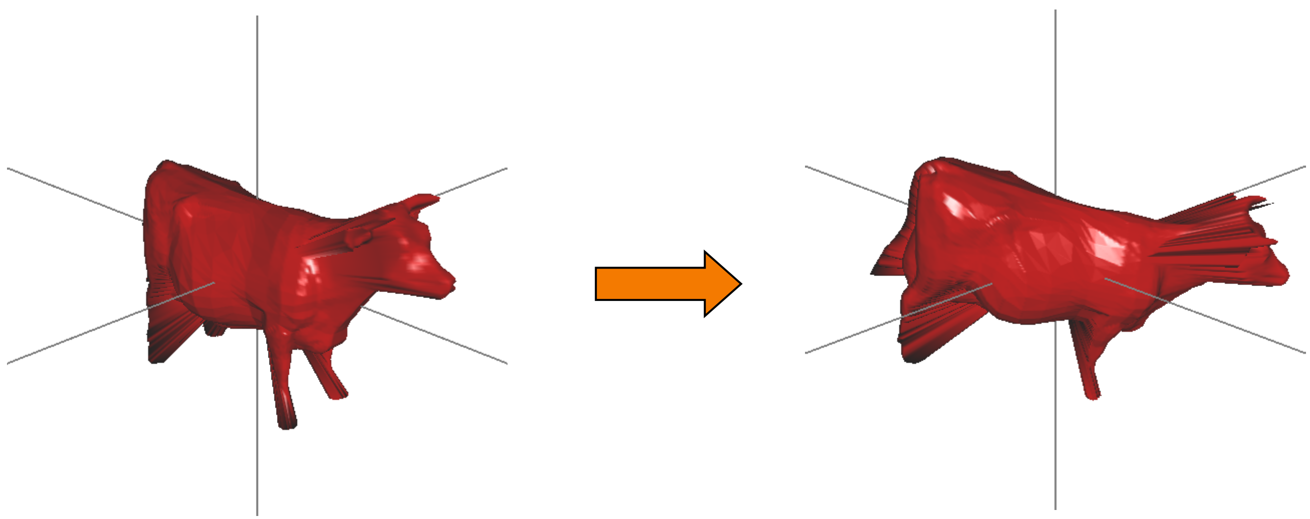

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