# Motion Planning

O'Rourke, Chapter 8

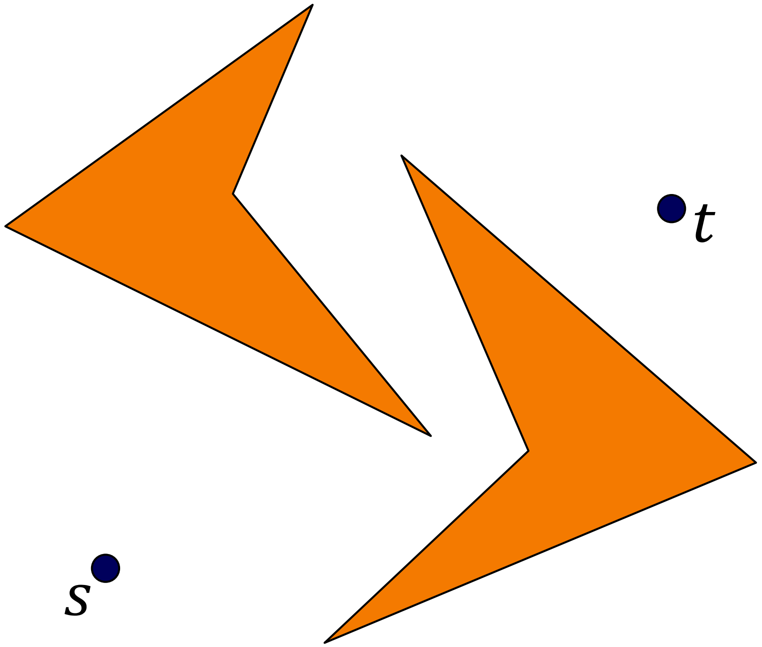
# Outline

- Translating a polygon
- Moving a ladder

# Shortest Path (Point-to-Point)

Goal:

Given disjoint polygons in the plane, and given positions $s$ and $t$, find the **shortest** path from $s$ to $t$ that avoids the polygons.



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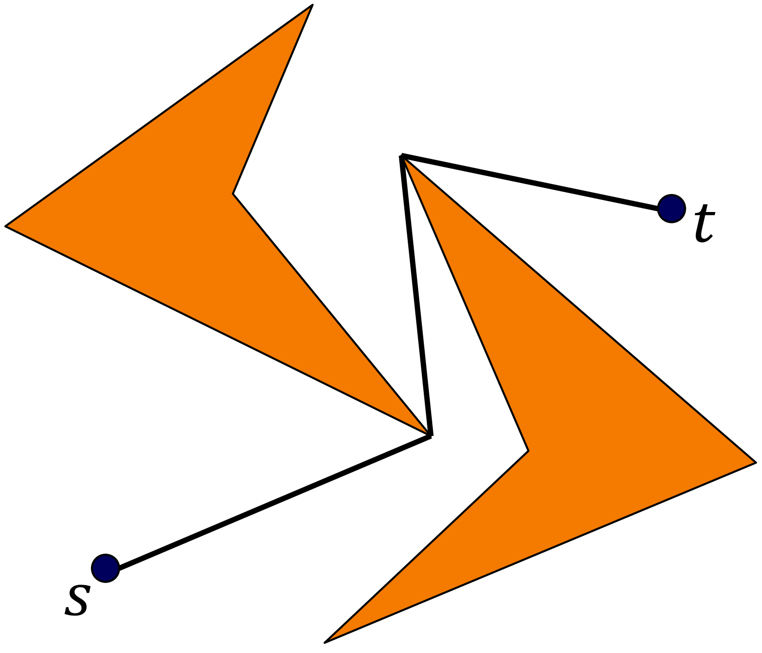
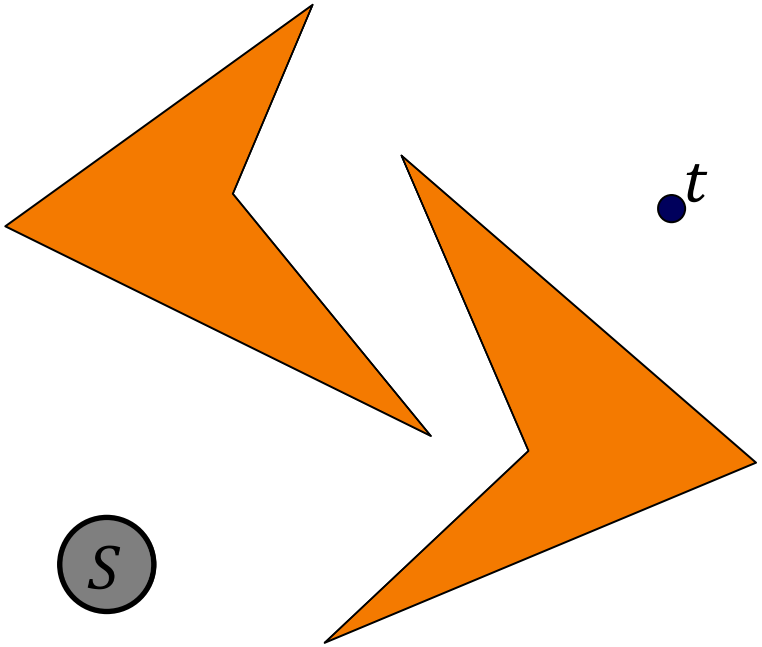


# Any Path (Shape-to-Point)

Goal:

Given disjoint polygons in the plane, and given a shape $S$ and a position $t$, determine if there is **any** path taking $S$ to $t$ avoiding the polygons.



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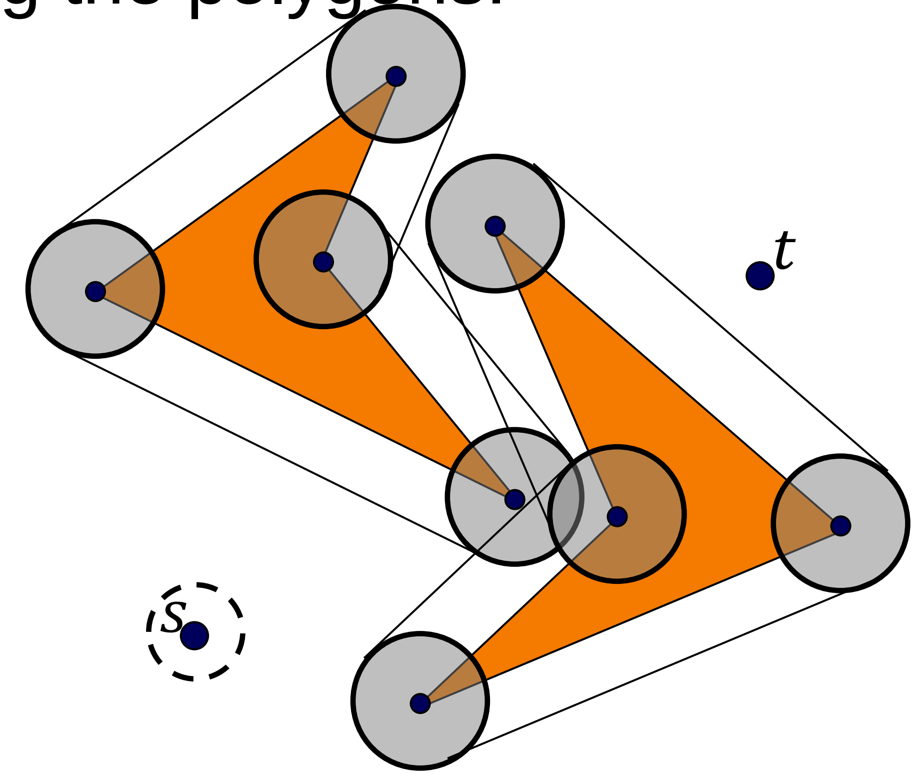
Dilate the polygons by $S$ to transform this into a point-to-point problem.



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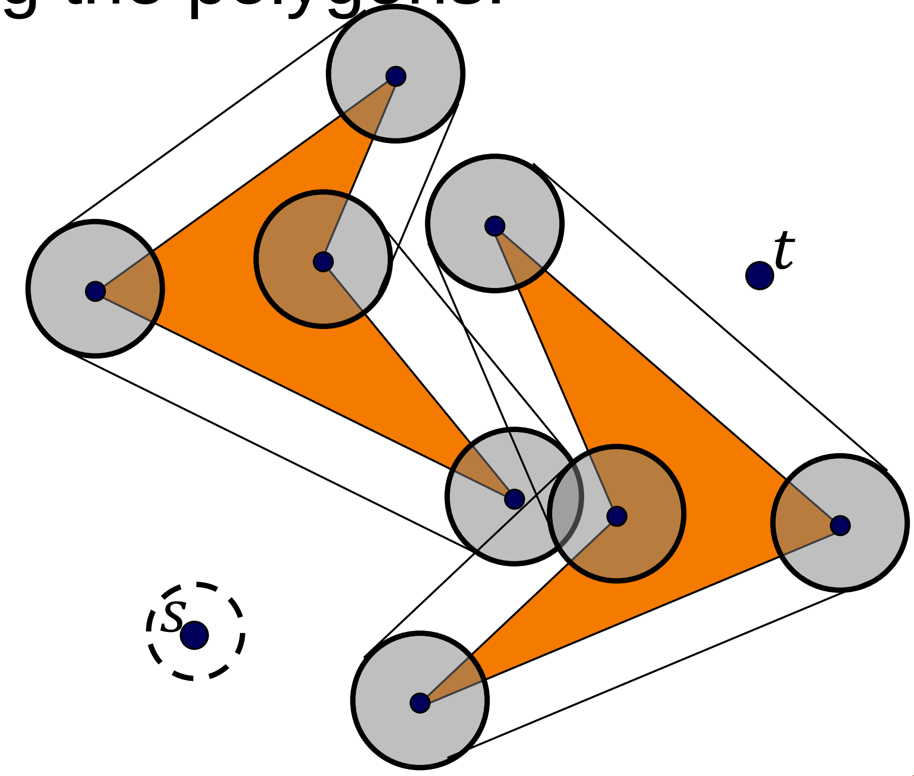



If $s$ can reach $t$ while avoiding the dilated polygons, $S$ can reach $t$.

# Any Path (Shape-to-Point)

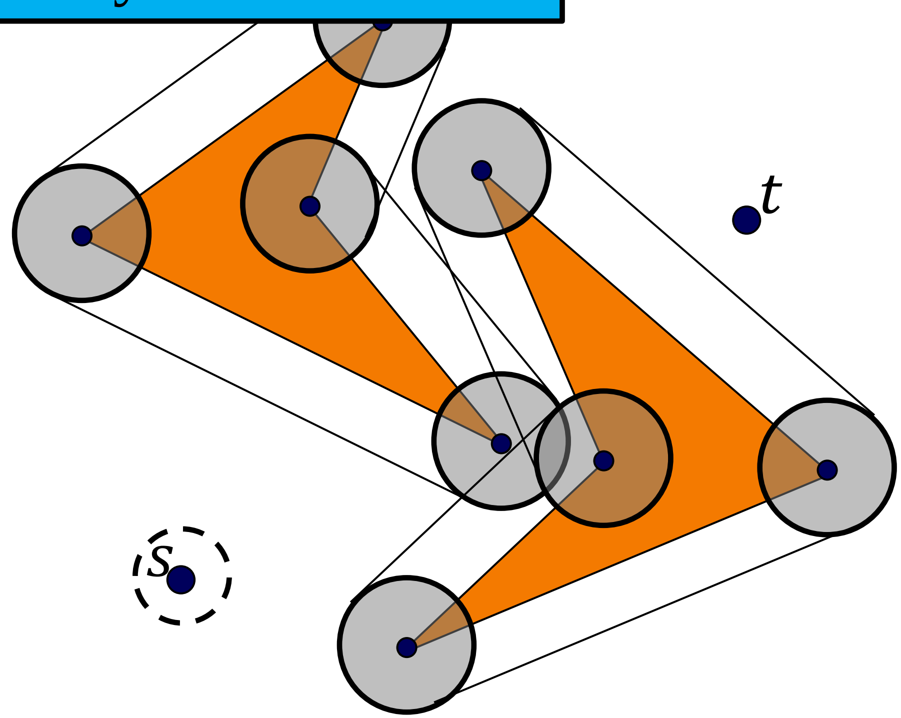
<u>Goal</u>:

Given di… …iven a shape $S$ … e is **any** path tak…

<u>Note that in this case</u>:
- $S$ is symmetric.
- $s$ is the center of $S$.
- The traced out boundary self-intersects.

<u>Approach</u>:

Dilate the polygons by $S$ to transform this into a point-to-point problem.

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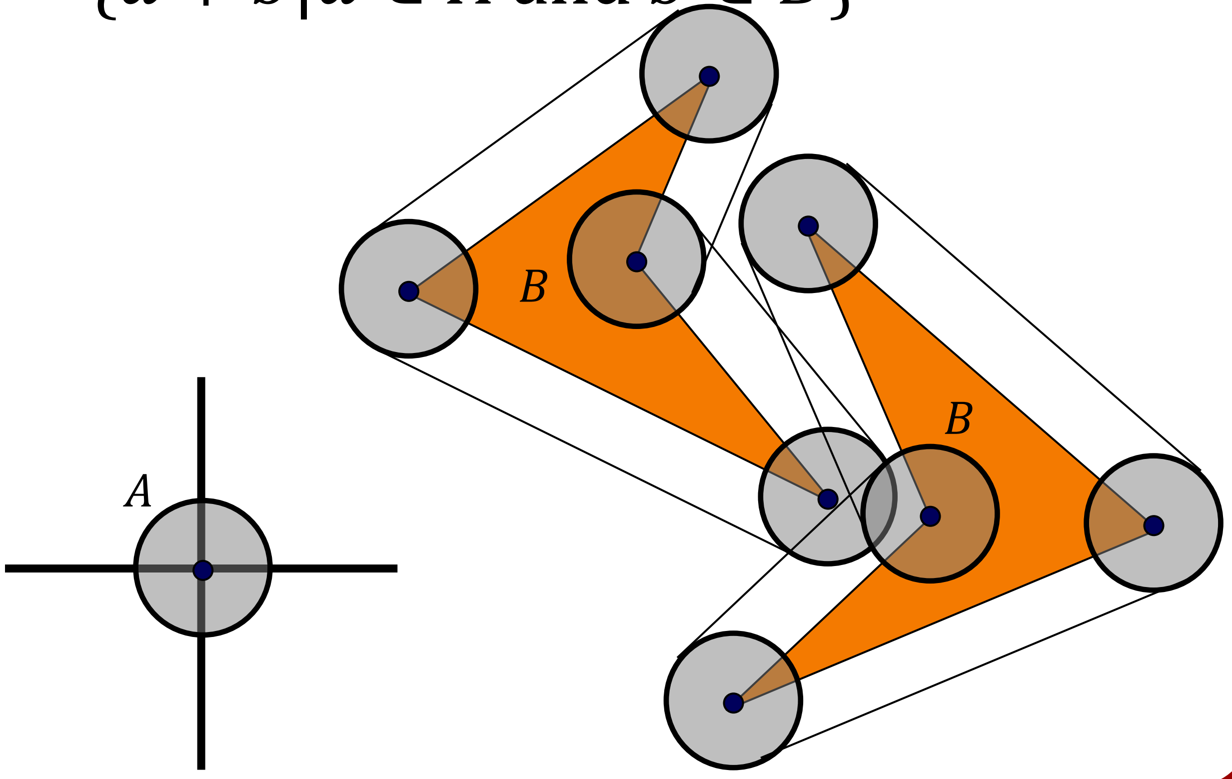
# Minkowski Sums

<u>Definition</u>:

Given two point-sets in the plane, $A$ and $B$, the *Minkowski Sum* is the set of points:

$$A \oplus B = \{a + b | a \in A \text{ and } b \in B\}$$



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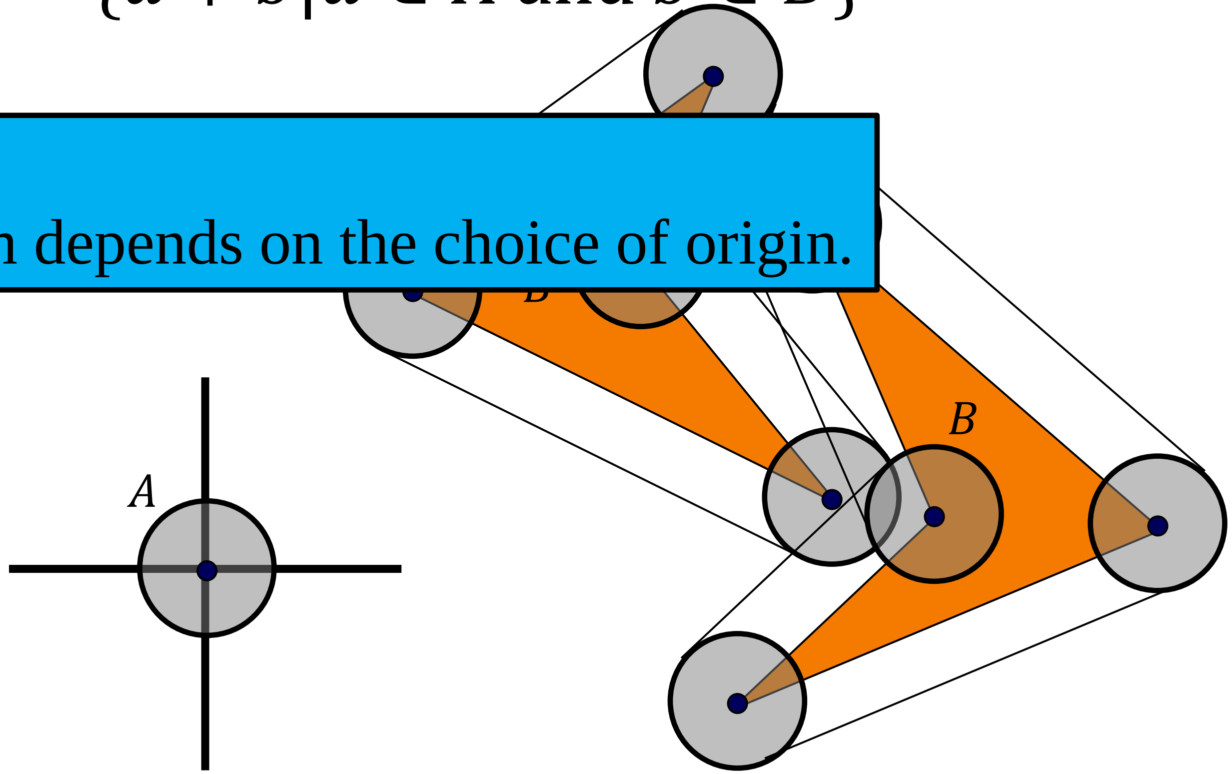


Note:
The sum depends on the choice of origin.

# Minkowski Sums

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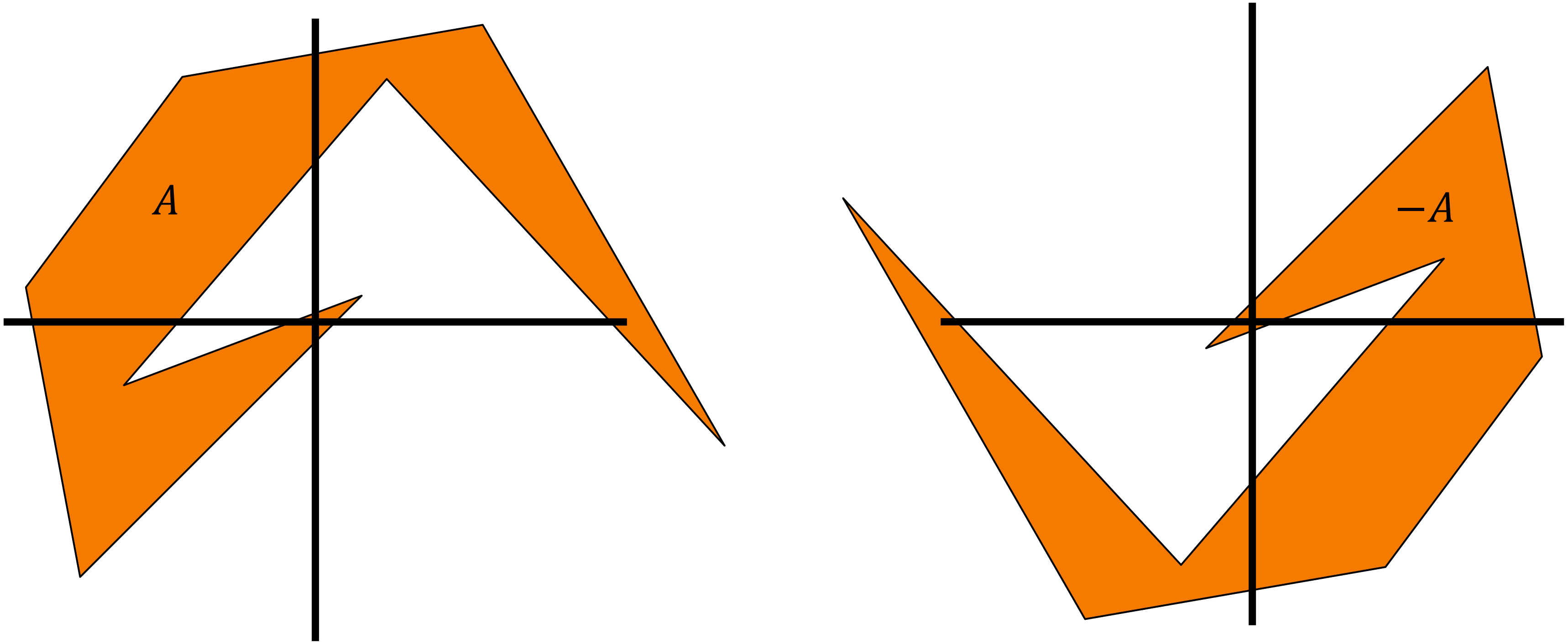
Given a point-set $A$ in the plane, its *negation* is the set of points:

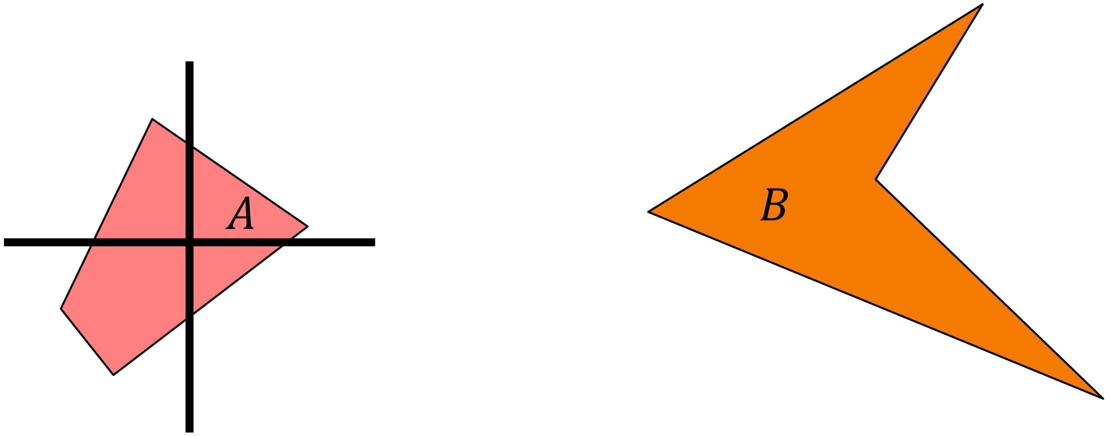
$$-A = \{-a | a \in A\}$$



# Motion Planning

<u>Claim</u>:

Given shapes $A$ and $B$, the translation of $A$ by $\tau$ intersects $B$ if and only if $\tau \in B \oplus (-A)$.



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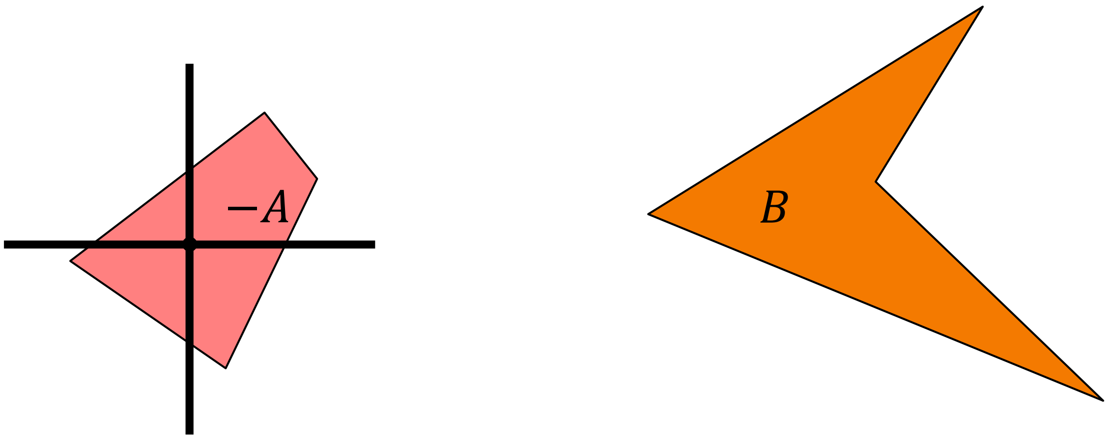

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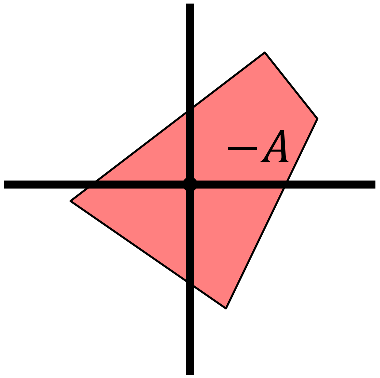


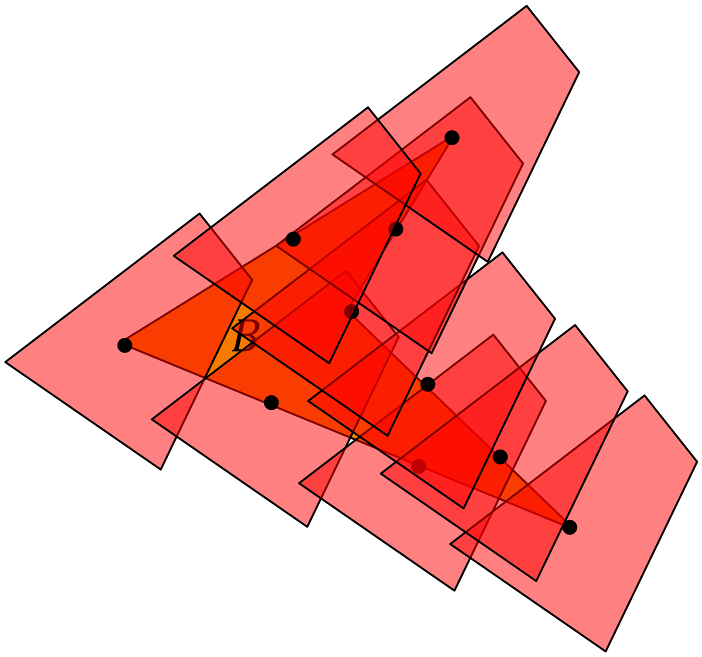

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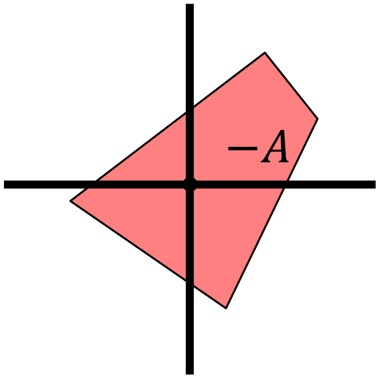


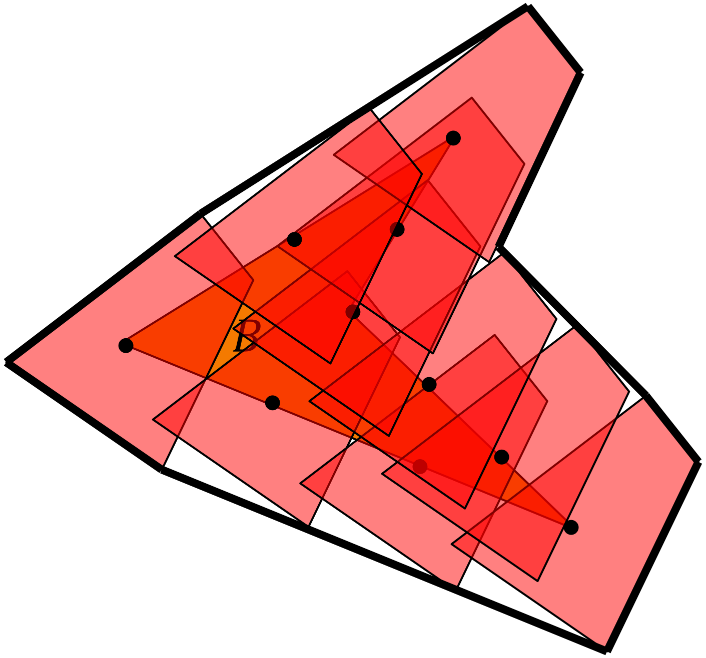

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Claim:

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Proof:

The translation of $A$ by $\tau$ intersects $B$.

$\Leftrightarrow \exists a \in A, b \in B$ such that $a + \tau = b$.

$\Leftrightarrow \exists a \in A, b \in B$ such that $\tau = b - \mathrm{a}$.

$\Leftrightarrow \tau \in B \oplus (-A)$.

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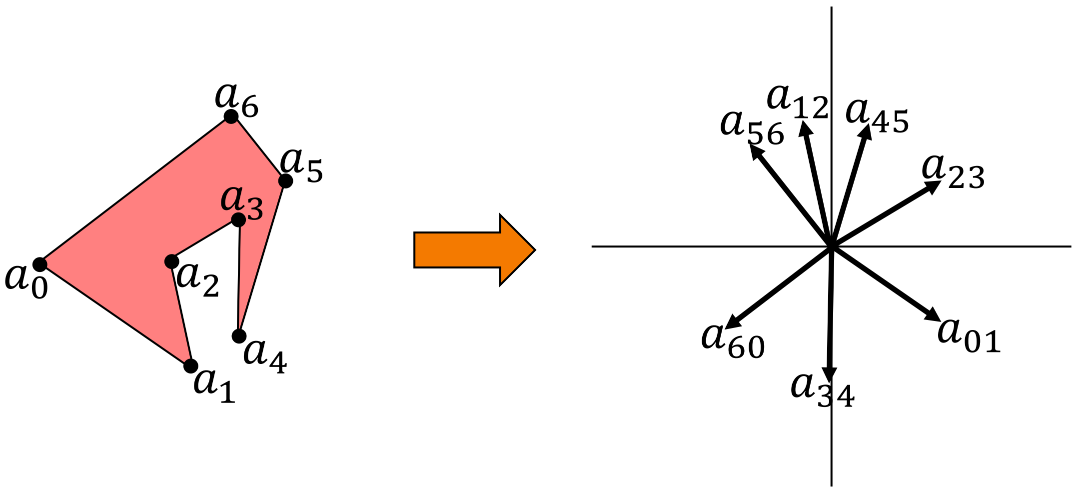
Definition:

Given a polygon $A=\{a_0,\ldots,a_{n-1}\}$, the *star diagram* is the mapping from the edges of $P$ to points on the unit circle (based on the direction of the edges).



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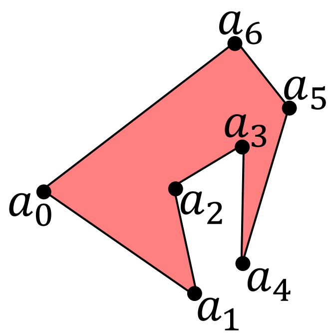
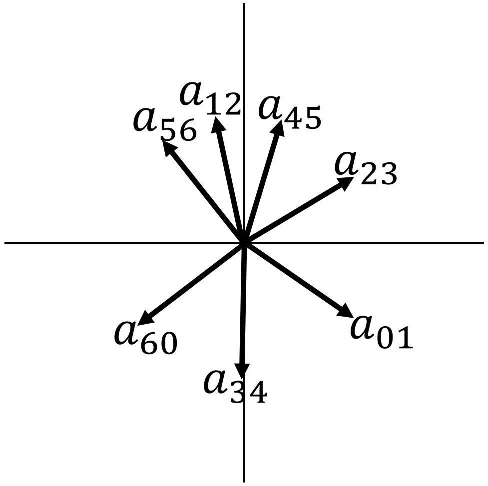
Definition:

Given a polygon $A = \{a_0, \ldots, a_{n-1}\}$, the *star diagram* is the mapping from the edges of $P$ to points on the unit circle (based on the direction of the edges).

Can think of this as a discrete Gauss map.



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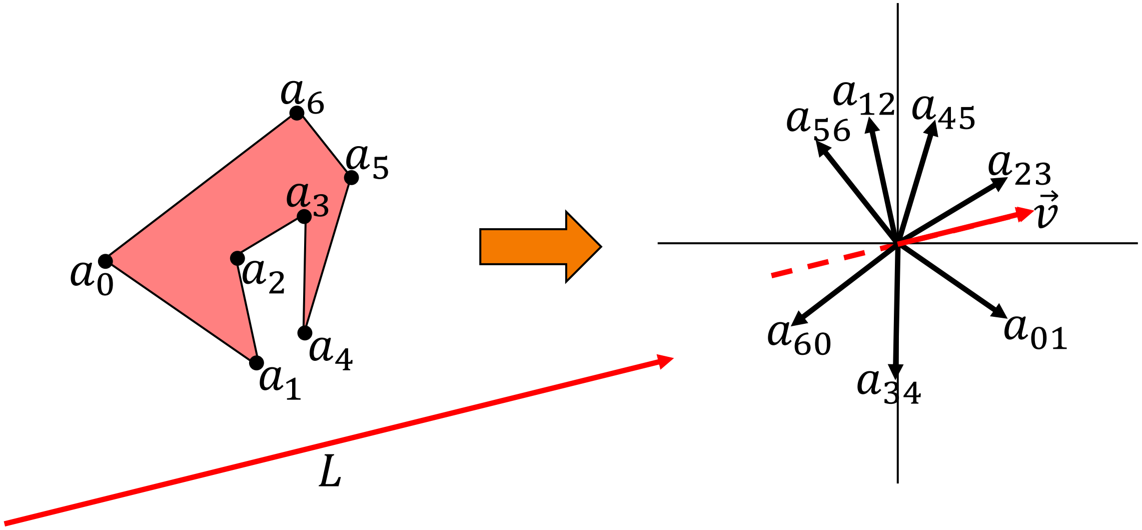
Note:

Given a (directed) line $L$ with direction $\vec{v}$, the vertex $a_i \in A$ is locally extreme w.r.t. $\vec{v}$ if the directions $a_{i-1,i}$ and $a_{i,i+1}$ are on opposite sides of $\vec{v}$.



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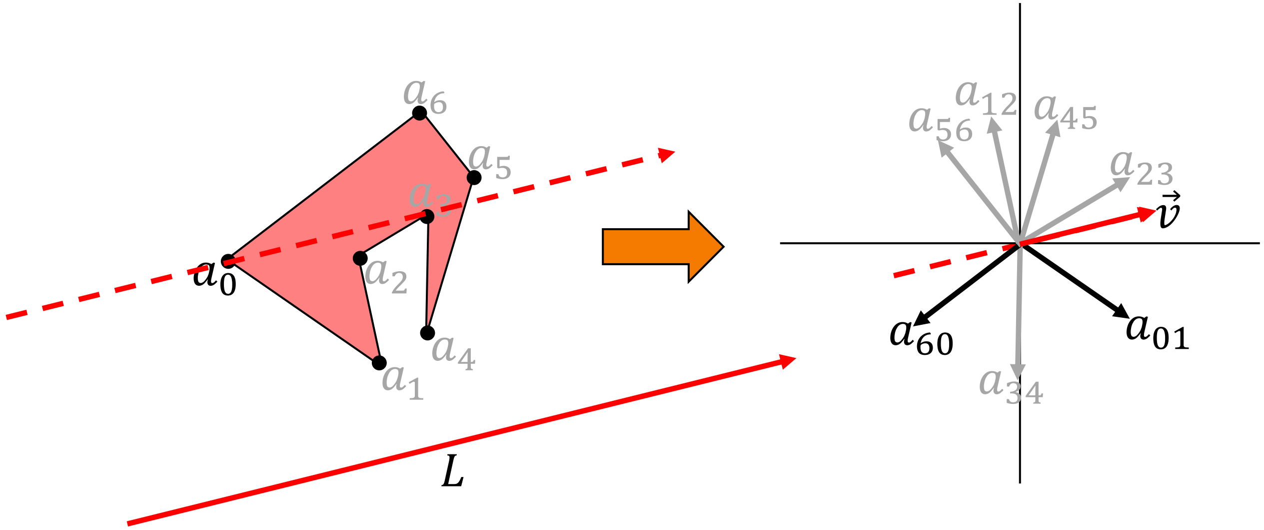

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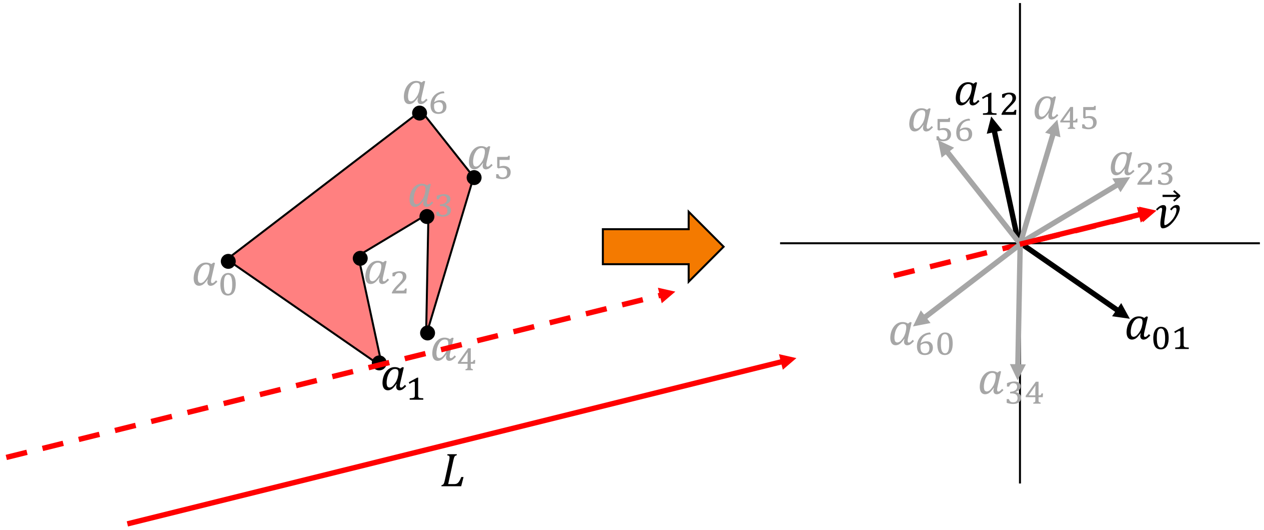

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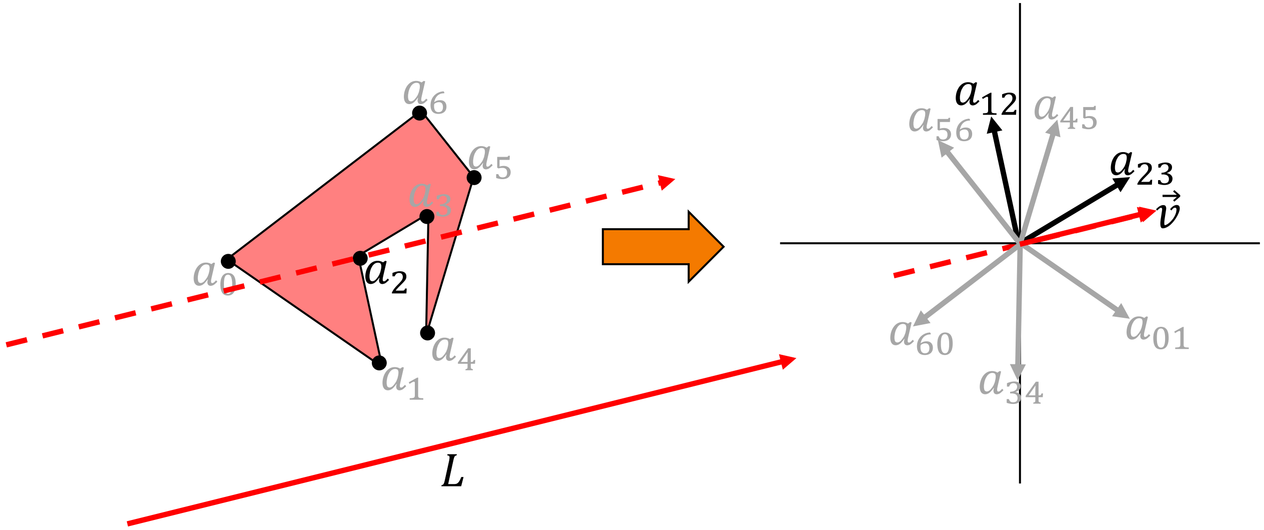

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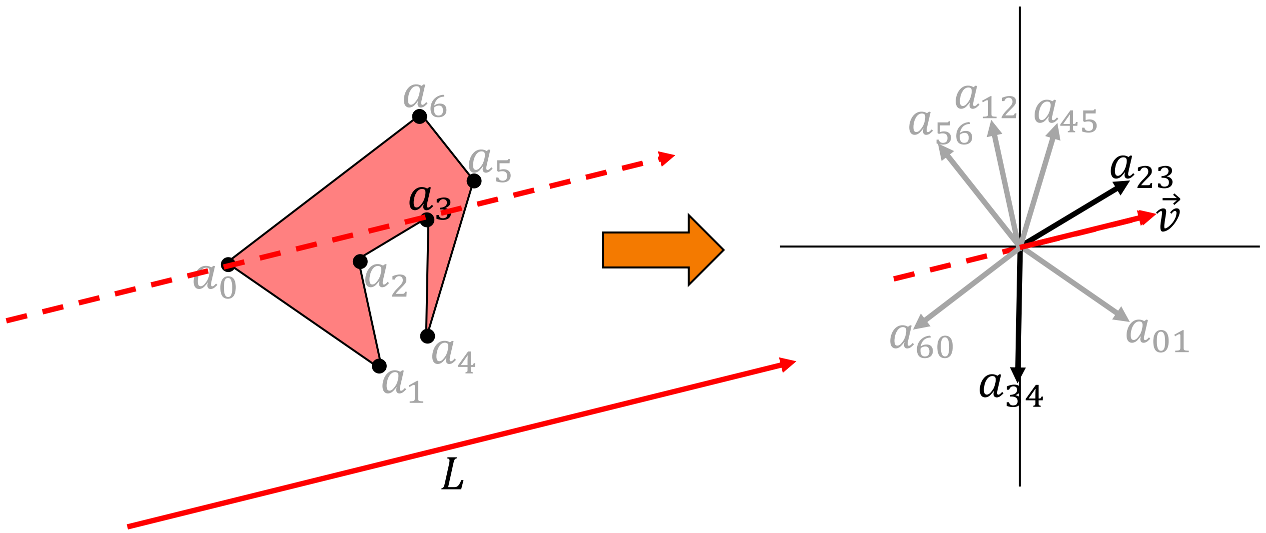

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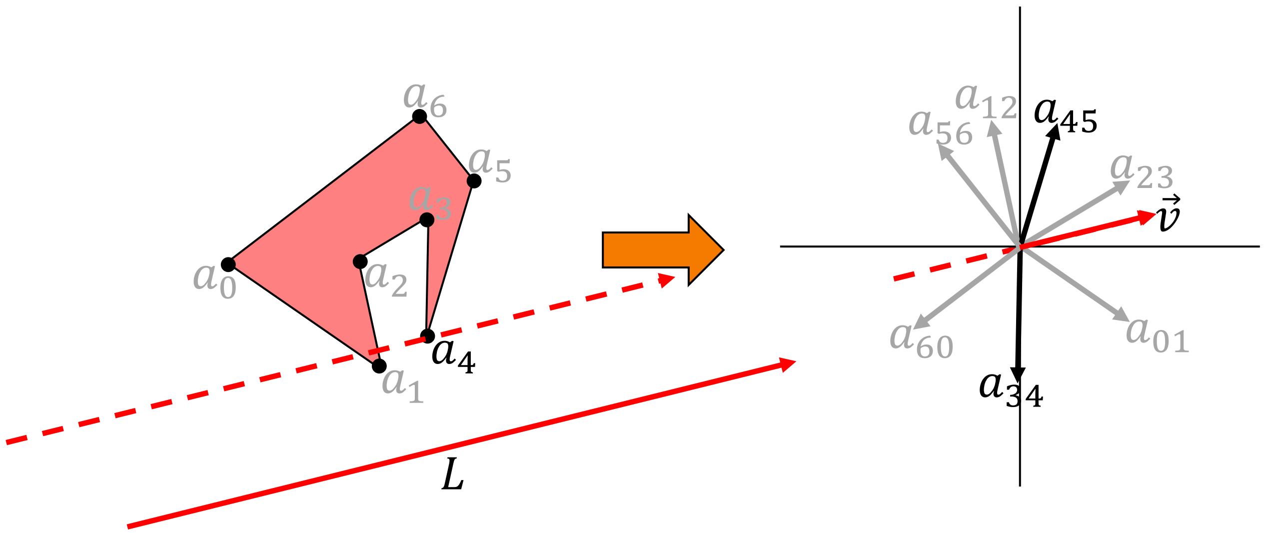

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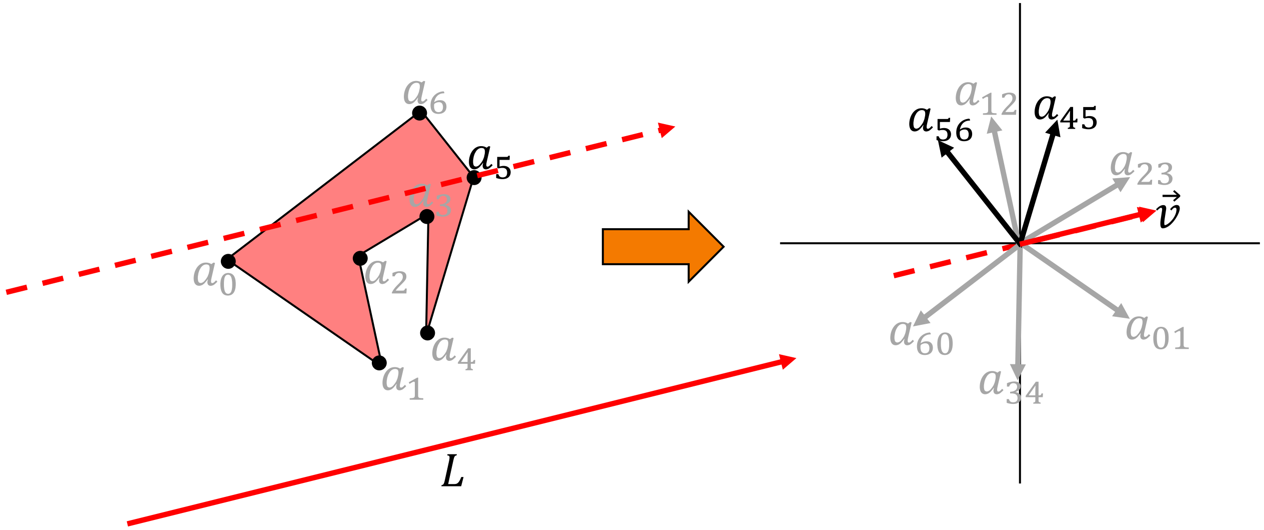

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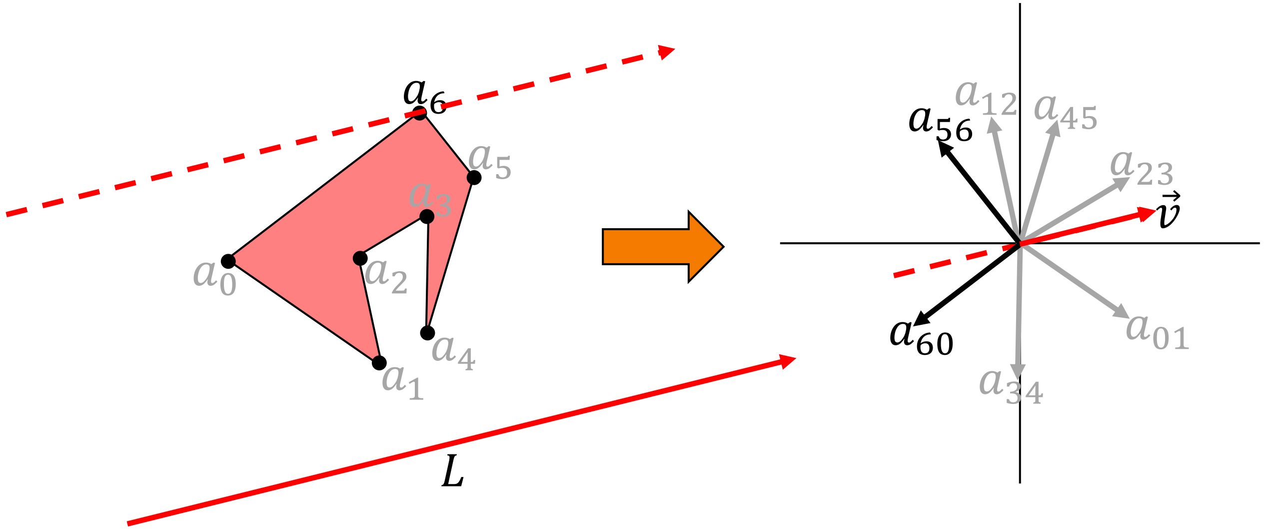

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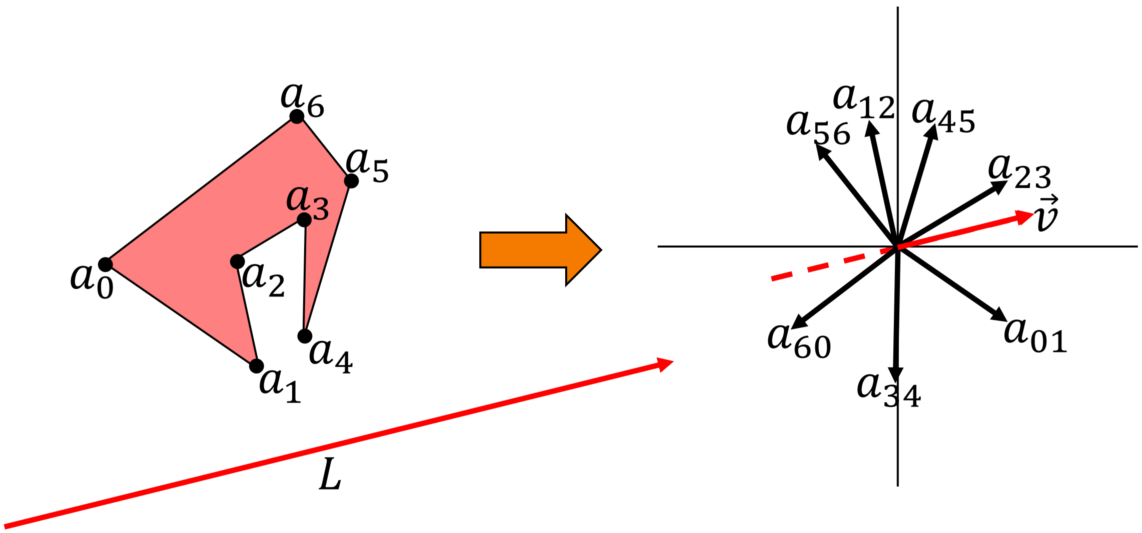
Given a (directed) line $L$ with direction $\vec{v}$, the vertex $a_i \in A$ is locally right-most w.r.t. $\vec{v}$ if it is locally extreme and the incoming edge $\overrightarrow{a_{i-1}a_i}$ points to the right of $\vec{v}$.



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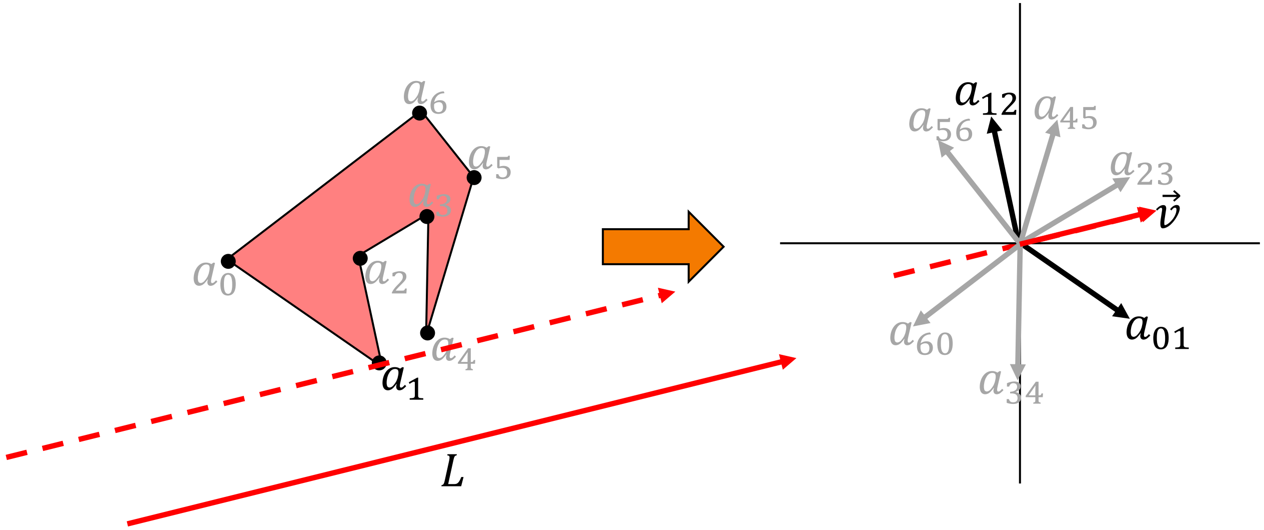

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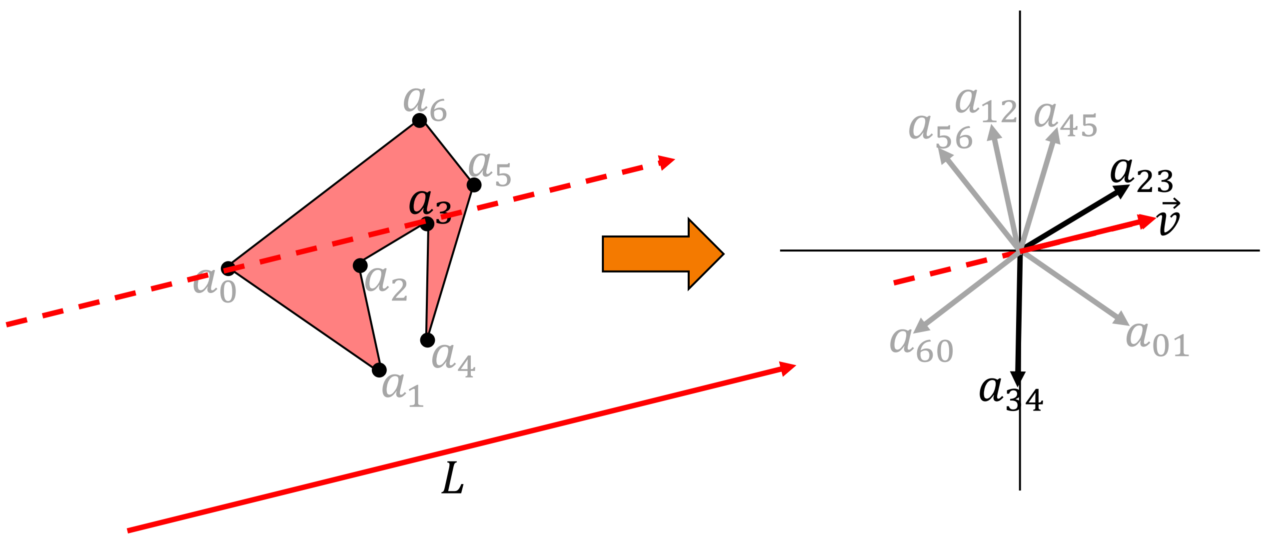

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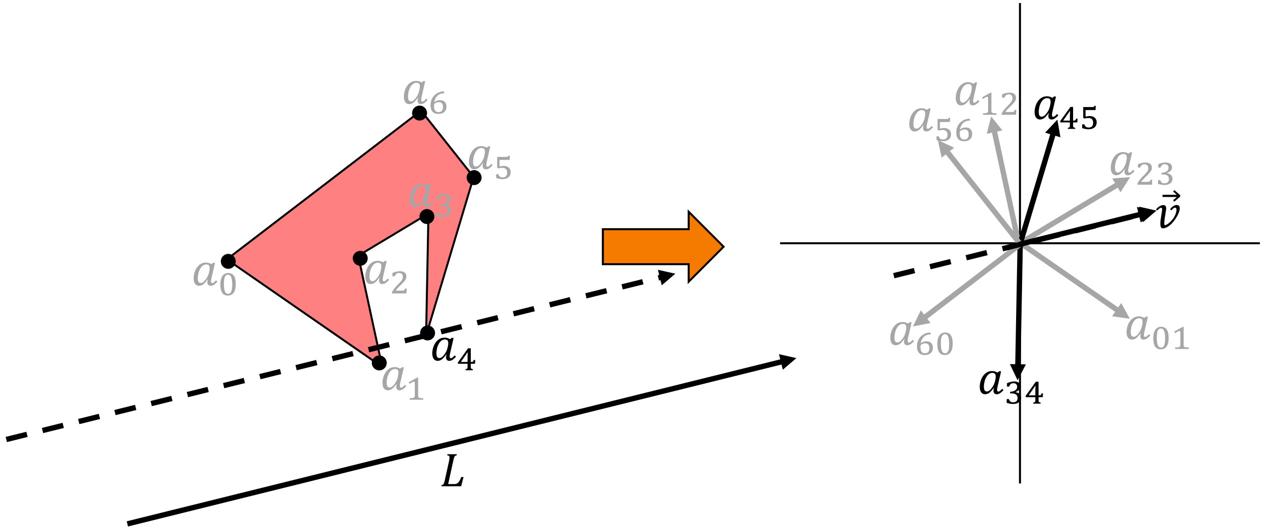

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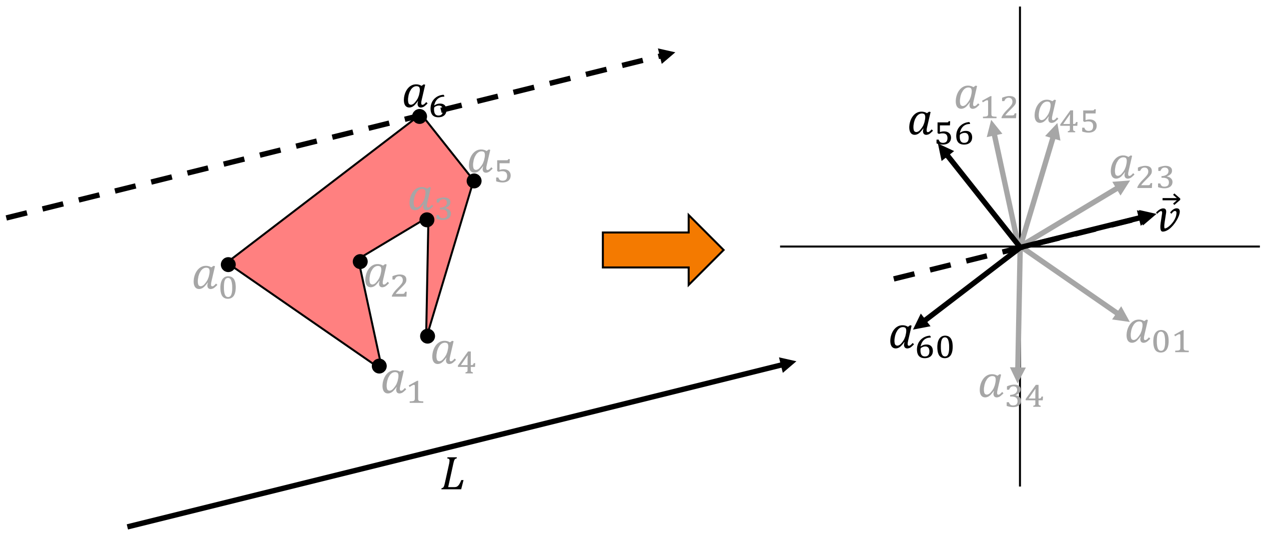

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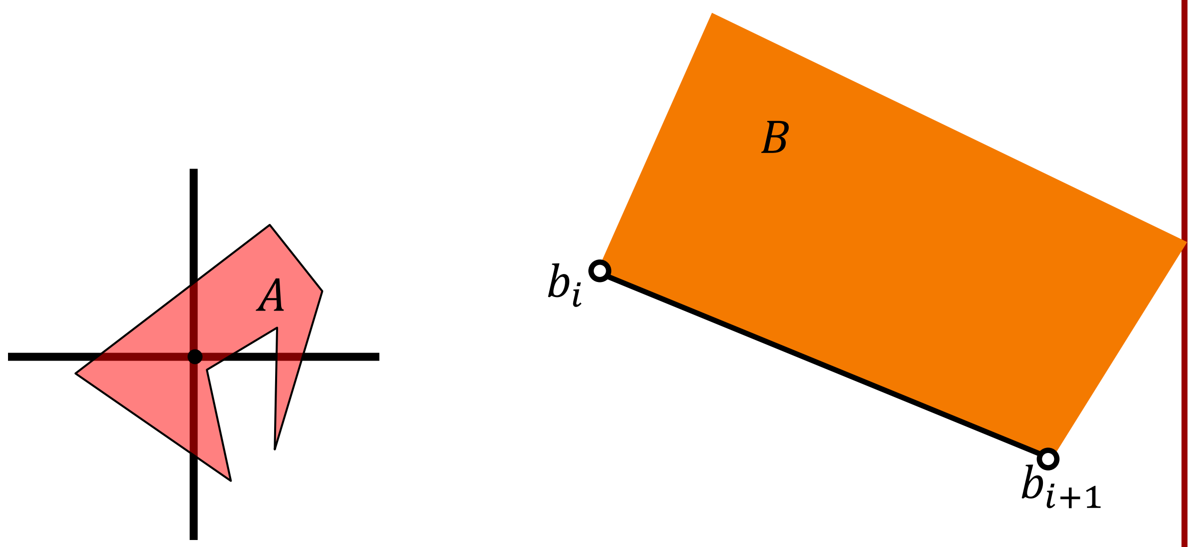
Key Idea 1:

When sweeping $-A$ along an edge $\overrightarrow{b_i b_{i+1}}$, the boundary of $B \oplus (-A)$ is swept out by the vertices of $-A$ that are locally right-most with respect to the segment $\overrightarrow{b_i b_{i+1}}$.



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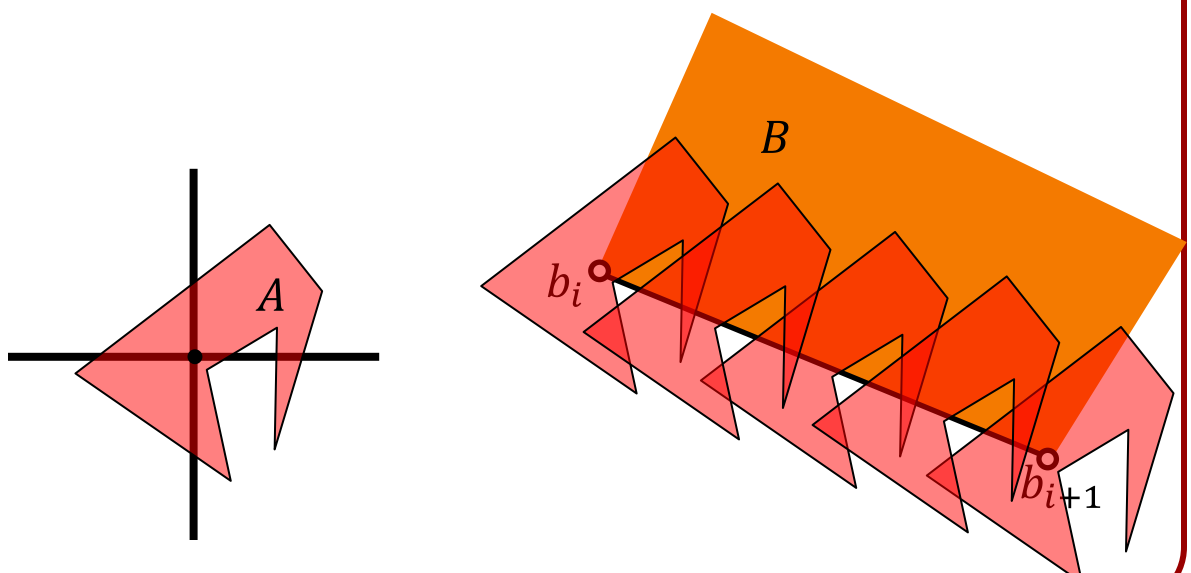

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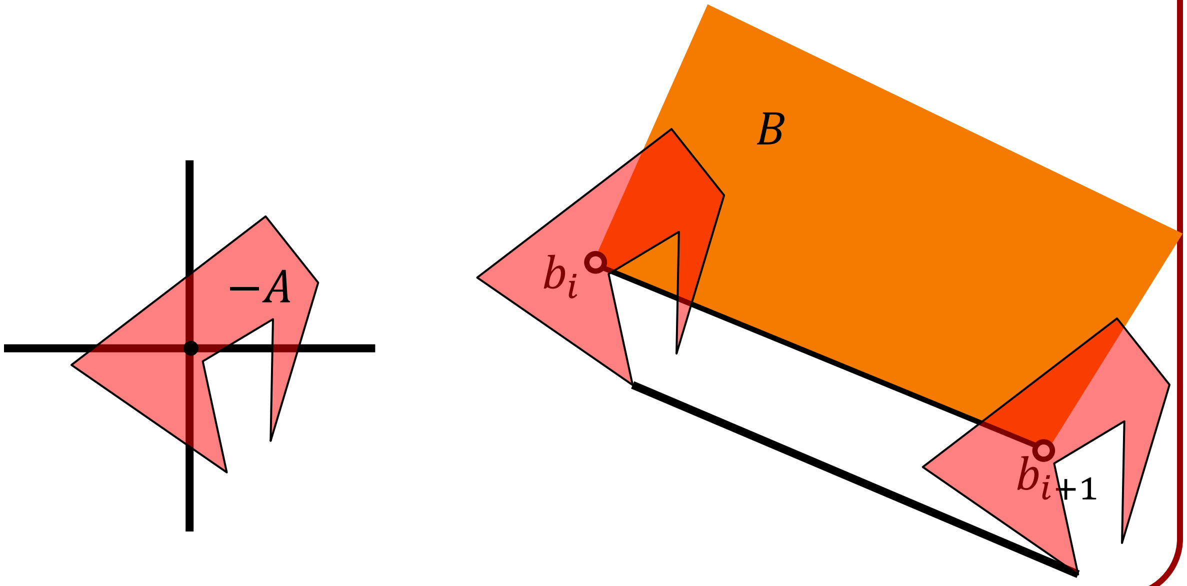

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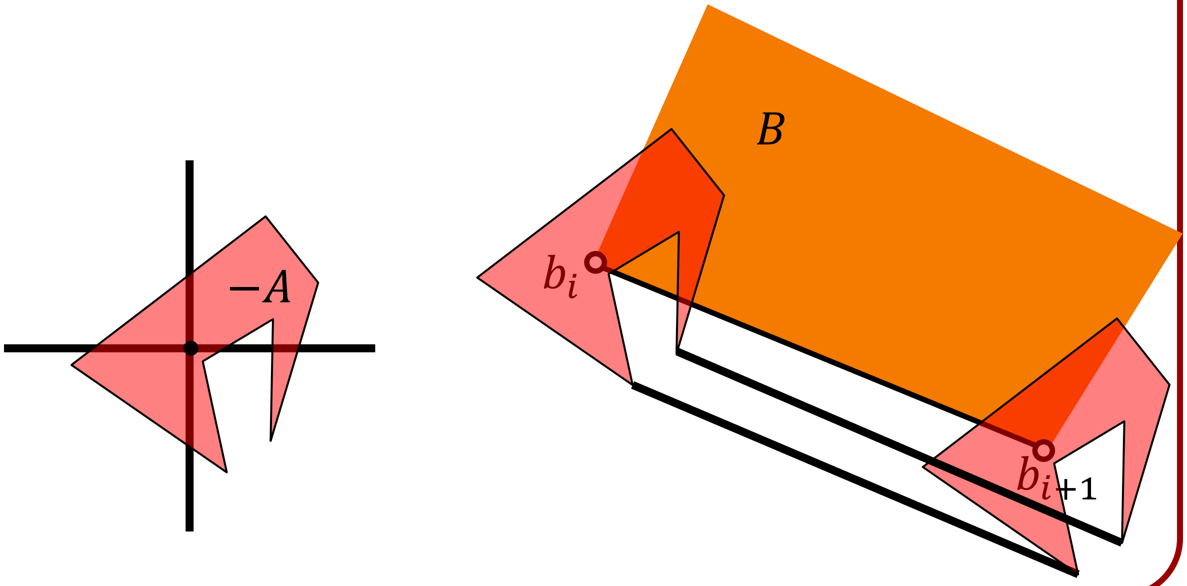

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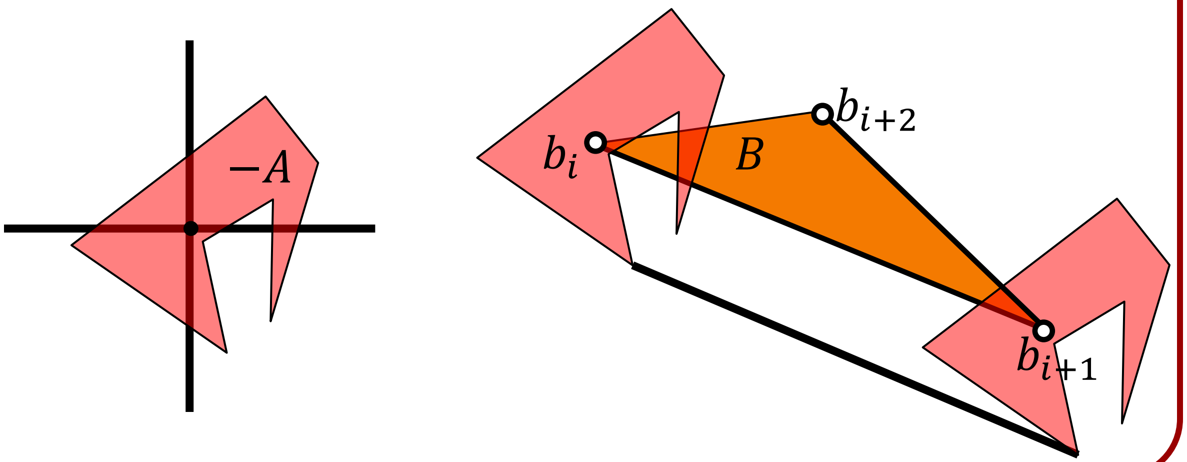
<u>Key Idea 2</u>:

As we turn the corner from $\overrightarrow{b_i b_{i+1}}$ to $\overrightarrow{b_{i+1} b_{i+2}}$, the boundary $B \oplus (-A)$ is swept by the part of the boundary of $-A$ between the two locally right-most vertices.



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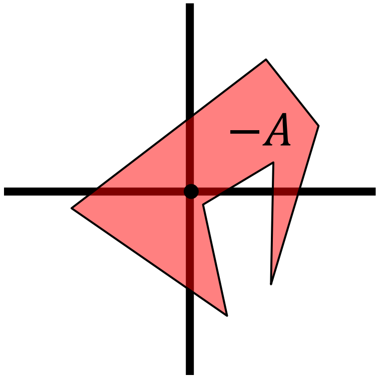


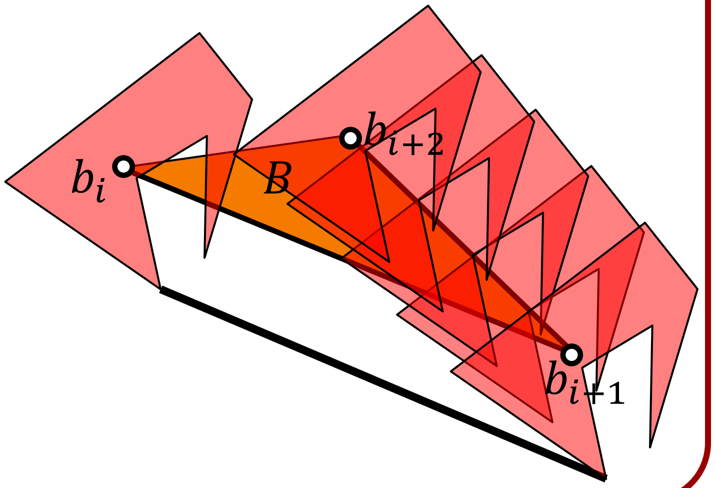

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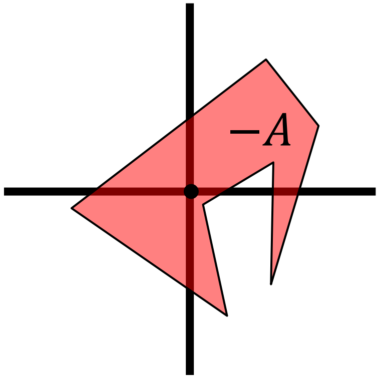


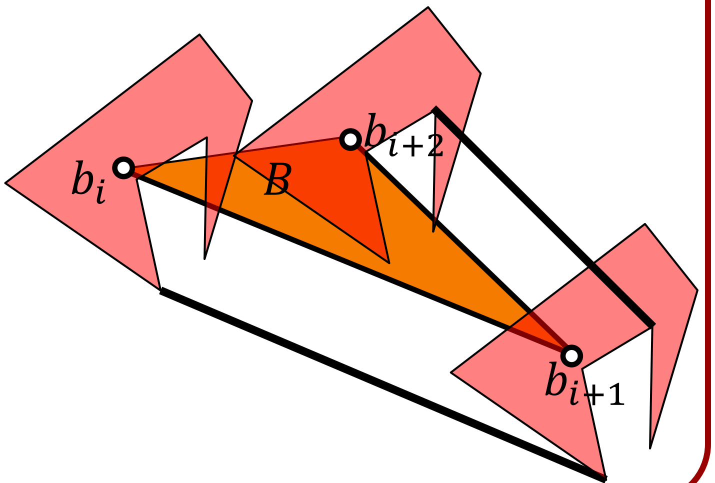

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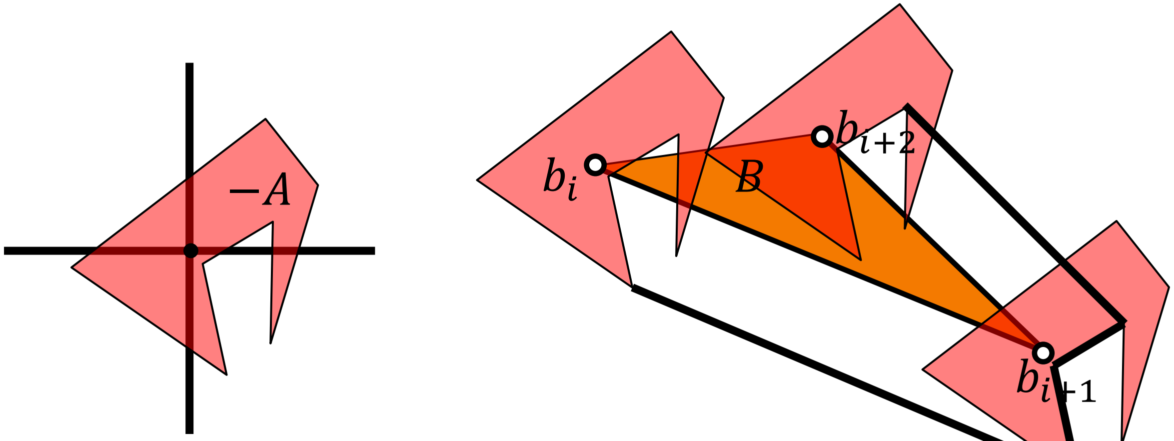

# Motion Planning

MinkowskiTrace( $A$ , $B$ )

- Place $-A$ at vertex $b_0 \in B$.
- for each $a \in$ LocallyRMost( $\overrightarrow{b_0 b_1}$ )
  - for $i \in [0, |B|)$:
    - Trace $a$ along edge( $\overrightarrow{b_i b_{i+1}}$ )
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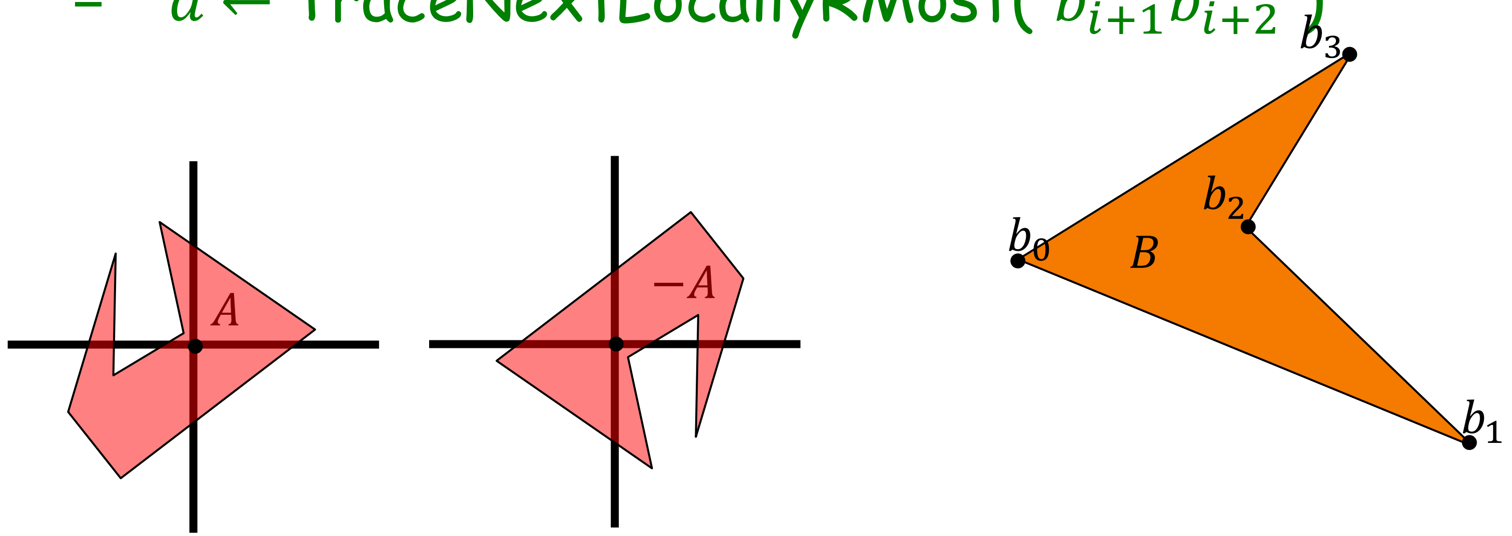

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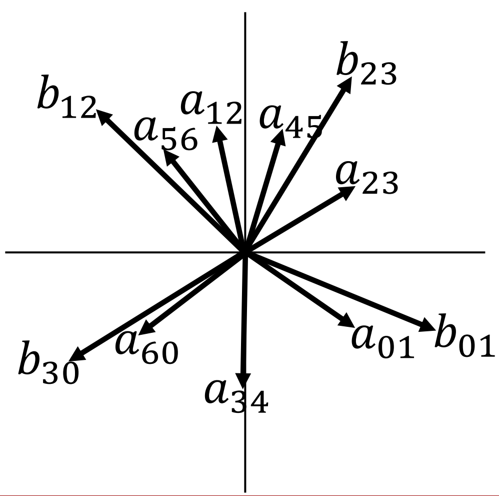


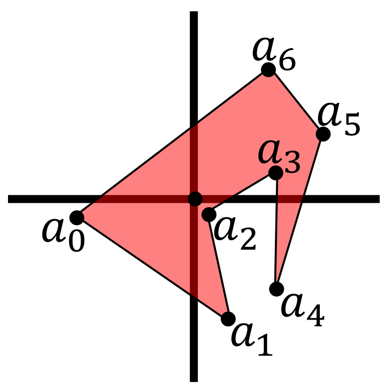


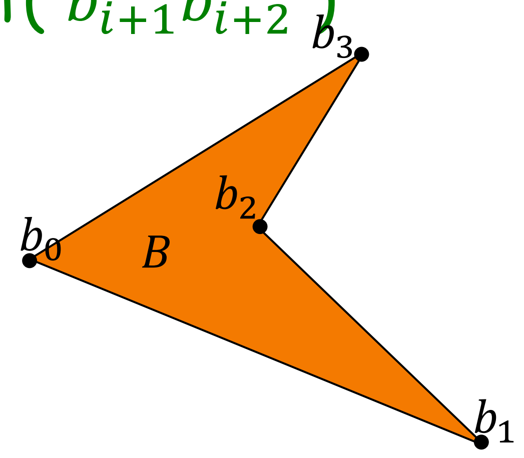

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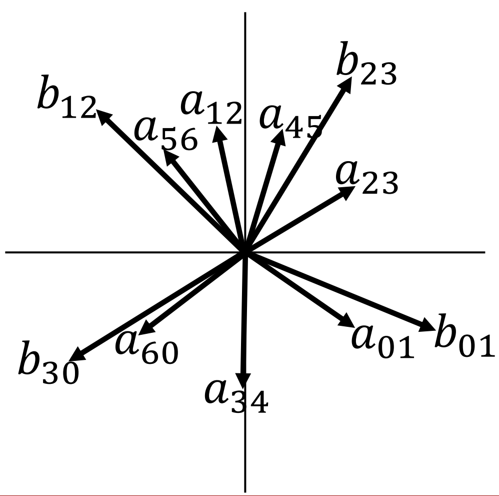


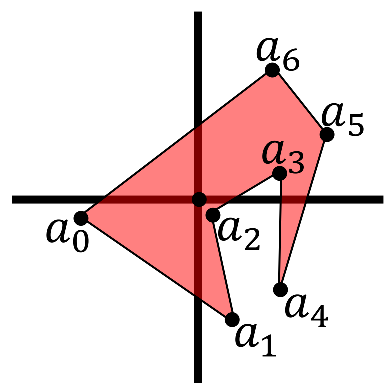


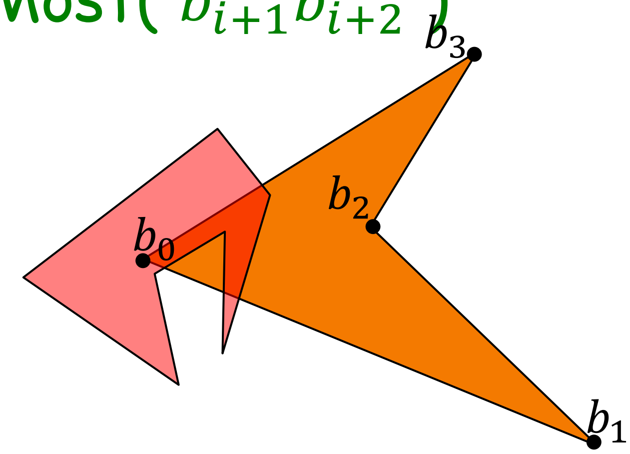

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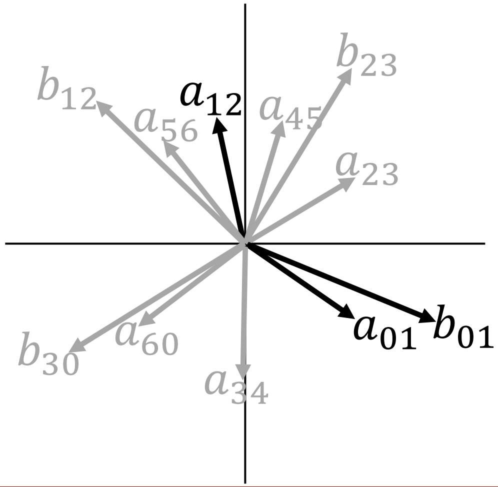


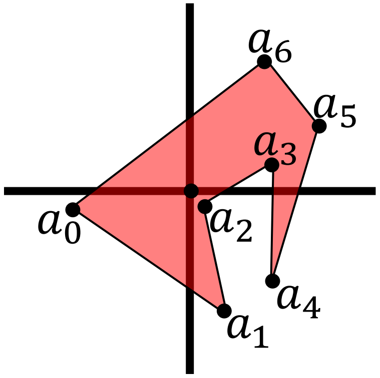


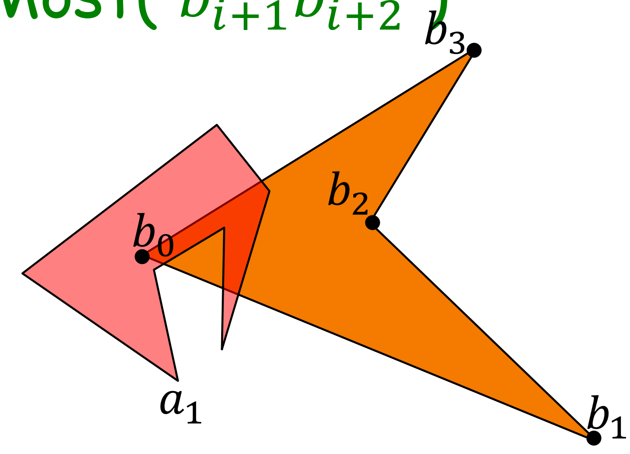

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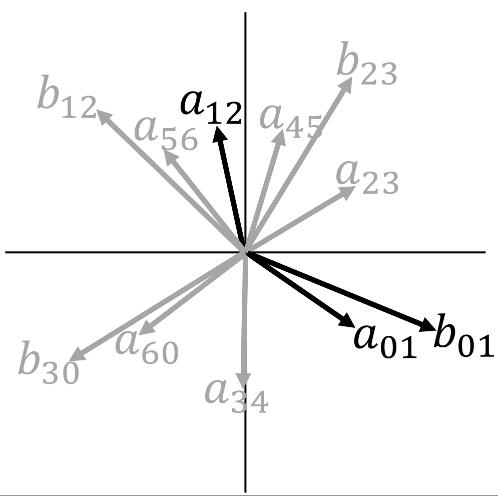


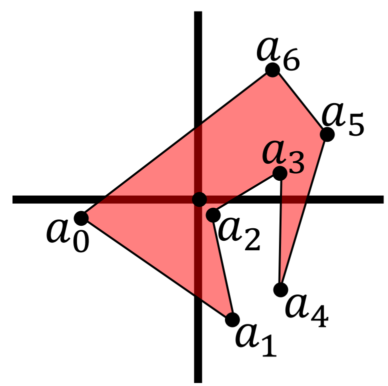


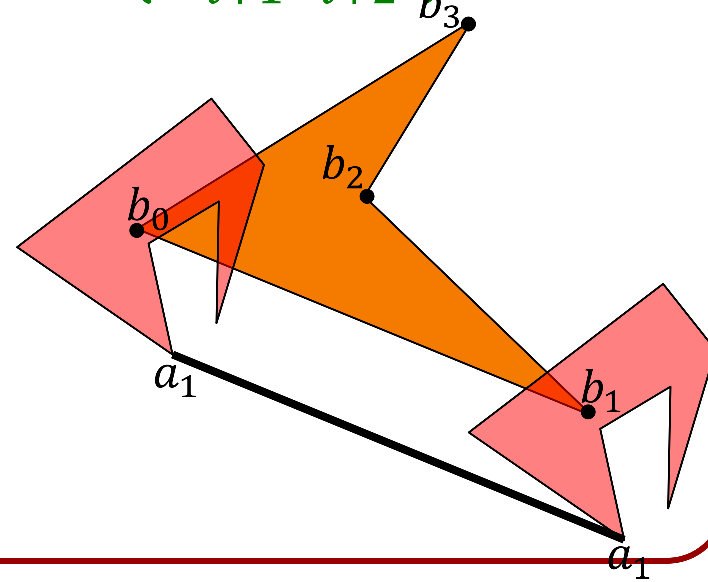

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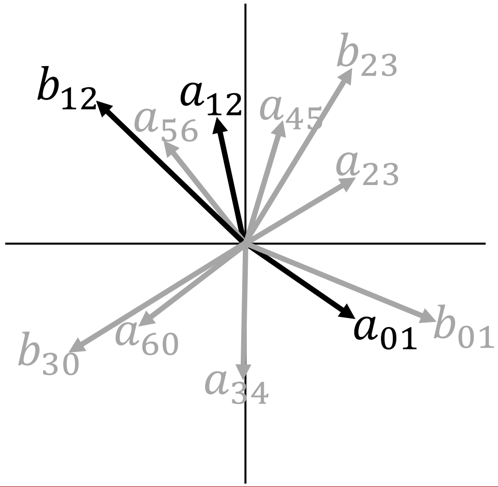


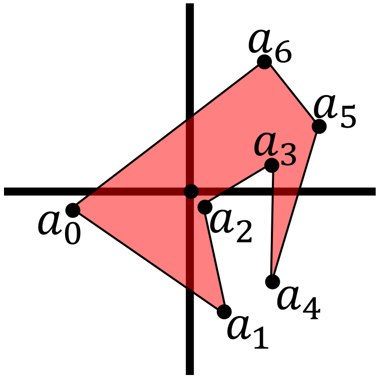


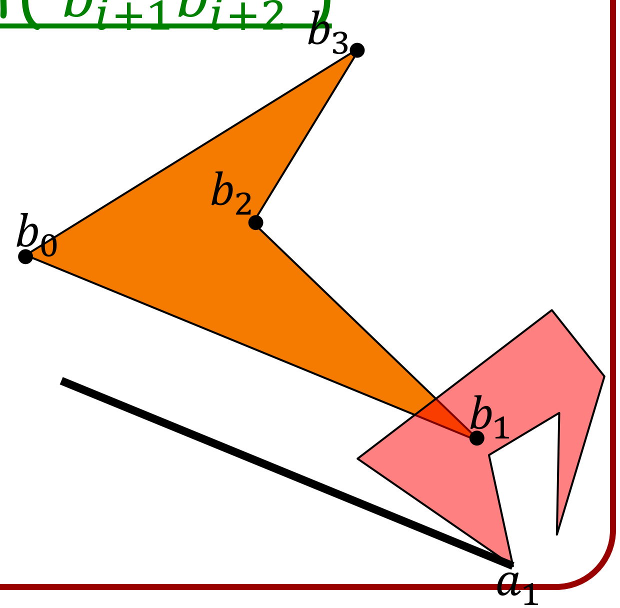

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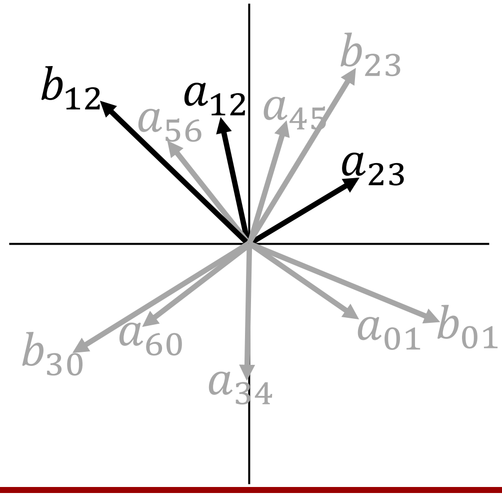


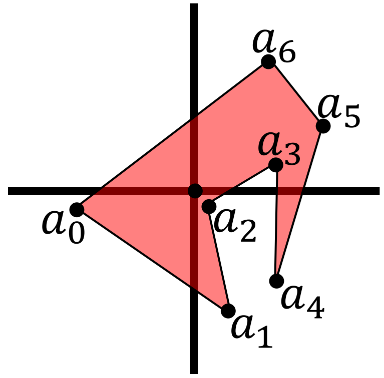


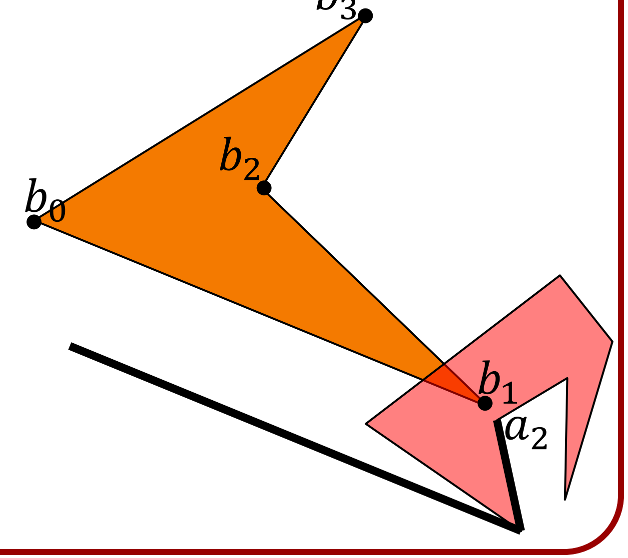

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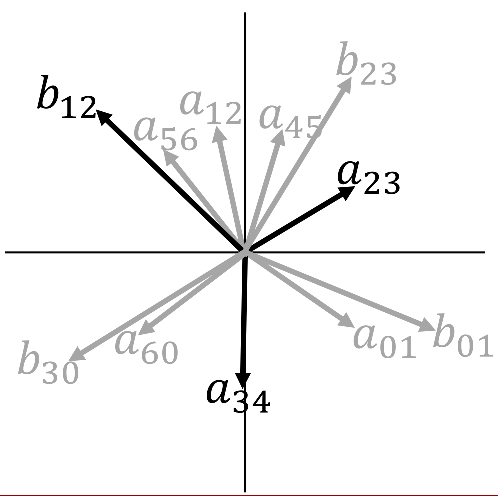


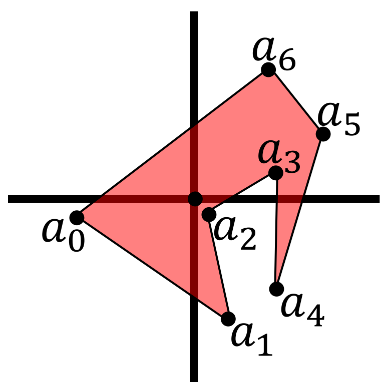


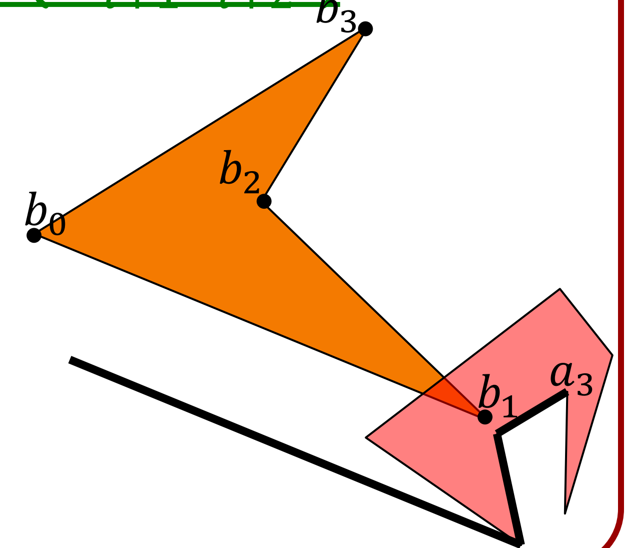

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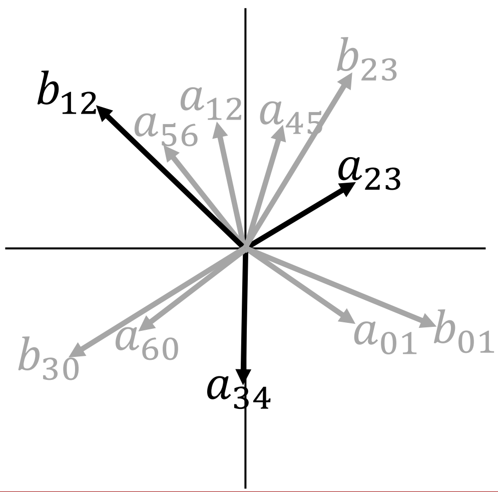


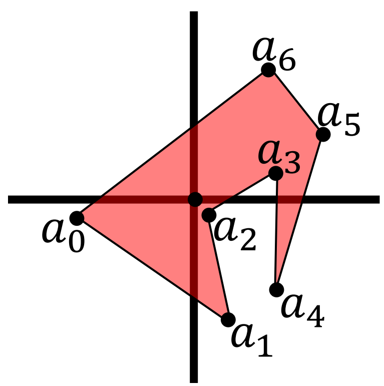


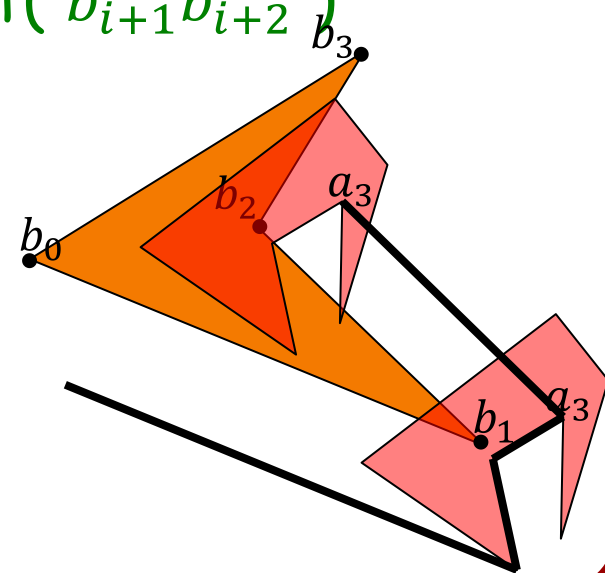

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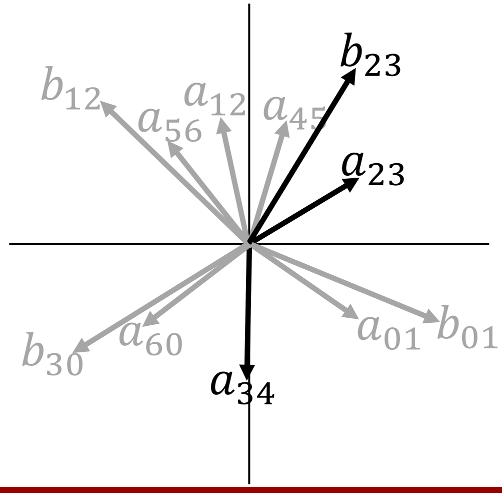


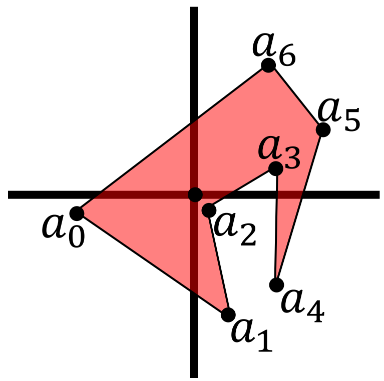


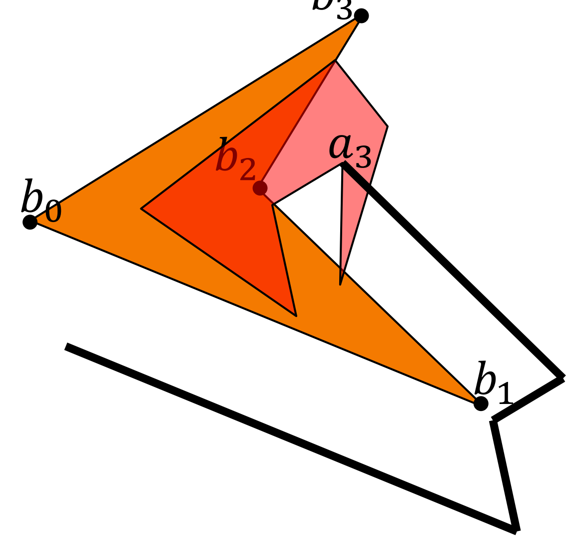

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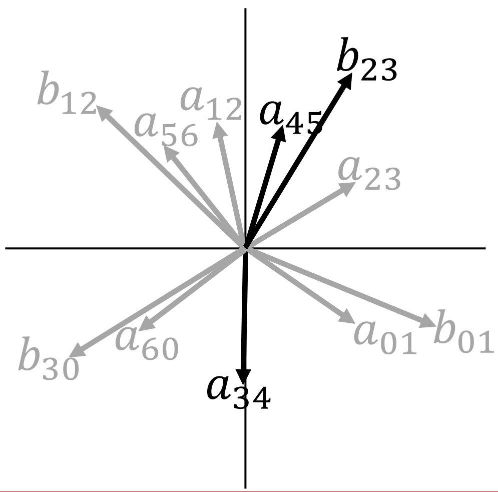


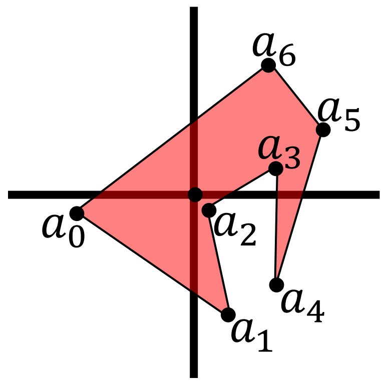


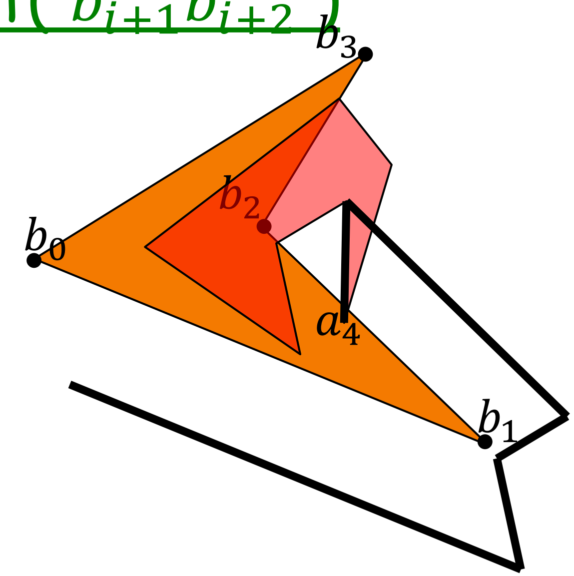

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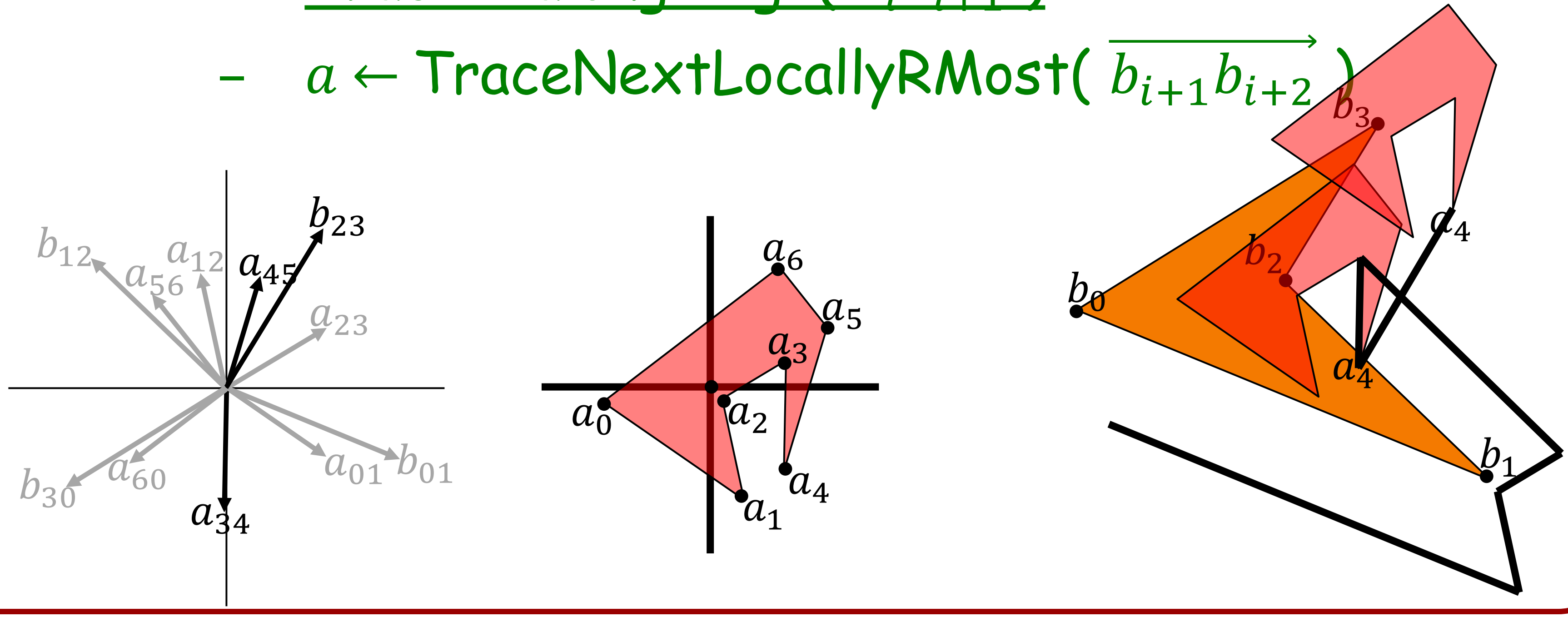

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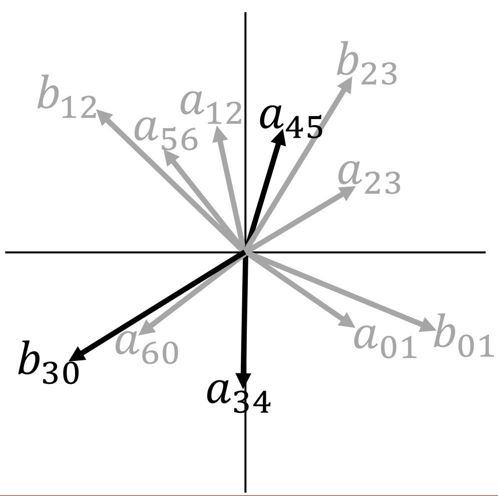


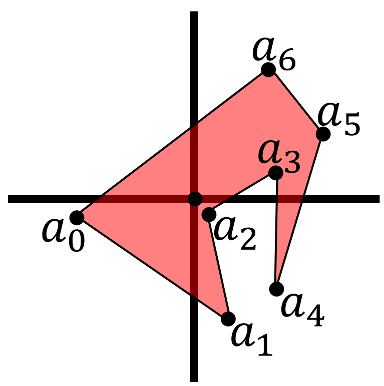


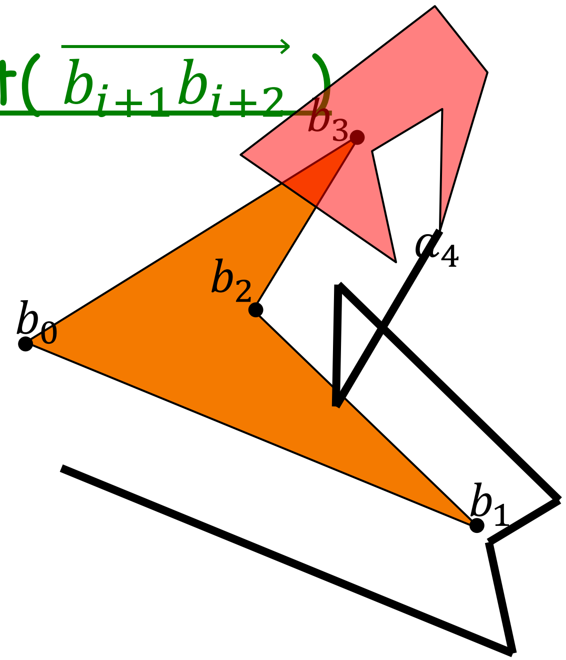

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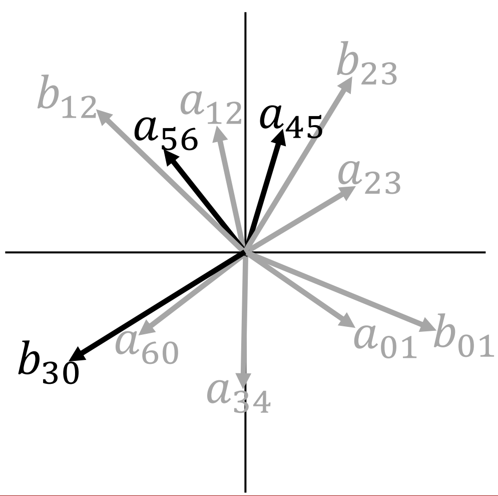


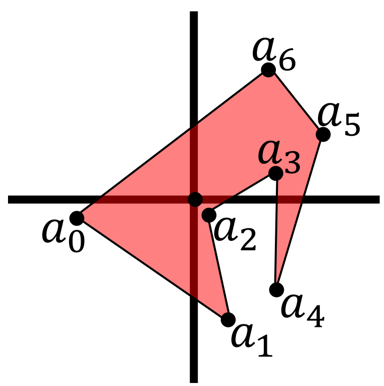


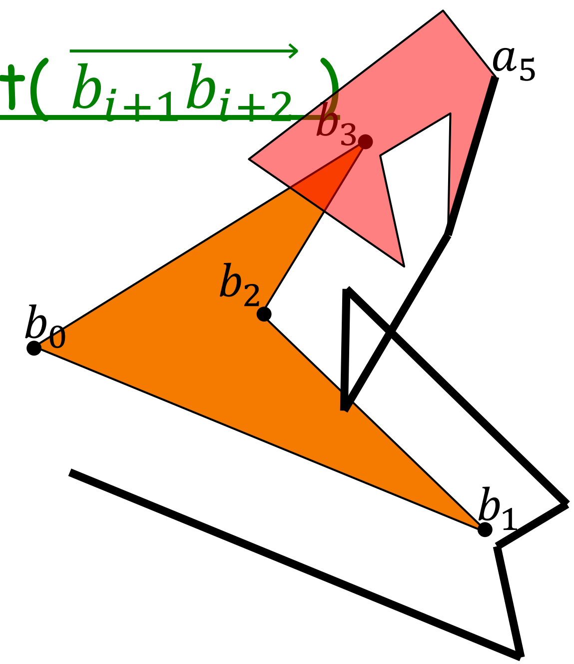

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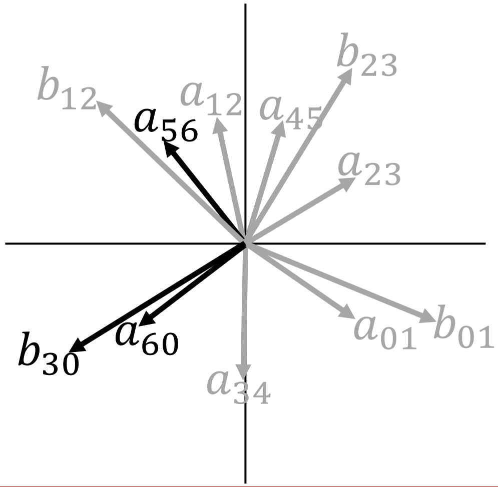


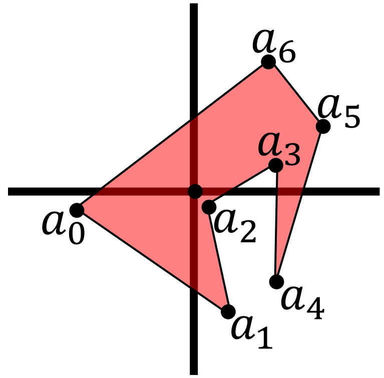


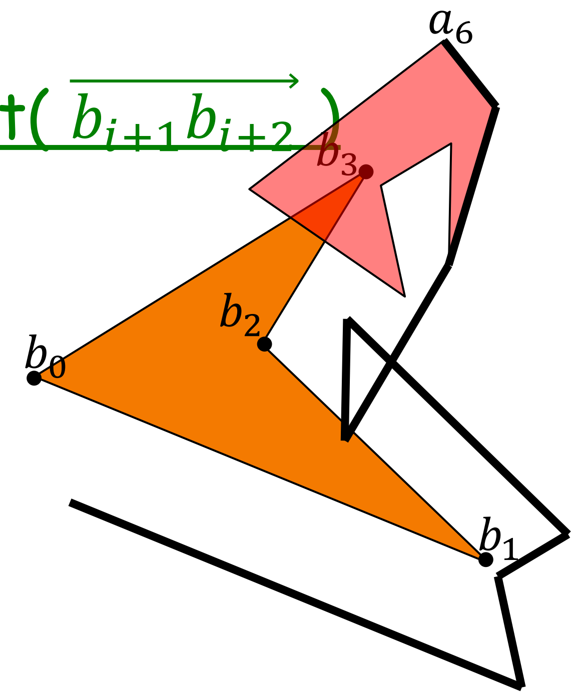

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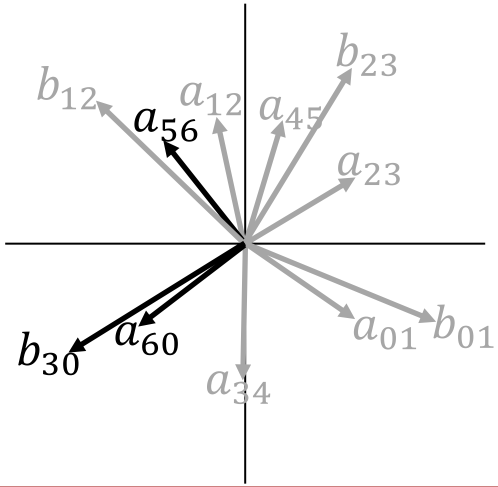


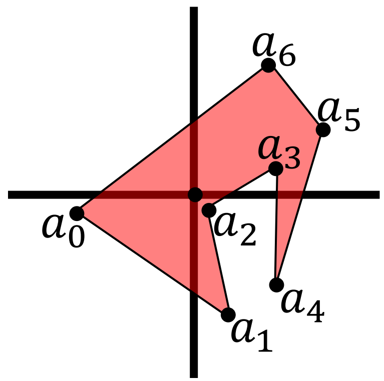


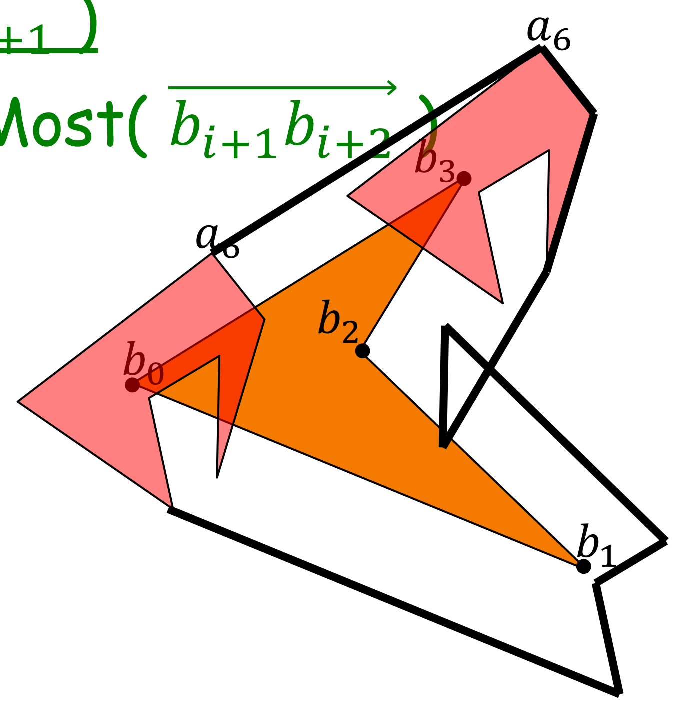

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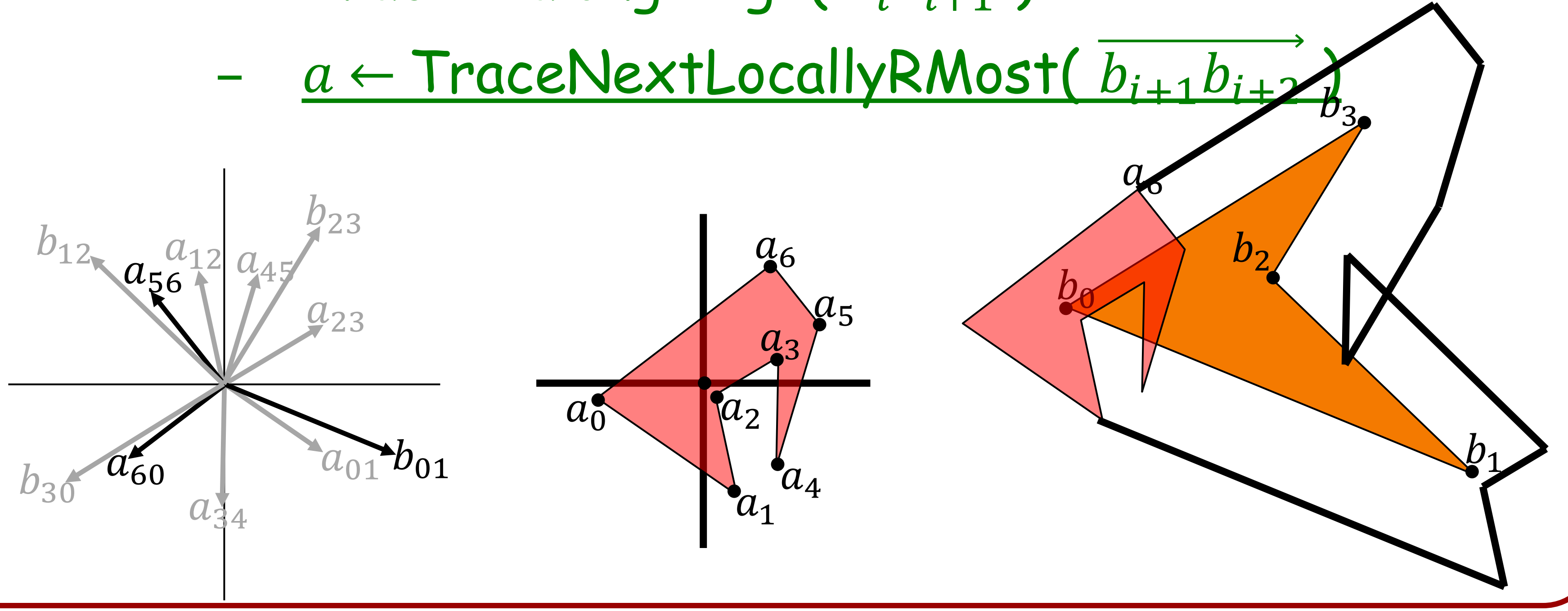

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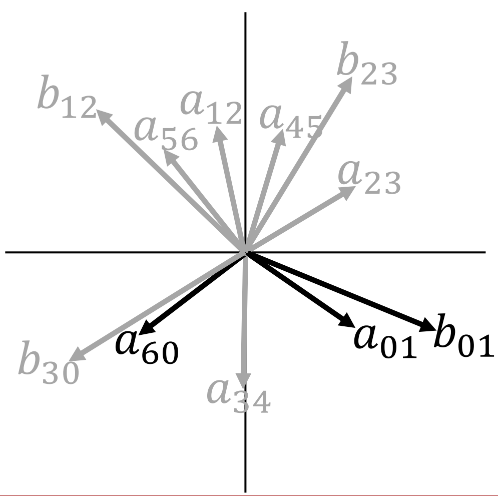


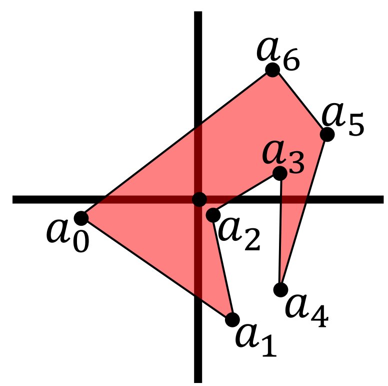


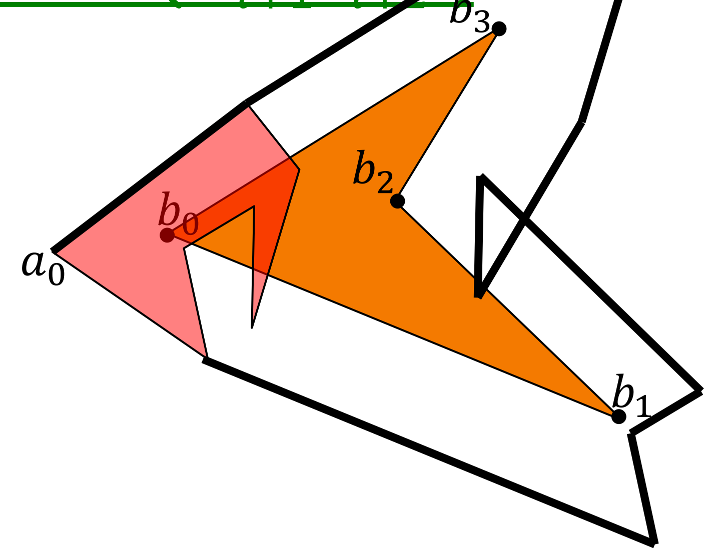

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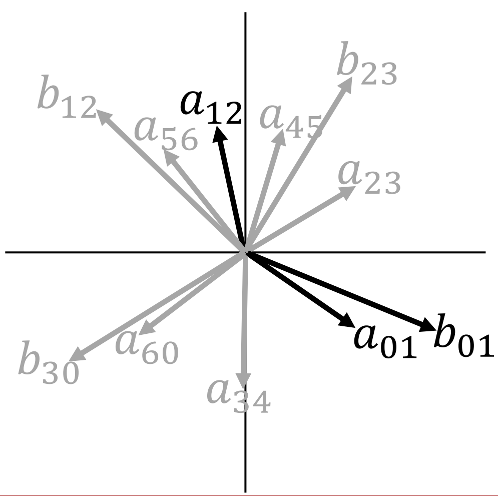


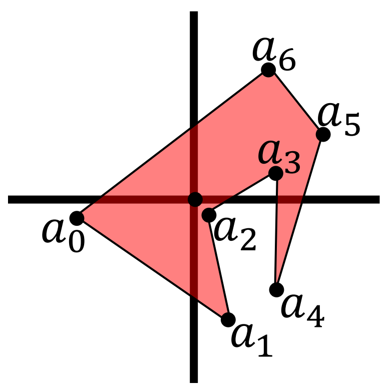


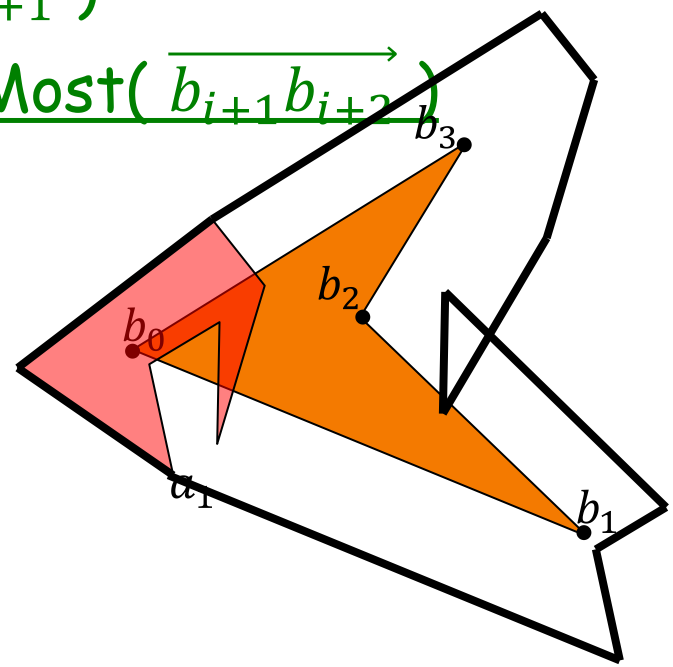

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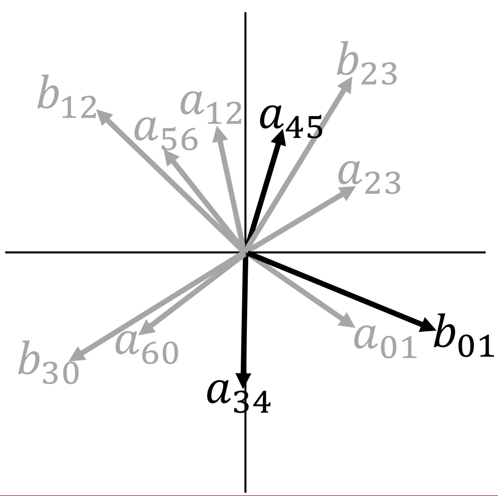


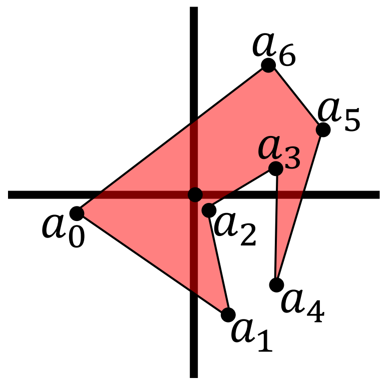


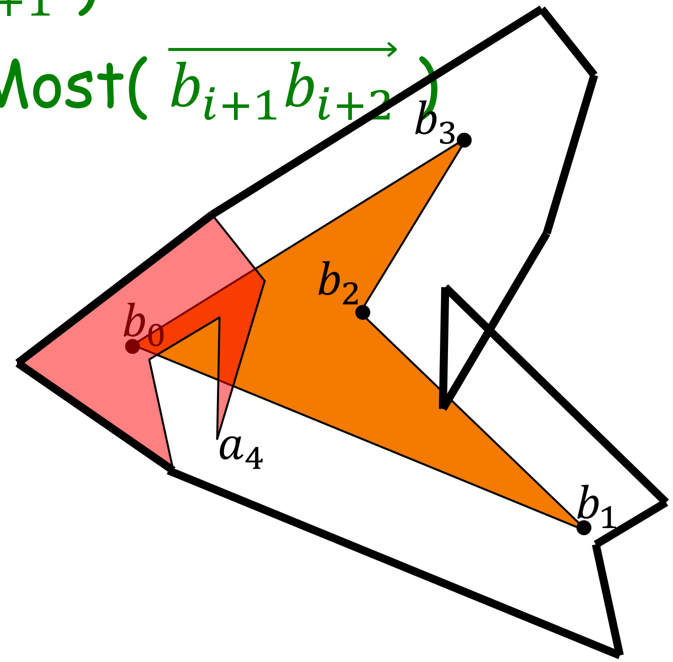

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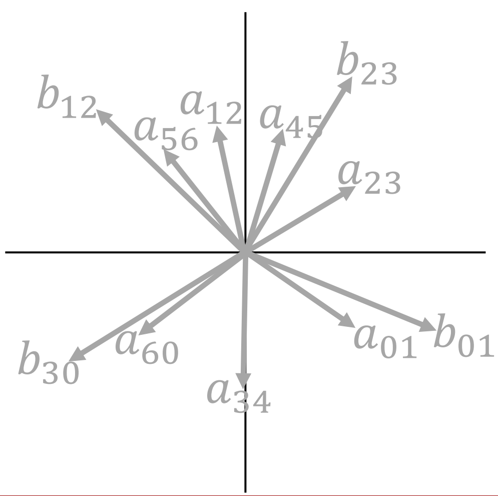


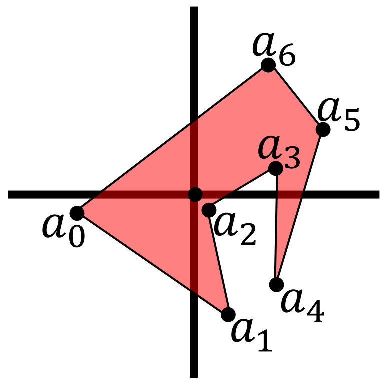


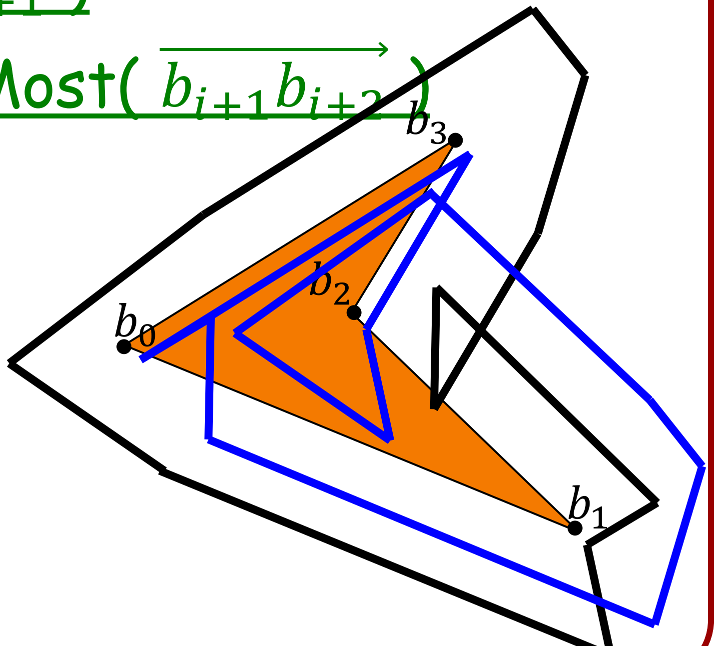

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<u>Note</u>:

- If $A$ is convex, the outer `for` loop cycles around the polygon $B$ once.
- Otherwise we can cycle as many times as there are different extremal vertices in $A$ -- $O(|A|)$.

# Motion Planning

MinkowskiTrace( $A$ , $B$ )

- Place $-A$ at vertex $b_0 \in B$.
- for each $a \in$ LocallyRMost( $\overrightarrow{b_0 b_1}$ )
  - for $i \in [0, |B|)$:
    - Trace $a$ along edge( $\overrightarrow{b_i b_{i+1}}$ )
    - $a \leftarrow$ TraceNextLocallyRMost( $\overrightarrow{b_{i+1} b_{i+2}}$ )

Note:

- In the worst case, tracing the locally maximal vertex can take $O(|A|)$ time.

# Motion Planning

MinkowskiTrace( $A$ , $B$ )

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Note:

- If $A$ and $B$ are convex, then in all these tracings, we cycle around $A$ once -- $O(1)$.

# Motion Planning

MinkowskiTrace( $A$ , $B$ )

- Place $-A$ at vertex $b_0 \in B$.
- for each $a \in$ LocallyRMost( $\overrightarrow{b_0 b_1}$ )
  - for $i \in [0, |B|)$:
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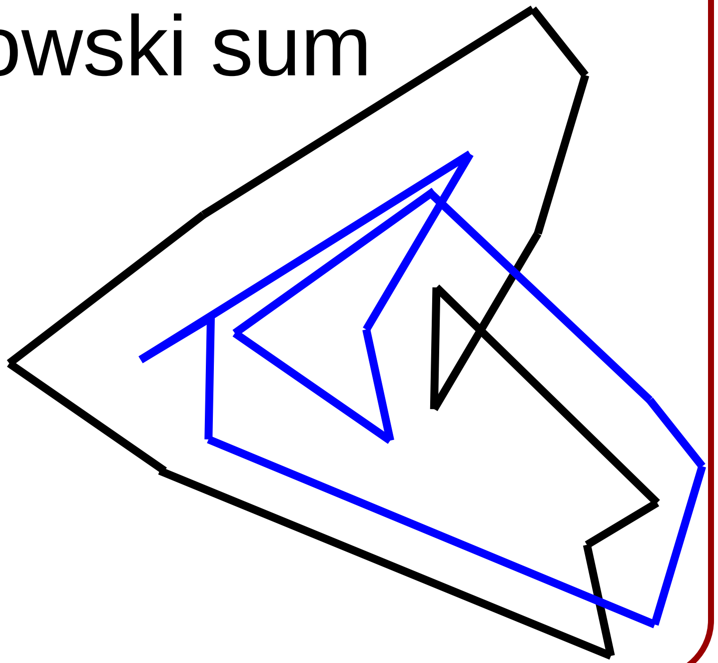
Complexity:

- If $A$ and $B$ convex: $O(|A| + |B|)$
- Else if $A$ convex: $O(|A| \cdot |B|)$
- Else: $O(|A|^2 \cdot |B|^2)$

# Motion Planning

- This only gives the *convolution of $A$ with $B$*.
- If $A$ or $B$ is not convex, this is not the same as the Minkowski sum.

Q: How do we get the Minkowski sum from the convolution?

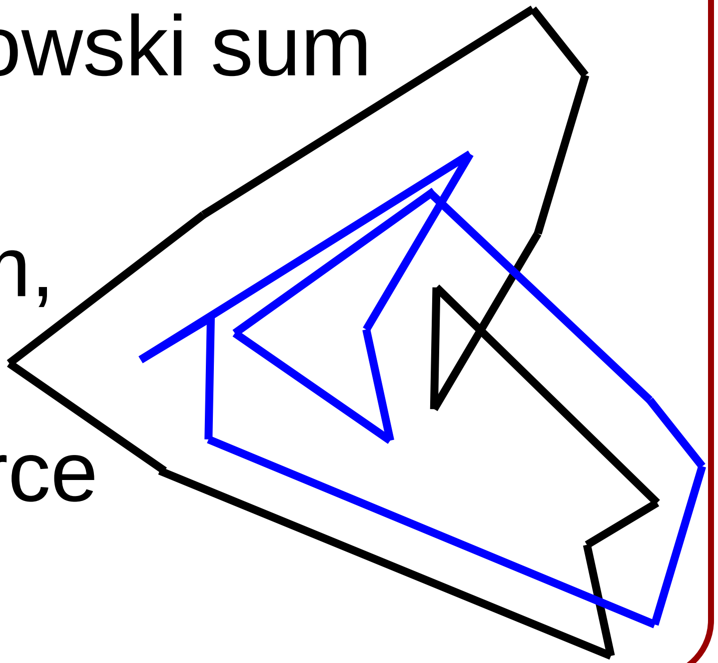


# Motion Planning

- This only gives the *convolution of $A$ with $B$*.
- If $A$ or $B$ is not convex, this is not the same as the Minkowski sum.

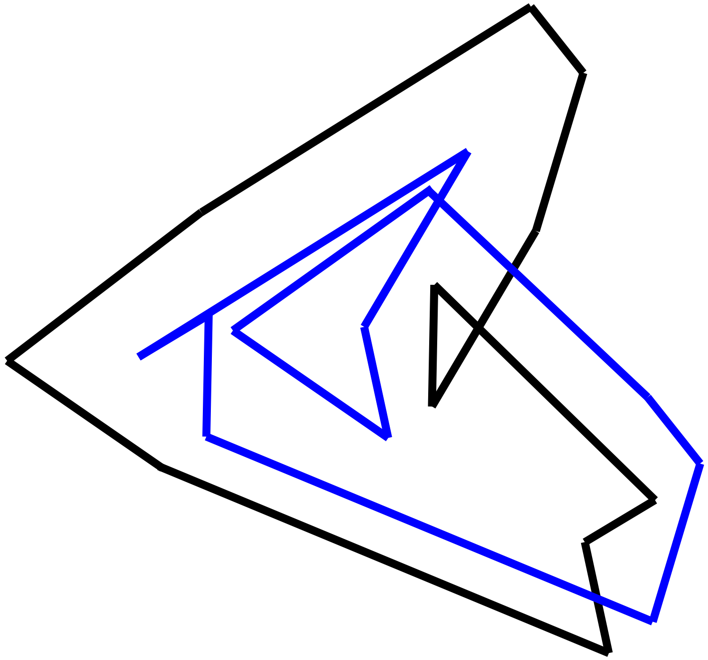
Q: How do we get the Minkowski sum from the convolution?

Q: Given the Minkowski sum, how do we determine if we can get from the source to the target?



# Convolution to Minkowski Sum

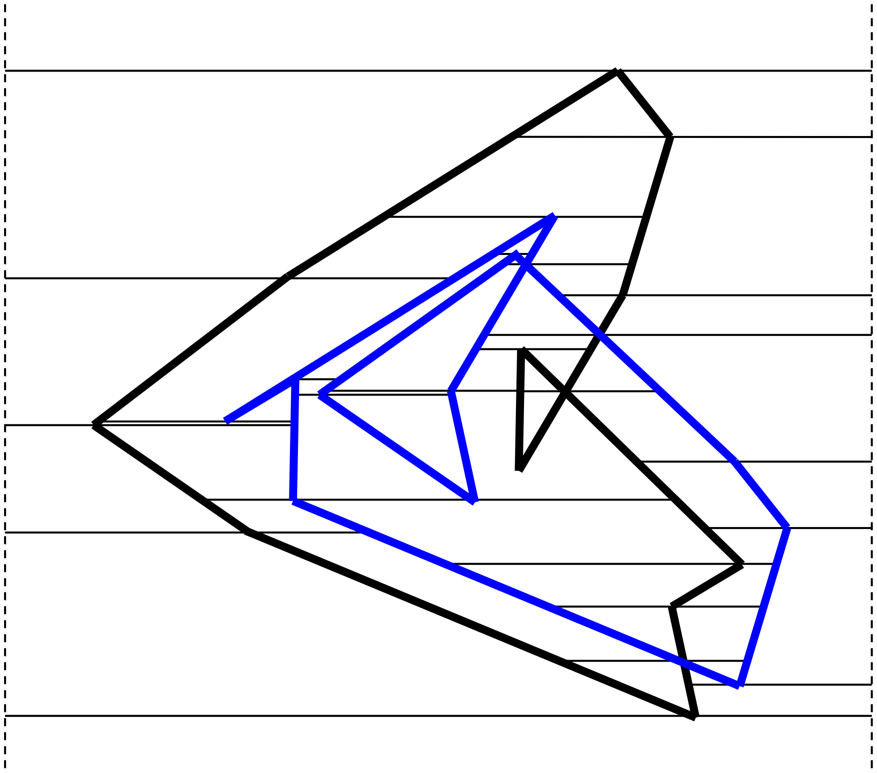
Approach:



# Convolution to Minkowski Sum

Approach:

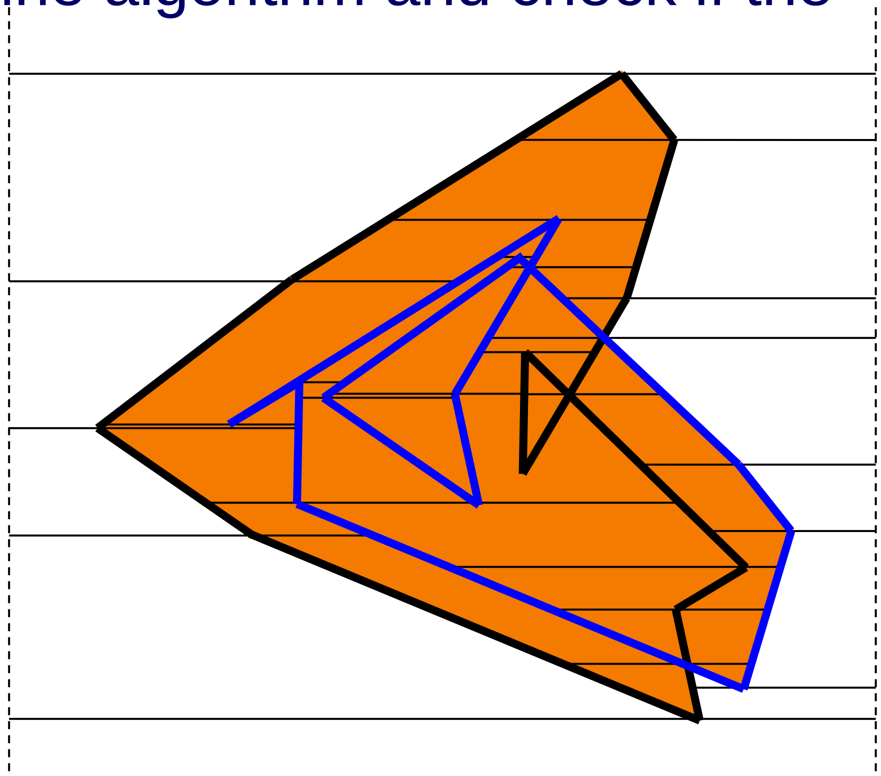
1. Compute the trapezoidalization



# Convolution to Minkowski Sum

Approach:

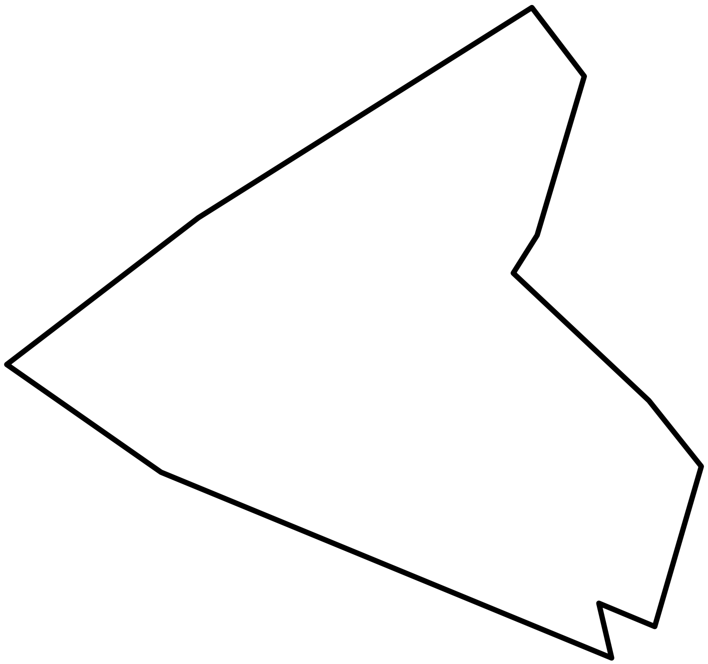
1. Compute the trapezoidalization
2. Use the winding number to identify interior trapezoids
   E.g. Run the sweep- line algorithm and check if the difference between the number of left and right edges crossed on one side of the sweep-line is non-zero.



# Convolution to Minkowski Sum

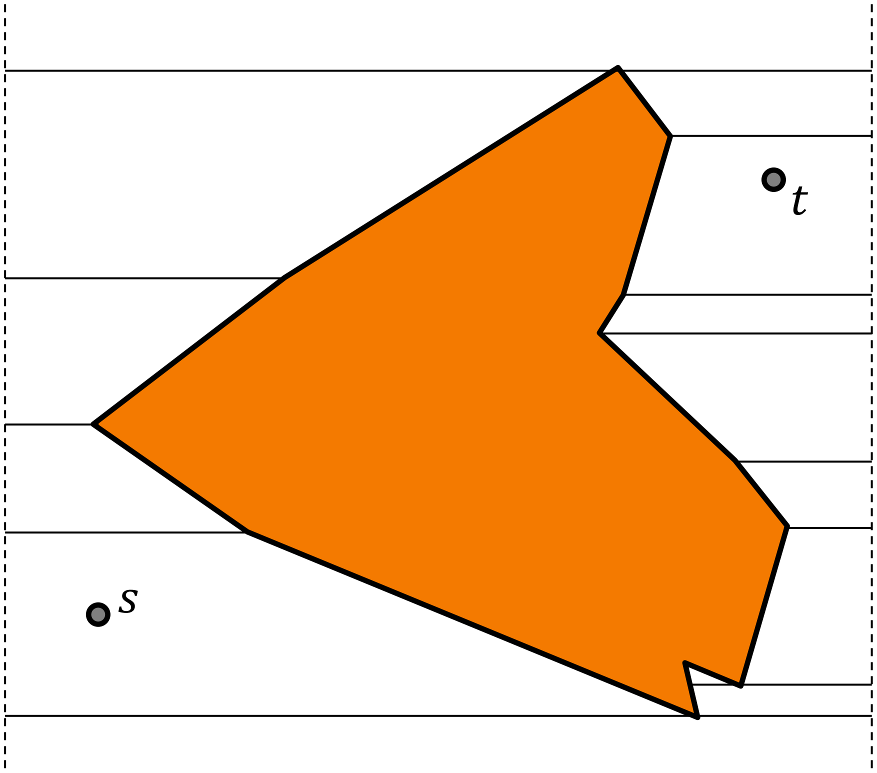
Approach:

1. Compute the trapezoidalization
2. Use the winding number to identify interior trapezoids
3. Compute the edges separating the interior trapezoids from the exterior ones



# Computing Accessibility

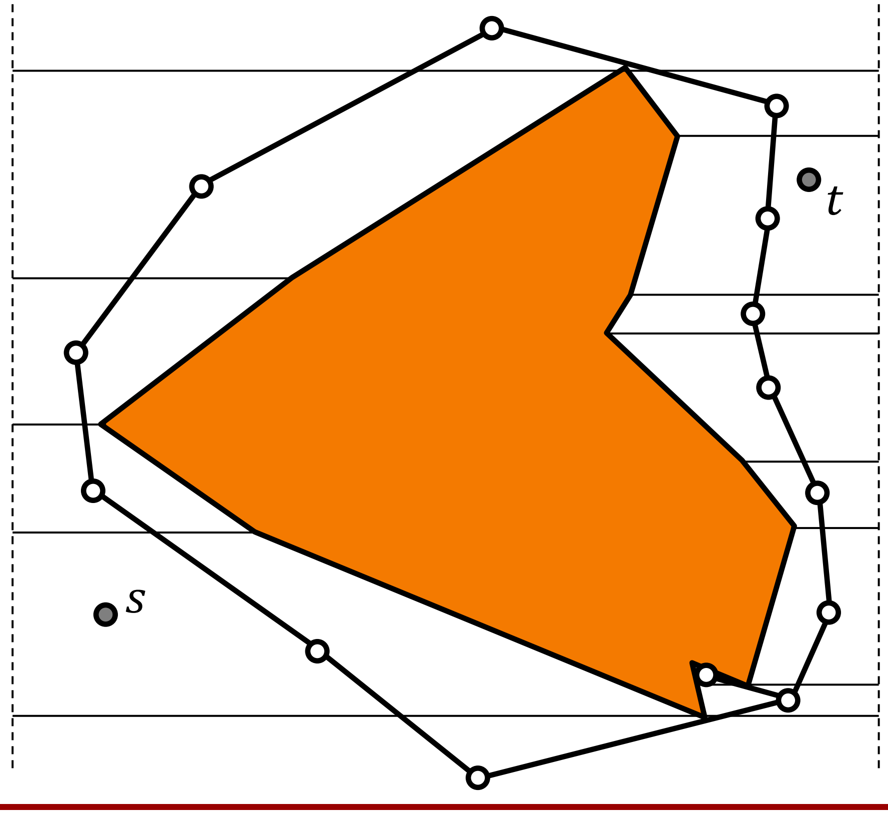
Given the trapezoidalization of the exterior of the convolution, and given source and target positions:



# Computing Accessibility

Given the trapezoidalization of the exterior of the convolution, and given source and target positions:

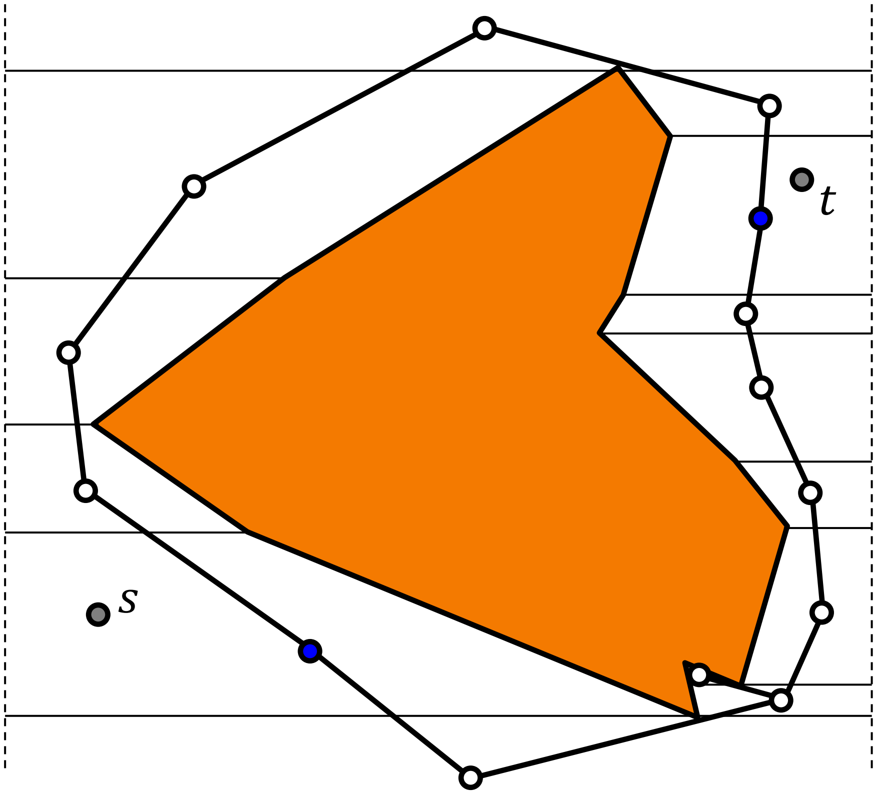
1. Compute the dual graph of the exterior trapezoids



# Computing Accessibility

Given the trapezoidalization of the exterior of the convolution, and given source and target positions:

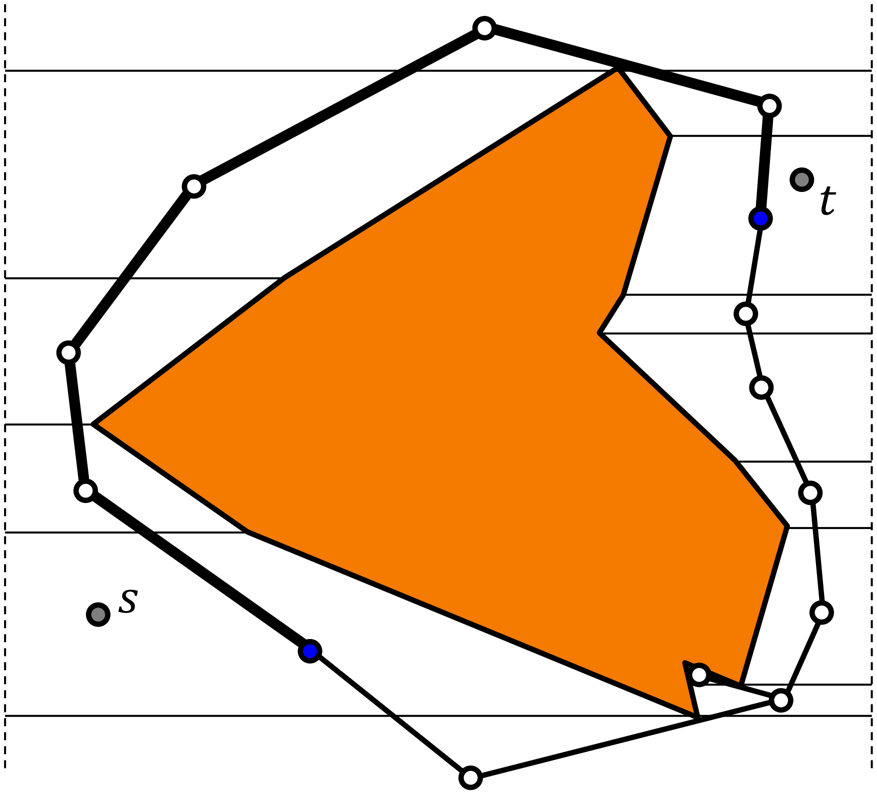
1. Compute the dual graph of the exterior trapezoids
2. Identify (dual) nodes corresponding to the trapezoids containing the source and target



# Computing Accessibility

Given the trapezoidalization of the exterior of the convolution, and given source and target positions:

1. Compute the dual graph of the exterior trapezoids
2. Identify (dual) nodes corresponding to the trapezoids containing the source and target
3. Check if there is a path in the dual graph between them



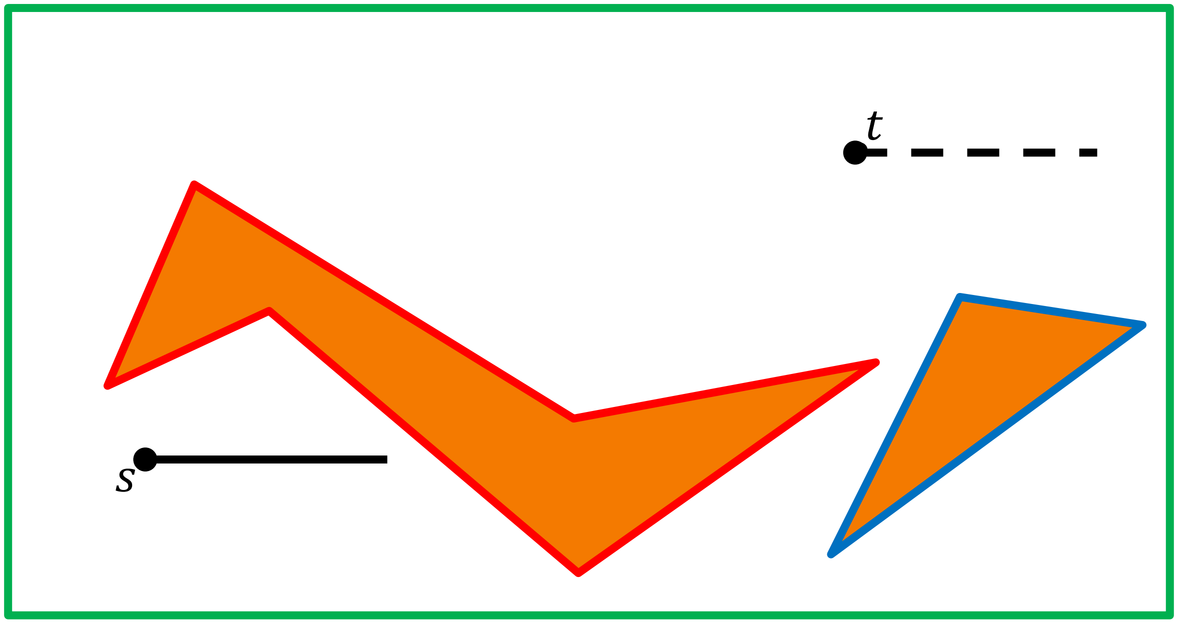
# Outline

- Translating a polygon
- Moving a ladder

# Moving a Ladder

<u>Goal</u>:

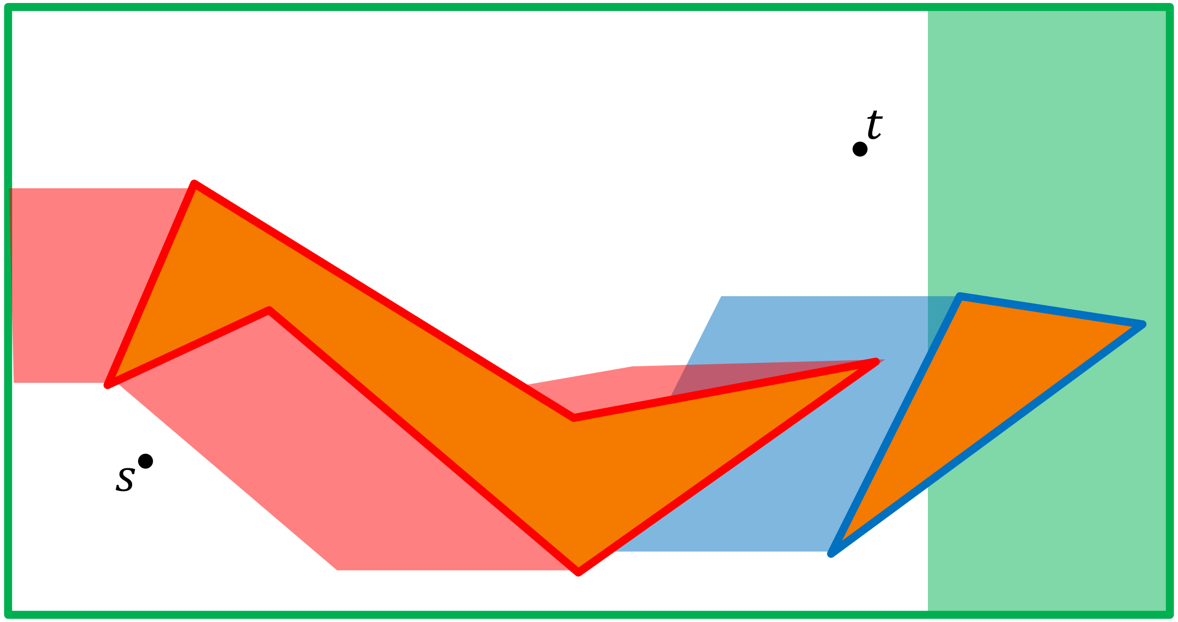
Given a line segment at position $s$, a set of polygons $P$, and a target position $t$, determine if the ladder can be translated <u>and rotated</u> to $t$ while avoiding the polygonal obstacles.



# Moving a Ladder

General Approach:

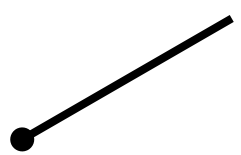
- Compute the Minkowski Sum of $P$ with $l$.

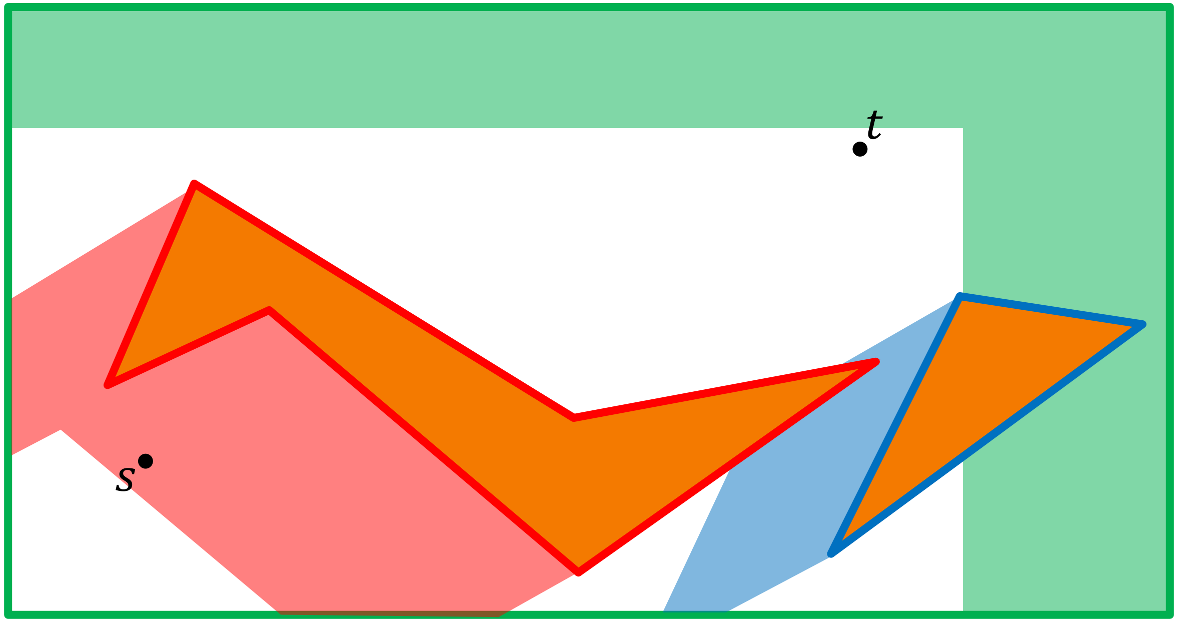


# Moving a Ladder

General Approach:

- Compute the Minkowski Sum of $P$ with the ladder.
- Do this for every rotation of the ladder.





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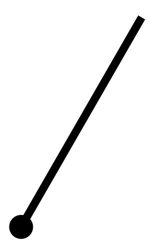

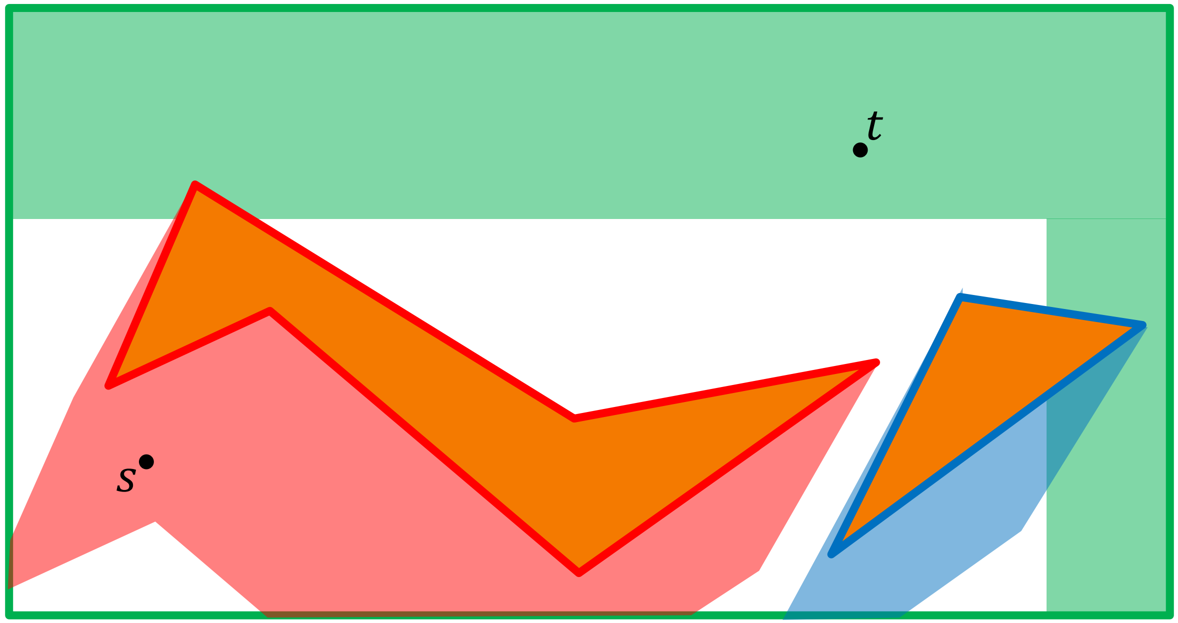

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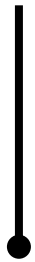
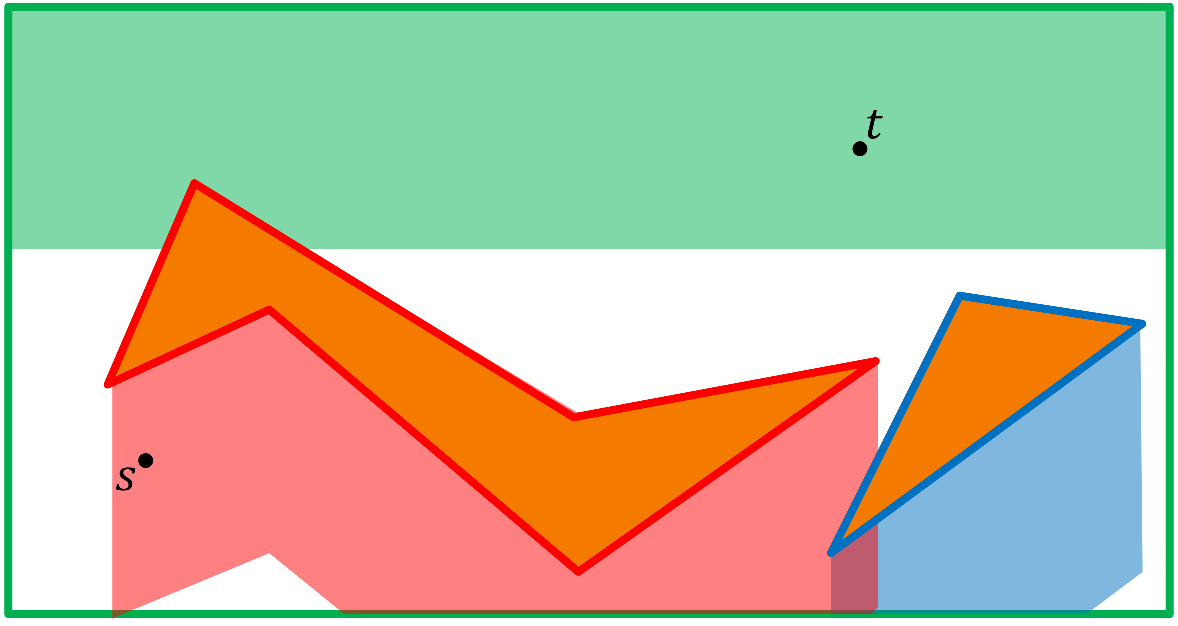
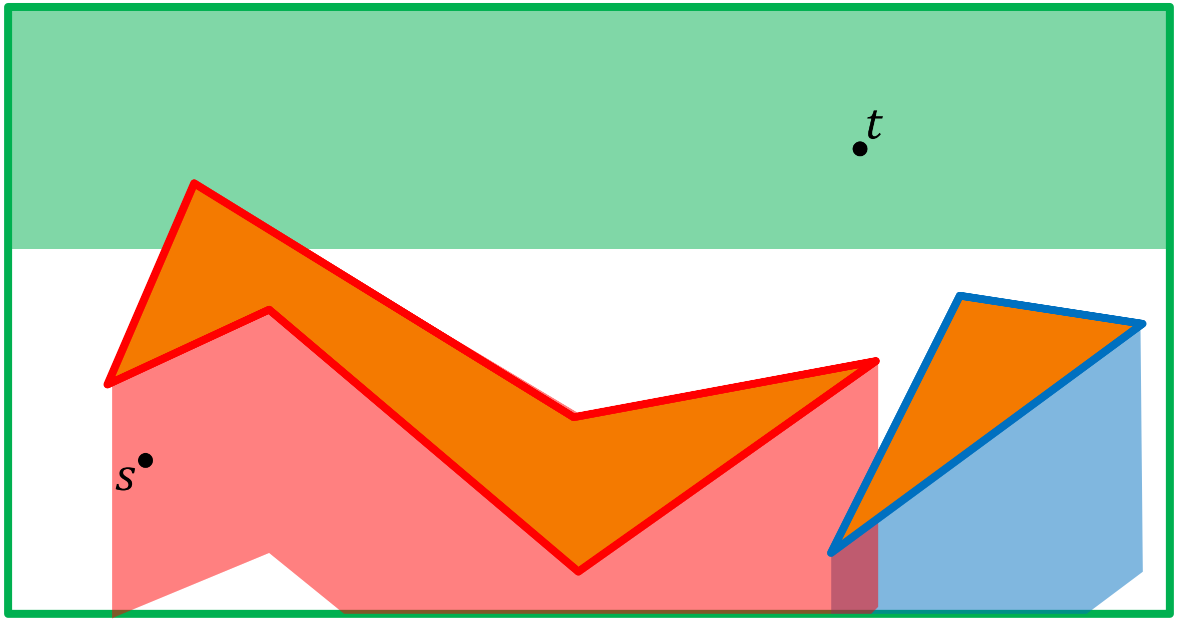




# Moving a Ladder

General Approach:

- Compute the Minkowski Sum of $P$ with the ladder.
- Do this for every rotation of the ladder.
- Stack into a 3D volume and test if there is a path from $s$ to $t$, in the 3D space, that avoids all the polygons.



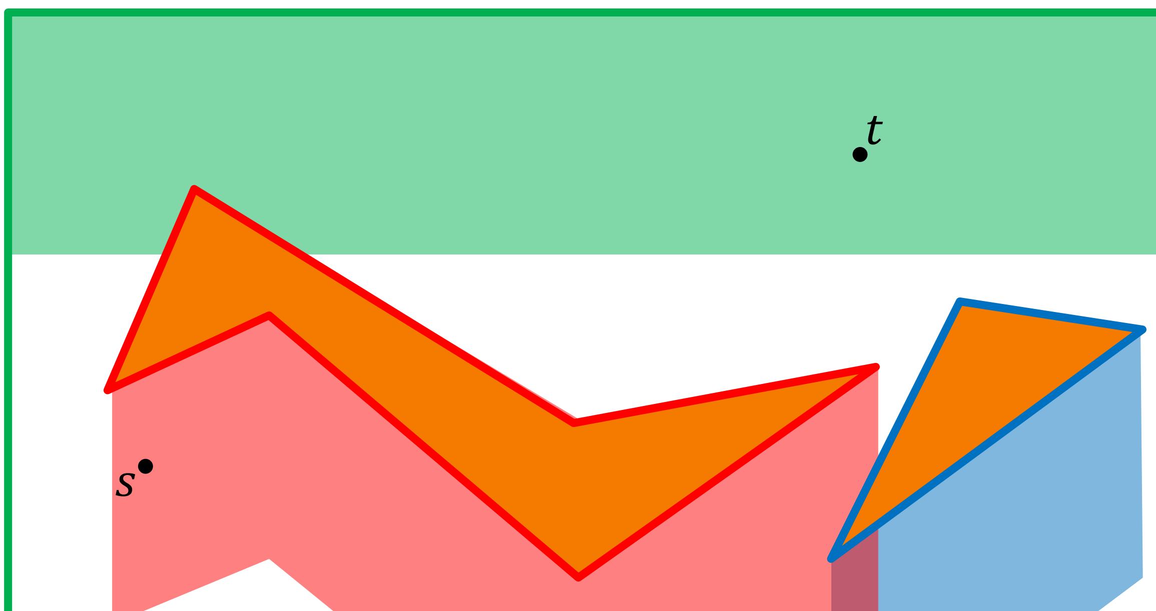
# Moving a Ladder

General Approach:

- Compute the Minkowski Sum of $P$ with the ladder.
- Do t
- Stac … a path from $s$ to $t$, in the 3D space, that avoids all the polygons.

Challenges:

- There are infinitely many angles.
- The boundaries of the 3D sums are not planar.



# Moving a Ladder

Implementation:

Partition empty region into trapezoids:

- Base aligned with the ladder direction.
- Sides defined by the edges of the polygons.

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In general, as we rotate the ladder, the shape of the cells change but the topology of the partition (i.e. dual graph) does not.

At discrete events, cells may be inserted, deleted, or merged.

# Moving a Ladder

<u>Implementation</u>:

Partition empty region into trapezoids:

These events can only occur when the rotated ladder aligns with a segment between two vertices of the polygons.

In ge … of the

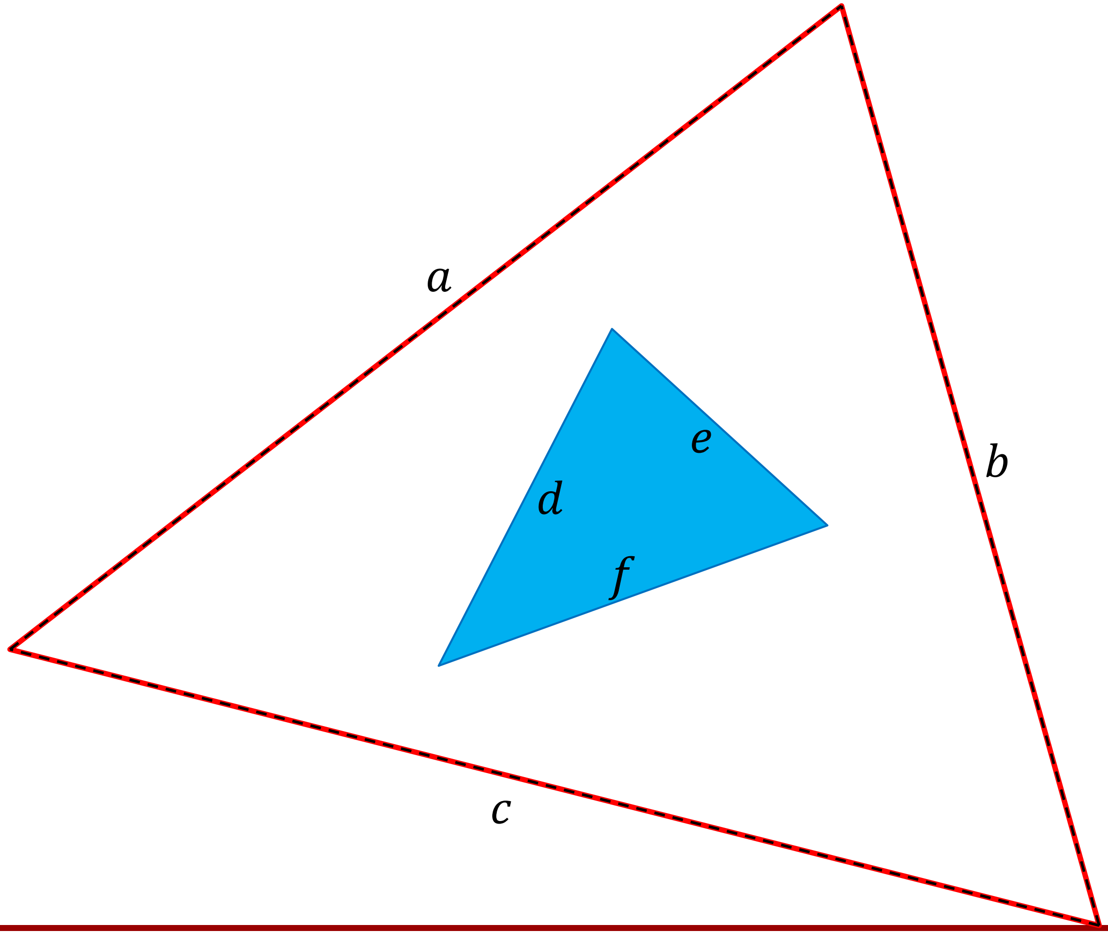
cells … .e.

dual graph) does not.

Instead of trying all possible rotations, we only need to consider the $O(|P|^2)$ cases.

At discrete events, cells may be inserted, deleted, or merged.

# Moving a Ladder



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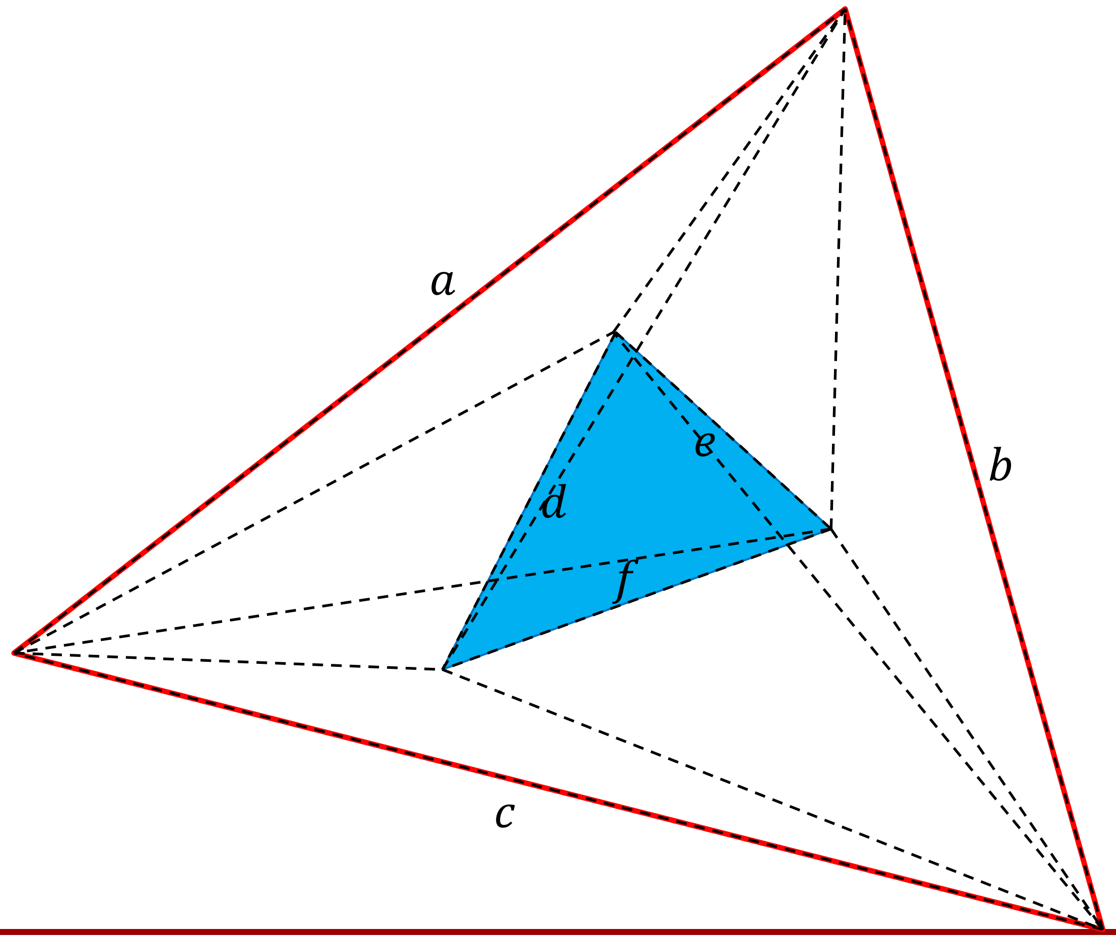

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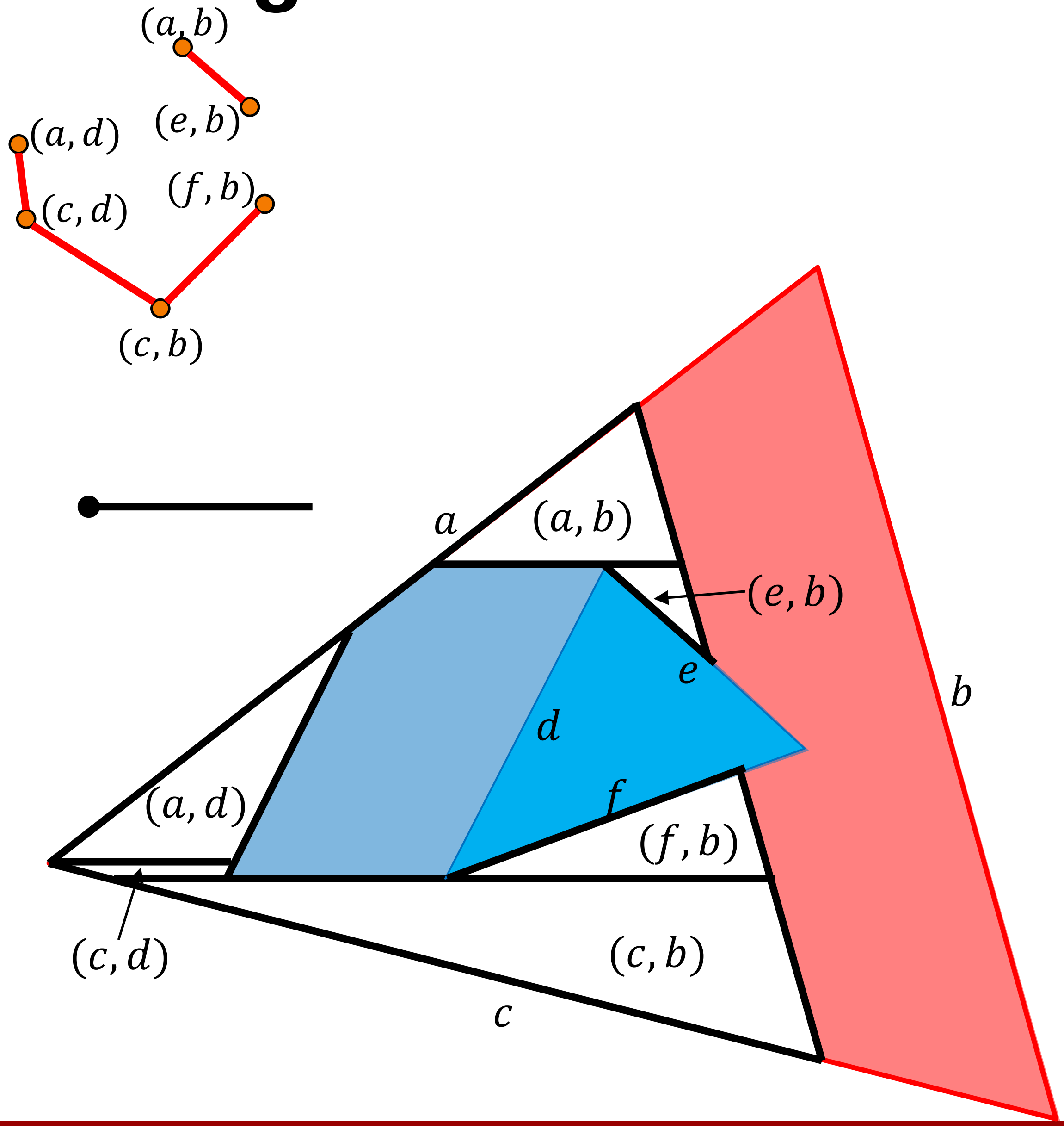

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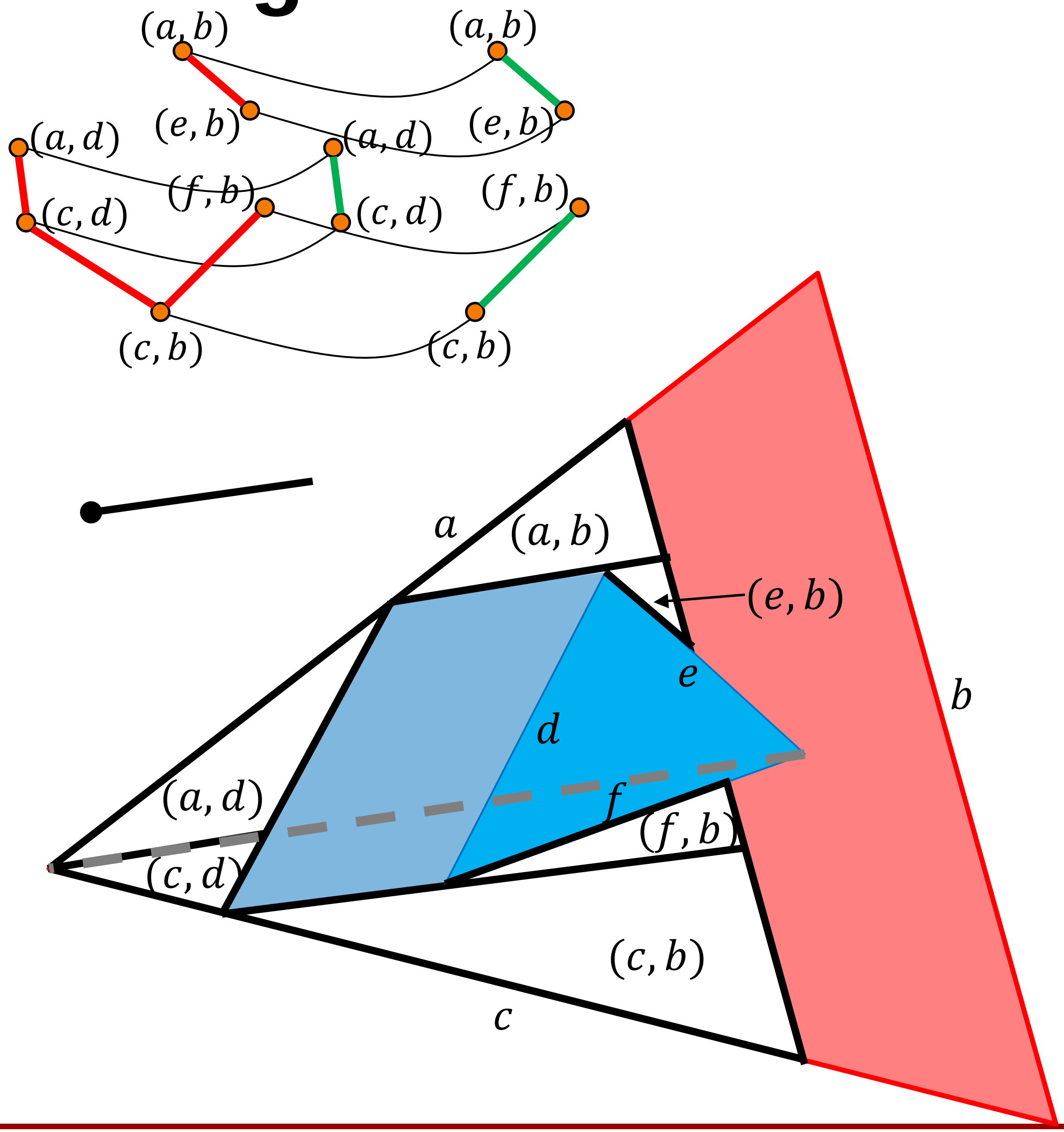

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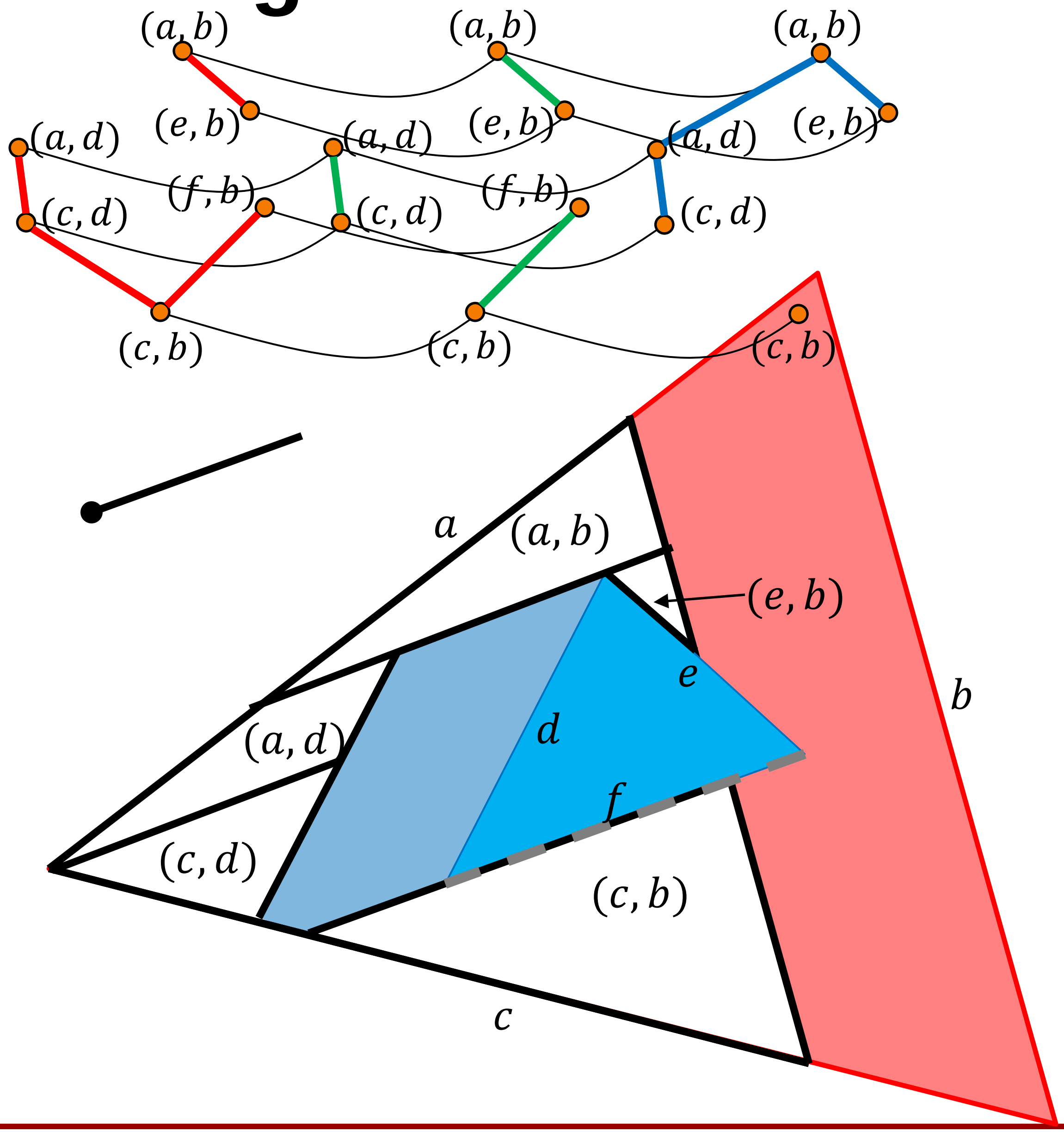

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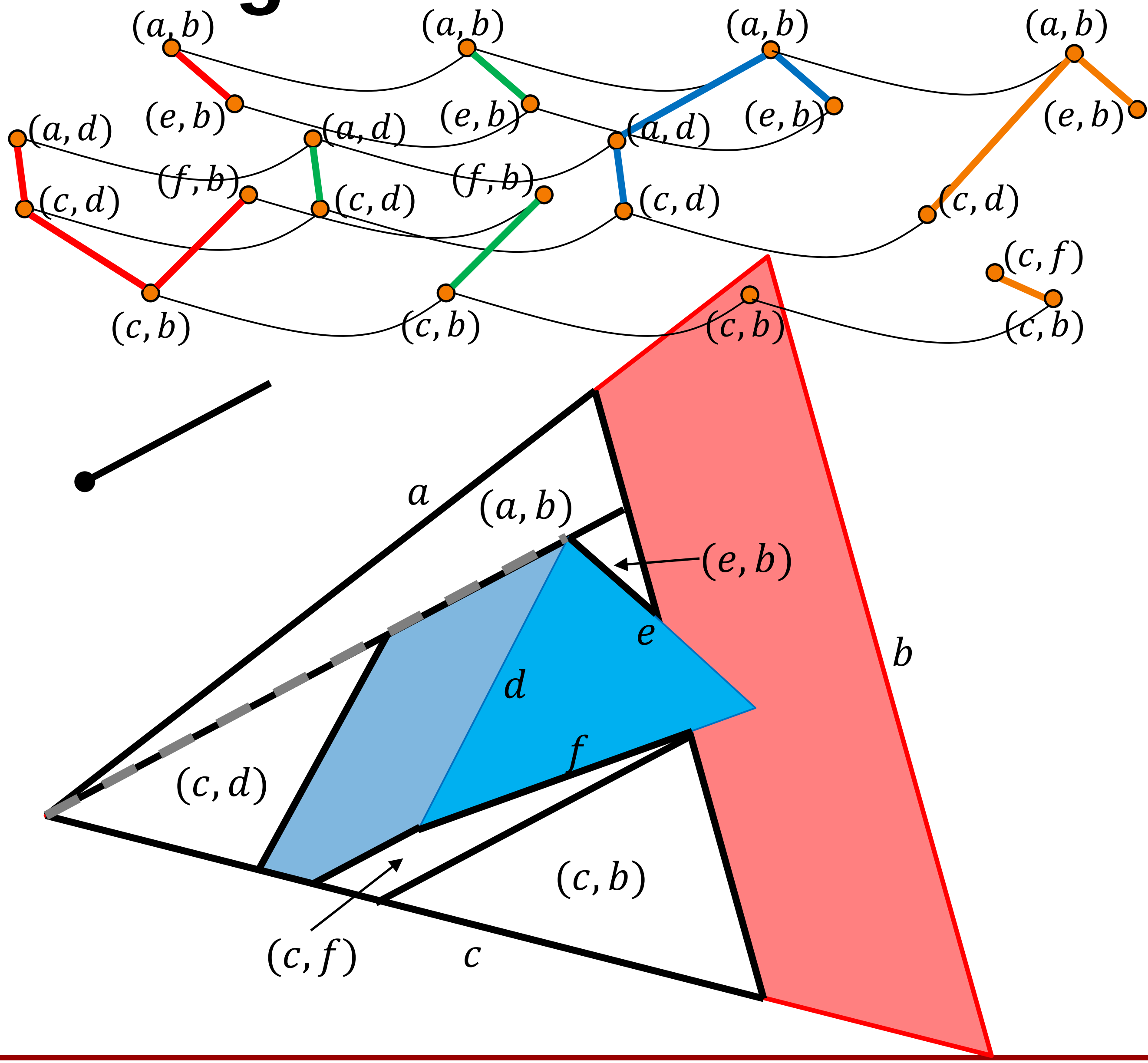

# Moving a Ladder

Approach:

- Set $\Theta = \{\theta_0 = 0, \theta_1, \ldots, \theta_n\}$ to be sorted angles, with $\theta_i$ the angle of the $i$-th segment.
- For $0 \leq i \leq n$:
  - » Compute the Minkowski Sum of $P$ with the rotation of the ladder by angle $\theta_i + \varepsilon$.
  - » Compute the dual graph of the trapezoidalization.
  - » Link the $(i-1)$-st and $i$-th dual graphs.
- Test if there is a path between the cells containing $s$ (at $\theta = 0$) and $t$ in the linked graph.