SIAM J. COMPUT.
Vol. 16, No. 1, February 1987

006

POWER DIAGRAMS: PROPERTIES, ALGORITHMS AND APPLICATIONS*

F. AURENHAMMER†

Abstract. The power pow (x, s) of a point x with respect to a sphere s in Euclidean d-space E^d is given by $d^2(x, z) - r^2$, where d denotes the Euclidean distance function, and z and r are the center and the radius of s. The power diagram of a finite set S of spheres in E^d is a cell complex that associates each $s \in S$ with the convex domain $\{x \in E^d \mid \text{pow}(x, s) < \text{pow}(x, t), \text{ for all } t \in S - \{s\}\}$.

The close relationship to convex hulls and arrangements of hyperplanes is investigated and exploited. Efficient algorithms that compute the power diagram and its order-k modifications are obtained. Among the applications of these results are algorithms for detecting k-sets, for union and intersection problems for cones and paraboloids, and for constructing weighted Voronoi diagrams and Voronoi diagrams for spheres. Upper space bounds for these geometric problems are derived.

Key words. Voronoi diagrams, Laguerre metric, cell complexes, convex hulls, hyperplane arrangements, duality, efficient algorithms, concrete complexity

AMS(MOS) subject classifications. F.2.3, E.2.2

1. Introduction. Voronoi diagrams for point-sets in d-dimensional Euclidean space E^d have been studied by a number of people in their original as well as in generalized settings. For a finite set $M \subsetneq E^d$, the (*closest-point*) *Voronoi diagram* of M associates each $p \in M$ with the convex region $R(p)$ of all points closer to p than to any other point in M. More formally, $R(p) = \{x \in E^d \mid d(x, p) < d(x, q), \forall q \in M - \{p\}\}$, where d denotes the Euclidean distance function. Voronoi diagrams are of importance in a variety of areas other than computer science whose enumeration exceeds the scope of this paper. Shamos and Hoey [35] were the first to introduce the planar diagram to computational geometry and also demonstrated how to construct it efficiently. Using a dual correspondence to convex hulls discovered by Brown [7], its higher-dimensional analogues can be obtained using methods in Seidel [32].

As the variety of applications of the Voronoi diagram was recognized, people soon became aware of the fact that many practical situations are better described by some modification than by the original diagram. Following these requirements, diagrams under more general metrics [21], [23], for more general objects than points [19], [9], and of higher order [35], [22], [10] have been investigated. A different generalization stems from the concept of weighting the given points. Each $p \in M$ has assigned an individual real number $w(p)$, the *weight* of p, and the distance of a point $x \in E^d$ is measured as a (suitable) function of $d(x, p)$ and $w(p)$. The diagrams induced by the distance function $d(x, p)/w(p)$ are studied in [3] and [2] for $M \subsetneq E^2$ and $M \subsetneq E^1$. The formula $d(x, p) - w(p)$ leads to diagrams whose planar instances can be interpreted as Voronoi diagrams for circles with centers p and radii $w(p)$, see e.g. [9]. The practical relevance of these weighting schemes is reported in [5] and [17] which describe applications in geography and economics.

This paper is concerned with the distance function $d(x, p)^2 - w(p)$ and the resulting diagrams in E^d. Again, this particular weighting allows a nice geometric interpretation: Viewing a point p with weight $w(p) \geqq 0$ as a sphere $s = \{x \in E^d \mid d(x, p) = \sqrt{w(p)}\}$, the above distance function is known as the *power* of a point $x \in E^d$ with respect to s, abbreviated pow (x, s). Let S denote a finite set of spheres in E^d. For $s \in S$, we call

* Received by the editors April 19, 1984, and in final revised form December 13, 1985.

† Institutes for Information Processing, Technical University of Graz and Austrian Computer Society, Schiesstattgasse 4a, A-8010 Graz, Austria.

the set $\text{cell}(s)=\{x\in E^d \mid \text{pow}(x,s)<\text{pow}(x,t),\ \forall t\in S-\{s\}\}$ the *power cell* of s and the collection of all $\text{cell}(s)$, for $s\in S$, the *power diagram* of S, or $\text{PD}(S)$ for short. (Also, *Laguerre diagram* or *Dirichlet cell complex* are used synonymously for $\text{PD}(S)$.)

Power diagrams have been studied for a long time in mathematics and other areas of science. The power function was mentioned as a generalized distance function by Laguerre (see e.g. [4]) and by Voronoi [36]. PDs play an important role in packing and covering spheres [31], [14], and in illuminating balls [15], [25], and are applicable to particular problems in number theory [24]. Independently of the author, the equivalence of PDs and the projections of the boundaries of convex polyhedra is shown in [28]. This equivalence implies a dual correspondence between PDs and polytopes. As a matter of fact, the PD for congruent spheres is the Voronoi diagram of their centers. References [26] and [33] prove that there exist Voronoi diagrams dual to polytopes which realize a maximal number of facets. Practical applications of PDs can be found in [20] (crystallography), [34] (metallurgy), and [17] (economics).

Surprisingly, power diagrams (in their general form) have not yet received attention in computational geometry.[1] We shall show that PDs in E^d, for $d\geqq 1$, can be constructed using known algorithms, and are applicable to seemingly unrelated problems. In fact, the contributions of this paper are of algorithmic as well as of mathematical nature. Properties of PDs and their generalizations to higher order are described in §§ 2 and 3. In § 4, the relationship between arbitrary PDs in E^d and convex hulls and arrangements of hyperplanes in E^{d+1} is investigated by exploiting a known geometric transform. Efficient algorithms that compute PDs, PDs of higher order, and particular power cells are outlined in § 5. Based on these results, computational solutions of several geometric problems are presented and their space complexity is analyzed. Finally, conclusions and open problems are given in § 7.

2. Analyzing power diagrams. In this section we discuss several properties of power diagrams in E^2 and E^d, for $d\geqq 3$. While § 2.1 introduces some notations, § 2.2 is mainly a review of known geometrical facts that provide the basis for our investigations in §§ 3 to 6.

2.1. Geometrical notations. We first recall the generalizations of lines, circles, and polygons in higher dimensions.

Let $p\in E^d$ be different from the origin and have positive dth coordinate, and let c be real. Then $h=\{x\in E^d \mid xp=c\}$ is a *hyperplane* and $h^+=\{x\in E^d \mid xp>c\}$ and $h^-=\{x\in E^d \mid xp<c\}$ is an *upper* and a *lower* (open) *halfspace* in E^d, respectively (xp denotes the inner product of x and p). $q\in E^d$ lies *above* (*in*, *below*) h if $q\in h^+$ ($q\in h$, $q\in h^-$). A *j-flat* is a set expressible as the intersection of $d-j$, but no fewer hyperplanes. For example, lines (planes) are hyperplanes in E^2 (E^3). In E^3, there exist 0-flats (points), 1-flats (lines), 2-flats (planes), and a 3-flat (E^3 itself).

For $p\in E^d$ and $c\geqq 0$, $\{x\in E^d \mid d(x,p)=c\}$ is a *sphere* and $\{x\in E^d \mid d(x,p)<c\}$ is an (open) *ball* in E^d with *center* p and *radius* c. Two spheres are *co-centric* (*congruent*) if they have the same center (radius).

A *polyhedron* f in E^d is the nonempty intersection of a finite number of halfspaces. We call f a *j-polyhedron* if there is a j-flat, but no $(j-1)$-flat that contains f. Examples for 0-, 1-, 2- and 3-polyhedra are points, line segments, polygons and polyhedra in E^3.

A j-polyhedron f is a (relatively) open convex set and its boundary consists of a finite number of i-polyhedra ($0\leqq i<j\leqq d$), also called *i-faces* of f. We use the terms

[1] The author recently learned that Imai, Iri and Murota have outlined a divide-and-conquer algorithm for planar PDs.

vertex, *edge* and *facet* for 0-, 1- and $(j-1)$-face, respectively. Two polyhedra f and g are *incident* if f is a facet of g, and *adjacent* if they are incident to the same facet. f is *bounded* (f is a *polytope*) if there is some ball that contains f. The *convex hull* of a finite point-set M in E^d is defined as the intersection of all halfspaces containing M and thus is a polytope. Two d-polyhedra P and Q are *dual* if there is a one-to-one correspondence ψ between the j-faces of P and the $(d-j-1)$-faces of $Q(0 \leqq j \leqq d-1)$ such that $f \subseteq g$, for any two faces f and g of P, exactly if $\psi(f) \supseteq \psi(g)$. For an extensive treatment of polyhedra, the reader is referred to [16] or to [6].

A *cell complex* C in E^d is a partition of E^d into finitely many j-polyhedra, the *j-faces* of $C(0 \leqq j \leqq d)$. d-faces of C will be also called *cells*. Let C be contained in some hyperplane of E^{d+1}. C and a $(d+1)$-polyhedron P are said to be *affinely equivalent* if there exists a central or parallel projection φ such that, for each face f of $C, f = \varphi(g)$ holds for some face g of P. Cell complexes which are affinely equivalent to some polyhedron will be called *polytopal*. Clearly, affine equivalence between C and P implies *combinatorial equivalence*, i.e. the graphs induced by the edges and vertices of C and P are isomorphic, but the reverse, in general, is not true.

2.2. Basic observations on power diagrams. We continue with a description of basic properties of power diagrams in E^d.

Recall that the power of a point x with respect to a sphere $s \subsetneq E^d$ with center z and radius r is defined as $\operatorname{pow}(x, s) = d^2(x, z) - r^2$. Thus $\operatorname{pow}(x, s) < 0$ if x lies in the ball bounded by s, $\operatorname{pow}(x, s) = 0$ if $x \in s$, and $\operatorname{pow}(x, s) > 0$ otherwise. In the last case, the Pythagorean Theorem implies that $\operatorname{pow}(x, s)$ equals the square of the distance of x to the touching point of a line tangent to s and through x. A fundamental property is expressed in the following observation.

Observation 1. Let s and t be spheres in E^d with centers $z_s \neq z_t$ and radii r_s and r_t. The points x satisfying $\operatorname{pow}(x, s) = \operatorname{pow}(x, t)$ describe the hyperplane

$$h: \quad 2x(z_s - z_t) = r_t^2 - r_s^2 - z_t^2 + z_s^2$$

which is perpendicular to the line joining z_s and z_t.

For circles s and t, h is known as the *power line*, *radical axis*, or *chordale*. Accordingly, we will term h the *chordale of s and t*, or $\operatorname{chor}(s, t)$ for short. Note that $\operatorname{chor}(s, t)$ is not defined for $z_s = z_t$. It seems reasonable to assume the chordale of two co-centric spheres to be at infinity. For $z_s \neq z_t$, we have the following nice properties: If $s \cap t \neq \varnothing$ then $s \cap t \subsetneq \operatorname{chor}(s, t)$. Otherwise, s and t are contained in the same open halfspace bounded by $\operatorname{chor}(s, t)$ if and only if s encloses (or is enclosed in) t. If $r_s = r_t$ then $\operatorname{chor}(s, t)$ is the perpendicular bisector of the line joining z_s and z_t.

It is well known that the chordales defined by three circles whose centers are not collinear intersect in a common point. This fact generalizes nicely to higher dimensions.

Observation 2. Let s, t and u be spheres in E^d. If z_s, z_t and z_u are not collinear then $f = \operatorname{chor}(s, t) \cap \operatorname{chor}(s, u) \cap \operatorname{chor}(t, u)$ is a $(d-2)$-flat. Otherwise, the three chordales are parallel.

f is termed the *radical center* of s, t and u. In contrast to three chordales, three hyperplanes in E^d intersect, in general, in a common $(d-3)$-flat.

Now let S be a finite set of spheres in E^d. By definition the power cell, $\operatorname{cell}(s)$ of $s \in S$, is the intersection of $n-1$ halfspaces bounded by chordales (Observation 1) and hence is a d-polyhedron with at most $n-1$ facets. Obviously, $\operatorname{cell}(s) \cap \operatorname{cell}(t) = \varnothing$ for $s, t \in S$, $s \neq t$, and there is some $s \in S$ which minimizes $\operatorname{pow}(x, s)$ for each $x \in E^d$. This implies that PD(S) is a partition of E^d into polyhedra (namely the power cells and the lower-dimensional faces bounding them), that is, PD(S) is a cell complex in E^d. Figure 1 illustrates these observations by depicting the PD of 6 circles in E^2. The

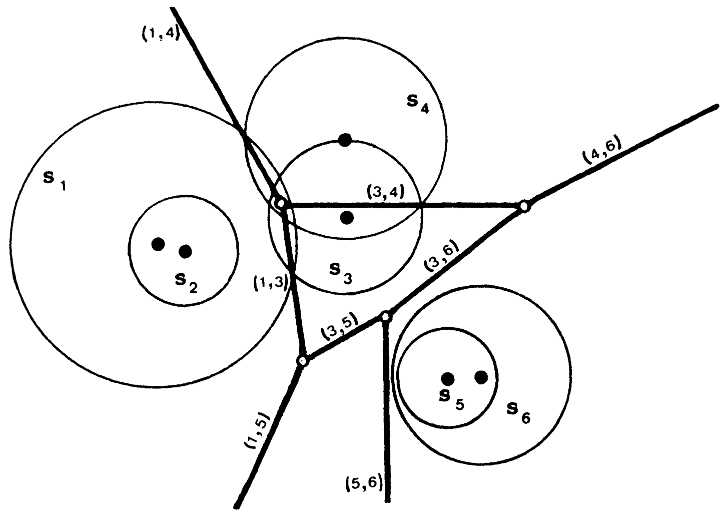

FIG. 1. *Power diagram for* 6 *circles.*

plane is partitioned into 5 cells, 8 edges and 4 vertices. Each edge e is labelled by (i, j) if e separates cell (s_i) and cell (s_j). Note the possible occurrence of empty power cells if a sphere is contained in the union of the balls bounded by the remaining spheres (for example, see circle s_2). However, circle s_5 shows that this condition does not suffice. It should be mentioned here that PDs are the generalized Voronoi diagrams that have the strongest similarities to the original diagrams.

Next we concentrate on the *size* of a PD for $n \geqq 3$ spheres in E^d, i.e., the number of its j-faces, for $0 \leqq j \leqq d$.

For $S \subsetneq E^2$, PD(S) can be interpreted as a *planar graph* consisting of *regions, edges,* and *vertices.* Each vertex is in the closure of at least three edges (Observation 2) and the number of regions does not exceed n. It is easy to verify that, in a PD(S) of maximal size, there are n regions and exactly three edges emanating from each vertex. Thus a dual graph of PD(S) is a triangulation on n vertices. (A *dual graph* $D(G)$ of a graph G contains exactly one vertex for each region of G. Two vertices of $D(G)$ are connected by an edge if and only if the boundaries of the corresponding regions of G have an edge in common. If S degenerates to a set of n points and $D(\text{PD}(S))$ has straight edges then $D(\text{PD}(S))$ is known as the *Delaunay triangulation* of S.) But a triangulation on n vertices cannot have more than $2n-4$ regions (including the only not simply-connected region which does not give rise to a vertex of PD(S) here) and $3n-6$ edges; consult e.g. [18]. We conclude as follows.

LEMMA 1. *Let S be a set of $n \geqq 3$ circles in E^2. Then* PD(S) *consists of at most n cells, $3n-6$ edges, and $2n-5$ vertices.*

Let us return to the d-dimensional case. The following assertion results from the affine equivalence of PDs in E^d and $(d+1)$-polyhedra established in Theorem 4. The formula on the size of a polyhedron is taken from [6]. McMullen [27] succeeded in proving its maximality (upper bound theorem).

THEOREM 1. *Let S be a set of $n > d > 0$ spheres in E^d, and let f_j denote the maximal number of j-faces of a $(d+1)$-polyhedron with n facets. Then*

$$f_j = \sum_{i=0}^{a} \binom{i}{j}\binom{n-d+i-2}{i} + \sum_{i=0}^{b} \binom{d-i+1}{j}\binom{n-d+i-2}{i}$$

for $a = \lceil d/2 \rceil$ *and* $b = \lfloor d/2 \rfloor$

(f_j is in $O(n^{\lceil d/2 \rceil})$), and PD(S) *contains at most n cells, f_j j-faces, for* $1 \leqq j \leqq d-1$, *and* $f_0 - 1$ *vertices. These bounds are tight.*

The diagram induced by 7 spheres in E^3 is shown in Fig. 2. For the sake of clearness, only the (parts of) faces above the plane that intersects the spheres in the dashed circles are drawn.

3. Power diagrams of higher order. Power diagrams are generalized in this section to higher order. Let S be a set of n spheres in E^d and let $T \subsetneq S$. The *power cell* of T, abbreviated cell (T), is defined as $\{x \in E^d \mid \text{pow}(x, t) < \text{pow}(x, s),\ \forall t \in T,\ \forall s \in S - T\}$. Clearly, an arbitrary T may give rise to an empty cell (T). The set of nonempty cells, defined by subsets of S with cardinality k, is termed the *order-k power diagram* of S, or k-PD(S) for short. This extension to order k is analogous to the concept of order-k Voronoi diagrams first considered in [35] which represent the special case of k-PDs for congruent spheres.

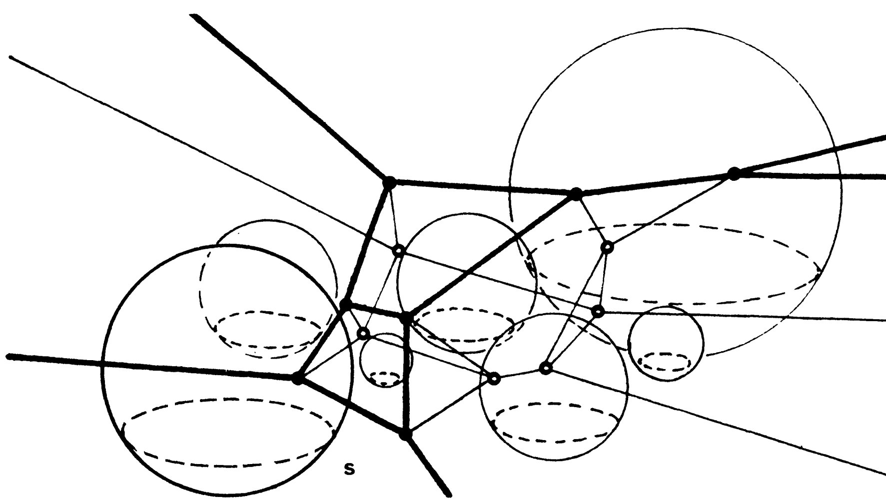

FIG. 2. *View of* PD (S) *from* cell (s).

In the following, several properties of k-PDs are analyzed. Our main interest is to determine the size of these diagrams. As easily seen, the cells of k-PD(S) are d-polyhedra which (together with their bounding faces) define a cell complex in E^d. Trivially, 1-PD(S) = PD(S) holds. Concerning $(n-1)$-PD(S), each cell Z belongs to $n-1$ spheres, and the power of the points in Z w.r.t. the nth sphere s is maximal. Z is named the *maximal power cell* of s, for short cell$_m(s)$, and $(n-1)$-PD(S) is the *maximal power diagram* of S, for short PD$_m(S)$. If we generalize PD$_m(S)$ to order k then obviously k-PD$_m(S) = (n-k)$-PD(S) holds for $k = 1, \cdots, n-1$. PD$_m$ for congruent spheres is the *farthest-point Voronoi diagram* [35] for their centers.

THEOREM 2. *Let S be a set of n spheres in E^d. The maximal number of j-faces of a* 1-PD *with n cells is a tight upper bound for the number of j-faces of* PD$_m(S)$ $(0 \leqq j \leqq d)$.

This assertion is a consequence of the identity of k-PDs and $(n-k)$-PDs (see Lemma 4). We next establish bounds on the number of faces attained by all power diagrams of higher order of a finite set of spheres. The result relies on a correspondence to arrangements of hyperplanes and is proved in § 4.3.

THEOREM 3. *For a set S of n spheres in E^d, the total number of j-faces* $(0 \leqq j \leqq d)$ *of* 1-PD(S), $\cdots$, $(n-1)$-PD(S) *is in* $\Theta(n^{d+1})$.

In spite of this general result, the author did not succeed in deriving improved bounds for individual k-PD(S)s, $2 \leqq k \leqq n-2$. Let us briefly consider some special cases. For $d=1$, the number of edges and vertices of k-PD(S) is, respectively, $\Omega(n \log k)$ and $O(n\sqrt{k})$. This follows from results in [13] or [11], and a dual correspondence between configurations of points and arrangements of lines [12]. As shown in [22], the planar order-k Voronoi diagram contains $O(k(n-k))$ cells, edges and vertices. Reference [10] mentions that the bound $\Omega(n^2 \log n)$ is achieved by $(n/2)$-PD(S) in E^2.[2] For $d \geqq 3$, no nontrivial bounds are currently known.

Let $T \subsetneq S$ with $|T| = k \geqq 1$. The remainder of this section gives some results about cell (T).

LEMMA 2. *Let* $t \in T$ *and* $s \in S-T$. *Further, let* $Z_t = \mathrm{cell}_m(t)$ *in* $\mathrm{PD}_m(T)$ *and* $Z_s = \mathrm{cell}(s)$ *in* PD($S-T$). *If* chor (t, s) *contains a facet* f *of* cell (T) *then* $f = \mathrm{chor}(t, s) \cap Z_t \cap Z_s$.

Proof. For any $x \in f$, $\mathrm{pow}(x, t) = \mathrm{pow}(x, s)$ and $\mathrm{pow}(x, t') < \mathrm{pow}(x, s')$ holds for all $t' \in T$ and $s' \in S-T$. Thus pow (x, t) is maximal for all spheres in T and minimal for all spheres in $S-T$. This implies $x \in \mathrm{chor}(t, s) \cap Z_t \cap Z_s$. But each x in this intersection clearly is in cell (T) which completes the argument. □

cell (T) is the intersection of $k(n-k)$ halfspaces and hence a polyhedron with at most $k(n-k)$ facets. It can be shown by example that this bound is tight for $d \geqq 3$. However, we are able to improve it for $d=2$.

LEMMA 3. *Let* S *be a set of* n *circles in* E^2 *and let* $T \subsetneq S$ *with* $|T| = k \geqq 1$. *Then* cell (T) *in* k-PD(S) *is a polygon with at most* n *edges and vertices.*

Proof. We define an *inner segment* as the union of all edges of cell (T) contained in the same cell of $\mathrm{PD}_m(T)$, and an *outer segment* as the union of all edges of cell (T) contained in the same cell of PD ($S-T$). Thus we have at most k inner and $n-k$ outer segments since these diagrams do not have more than k and $n-k$ cells. Due to Lemma 2, each edge of cell (T) is the intersection of an inner and an outer segment. As the union of the inner segments (as well as of the outer segments) is the union of the edges of cell (T), at most $k+n-k$ such intersections are possible. See Fig. 3 for a cell of higher order with n edges. Since cell (T) is a polygon, there are at most n vertices, also. □

Figure 3 illustrates a cell of 3-PD (S) for a set S of 8 congruent circles. Since their radii have no influence, only their centers are marked with ▲ (●) for circles in T ($S-T$). $\mathrm{PD}_m(T)$ is drawn with dashed lines and PD ($S-T$) with full lines. The thick lines indicate the boundary of cell (T).

Note that Lemma 3 is also valid for planar order-k Voronoi diagrams. As far as the author knows, no such result has been given even for these diagrams.

As previously observed, not every $T \subsetneq S$ need have a cell $(T) \neq \emptyset$. It is of interest to find sufficient conditions to guarantee the existence of cell (T). To this end, we give the criterion that follows immediately from Lemma 2.

Observation 3. cell $(T) \neq \emptyset$ if and only if there exists a point x with $x \in \mathrm{cell}_m(t)$ in $\mathrm{PD}_m(T)$, $x \in \mathrm{cell}(s)$ in PD ($S-T$), and $\mathrm{pow}(x, t) < \mathrm{pow}(x, s)$.

4. Embedding in $d+1$ dimensions. This section investigates the close relationship between power diagrams in E^d and objects in E^{d+1}. Hereby, we make use of a known geometric transform which relates spheres in E^d to hyperplanes and points in E^{d+1}

[2] A referee has kindly informed the author that the overall number of faces of the diagrams 1-PD (S), $\cdots$, k-PD (S) in E^2 is bounded by $O(nk^5)$, based on a result in B. Chazelle and F. P. Preparata, *Halfspace range search: an algorithmic application of k-sets*, Proceedings of the Symposium on Computational Geometry, 1985, pp. 107–115.

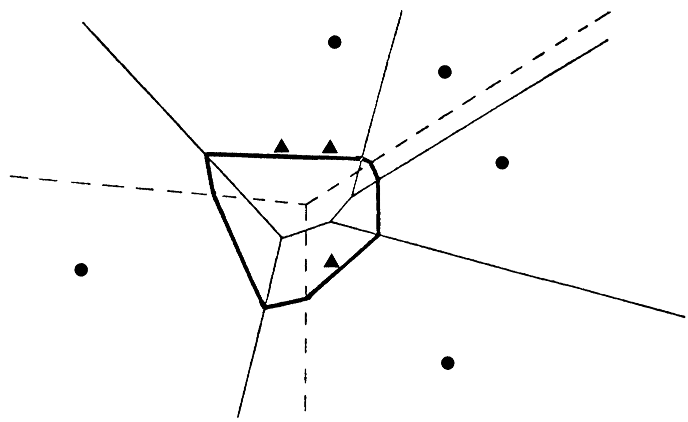

FIG. 3. cell (T) *for* $|T|=3$.

in an elementary and intuitive way. The concept of PDs turns out to be very general and allows a connection between various well-known geometric structures.

Subsection 4.1 formulates the basic geometric ideas and discusses some elementary consequences. In § 4.2, the equivalence to polyhedra and the dual correspondence to convex hulls is exhibited. These results can be viewed as a generalization of the results in [7] and [33] concerning Voronoi diagrams. In fact, we show that power diagrams are exactly those generalized Voronoi diagrams that correspond to polyhedra and thus, in the dual environment, to convex hulls. The equivalence to polyhedra has been independently perceived in [28]. Section 4.3 describes the relationship of higher-order power diagrams to arrangements of hyperplanes. For the particular instance of Voronoi diagrams, this relationship was first exploited in [10].

4.1. The transform. That PDs can be embedded in one dimension higher relies on the fact that the power function of a sphere in E^d can be described by a hyperplane in E^{d+1}.

Let E^{d+1} be spanned by the coordinate axes $x_1, \cdots, x_{d+1}$ and let h_0 denote the hyperplane $x_{d+1}=0$. Furthermore, let U denote the paraboloid $x_{d+1}=x^2$, for $x=(x_1, \cdots, x_d)$. We define the transform Π that maps a sphere $s \subsetneq h_0$ with center z and radius r into the hyperplane

$$\Pi(s): \quad x_{d+1}=2xz-z^2+r^2$$

and vice versa.

It is an easy analytic exercise to prove that Π is a bijective mapping between arbitrary spheres in h_0 and nonvertical (i.e. not parallel to the x_{d+1}-axis) hyperplanes in E^{d+1} that intersect U. In fact, $\Pi(s) \cap U$ is the *vertical projection* of s onto U. The next observation shows that $\Pi(s)$ expresses the power function of s in an elegant way.

Observation 4. Let s and x be a sphere and a point in h_0, and let x' (x'') denote the vertical projection of x onto U $(\Pi(s))$. Then $\text{pow}(x, s) = d(x, x') - d(x, x'')$ holds.

Proof. Let z and r denote the center and the radius of s, respectively. From the equations describing U and $\Pi(s)$ we obtain $d(x, x')=x^2$ and $d(x, x'')=2xz-z^2+r^2$, and thus the difference $(x-z)^2-r^2=\text{pow}(x, s)$. $\square$

Observation 4 implies the following main assertion.

LEMMA 4. *Let s and t be nonco-centric spheres in h_0. Then* chor (s, t) *is the vertical projection of* $\Pi(s) \cap \Pi(t)$ *onto* h_0.

Figure 4 displays the correspondence between the radical axis of two circles in the x_1x_2-plane and the intersection line of their associated planes in E^3. Observe that the plane of a circle can never be parallel to the x_3-axis.

Our next aim is to derive a dual correspondence between hyperplanes and points in E^{d+1}. To this end, the *polarity* Δ w.r.t. U is defined to map a hyperplane h: $x_{d+1}=a_1x_1+\cdots+a_dx_d+a_{d+1}$ into the point

$$\Delta(h)=(a_1/2,\cdots,a_d/2,-a_{d+1})$$

and a flat f in E^{d+1} into $\Delta(f)=\bigcup_{h\supseteq f}\Delta(h)$. Simple analytic calculations imply that Δ is a bijection between j-flats and $(d-j)$-flats $(0\leqq j\leqq d)$. Moreover, Δ is *involutory*, i.e., its own inverse. In particular, $\Delta(\Delta(p))=p$ for $p\in E^{d+1}$, where p is called the *pole* of $\Delta(p)$ w.r.t. U, and $\Delta(p)$ is called the *polar hyperplane* of p w.r.t. U. An additional characteristic property of Δ is that it preserves the relative position of points and hyperplanes.

Observation 5. Let p and h be a point and a nonvertical hyperplane in E^{d+1}. Then p is above (in, below) h exactly if $\Delta(h)$ is above (in, below) $\Delta(p)$.

It is worthwhile to mention that spheres, hyperplanes and points can be also related to each other without using the paraboloid U. For example, U may be replaced by any other paraboloid in E^{d+1} with vertical rotation axis, and *stereographic projection* (*inversion*, see [8]) may be used instead of vertical projection to map spheres into hyperplanes. Further, $T(h)=p$ for h: $x_{d+1}=p_1x_1+\cdots+p_dx_d+p_{d+1}$ and $p=(p_1,\cdots,p_{d+1})$, and $T(p)=h'$, for $h'=-p_1x_1-\cdots-p_dx_d+p_{d+1}$, is also a dual correspondence [8]. For the above mappings, Lemma 4 and Observation 5 remain valid.

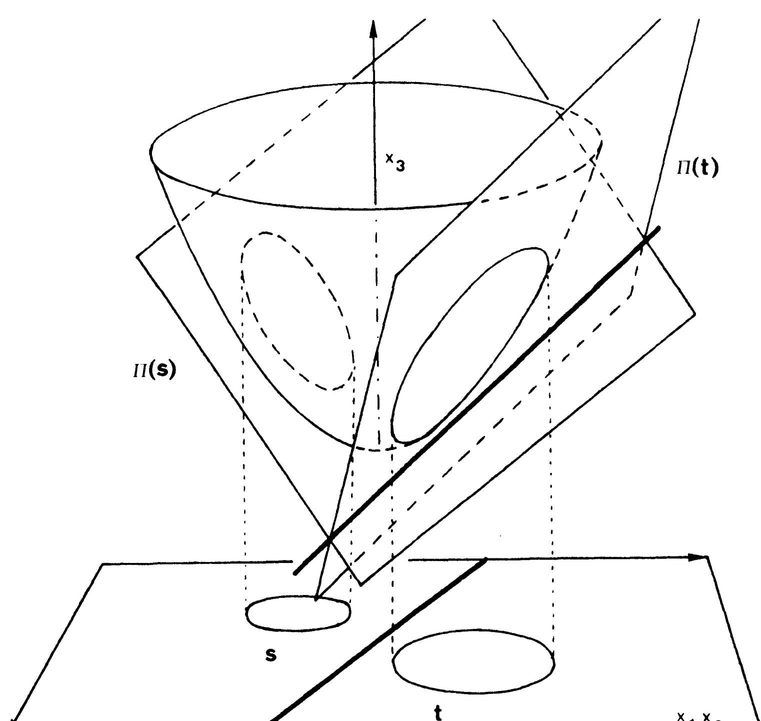

FIG. 4. *Spatial interpretation of* chor (s, t).

The reasons for using Π and Δ are their simple geometric interpretation and that they result in a simple analysis.

This subsection closes with a necessary and sufficient condition for a set of hyperplanes to be the chordales defined by n spheres.

LEMMA 5. *Let H be a set of hyperplanes in E^d. Then there exists a set S of n spheres in E^d such that H is the set of chordales for S if and only if there is a mapping $\gamma: \{1, \cdots, n\}^2 \to H \cup \emptyset$ with the following properties*:

(1) $\gamma(i,j) = \gamma(j,i) \neq \emptyset$ *for* $1 \leqq i < j \leqq n$,

(2) $\gamma(i,i) = \emptyset$ *for* $1 \leqq i \leqq n$,

(3) $\gamma(i,j) \cap \gamma(i,k) = \gamma(i,j) \cap \gamma(j,k) = \gamma(i,k) \cap \gamma(j,k)$ *for* $1 \leqq i < j < k \leqq n$.

Proof. Without loss of generality, we identify E^d with the hyperplane h_0 of E^{d+1}. Then γ completely characterizes a set H of $(d-1)$-flats in h_0 which are the vertical projections of the pairwise intersections of n nonvertical and nonparallel hyperplanes $g_1, \cdots, g_n$ in E^{d+1}: If H represents such a projection then γ obviously exists. If γ exists, then $g_1, \cdots, g_n$ can be constructed as follows. Let $f_{i,j}$ denote the vertical projection of $\gamma(i,j)$ onto g_i, for $1 \leqq i < j \leqq n$. g_1 is chosen to be arbitrary but nonvertical, g_2 is chosen to be nonvertical, to pass through $f_{1,2}$, and to be distinct from g_1, and g_k, for $k = 3, \cdots n$, through $f_{1,k}$ and $f_{2,k}$. According to (3), g_k exists and $g_k \cap g_i = f_{i,k}(2 < i < k)$, since $g_1 \cap g_i \cap g_k \subsetneq f_{i,k}$ and $g_2 \cap g_i \cap g_k \subsetneq f_{i,k}$. The reader may convince himself that g_k can be chosen such that $g_k \cap U \neq \emptyset$, for $1 \leqq k \leqq n$. Then each g_k corresponds to a unique sphere $\Pi(g_k)$, and by Lemma 4, we deduce that H is the set of chordales for $S = \{\pi(g_k) \mid k = 1, \cdots, n\}$. □

4.2. Power diagrams and polyhedra. The geometrical results in § 4.1 enable us to prove a main theorem of this paper. It asserts that power diagrams in E^d are, essentially, the same as the boundaries of $(d+1)$-polyhedra.

THEOREM 4. *For any $(d+1)$-polyhedron P, which is expressible as the intersection of upper halfspaces, there exists an affinely equivalent power diagram in h_0, and vice versa.*

Proof. Let $P = \bigcap_{i=1}^n h_i$, such that the upper halfspace h_i determines a facet f_i of P and, w.l.o.g., assume $h_i \cup U \neq h_i$, for $i = 1, \cdots, n$. There is a set $S = \{s_1, \cdots, s_n\}$ of spheres in h_0 such that $\Pi(s_i)$ bounds h_i below. Observation 4 now implies that the vertical projection of $x \in h_0$ onto P is in f_i exactly if $x \in \text{cell}(s_i)$. In other words, cell (s_i) is the vertical projection of f_i onto h_0, for $i = 1, \cdots, n$. This shows the existence of an affinely equivalent power diagram PD (S) for arbitrary P. Conversely, the existence of P for arbitrary PD (S) in h_0 results from the existence of $\Pi(s_i)$ for any $s_i \subsetneq h_0$. □

Observe that the reflection P' of P through h_0 also is affinely equivalent to PD (S). Observation 4 now implies that PD (S) is the maximal power diagram of some set of spheres. We thus have the following.

COROLLARY 1. *For any P in Theorem 4, there is an affinely equivalent maximal power diagram in h_0, and vice versa.*

Another straightforward consequence of Theorem 4 is given.

COROLLARY 2. *A cell complex C in E^d is polytopal if and only if C is a power diagram.*

We proceed to demonstrate the dual correspondence between power diagrams and convex hulls. Let M denote a finite subset of E^{d+1}. In a natural way, the boundary of the convex hull CH (M) of M splits into two parts: The *top part* $\text{CH}_t(M)$ (the *bottom part* $\text{CH}_b(M)$) of M consists of all facets f of CH (M) (and their bounding faces), such that CH (M) lies in the lower (upper) halfspace bounded by the hyperplane of E^{d+1} that contains f. Vertical facets, if some exist, are assigned neither to $\text{CH}_t(M)$ nor to $\text{CH}_b(M)$. We shall prove the following.

THEOREM 5. *For any finite set* $M \subsetneq E^{d+1}$, *there exists a set* S *of spheres in* h_0 *such that* $\mathrm{PD}(S)$ *is dual to* $\mathrm{CH}_b(M)$ *and* $\mathrm{PD}_m(S)$ *is dual to* $\mathrm{CH}_t(M)$. *The reverse of the assertion also holds.*

Proof. By Theorem 4, there is an affinely equivalent polyhedron P for each $\mathrm{PD}(S)$ in h_0 and vice versa. P is the intersection of the halfspaces above the hyperplanes in $H=\{\Pi(s) \mid s\in S\}$. We let $M=\{\Delta(h) \mid h\in H\}$ and show the duality of P and $\mathrm{CH}_b(M)$. Clearly, there exists exactly one $p_i\in M$ for each facet f_i of P. Let h_i be the hyperplane containing f_i, and let f_i and f_j be adjacent in a $(d-1)$-face g $(1\leqq i<j\leqq |H|)$. Then each $x\in g$ is in $h_i\cap h_j$ and above each $h\in H-\{h_i, h_j\}$. Observation 5 now implies that exactly then $p_i, p_j\in\Delta(x)$, and each $p\in M-\{p_i, p_j\}$ is above $\Delta(x)$, that is, p_i and p_j define an edge of $\mathrm{CH}_b(M)$. The duality of P, and thus of $\mathrm{PD}(S)$, and $\mathrm{CH}_b(M)$ immediately follows. By Corollary 1, the proof of the duality of $\mathrm{PD}_m(S)$ and $\mathrm{CH}_t(M)$ is completely analogous. □

For $d+1=3$, $\mathrm{CH}_b(M)$ has the following interpretation. The vertical projection D of $\mathrm{CH}_b(M)$ onto the x_1x_2-plane is a planar graph which is dual to $\mathrm{PD}(S)$. The vertices of D are the centers of the circles in S, and dual edges are perpendicular. (The interested reader may verify this fact.) If S is a finite set of points then D is the Delaunay triangulation of S.

4.3. Power diagrams and arrangements. A finite set H of hyperplanes in E^{d+1} dissects E^{d+1} into polyhedra which define a cell complex, often called the *arrangement* $A(H)$ of H. In this subsection, the correspondence of higher-order power diagrams to arrangements is described.

Let $S=\{s_1,\cdots,s_n\}$ be a set of spheres in h_0, and let $H=\{\Pi(s_1),\cdots,\Pi(s_n)\}$. We start by showing that each cell of k-$\mathrm{PD}(S)$ is the projection of a cell of $A(H)(1\leqq k\leqq n-1)$. For $x\in h_0$, let r_x denote the vertical ray in E^{d+1} that contains x and is directed to $-\infty$. Since the hyperplanes in H are nonvertical, they can be assigned to x in the same order as they are intersected by r_x. According to Observation 4, r_x intersects $\Pi(s_1),\cdots,\Pi(s_k)$ first and then $\Pi(s_{k+1}),\cdots,\Pi(s_n)$ exactly if $\mathrm{pow}(x,s_i)<\mathrm{pow}(x,s_j)$, for $i=1,\cdots,k$ and $j=k+1,\cdots n$. By definition, $x\in\mathrm{cell}(\{s_1,\cdots,s_k\})$ then. This yields the following.

LEMMA 6. *Let* $Z=Z^-\cap Z^+$, *where* Z^- *is the intersection of the halfspaces below* $\Pi(s_1),\cdots,\Pi(s_k)$ *and* Z^+ *is the intersection of the halfspaces above* $\Pi(s_{k+1}),\cdots,\Pi(s_n)$. $\mathrm{cell}(T)$, *for* $T=\{s_1,\cdots,s_k\}$, *is the image of* Z *under vertical projection onto* h_0.

The existence of $\mathrm{cell}(T)$ thus depends on the existence of Z. The relationship to Lemma 2 is worth mentioning. As proved in § 4.2, the faces of the unbounded polyhedra Z^- and Z^+ project to the faces of $\mathrm{PD}_m(T)$ and $\mathrm{PD}(S-T)$. The intersection of a facet f_i of Z^- in $\Pi(s_i)$ with a facet f_j of Z^+ in $\Pi(s_j)$ has dimension $d-1$ if and only if $\mathrm{chor}(s_i, s_j)$ defines a facet of $\mathrm{cell}(T)$. Figure 5 illustrates the two 3-polyhedra corresponding to the power diagram and the maximal power diagram in Fig. 3, as well as their intersection Z. The thick-lined edges of Z project to the edges of $\mathrm{cell}(T)$ onto the x_1x_2-plane.

Lemma 6 makes it clear that k-$\mathrm{PD}(S)$ is the projection of certain faces of $A(H)$. To characterize this relationship more precisely, we introduce particular subsets of $A(H)$. The *k-level* L_k of $A(H)$ consists of all d-faces f of $A(H)$ (and the faces bounding f) such that f is below exactly k hyperplanes in H. It is not difficult to see that L_k is a connected subset of $A(H)$, and that $L_k\cap L_j$ consists of faces of dimension less than d $(1\leqq k<j\leqq n)$. Furthermore, L_1 (L_n) bounds the intersection of the upper (lower) halfspaces defined by H. As a consequence, L_1 projects to $\mathrm{PD}(S)$ and L_n projects to $\mathrm{PD}_m(S)$. Moreover, we have the following.

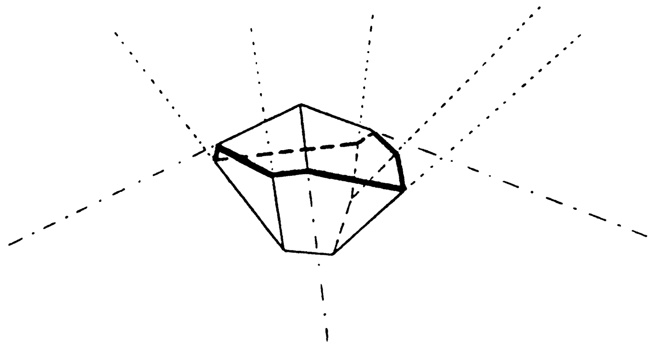

FIG. 5. *The polyhedron Z exists.*

THEOREM 6. *Let h_0, S, and L_k be defined as above. The vertical projection of $L_k \cap L_{k+1}$ onto h_0 represents the collection of j-faces of k-PD (S), for $j = 0, \cdots, d-1$ and $1 \leqq k \leqq n-1$.*

Proof. By Lemma 6, each cell Z of $A(H)$ (except the two polyhedra with boundary L_1 and L_n) projects to a cell Z' of k-PD (S), for some k. Z is the intersection of k lower and $n-k$ upper halfspaces and thus is enclosed between L_k and L_{k+1}. Let Z^- (Z^+) denote the intersection of these lower (upper) halfspaces. Then the intersection of the boundaries of Z^- and Z^+ is a subset of $L_k \cap L_{k+1}$ and, on the other hand, it projects to the boundary of Z' in h_0. □

As a consequence, the information inherent in all k-PD (S)s, for $k = 1, \cdots, n-1$, is inherent in $A(H)$. Note also that every arrangement in E^{d+1} that does not contain a vertical hyperplane (and whose hyperplanes intersect the paraboloid U, which can always be achieved by translating the arrangement vertically) corresponds to the k-PDs of some set of spheres in h_0. Furthermore, the following interesting property holds.

LEMMA 7. *Let S be a set of n spheres in h_0. There is a set S^* of spheres in h_0 such that k-PD $(S) = (n-k)$-PD (S^*), for $k = 1, \cdots, n-1$.*

Proof. Let r be a fixed vertical vector such that $h_s + r \cap U \neq \emptyset$, for each $s \in S$ and let h_s be the mirror image of $\Pi(s)$ through h_0. Then $S^* = \{s^* | \Pi(s^*) = h_s, s \in S\}$ has the desired properties by Theorem 6 and the definition of a k-level. □

It should be noted that Lemma 7, in general, is not true for Voronoi diagrams. More precisely, we cannot always find a point-set S^* for any set S of n points in E^d such that the order-k Voronoi diagram of S coincides with the order-$(n-k)$ Voronoi diagram of S^*.

The maximal size of an arrangement (see [37] or [1] for a deduction of the following formula) implies asymptotically tight bounds on the number of faces of all k-PDs for a fixed set of spheres.

LEMMA 8. *Let H be a set of n hyperplanes in E^d. Then the number of j-faces of $A(H)$ is less than or equal to*

$$\sum_{i=d-j}^{d} \binom{i}{d-j}\binom{n}{i}$$

which is in $\Theta(n^d)$, for $0 \leqq j \leqq d$.

Proof of Theorem 3. It follows from Lemma 8 in conjunction with Theorem 6 that 1-PD $(S), \cdots, (n-1)$-PD (S) together have $O(n^{d+1})$ j-faces, for $0 \leqq j \leqq d$. The bound is asymptotically tight since each j-face of $A(H)$, for $j<d$, projects to some j-face, and each $(d+1)$-face of $A(H)$ projects to some d-face, of a particular k-PD (S). □

5. How to construct power diagrams. We turn our attention from geometry to the design of algorithms for constructing power diagrams and order-k power diagrams. By using the correspondences proven in § 4, we obtain straightforward methods. We only give a rough idea of how to proceed since the expensive parts of the algorithms are known geometric tasks. Throughout this section, let S denote a set of n spheres in E^d, and let E^d be identified with h_0: $x_{d+1}=0$ in E^{d+1}.

5.1. Constructing PD (S) and PD$_m$ (S). First, we concentrate on the construction of PD (S) and PD$_m$ (S). By construction we mean the computation of a data structure that represents the induced cell complex C in an appropriate way. Clearly, the data structure depends on the kind of inherent information required. A data structure adaptable to most requirements is the so-called *incidence lattice* $I(C)$ of C (see e.g. [3] or [10]): Each j-face of C is represented by a node $(0 \leqq j \leqq d)$, and nodes of incident faces are associated via pointers. Beside the combinatorial structure of C, we have to fix the position of C in E^d. It suffices to store the coordinates of the vertices or the positions of the hyperplanes through the facets of C.

According to Theorem 5, $I(\mathrm{PD}(S))$ and $I(\mathrm{PD}_m(S))$ can be obtained from the bottom and the top part of a certain convex hull in E^{d+1}. Thus the main part of the algorithm is the determination of a convex hull. We refer to [32] for a detailed description of a convex hull algorithm for finite point-sets in arbitrary finite dimensions. Since the method in [32] is optimal only in even dimensions, the algorithm in [29] is preferable for point-sets in E^3. We state their results explicitly as follows.

PROPOSITION 1. *Let M be a set of n points in E^d. The convex hull of M can be determined in $O(n \log n)$ time for $d=3$ and in $O(n \log n + n^{\lceil d/2 \rceil})$ time for $d \neq 3$ and requires $O(n+n^{\lfloor d/2 \rfloor})$ space.*

In what now follows, a more formal and detailed description of the algorithm to construct the (maximal) power diagram is given.

ALGORITHM (MAXIMAL) POWER DIAGRAM.

Let $S=\{s_1, \cdots, s_n\}$. The computation of PD (S) and PD$_m$ (S) requires three steps.

Step 1. For $i=1, \cdots, n$, compute the hyperplanes $\Pi(s_i)$ and their poles $p_i=\Delta(\Pi(s_i))$, using the formulae in § 4.1.

Step 2. Determine the intersection Z^+ of the halfspaces above $\Pi(s_1), \cdots, \Pi(s_n)$ and the intersection Z^- of their complements.

Step 2.1. Construct the convex hull CH of $\{p_1, \cdots, p_n\}$ (by means of the algorithm in [29] for $d+1=3$ and the algorithm in [32], otherwise) such that CH is represented by $I(\mathrm{CH})$.

Step 2.2. Split $I(\mathrm{CH})$ into I_t for the top part and I_b for the bottom part of CH. The facets f of CH (and the faces in the boundary of f) are assigned to I_t or I_b according to the slope of the hyperplane containing f.

Step 2.3. Replace each j-face in I_t and I_b by a $(d-j)$-face, for $j=0, \cdots, d$. For each vertex that arises from a facet f, we store the coordinates of the pole $\Delta(h)$ of the hyperplane h through f. The coordinates of $p_1, \cdots, p_n$ are deleted. We now have $I_t=I(Z^-)$ and $I_b=I(Z^+)$.

Step 3. Project Z^+ and Z^- vertically onto h_0 (that is, ignore the $(d+1)$st coordinates of their vertices). This finally yields PD (S) and $\mathrm{PD}_m(S)$ in h_0.

Let us analyze the time used by this algorithm. Obviously, Step 1 can be performed in $O(n)$ time. As a consequence of the properties of an incidence lattice, Steps 2.2, 2.3 and 3 require time proportional to the size of PD (S) and $\mathrm{PD}_m(S)$. Since the size of these diagrams implies a trivial lower time bound for their construction, the time complexity essentially depends on Step 2.1. This yields the following.

THEOREM 7. *Let S be a set of n spheres in E^d, and let $T_d(n)$ $(S_d(n))$ denote the amount of time (space) necessary to compute the convex hull of n point in E^d. Algorithm* (MAXIMAL) POWER DIAGRAM *constructs* PD (S) *and* $\mathrm{PD}_m(S)$ *in* $O(T_{d+1}(n))$ *time and* $O(S_{d+1}(n))$ *space.*

The time optimality of the algorithm thus depends on the optimality of convex hull algorithms. According to Proposition 1 and Theorem 1, the method is asymptotically time-optimal for $d=2$ and every odd d, and space-optimal for arbitrary d (in the worst case).

5.2. Building up all higher-order PDs. This subsection deals with the construction of all higher-order power diagrams of S. Theorem 6 provides the strategy: Construct the arrangement $A(H)$ induced by $H=\{\Pi(s)|s\in S\}$, determine the faces contained in $L_k\cap L_{k+1}$, and project them vertically onto h_0. This yields the faces bounding the cells of k-PD (S), for $k=1,\cdots,n-1$. Evidently, the major part of the algorithm is building $A(H)$. An asymptotically time- and space-optimal method for constructing arbitrary arrangements in E^d by incrementally inserting the hyperplanes is presented in [10]. Their main result reads as follows.

PROPOSITION 2. *Let H be a set of n hyperplanes in E^d and let $h\in H$. Then $A(H)$ can be obtained from $A(H-\{h\})$ in $O(n^{d-1})$ time.*

We are left with the problem of projecting $L_k\cap L_{k+1}$ onto h_0, for $k=1,\cdots,n-1$. The representation of $A(H)$ by its incidence lattice $I(A(H))$ allows us to identify the j-faces $(0\leqq j\leqq d-1)$ in $L_k\cap L_{k+1}$ in time proportional to the size of L_k without asymptotically increasing the space requirement. See [10] for the augmentation of the nodes of $I(A(H))$ necessary to carry out this task. We conclude as follows.

THEOREM 8. *Let S be a set of n spheres in E^d. Then* 1-PD $(S),\cdots,(n-1)$-PD (S) *can be computed in $O(n^{d+1})$ time and space.*

By Theorem 3, the result is asymptotically optimal for any d. Note that the strategy can be used for on-line constructing of higher-order PDs by inserting the given spheres.

5.3. Computing particular power cells. In some situations it might be of interest to determine a particular power cell of k-PD (S) $(1\leqq k\leqq n-1)$ or to detect its existence. Let $T\subsetneq S$ and $|T|=k$. Trivially, cell (T) can be computed by intersecting $k(n-k)$ halfspaces in E^d which are bounded by chordales. While this method seems to be optimal e.g. in E^3 (Lemma 2 suggests that cell (T) may contain $k(n-k)$ facets), the number of bounding faces of cell (T) in E^2 is limited by n (compare Lemma 3). Thus this problem begs for a more efficient solution.

Indeed, Lemma 6 offers an alternative construction method for $S\subsetneq E^2$. The boundary of cell (T) is the vertical projection of the intersection of the boundaries of two 3-polyhedra Z^- and Z^+ which correspond to the intersection of k lower and $n-k$ upper halfspaces in E^3. References [30] and [33] give algorithms that compute the intersection of n arbitrary halfspaces in E^3 in $O(n\log n)$ time and $O(n)$ space. (It is worth mentioning that, similarly to Algorithm (MAXIMAL) POWER DIAGRAM, these algorithms apply convex hull algorithms to point-sets obtained from the set of half-spaces via dual transforms.)

ALGORITHM POWER CELL.

Let $S \subsetneq h_0$, $T \subsetneq S$, and $T \neq \emptyset$. cell (T) is constructed as follows:

Step 1. Construct the 3-polyhedron $Z = Z^- \cap Z^+$ using the algorithm in [30] or [33].

Step 2. If $Z \neq \emptyset$ then compute the intersection of the boundaries of Z^- and Z^+. This yields a cyclic sequence Q of adjacent edges.

Step 3. Project Q vertically onto h_0. This yields the edges of cell (T).

From the preceding discussion, we obtain the following.

THEOREM 9. *For a set S of n circles in E^2, Algorithm* POWER CELL *computes* cell (T) *for arbitrary fixed $T \subsetneq S$ in $O(n \log n)$ time and $O(n)$ space.*

Clearly, the space requirement is optimal to within a constant factor since cell (T) can have up to n edges. We shall prove the time bound to be optimal, too. Let M and M' be the set of centers of the circles in T and $S - T$, respectively. Lemma 2 makes it clear that M and M' can be obtained, sorted by polar angles w.r.t. a point p interior to the convex hull of M', in $O(n)$ time from the cyclic order of the edges of cell (T). (A suitable point p can be found in $O(1)$ time by determining the mass center of three points in M'.) Since $\Omega(k \log k) + \Omega((n-k) \log (n-k)) = \Omega(n \log n)$, for $k = |T|$, is a lower time bound for this sorting problem, the same bound holds for the construction of cell (T), too.

Note that Algorithm POWER CELL can be used to detect the existence of cell (T). Compare this with Observation 3 that suggests a different approach to the detection problem.

6. Applications. Power diagrams are applicable to several problems involving spheres and polyhedra in E^d. In this section we give a brief description of four situations where PDs lead to an efficient computational solution, and serve to analyze the space complexity. We mention only marginally that PDs apply naturally to computing the union or intersection of balls in E^d, since these tasks have been solved in [8] with the same time and space complexity that would result from using PDs. The efficiency of many of the algorithms given below depends only on convex hull algorithms. So throughout this section, let $T_d(n)$ $(S_d(n))$ denote the time (space) required to compute the convex hull of n points in E^d (compare Proposition 1).

6.1. Illumination of balls. Let V and B be subsets of E^d. We say that *V illuminates* B if, for each point x on the boundary of B, there is a point y in V such that the relatively open line segment joining x and y does not intersect B. As reported in [25], the union B of n balls in E^d can be illuminated by the vertices of the power diagram for the corresponding spheres. Thus finding illumination points for B is reduced to computing a PD. From Theorem 7, we conclude as follows.

THEOREM 10. *Let B be the union of n balls in E^d. Illumination points for B can be found in $O(T_{d+1}(n))$ time and $O(S_{d+1}(n))$ space.*

Since the minimal number of points necessary to illuminate B is $\Theta(S_{d+1}(n))$ in the worst-case [25], the space bound is optimal for any d, and the time bound is optimal for odd $d \geqq 3$. By reducing the illumination problem to sorting $O(n)$ reals, we deduce that the time bound is optimal for $d = 1, 2$, also.

6.2. Detecting *k*-sets. Let M denote a set of n points in E^d, and let $M' \subsetneq M$ with $|M'| = k(1 \leqq k \leqq n-1)$. M' is called a *k-set* of M if there exists a halfspace h such that $h \cap M = M'$.

The k-sets of M can be characterized in terms of power diagrams by using the geometrical transforms Π and Δ introduced in § 4.1. (Without loss of generality, assume that the sphere $\Pi(h)$ exists for each halfspace h below.)

LEMMA 9. *M' is a k-set of M if and only if* cell (T) *in* k-PD (S) *is nonempty, for* $T = \{\Pi(\Delta(q)) | q \in M'\}$ *and* $S = \{\Pi(\Delta(p)) | p \in M\}$.

Proof. Assume that M' is a k-set of M, i.e., there exists a hyperplane h such that, w.l.o.g., each $q \in M'$ is below h, and each $p \in M - M'$ is above h. By Observation 5, this is equivalent to the fact that point $\Delta(h)$ is below all hyperplanes $\Delta(q)$, and above all hyperplanes $\Delta(p)$. Thus $\Delta(h)$ is contained in the intersection Z of the halfspaces below all $\Delta(q)$ and above all $\Delta(p)$, that is, $Z \neq \emptyset$. Since cell (T) in k-PD (S) is the vertical projection of Z(see Lemma 6), the assertion follows directly. □

The above discussion implies that M' can be recognized to be a k-set of M by intersecting either n halfpaces in E^d or $k(n-k)$ halfpaces in E^{d-1}. Since intersecting halfspaces is essentially equivalent to determining a convex hull [8], [32] we obtain the following.

THEOREM 11. *Let M' be a subset of cardinality k of a set M of n points in E^d. Then* $O(\min\{T_d(n), T_{d-1}(k(n-k))\})$ *time suffices to detect whether or not M' is a k-set of M.*

Note that, for each $p \in Z$, M' is separated from $M - M'$ by $\Delta(p)$. A point p interior to Z can be computed in $O(1)$ time given Z (choose p as the mass center of $d+1$ boundary points of Z which do not belong to the same facet), and in $O(n)$ time given cell (T) (choose a point x interior to cell (T), intersect the parts of the vertical ray through x which are contained in the halfspaces determining Z, and choose p in this intersection). Thus a separating hyperplane for M' can be computed within the time bounds in Theorem 11.

6.3. Multiplicative weighted Voronoi diagrams. Let $p \in E^d$ have an associated *weight* $w(p) > 0$. The *multiplicative weighted distance* from $x \in E^d$ to p is defined as $d_m(x, p) = d(x, p)/w(p)$. Among a set M of n weighted points, the *region* (of dominating influence) of p is determined by $\{x \in E^d | d_m(x, p) < d_m(x, q), \forall q \in M - \{p\}\}$. The subdivision of E^d induced by the regions of the points in M is called the *multiplicative weighted Voronoi diagram* of M.

By easy analytic arguments, the regions of any distinct $p, q \in M$ are separated by a sphere sph (p, q). Moreover, sph (p, q), sph (p, r), and sph (q, r), for distinct $p, q, r \in M$, define a common chordale. Thus $\Pi(\text{sph}(p, q)) \cap \Pi(\text{sph}(p, r)) \cap \Pi(\text{sph}(q, r))$ is a $(d-1)$-flat (compare Lemma 4), such that $\{\Pi(\text{sph}(p, q)) | p, q \in M, p \neq q\}$ is the set of chordales of some set S of spheres in E^{d+1} (Lemma 5). It is not difficult to observe that there is a one-to-one correspondence σ between M and S such that the boundary of the region of $p \in M$ is the vertical projection of the intersection of cell $(\sigma(p))$ in PD (S) with the projection paraboloid U. S can be constructed iteratively from M in $O(n)$ time in a straightforward manner. Since projecting does not affect the asymptotic runtime, Theorem 7 implies the following.

THEOREM 12. *The multiplicative weighted Voronoi diagram for n points in E^d can be constructed in $O(T_{d+2}(n))$ time and $O(S_{d+2}(n))$ space.*

See [2] and [3] for algorithms yielding the same (optimal) bounds for $d = 1, 2$, respectively. For $d \geqq 3$, the result appears to be new. It tells us that the space complexity of the diagram is asymptotically the same as of a power diagram in E^{d+1}, and that the method is time-optimal for d even, and space-optimal for any d.

As a matter of fact, the weight of $p \in M$ can be expressed by a cone [2] or a paraboloid $\upsilon(p)$ with vertical rotation axis, such that the components of the boundary of the union of the point-sets below all $\upsilon(p)$ project vertically to the components of the weighted diagram of M. This yields the following.

COROLLARY 3. *The union (intersection) of n axis-parallel cones in E^d with apexes in a common hyperplane, or of n axis-parallel paraboloids in E^d, can be determined in $O(T_{d+1}(n))$ time and $O(S_{d+1}(n))$ space. This is time-optimal for $d=2$ and odd d, and space-optimal for any $d \geqq 2$.*

Similarly to PDs, the weighted diagram of M can be generalized to order k $(1 \leqq k \leqq n-1)$. (The intersection of the above objects then corresponds to the order-$(n-1)$ diagram.) Its regions then can be constructed by intersecting the cells of k-PD (S), for $S=\{\sigma(p) | p \in M\}$, with the paraboloid U. Hence all higher-order diagrams of M can be built up in $O(n^{d+2})$ time and space by Theorem 8. However, this strategy is not optimal since k-PD (S) contains much more information than is inherent in the weighted order-k diagram for $2 \leqq k \leqq n-2$.

6.4. Voronoi diagrams for spheres. Let T denote a set of n spheres in E^d. The *(minimum) distance* of $x \in E^d$ to $s \in T$ is defined as $d(x, s) = \min\{d(x, y) | y \in s\}$. If x is not contained in the ball bounded by s then $d(x, s) = d(x, z_s) - r_s$, where z_s and r_s denote the center and the radius of s. (We therefore could alternatively define $d(x, s)$ as the *additive weighted distance* of x from $p = z_s$ with $w(p) = r_s$.) The *Voronoi diagram of T* associates each $s \in T$ with its *region* $\{x \in E^d | d(x, s) < d(x, t), \forall t \in T - \{s\}\}$. This diagram (which is clearly different from PD (T)) can be related to a power diagram in E^{d+1} in the following way.

Let T be contained in h_0: $x_{d+1} = 0$ in E^{d+1}. Further, let $\kappa(s)$ denote the unique cone in E^{d+1} with vertical axis, apex below h_0, and $\kappa(s) \cap h_0 = s$, and let $\sigma(s)$ denote the unique sphere inscribed in $\kappa(s)$ with $\sigma(s) \cap \kappa(s) = s$. We have the following properties.

LEMMA 10. *Let $s, t \in T$ such that $r_s < r_t$.*

(1) hyp $(s, t) = \{x \in h_0 | d(x, t) = d(x, s)\}$ *is a hyperboloid with focus z_s and rotation axis through z_s and z_t.*

(2) hyp (s, t) *is the vertical projection onto h_0 of* $\kappa(s) \cap \kappa(t) = \kappa(s) \cap$ chor $(\sigma(s), \sigma(t))$.

We omit the proof since it follows from simple geometric calculations. From the above relationship it is obvious that the boundary of the region of s is the vertical projection of $\kappa(s) \cap$ cell $(\sigma(s))$ in PD (S), for $S = \{\sigma(s) | s \in T\}$. Hence Theorem 7 implies the following computational complexity.

THEOREM 13. *There exists an algorithm that constructs the Voronoi diagram of a set of n spheres in E^d in $O(T_{d+2}(n))$ time and requires $O(S_{d+2}(n))$ space.*

This is optimal only for $d=1$ where the problem degenerates to a sorting problem. In E^2, the diagram realizes (in contrast to the multiplicative weighting scheme) only $O(n)$ components; improvements (e.g., to $O(n \log^2 n)$ time [9]) exist. However, no methods had been known for $d \geqq 3$. Furthermore, our strategy leads to a worst-case optimal construction of the individual region R of a circle $s \subsetneq T \subsetneq E^2$. R may be bounded by $O(n)$ hyperbolic arcs, from which a certain ordering of T (compare Theorem 9) can be obtained.

THEOREM 14. *Let T be a set of n circles in E^2 and let $s \in T$. The region R of s in the Voronoi diagram of T can be computed in $O(n \log n)$ time and $O(n)$ space. This is worst-case optimal.*

If the radius of s is minimal among the circles in T, then R is the intersection of $n-1$ halfplanes bounded by hyperbolas with one common focus. This yields the following.

COROLLARY 4. *The common intersection of n hyperbolically bounded halfplanes with common foci can be determined in $O(n \log n)$ time and $O(n)$ space.*

7. Conclusion. In this paper, the model of power diagrams is introduced. The contributions fall into two parts: The analysis of power diagrams and their construction and application, and the exposition of the central role of power diagrams in geometric environments. As far as is known to the author, no methods for computing power cell complexes in E^d, for $d \geqq 3$ (in their general form), have been outlined before. The main results of this work rely on a fundamental property of the chordale of two spheres (Lemma 5). Similar correspondences between spheres, hyperplanes and points, but mainly expressed in other terms, have been observed, e.g., in [8], [28] and [33]. The various applications of PDs mentioned suggest that PDs serve as a tool for drawing connections between diverse areas in computational geometry.

The equivalence of PDs and convex polyhedra sheds new light into the interesting connection between Voronoi diagrams and particular types of polyhedra first perceived in [7], thus showing that power diagrams are, in some sense, the most natural generalizations of Voronoi diagrams. Consequently, PDs provide a new way of characterizing cell complexes, namely those that can be obtained by projecting the faces bounding a convex polyhedron. This is independently observed in [28], but is presented here in a more natural and simple way. In fact, many naturally arising cell complexes are PDs (see e.g. §§ 6.3 and 6.4), and hence are constructable via our algorithms. We have also shown that PDs can be related to convex hulls and arrangements of hyperplanes. It seems surprising that all these elementary structures are related by duality and affine equivalence. As a consequence, the computation of PDs and their order-k modifications can be efficiently carried out using known geometric algorithms. This work illustrates once more that geometric transforms are the key to new fruitful methods.

Let us summarize the algorithms outlined in this paper.

(1) Algorithm (MAXIMAL) POWER DIAGRAM constructs PD (S) as well as $\mathrm{PD}_m(S)$, for a set S of n spheres in E^d. Its complexity depends on that of convex hull algorithms and hence is optimal for $d=2$ and any odd d.

(2) An algorithm for simultaneously constructing 1-PD $(S), \cdots, (n-1)$-PD (S), for $n=|S|$, is outlined. It is based on the construction of arrangements of hyperplanes and is optimal for any d.

(3) Algorithm POWER CELL computes the power cell of a particular subset T of S in E^2. The method is worst-case optimal and can be used to decide whether cell (T) is empty.

Applications of PDs to the following problems are addressed:

(i) Finding illumination points for a finite set of spheres in E^d.

(ii) Recognizing k-sets of a finite set of points in E^d resp. determining a separating hyperplane.

(iii) Constructing the multiplicative weighted Voronoi diagram of a finite set of points in E^d (computing the union or intersection of certain types of cones or paraboloids in E^{d+1}), and constructing its order-k modifications.

(iv) Constructing the Voronoi diagram of a finite set of spheres in E^d, and determining the region of a particular circle of the given set in E^2 (determining the intersection of hyperbolically bounded halfplanes in E^2 with common foci).

In all four applications, power diagrams lead to an improved or to the first known solution, and serve to analyze the space requirement.

Finally, we state some open problems that arise in connection with PDs. First, fast algorithms for computing the convex hull of a finite set in E^d, for $d \geqq 5$, d odd, would be of importance. This paper shows once more the applicability of convex hull algorithms in higher dimensions to problems in E^2 and E^3. Further, the efficient construction of the individual order-k power diagram is still an open problem. It seems that the correspondence to k-levels (Theorem 6) can be exploited to obtain a satisfactory solution. In addition, improved lower and upper bounds on the number of faces of a k-PD are of interest. Establishing bounds on the size of a k-level would answer this question.

REFERENCES

[1] G. L. ALEXANDERSON AND J. E. WETZEL, *Simple partitions of space*, Math. Mag., 51 (1978), pp. 220–225.

[2] F. AURENHAMMER, *The one-dimensional weighted Voronoi diagram*, in Inf. Proc. Letters, to appear, also available: Rep. F 110, IIG, Tech. Univ. of Graz, Austria, 1983.

[3] F. AURENHAMMER AND H. EDELSBRUNNER, *An optimal algorithm for constructing the weighted Voronoi diagram in the plane*, Pattern Recognition, 17(2) (1984), pp. 251–257.

[4] W. BLASCHKE, *Vorlesungen ueber Differentialgeometric* III, Springer-Verlag, Berlin, 1929.

[5] B. N. BOOTS, *Weighting Thiessen polygons*, Econom. Geog. (1979), pp. 248–259.

[6] A. BRÖNSTED, *An Introduction to Convex Polytopes*, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[7] K. Q. BROWN, *Voronoi diagrams from convex hulls*, Inform. Process Lett., 9 (1979), pp. 223–228.

[8] ———, *Geometric transforms for fast geometric algorithms*, Ph.D. thesis, Rep. CMU-ČS-80-101, Dep. of Computer Science, Carnegie-Mellon Univ., Pittsburgh, 1980.

[9] R. L. DRYSDALE, III AND D. T. LEE, *Generalization of Voronoi diagrams in the plane*, this Journal, 10 (1981), pp. 73–87.

[10] H. EDELSBRUNNER, J. O'ROURKE AND R. SEIDEL, *Constructing arrangements of lines and hyperplanes with applications*, Proc. 24th. Ann. IEEE Symp. FOCS (1983), pp. 83–91.

[11] H. EDELSBRUNNER AND E. WELZL, *On the number of line separations of a finite set in the plane*, J. Combin. Theory Ser. A (1985), pp. 15–29.

[12] ———, *Halfplanar range estimation*, Rep. F 98, IIG, Tech. Univ. of Graz, Austria, 1982.

[13] P. ERDÖS, L. LOVASZ, A. SIMMONS AND E. G. STRAUS, *Dissection graphs for planar point sets*, in A Survey of Combinatorial Theory, J. N. Srivastava et al., eds., Amsterdam, 1973.

[14] L. FEJES TOTH, *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer-Verlag, Berlin, Heidelberg, 1972.

[15] ———, *Illumination of convex discs*, Acta Math. Acad. Sci. Hungar., 29, 3–4 (1977), pp. 355–360.

[16] B. GRÜNBAUM, *Convex Polytopes*, Wiley Interscience, London, 1967.

[17] P. HANJOUL, H. BEGUIN AND J. C. THILL, *Shapes of market areas under Euclidean distance*, Rep., Univ. de Louvaine-La-Neuve, Belgium, 1984.

[18] F. HARARY, *Graph Theory*, Addison-Wesley, Reading, MA, 1972.

[19] D. G. KIRKPATRICK, *Efficient computation of continuous skeletons*, Proc. 20th. Ann. IEEE Symp. FOCS, (1979), pp. 18–27.

[20] E. KOCH, *Wirkungsbereichspolyeder und Wirkungsbereichsteilungen zu kubischen Gitterkomplexen mit weniger als drei Freiheitsgraden*, Zeitschr. für Kristallographie, 138 (1973), pp. 196–215.

[21] D. T. LEE, *Two-dimensional Voronoi diagrams in the L_p metric*, J. Assoc. Comput. Mach., 27 (1980), pp. 604–618.

[22] ———, *On k-nearest neighbor Voronoi diagrams in the plane*, IEEE Trans. Comp., C-31, 6 (1982), pp. 478–487.

[23] D. T. LEE AND C. K. WONG, *Voronoi diagrams in L_1 (L_∞) metrics with two dimensional storage applications*, this Journal, 9 (1980), pp. 200–211.

[24] G. C. LEKKERKERKER, *Geometry of Numbers*, Wolters-Noordhoff, Groningen, 1969.

[25] J. LINHART, *Die Beleuchtung von Kugeln*, Geom. Dedicata, 10 (1981), pp. 145–154.

[26] ———, *Dirichletsche Zellenkomplexe mit maximaler Eckenanzahl*, Geom. Dedicata, 11 (1981), pp. 363–367.

[27] P. MCMULLEN, *The maximal number of faces of a convex polytope*, Mathematica, 17 (1970), pp. 179–184.

[28] I. PASCHINGER, *Konvexe Polytope und Dirichletsche Zellenkomplexe*, Dissertation, Inst. of Math., Univ. Salzburg, Austria, 1982.

[29] F. P. PREPARATA AND S. J. HONG, *Convex hulls of finite sets of points in two and three dimensions*, Comm. ACM, 20 (1977), pp. 87-93.

[30] F. P. PREPARATA AND D. E. MULLER, *Finding the intersection of n halfspaces in time $O(n \log n)$*, Theoret. Comput. Sci., 8 (1979), pp. 45-55.

[31] C. A. ROGERS, *Packing and Covering*, Univ. Press, Cambridge, 1964.

[32] R. SEIDEL, *A convex hull algorithm optimal for point sets in even dimensions*, M. S. thesis, Rep. 81-14, Dep. Computer Science, Univ. of British Columbia, 1981.

[33] ———, *On the size of closest point Voronoi diagrams*, Rep. F 94, IIG, Tech. Univ. of Graz, Austria, 1982.

[34] F. W. SMITH, *The structure of aggregates and the molecular viscosity of a Bernal liquid*, Canad. J. Phys., 42 (1964).

[35] M. I. SHAMOS AND D. HOEY, *Closest point problems*, Proc. 16th Ann. IEEE Symp. FOCS, (1975), pp. 151-162.

[36] G. F. VORONOI, *Nouvelles applications des paramètres continues à la théorie des formes quadratiques, Deuxième Mémoire: Recherches sur les paralléloedres primitifs*, J. Reine Angew. Math., 134 (1908), pp. 198-287, 136 (1909), pp. 67-181.

[37] T. ZASLAVSKY, *Facing up to arrangements: Face-count formulas of partitions of space by hyperplanes*, Memoirs Am. Math. Soc., 154, Providence, 1975.