Search and Intersection

O'Rourke, Chapter 7

Outline

- Trapezoidal Decomposition
- Extreme Points (2D)
- Extreme Points (3D)

Trapezoidal Decomposition

Goal:

Given a partition of 2D space into polygons, (efficiently) compute a (compact) data-structure that enables fast point-in-polygon queries.

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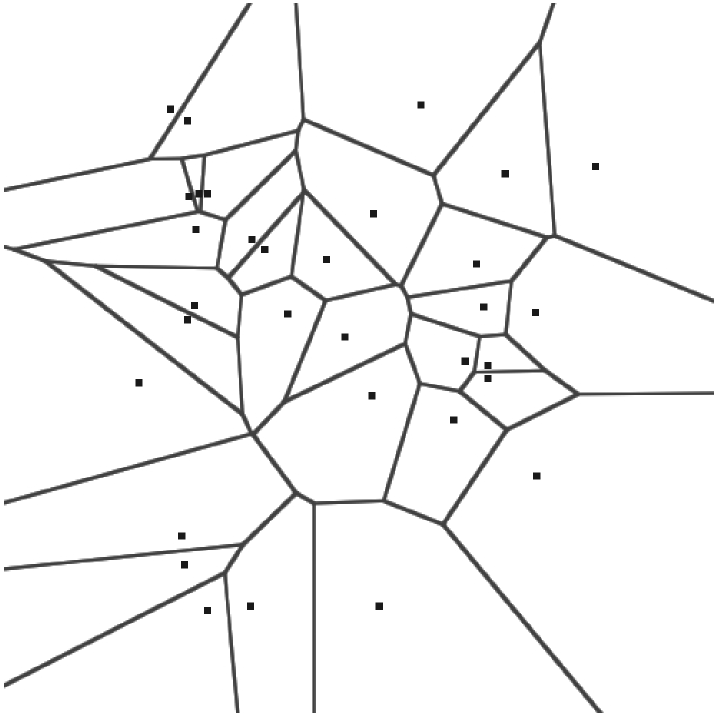

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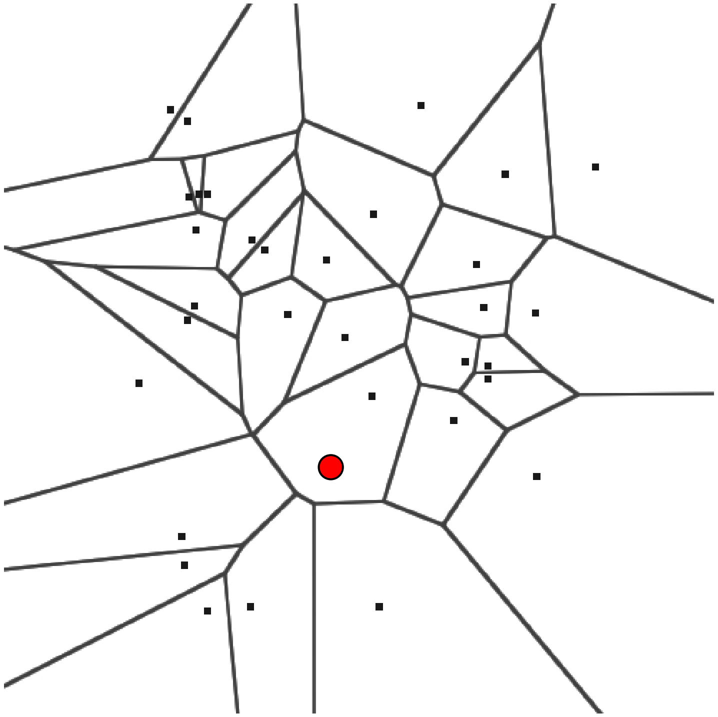

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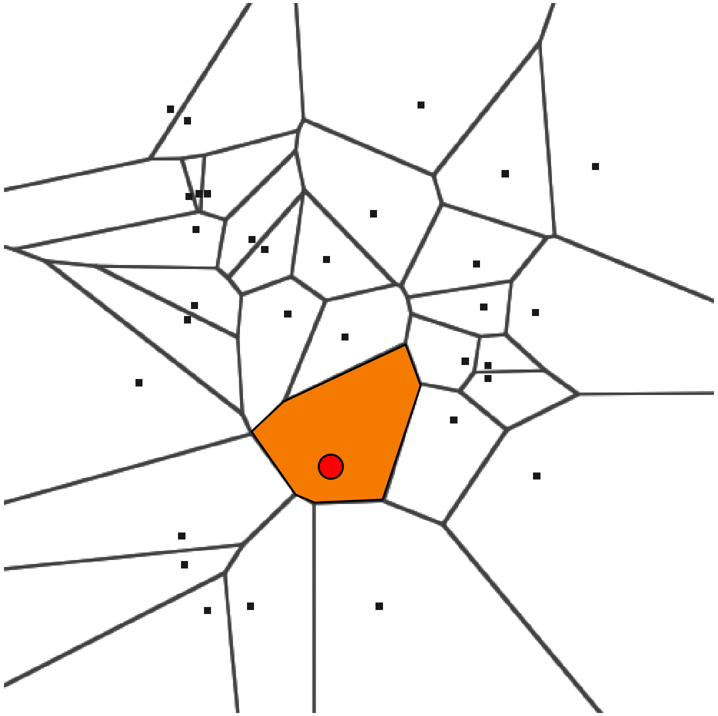
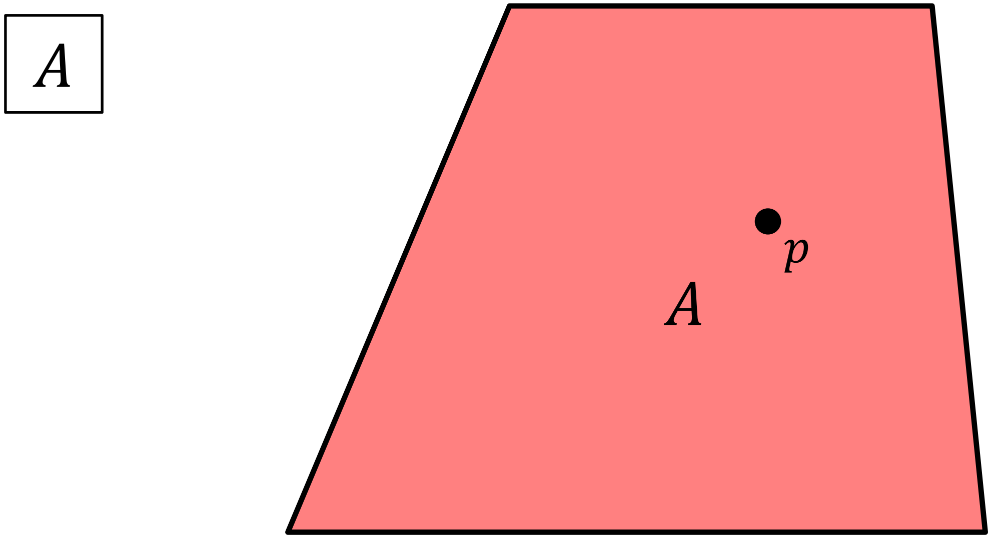


Trapezoidal Decomposition

Approach:

Construct the partition iteratively, adding new line-segments into an existing partition:

- Add end-point, performing top/bottom split of containing trapezoid.



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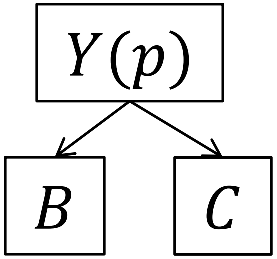

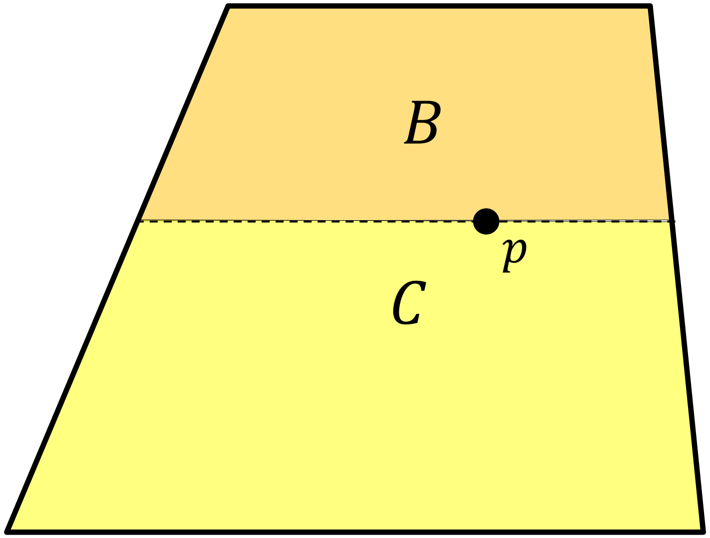

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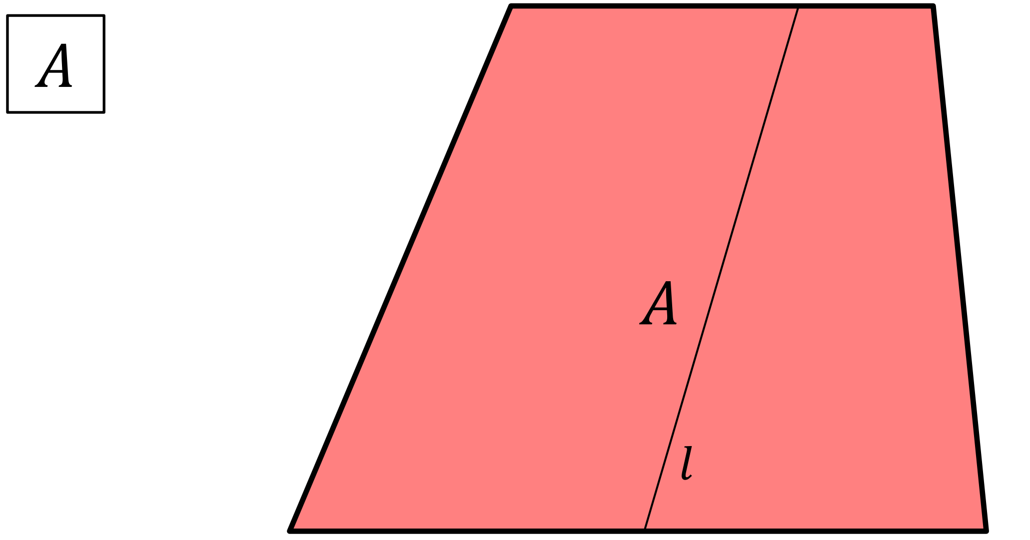

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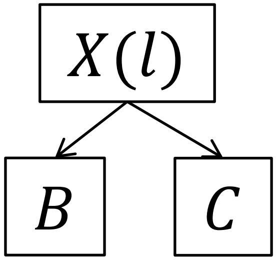

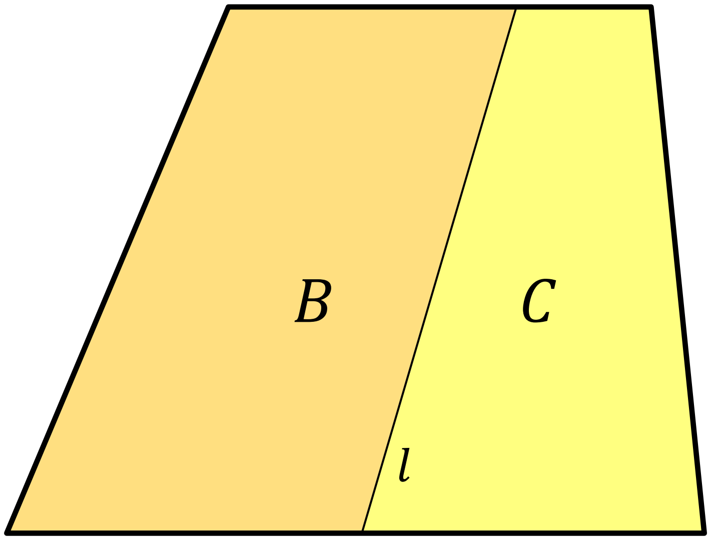

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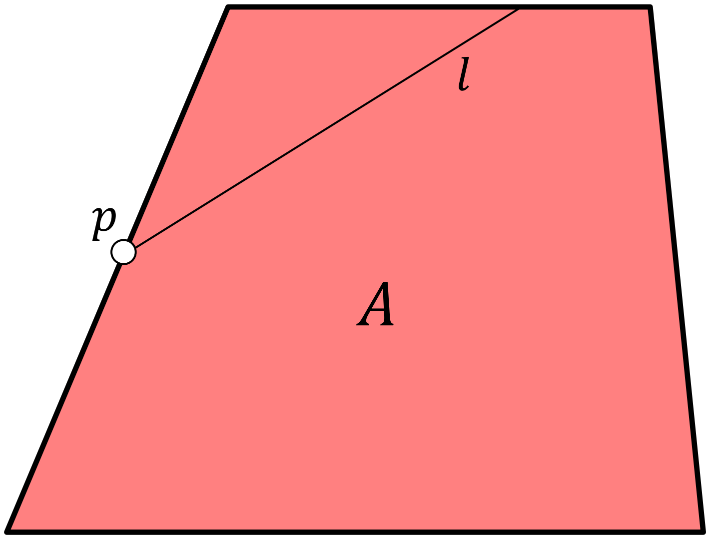

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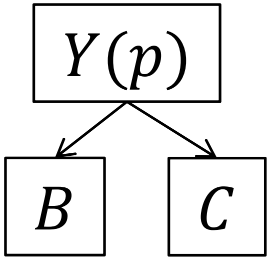

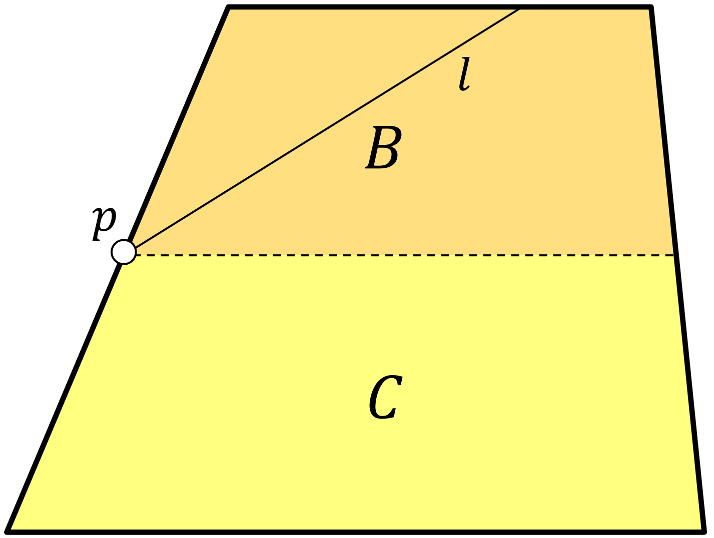

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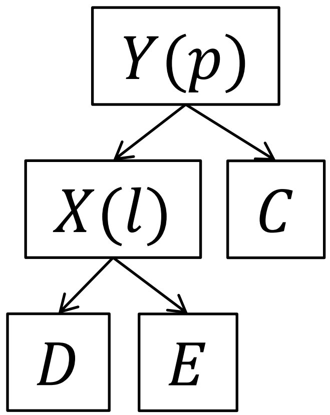

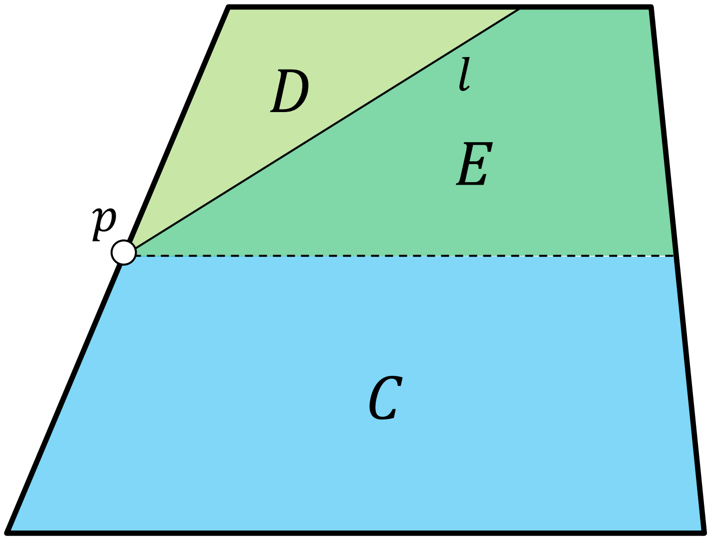

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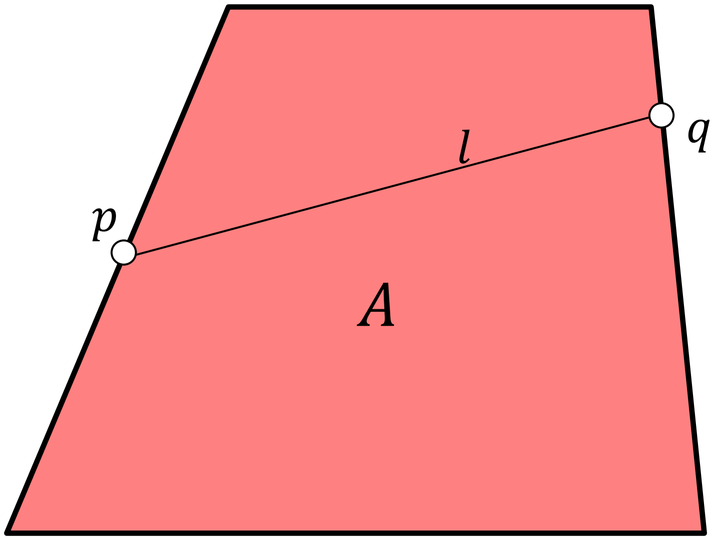

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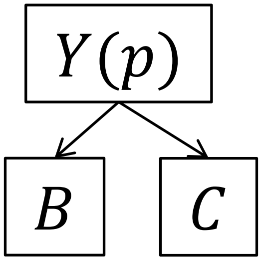

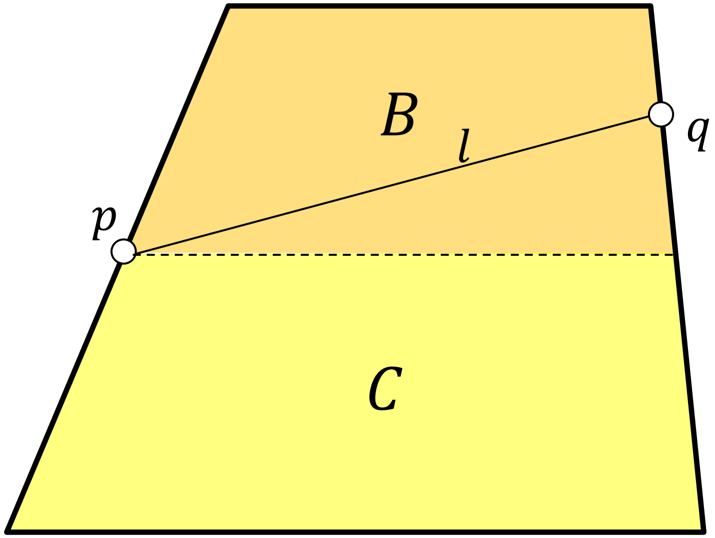

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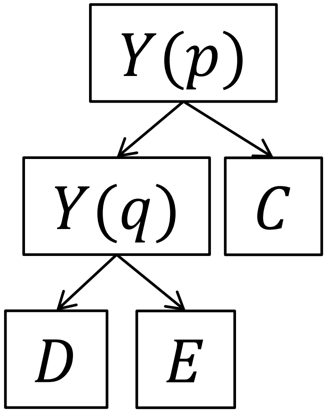

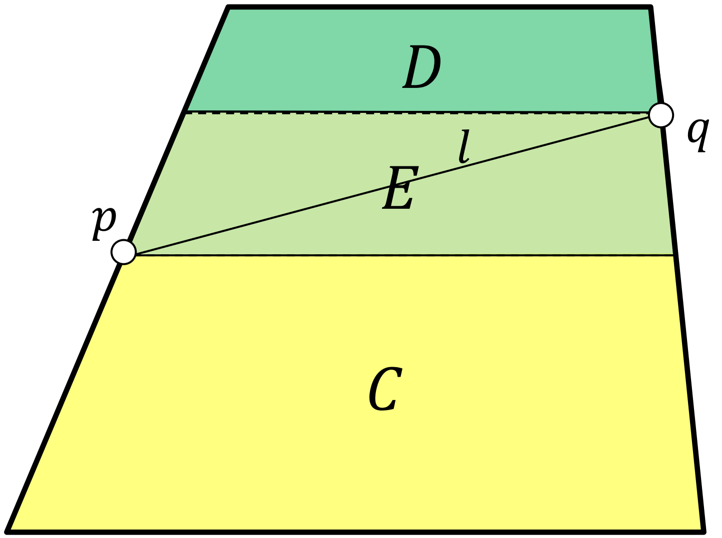

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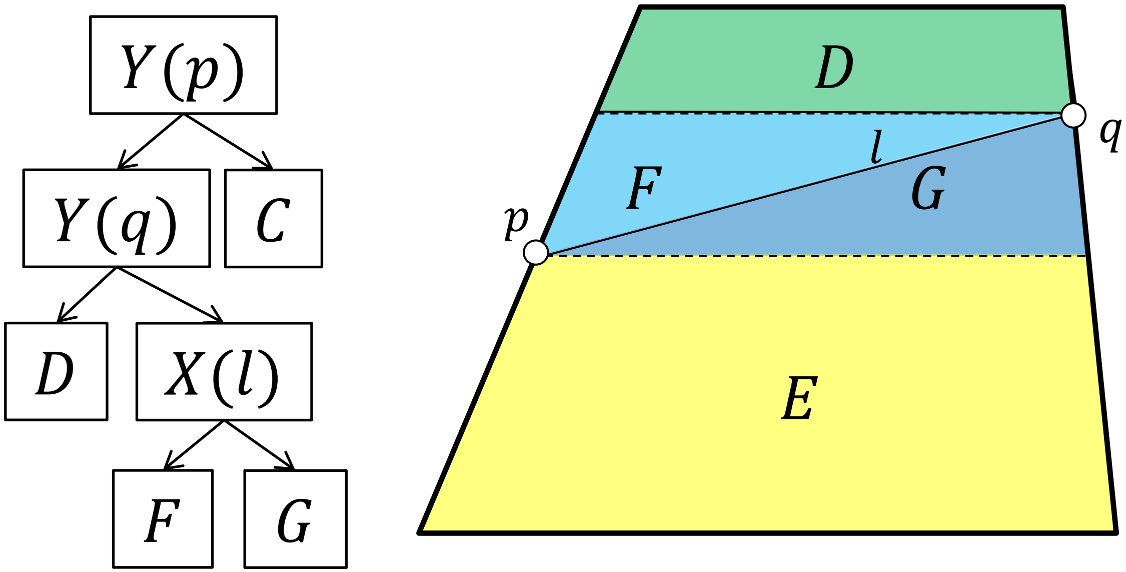

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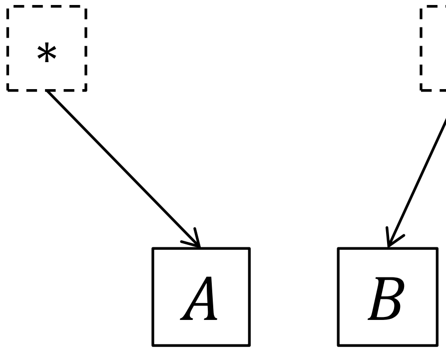

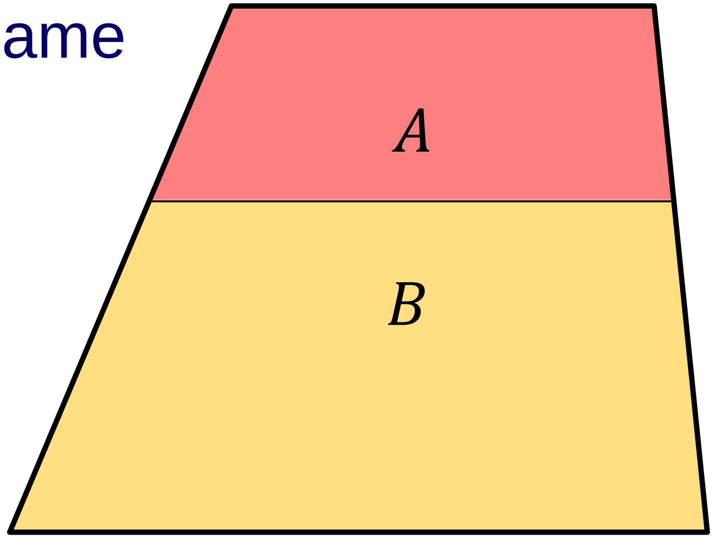

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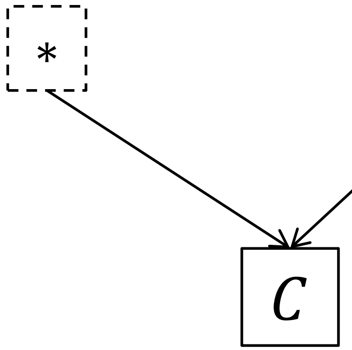
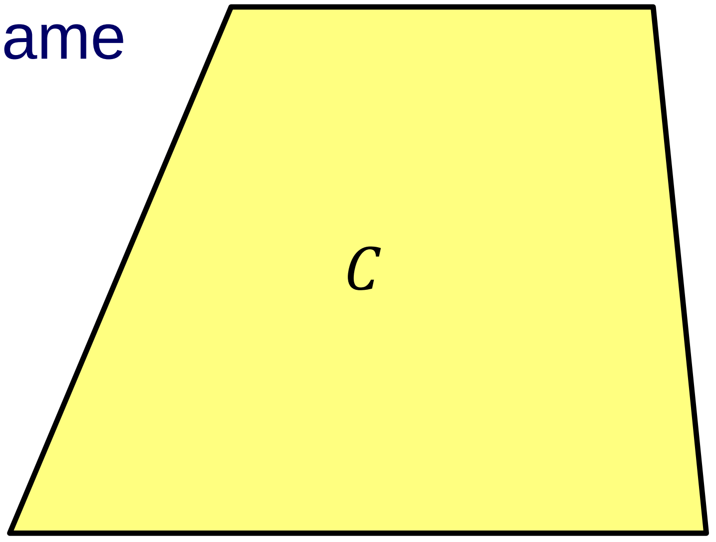




Trapezoidal Decomposition

Approach:

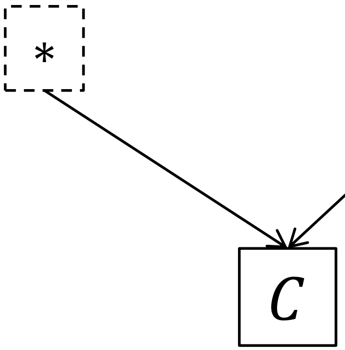
We get a directed binary tree with:

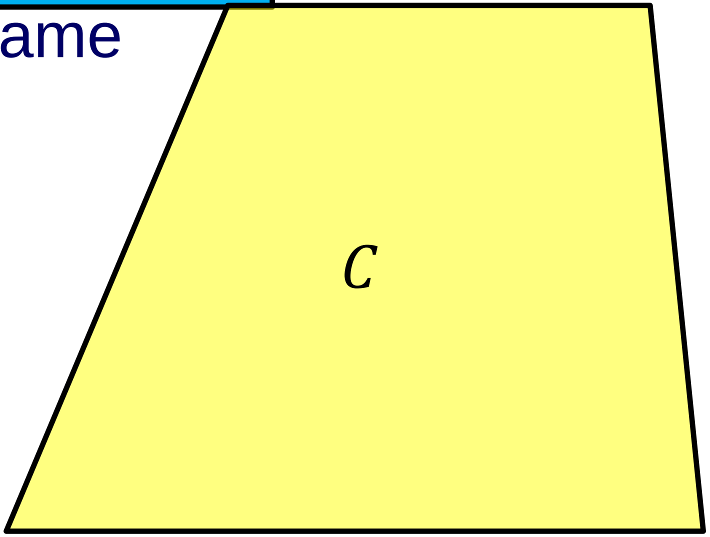
- interior nodes ⇒ left/right or top/bottom partitions
- leaves ⇒ trapezoids

  - Add line
  - Merge v
    if they are bounded by the same edges (in the same order).

Left child: above/left

Right child: below/right





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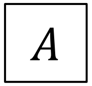

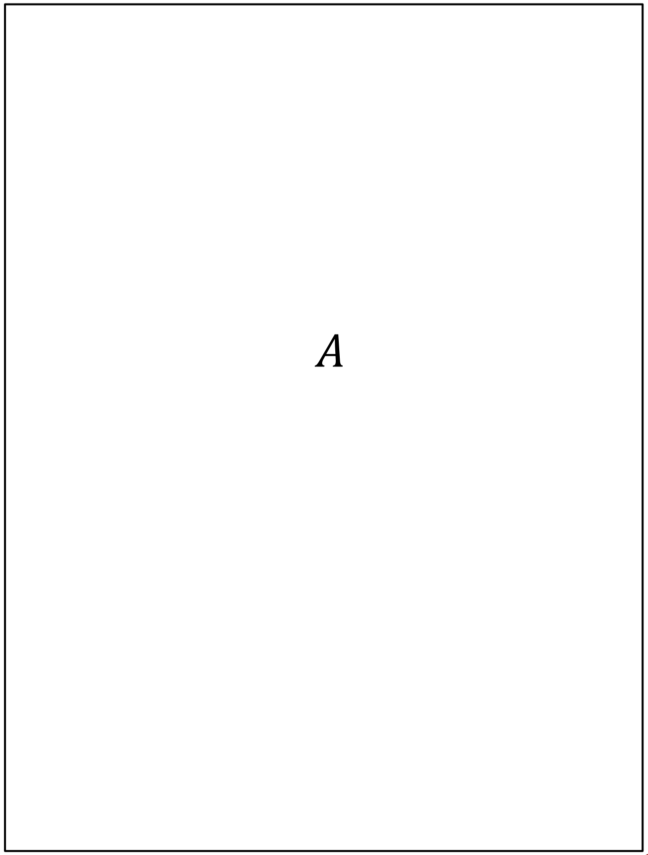

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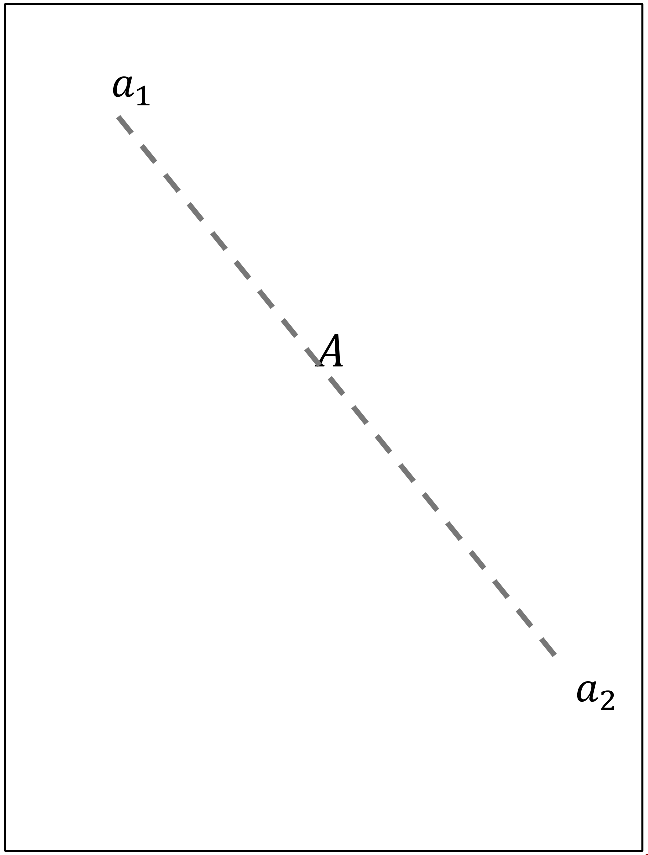

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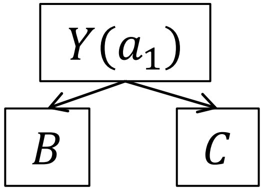

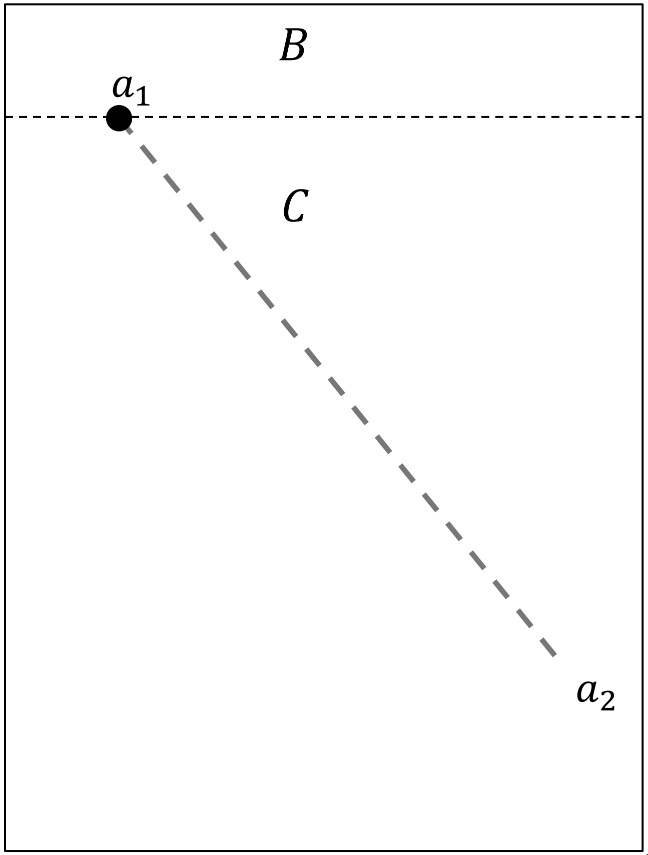

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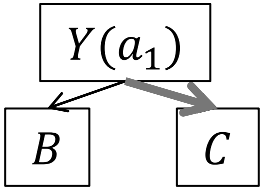

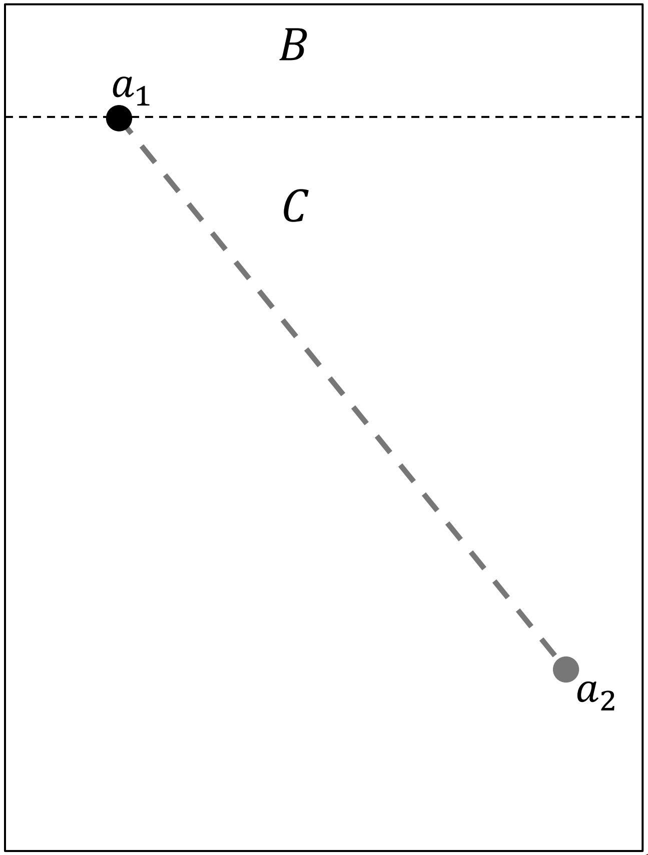

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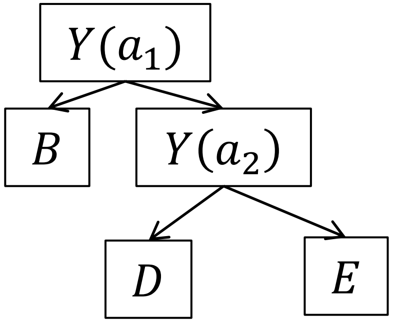

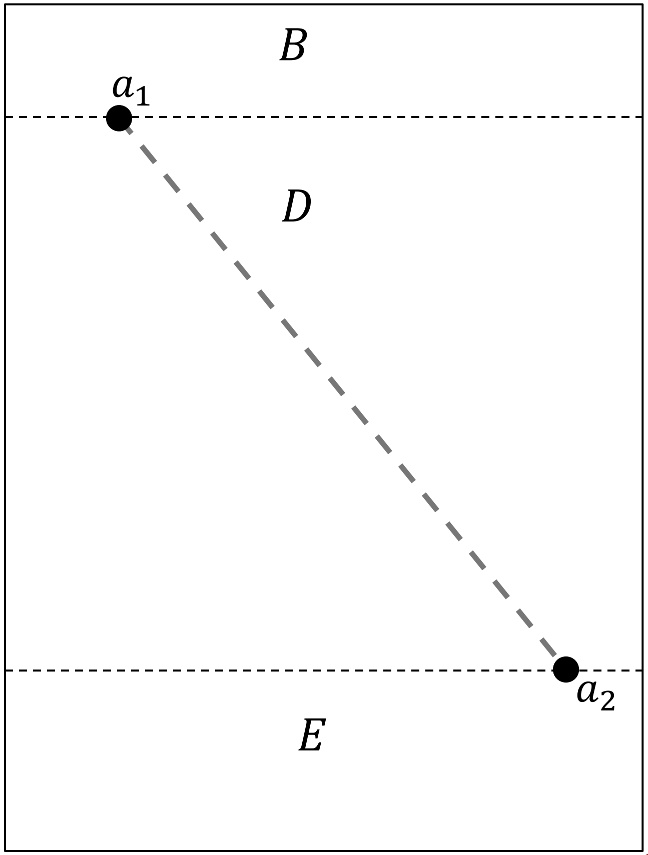

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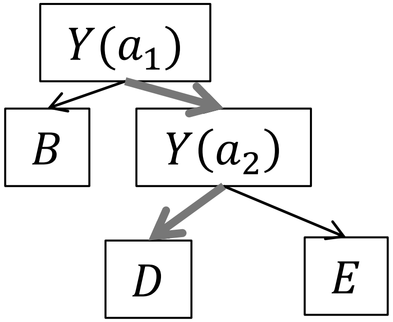

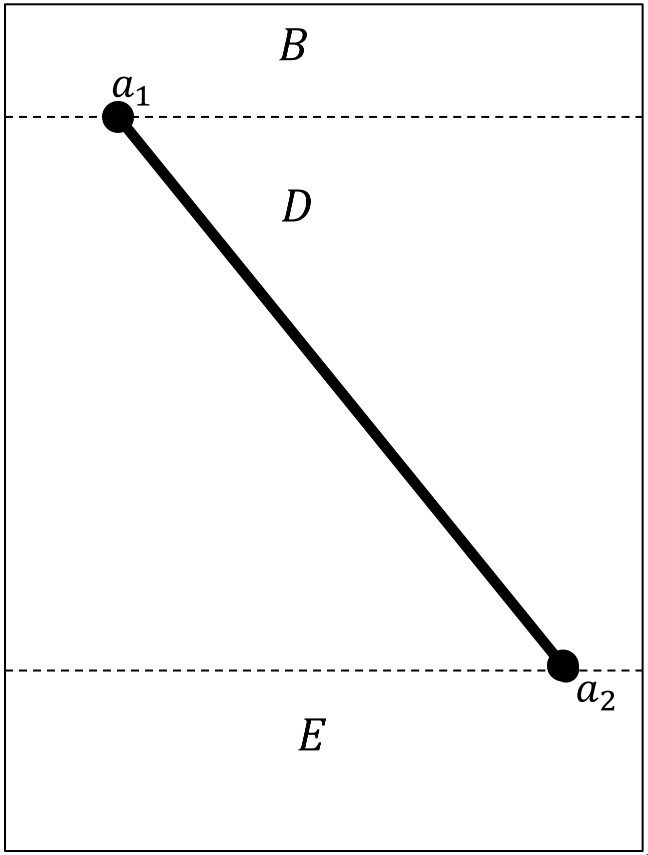

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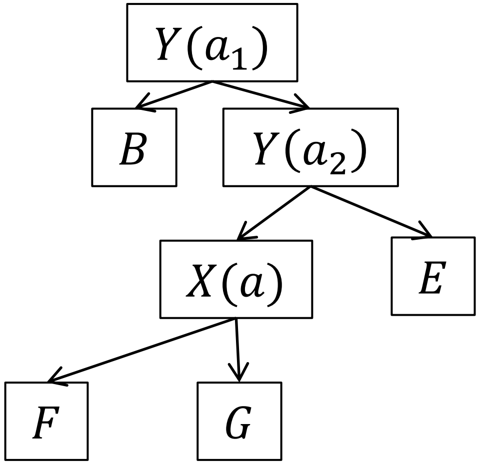

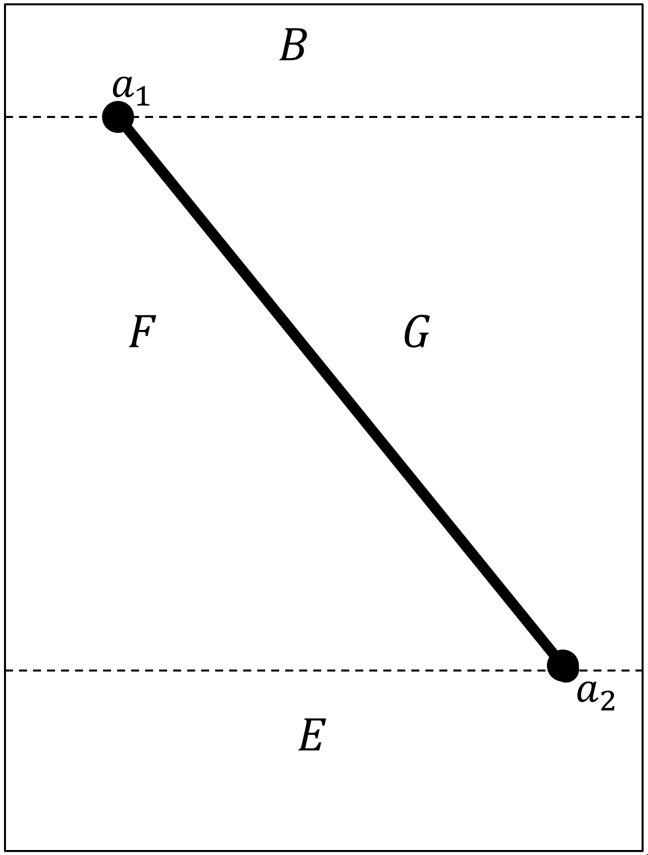

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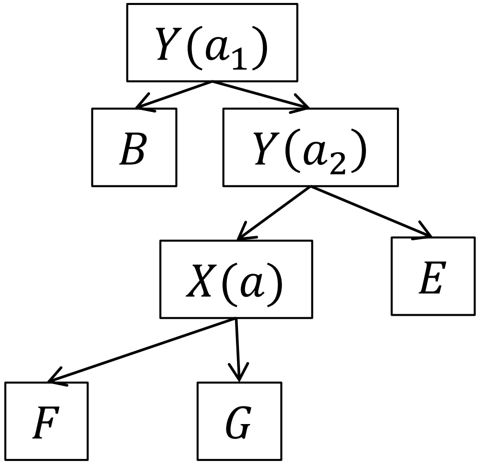

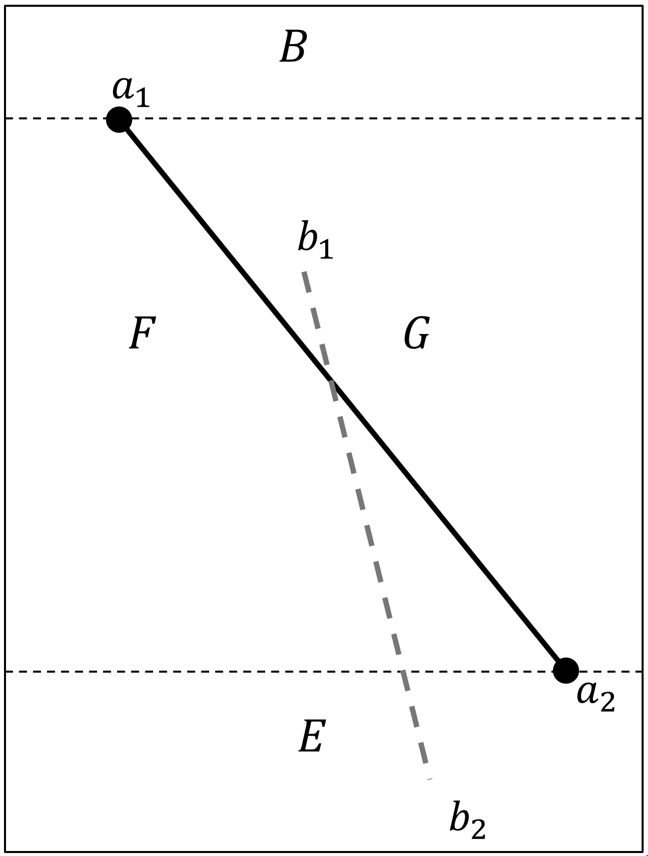

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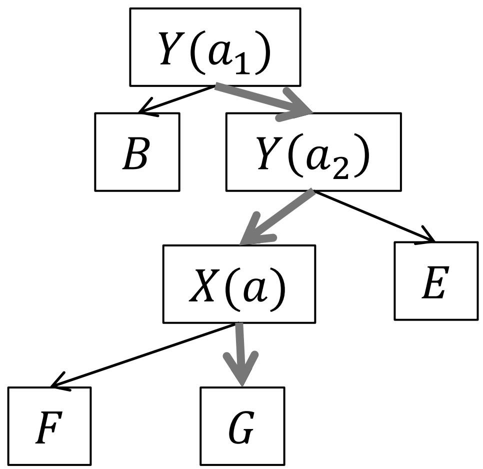

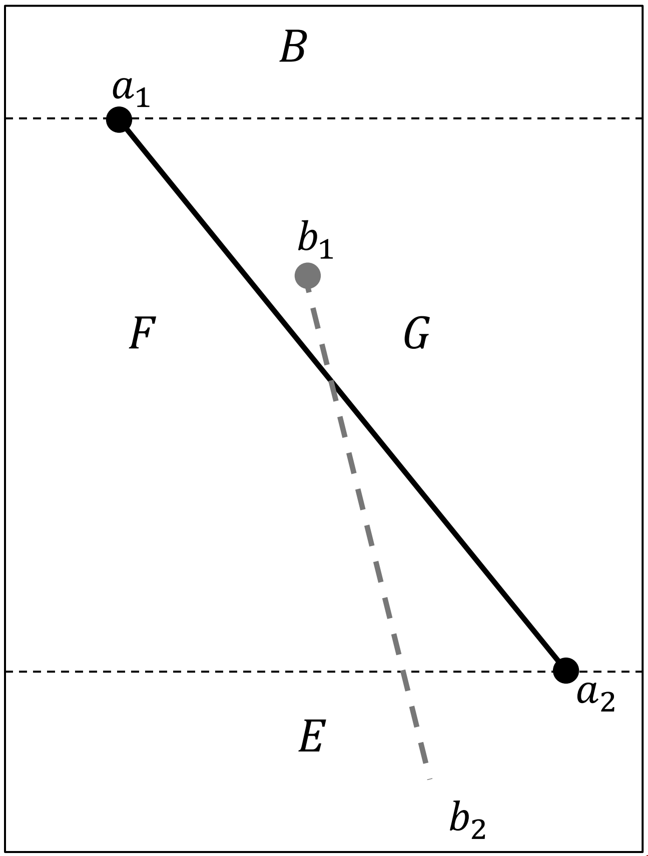

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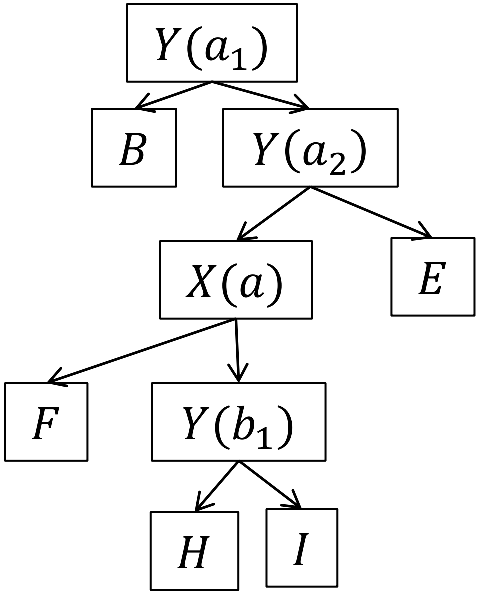

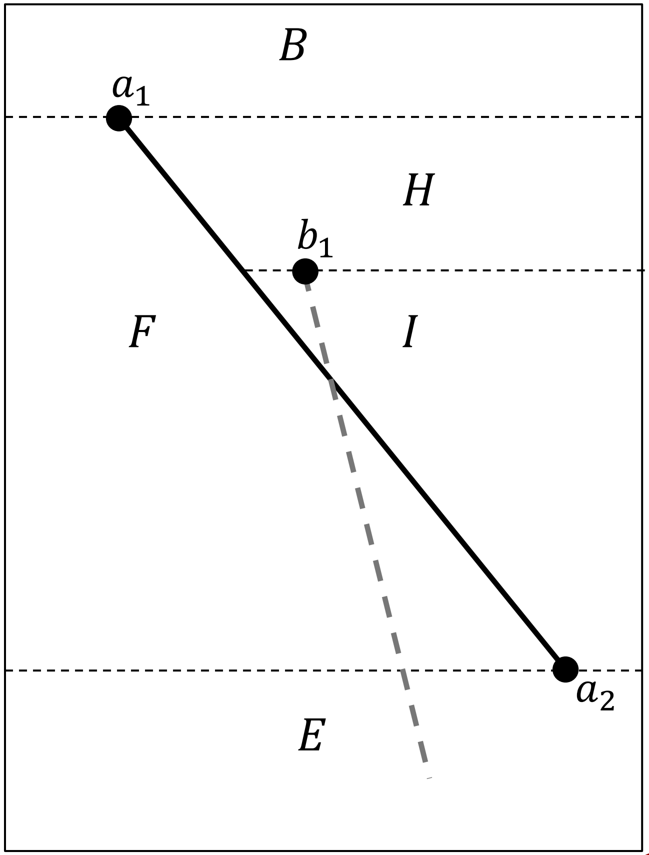

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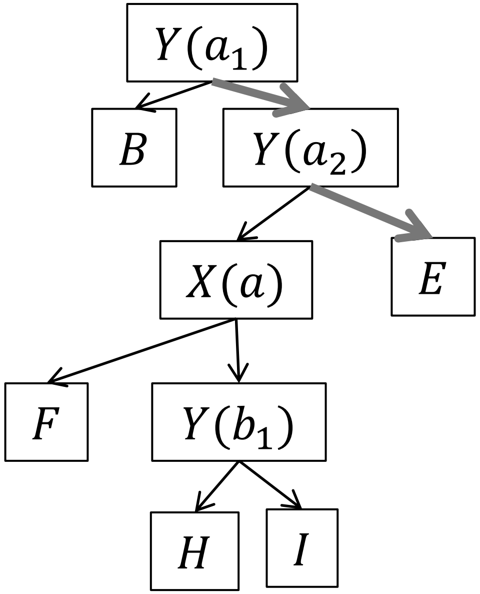

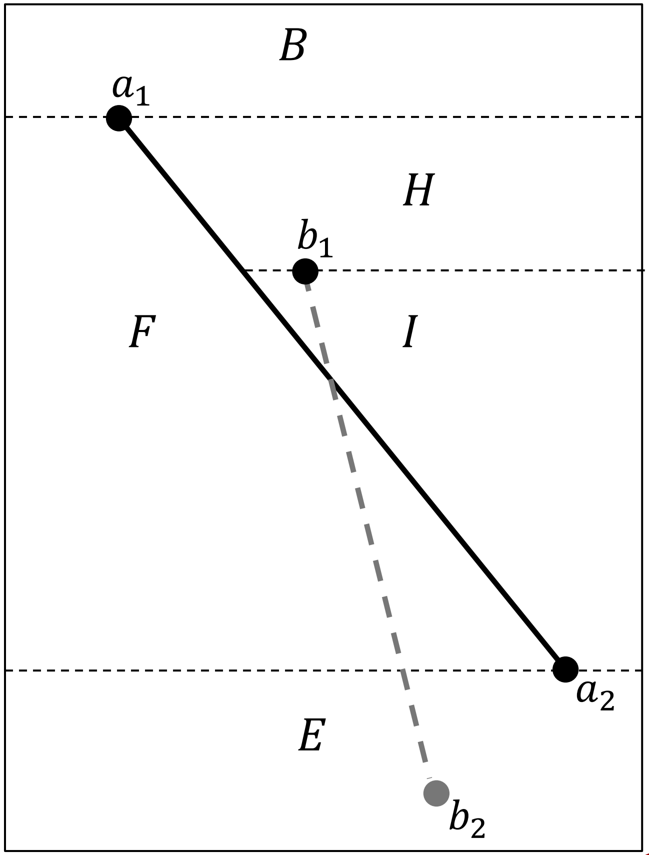

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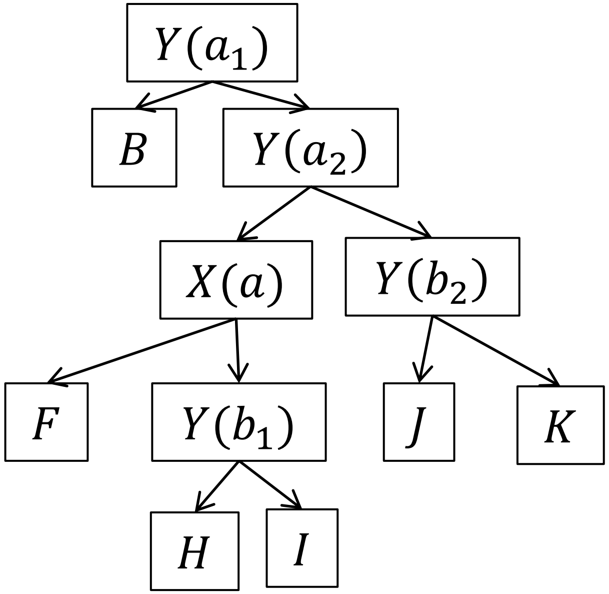

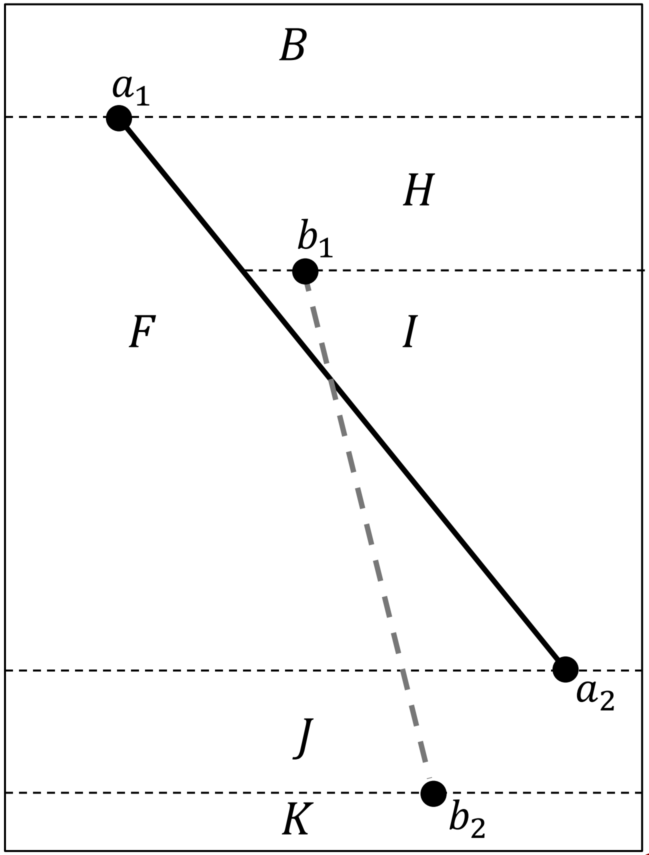

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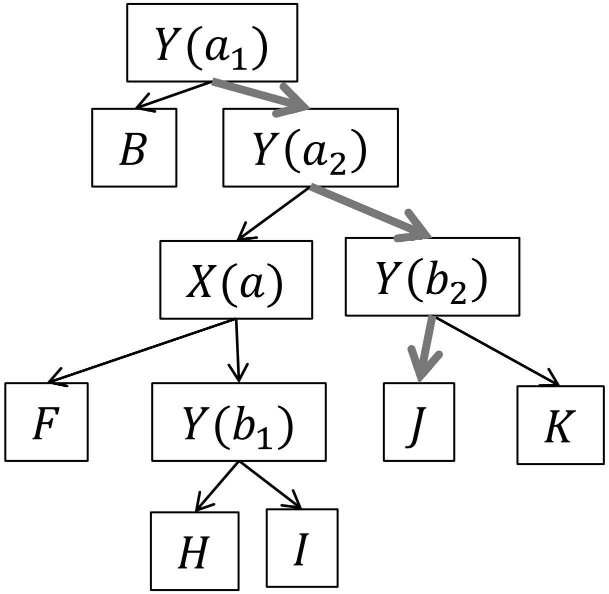

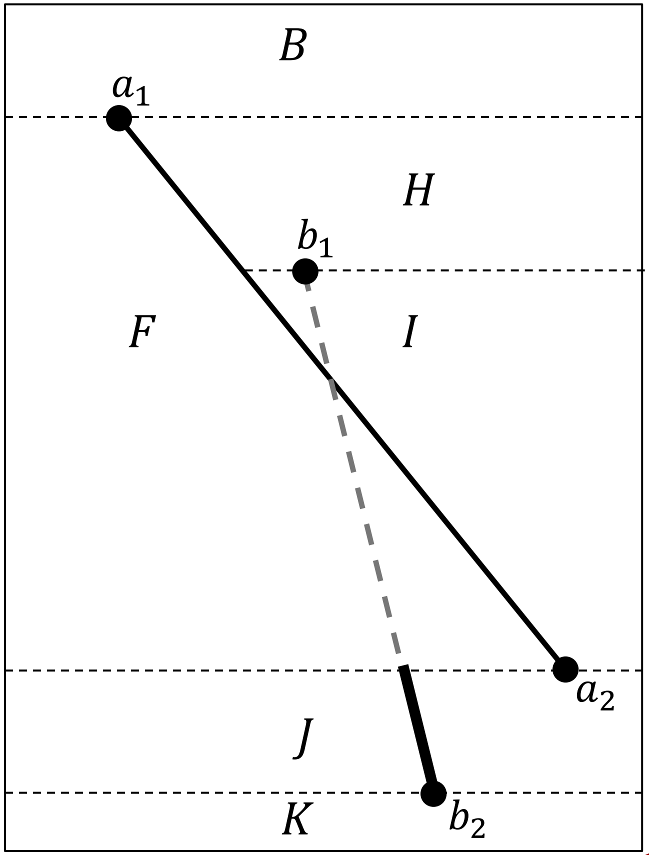

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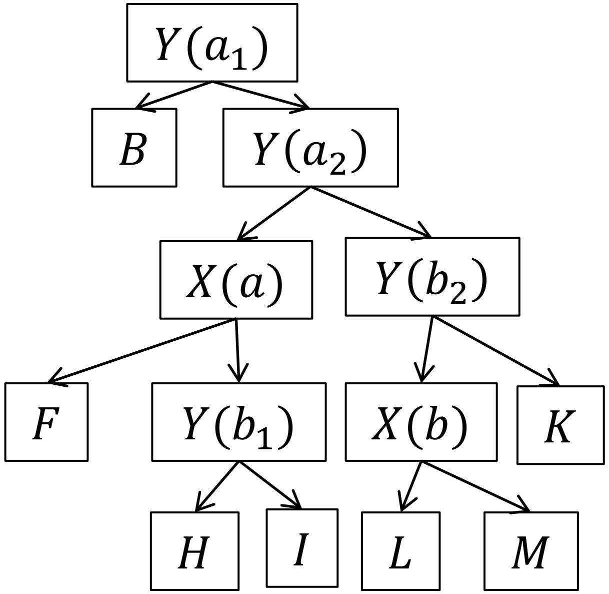

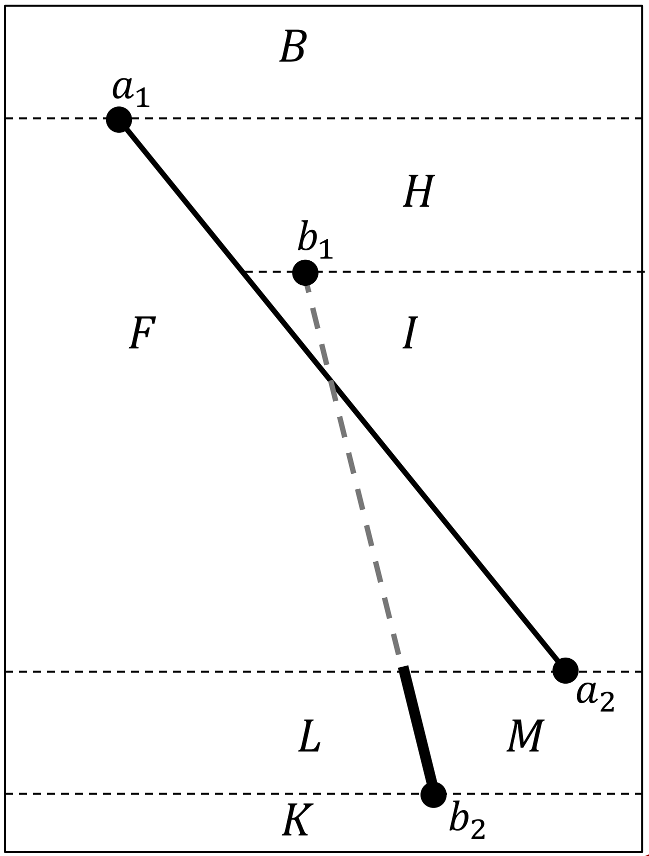

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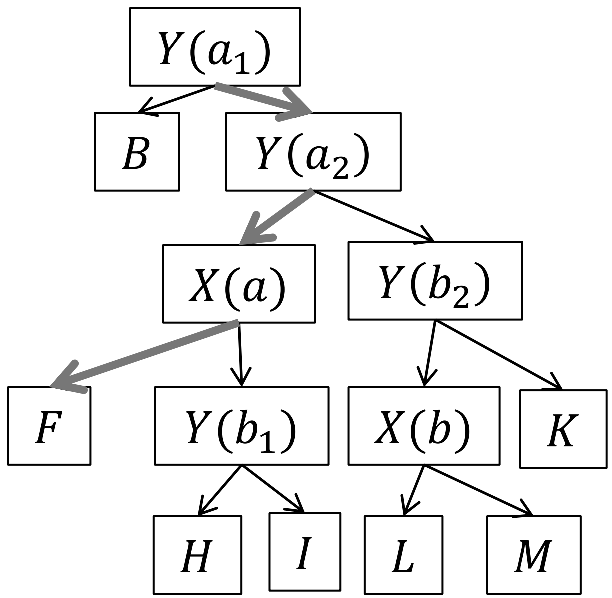

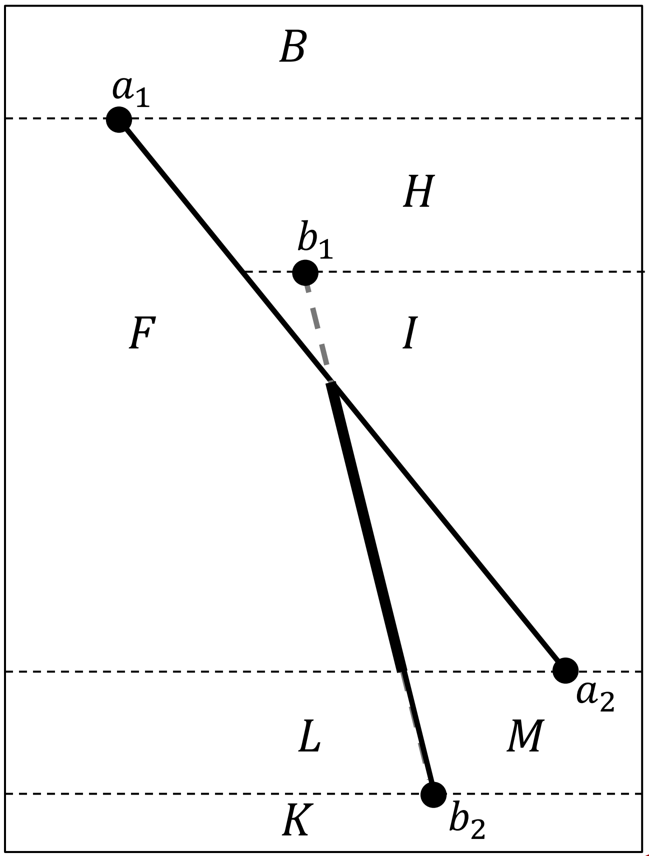

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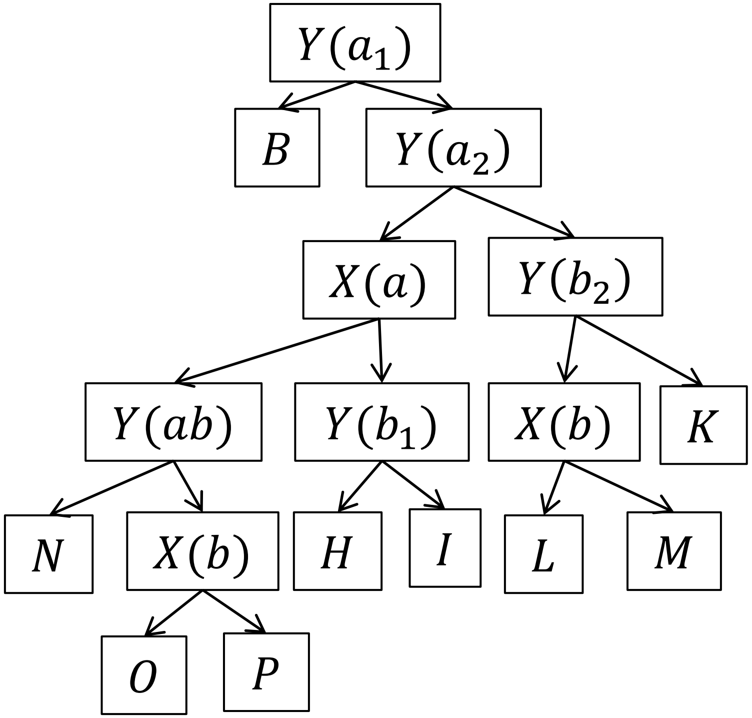

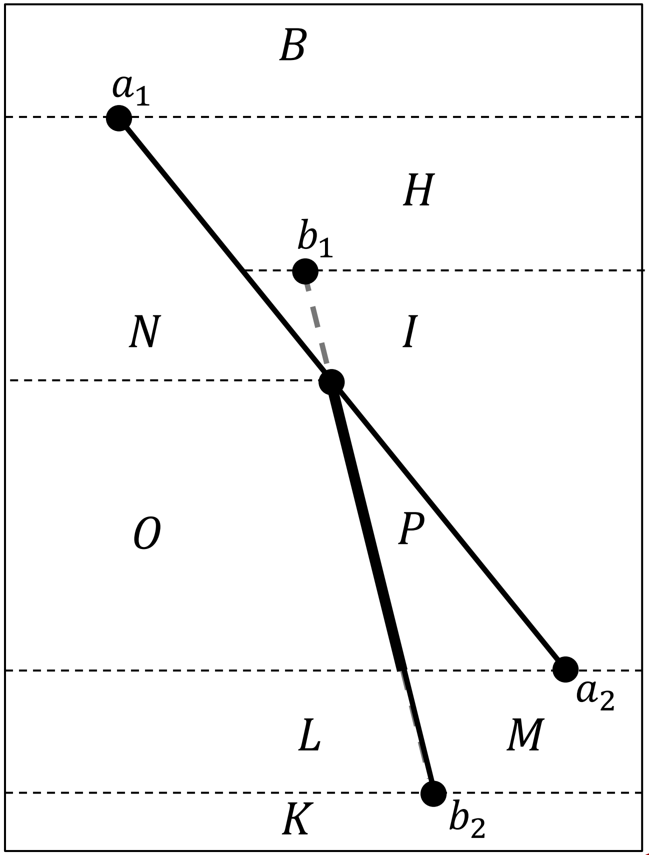

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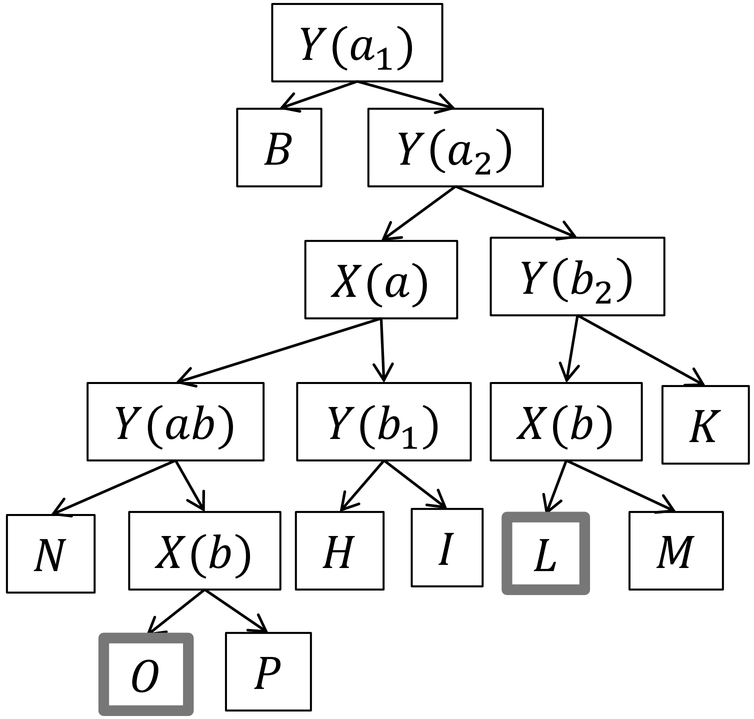

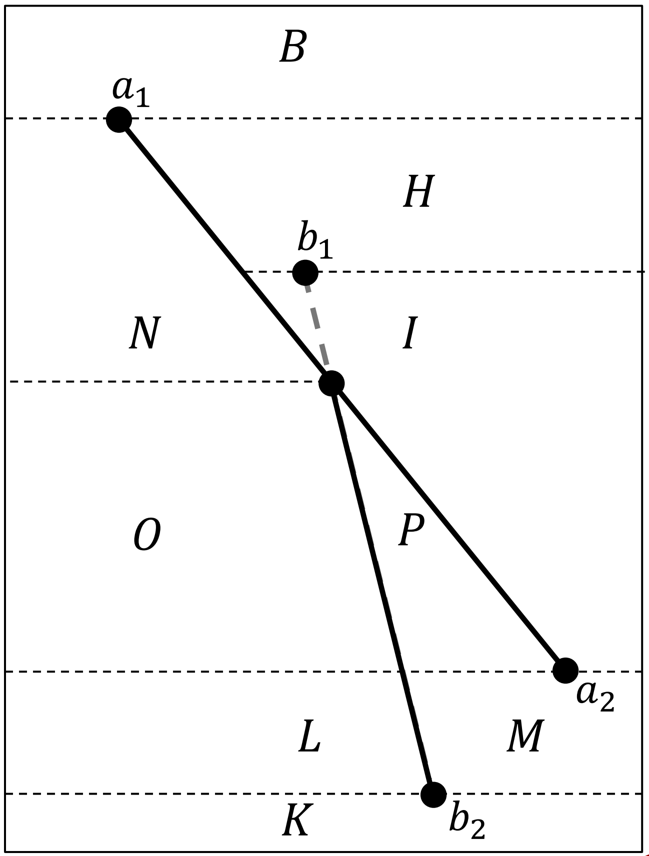

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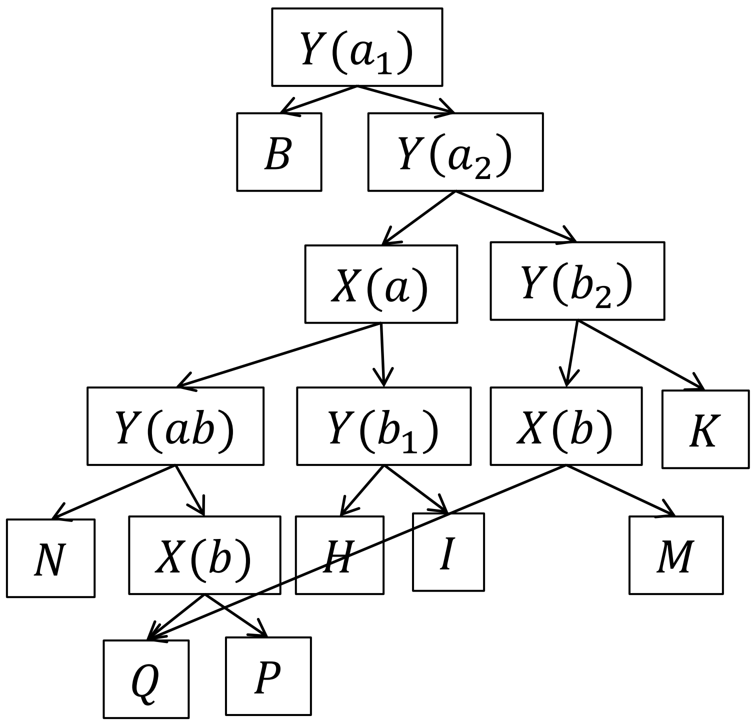
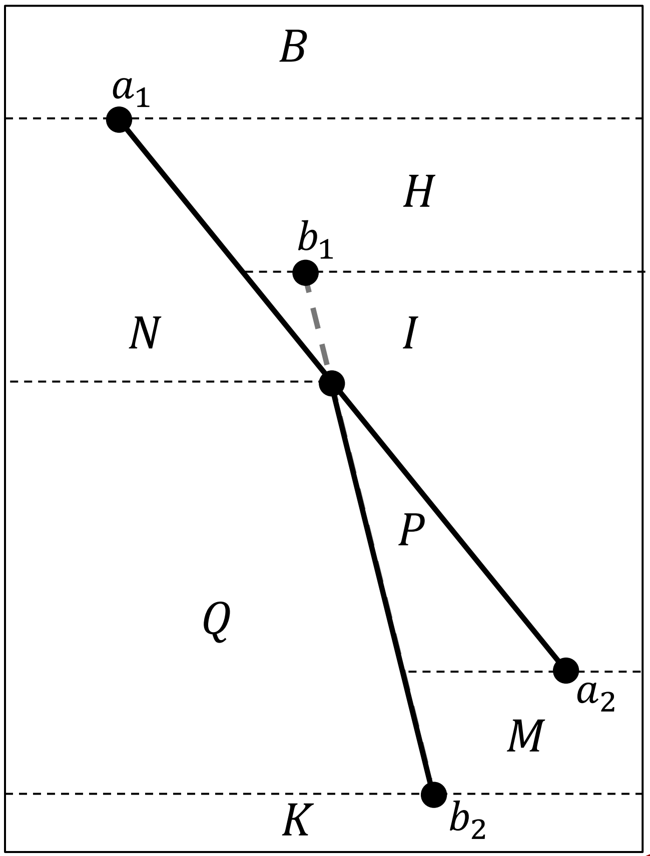
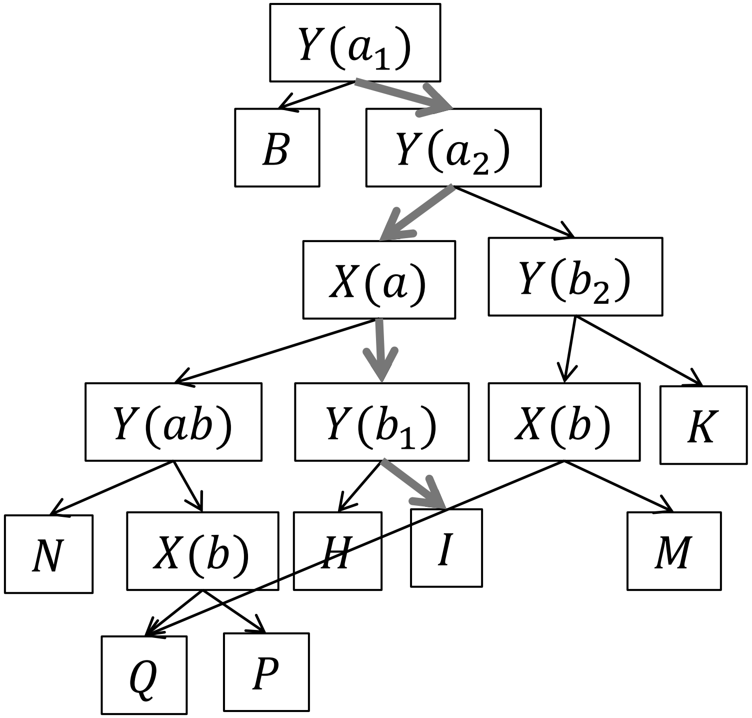
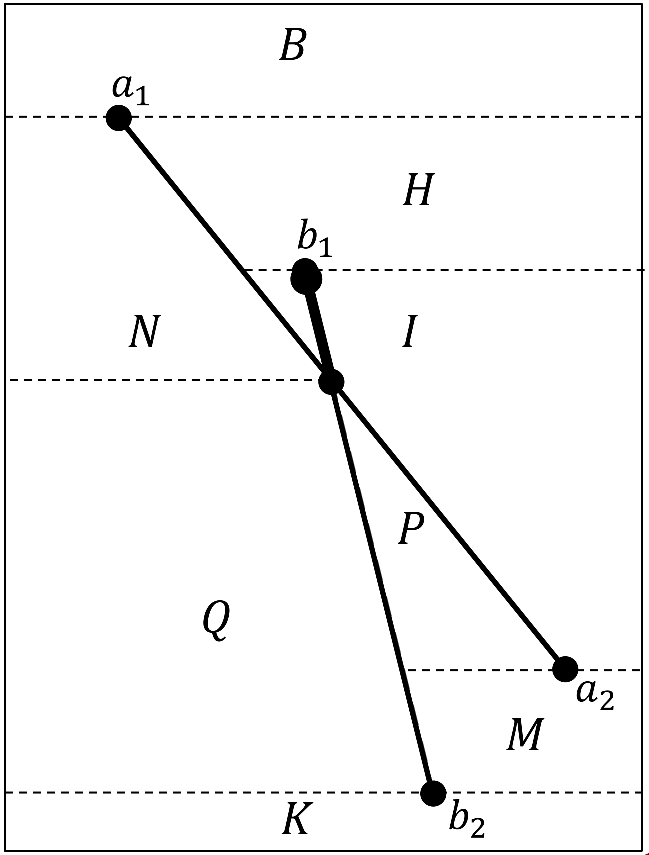




Trapezoidal Decomposition

- Add end-point
- **Add line segment**
- Merge adjacent trapezoids





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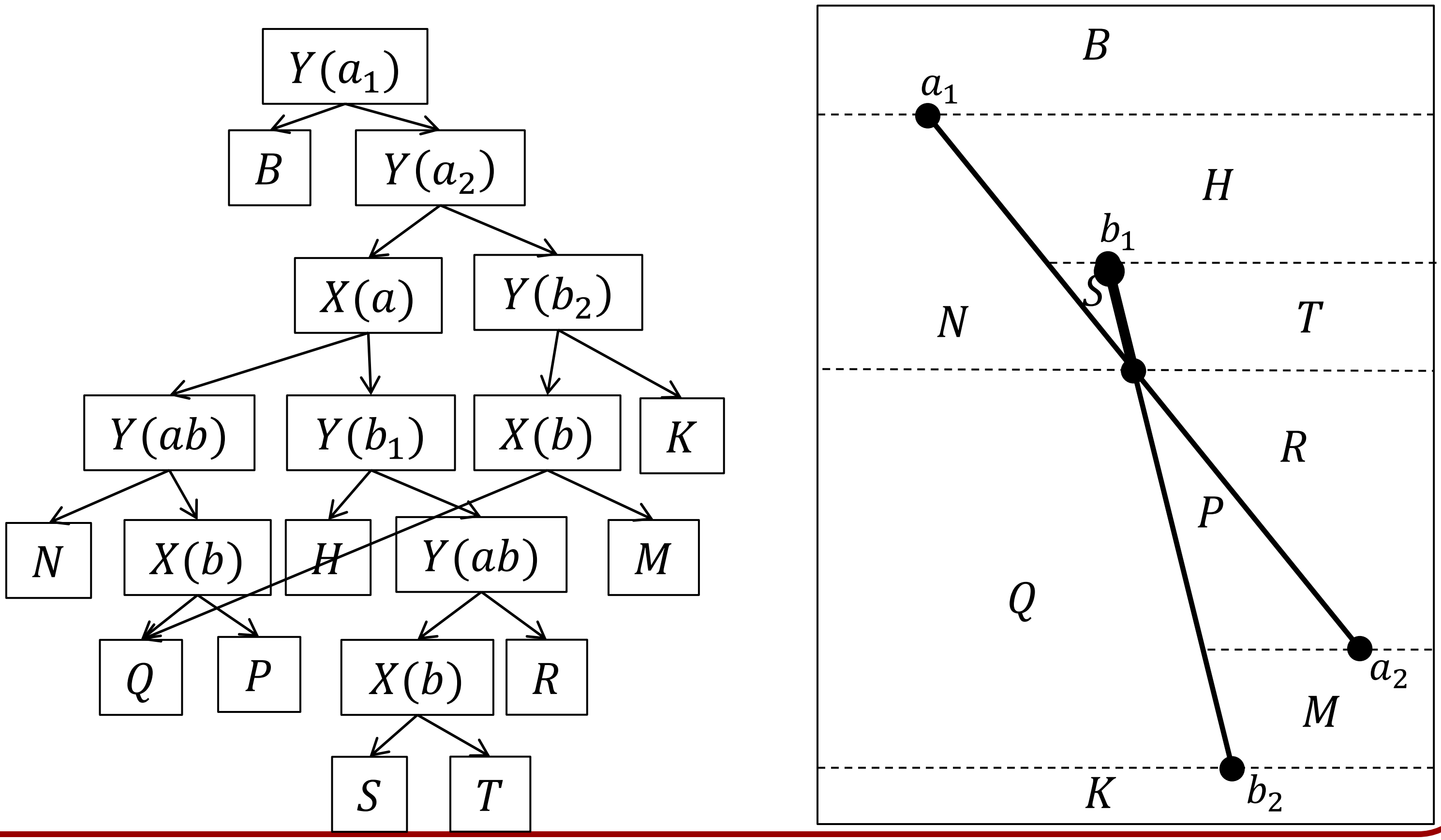
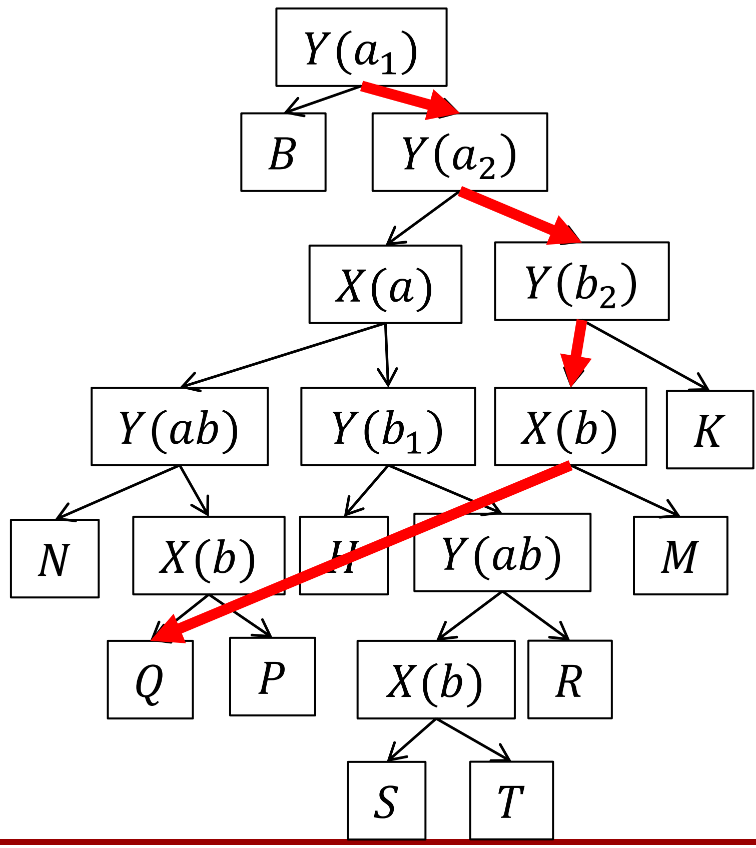
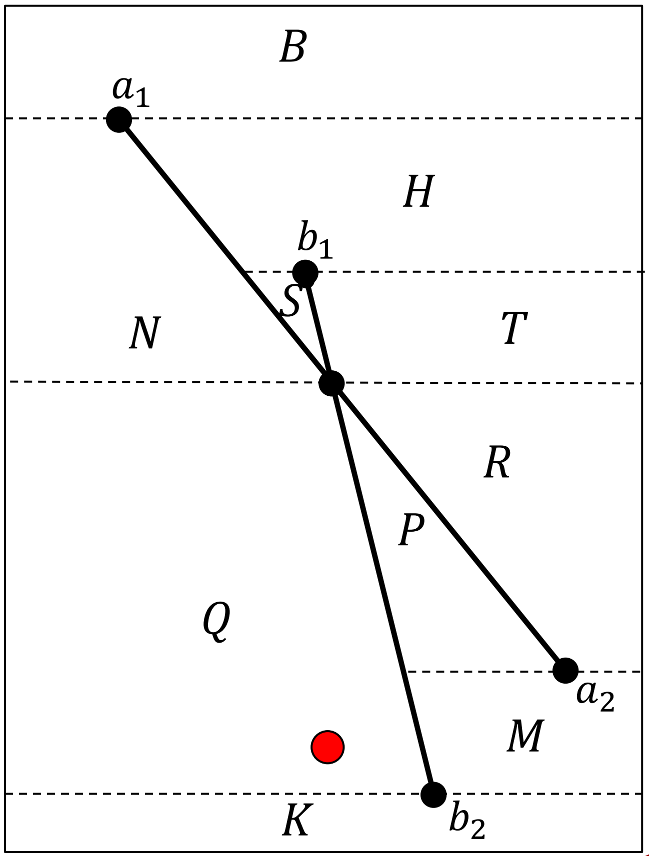


Trapezoidal Decomposition

Because of merging, we can have multiple paths into the same trapezoid.





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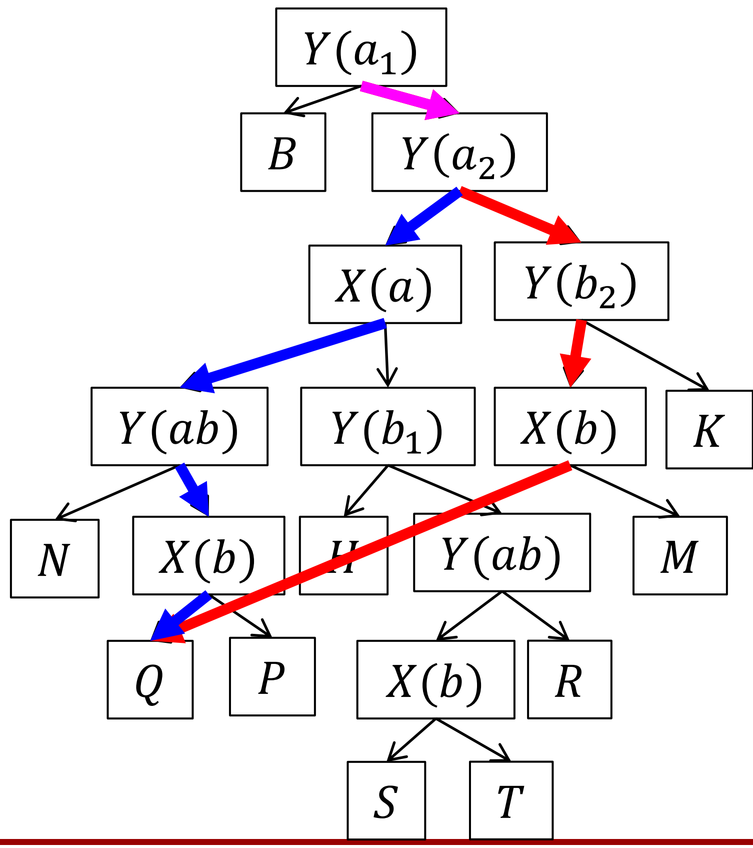
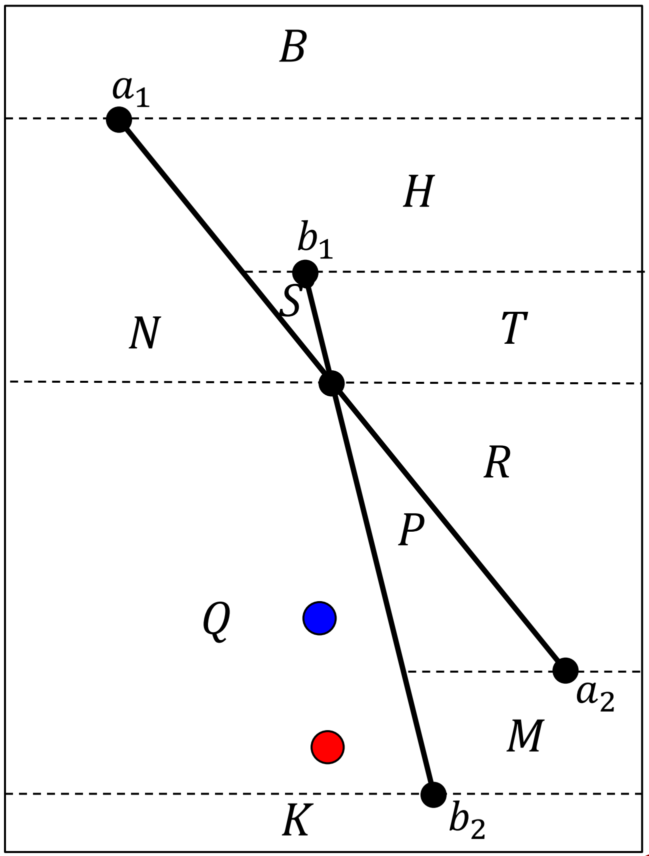


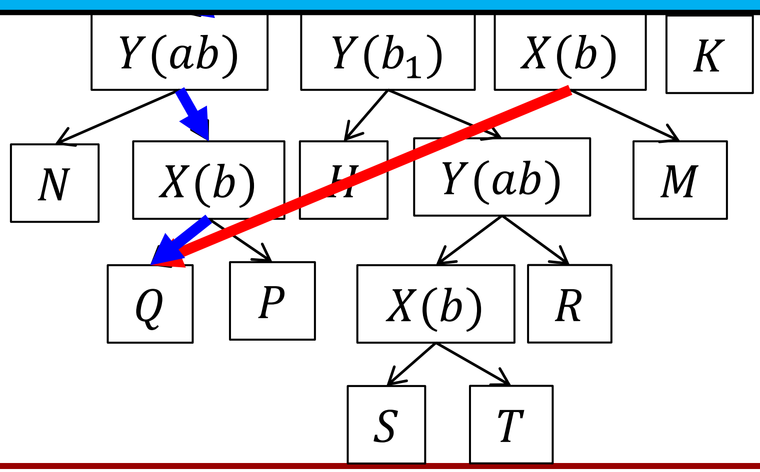


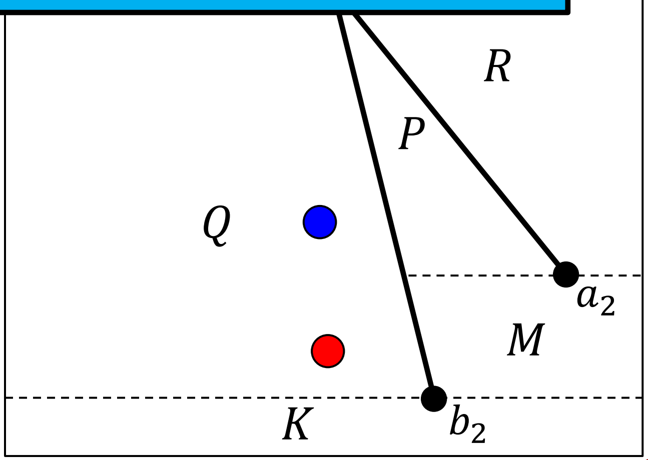
Trapezoidal Decomposition

Because of merging, we can have multiple paths into the same trapezoid.

Assuming the tree stays balanced, construction has complexity $O((n+k)\log n)$ and query has complexity $O(\log n)$.

If line segments are added in random order, the tree will be well-balanced, with high probability.





Outline

- Trapezoidal Decomposition
- Extreme Points (2D)
- Extreme Points (3D)

Extreme Points

Linear Programming:

Given a set of linear constraints:

$$C_i = \{p \mid \langle p, n_i \rangle \geq d_i\}$$

and a linear energy function:

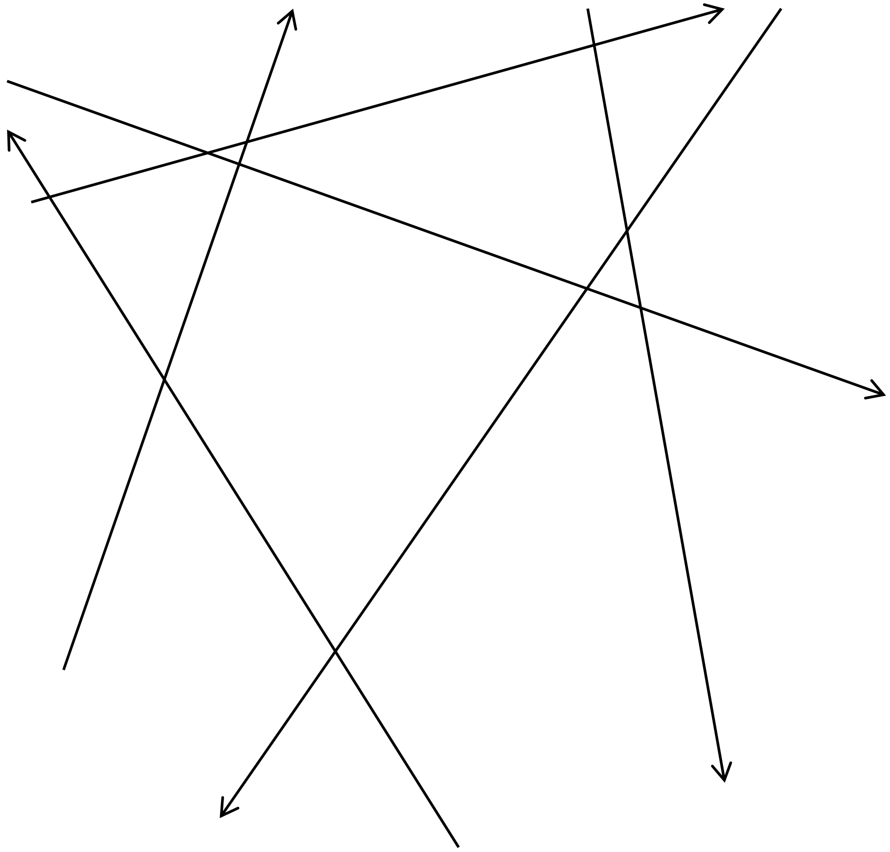
$$E(p) = \langle p, n \rangle + d$$

we would like to find the point p that satisfies the constraints and minimizes the energy.

Extreme Points

Linear Programming:

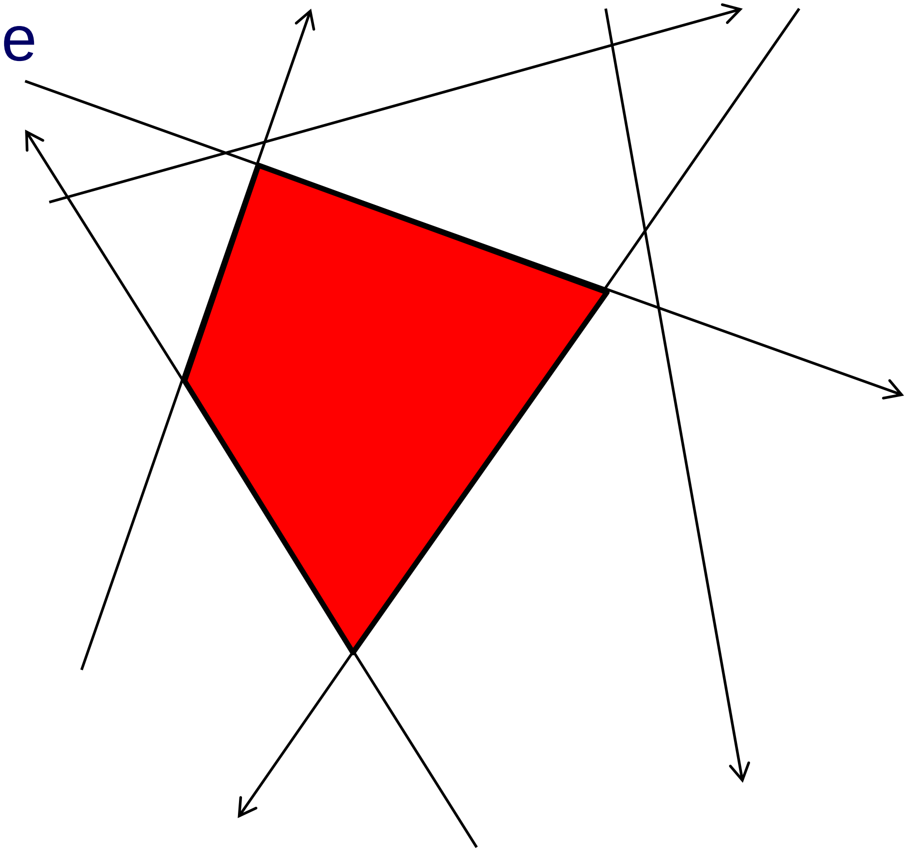
- Since the constraints are linear, each one defines a half-space of valid solutions.



Extreme Points

Linear Programming:

- Since the constraints are linear, each one defines a half-space of valid solutions.
- The intersection of these half-spaces is convex.



Extreme Points

Linear Programming:

- Since the constraints are linear, each one defines a half-space of valid solutions.
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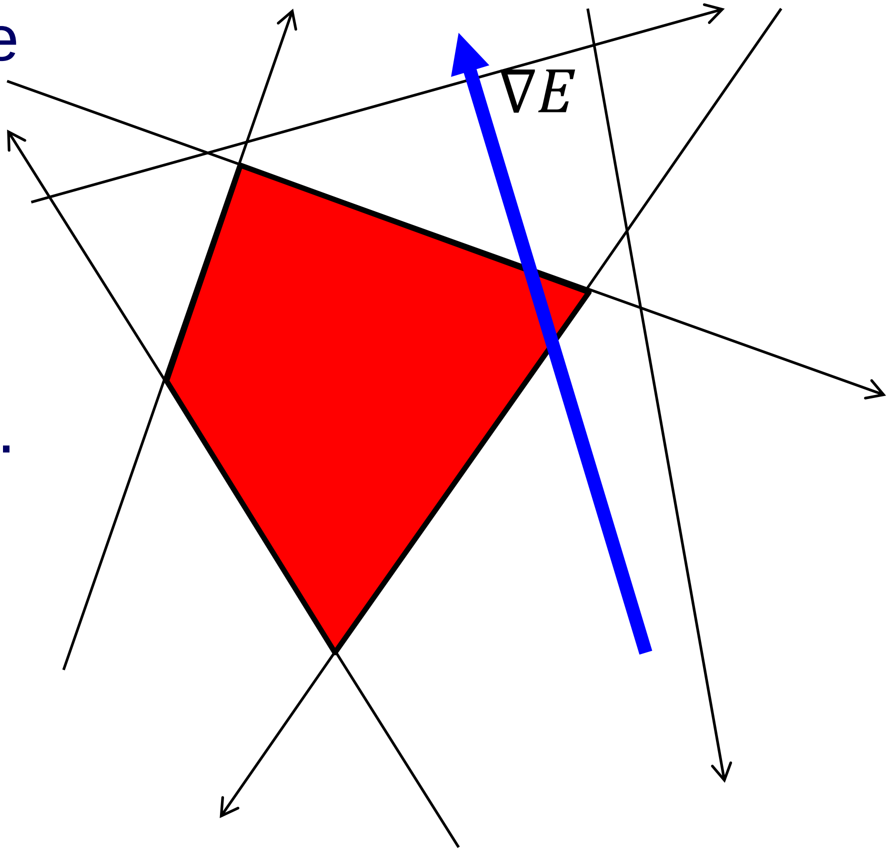
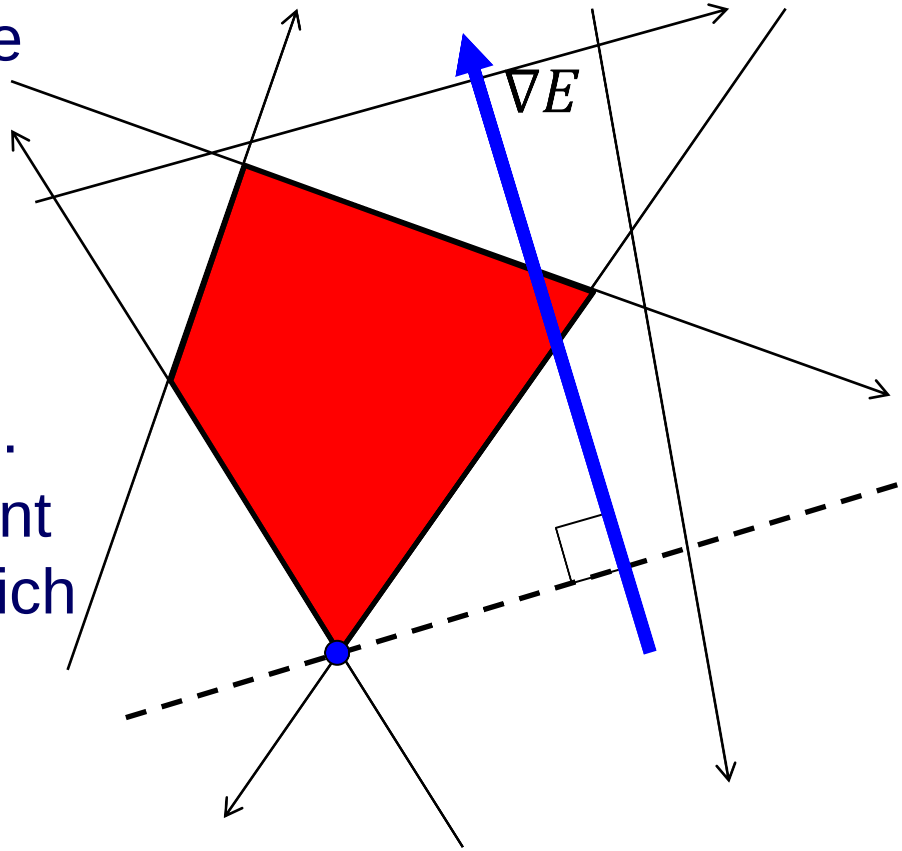
- Since the energy is linear, it has a constant gradient ∇E pointing away from the minimum.



Extreme Points

Linear Programming:

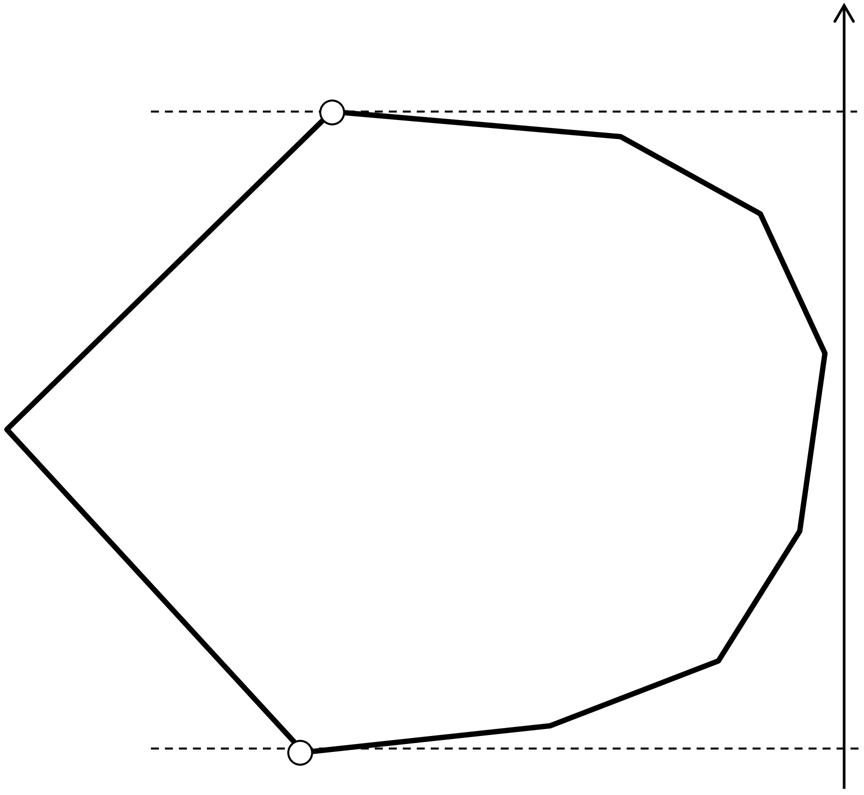
- Since the constraints are linear, each one defines a half-space of valid solutions.
- The intersection of these half-spaces is convex.
- Since the energy is linear, it has a constant gradient ∇E pointing away from the minimum.
- The minimizer is the point in the convex region which is extreme along the gradient direction, ∇E.



Extreme Points (2D)

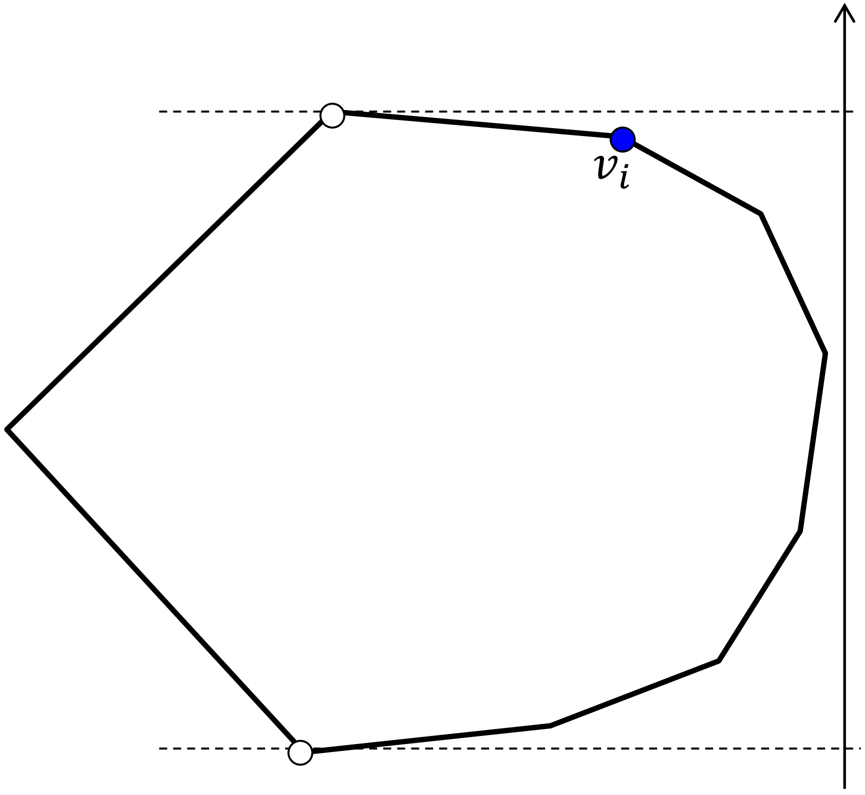
Given a convex polygon P, we would like to find the extreme points along a particular direction.

Without loss of generality, we can assume that the direction is vertical.



Extreme Points (2D)

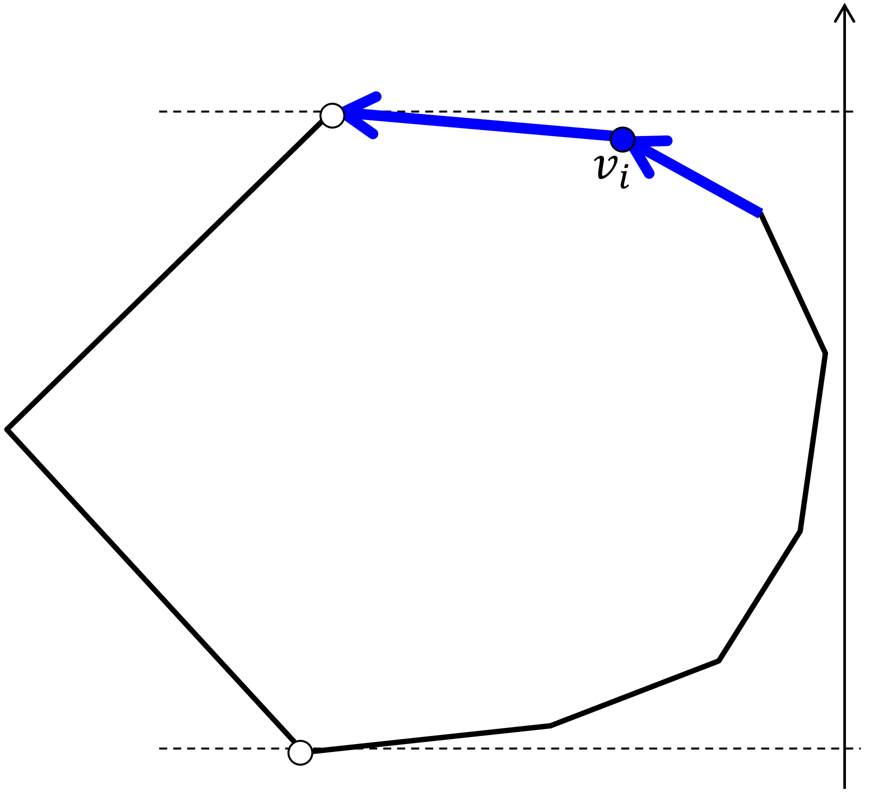
Pick a vertex v_i at random:



Extreme Points (2D)

Pick a vertex v_i at random:

- Incoming and outgoing edge rising ⇒ Right



Extreme Points (2D)

Pick a vertex v_i at random:

- Incoming and outgoing edge rising ⇒ Right
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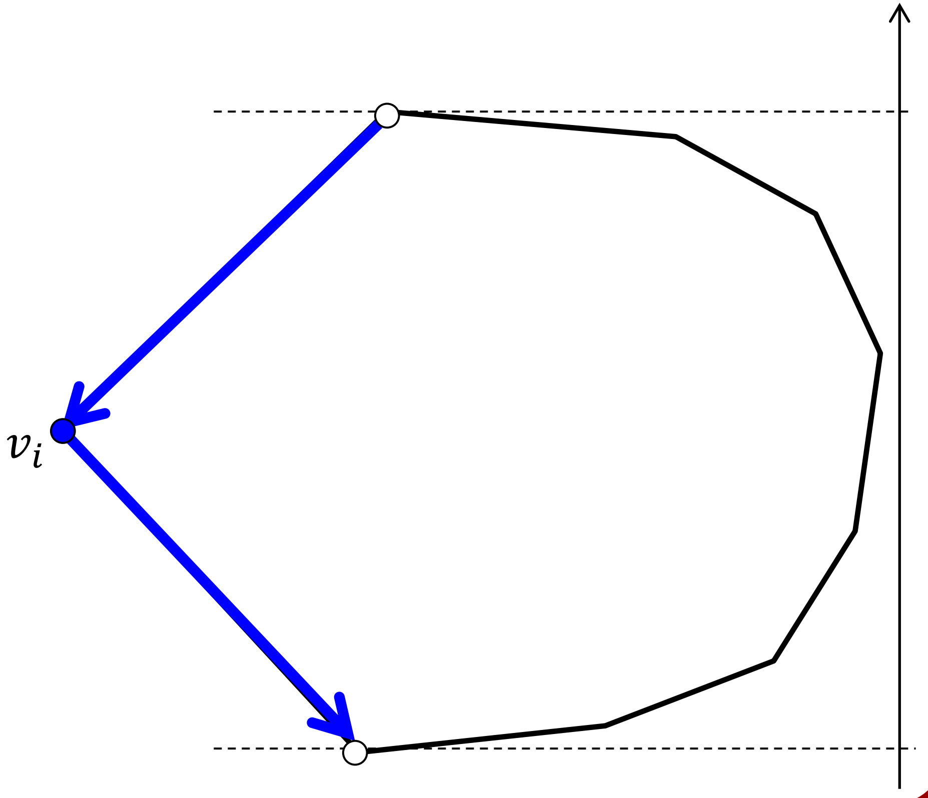

Extreme Points (2D)

Pick a vertex v_i at random:

- Incoming and outgoing edge rising ⇒ Right
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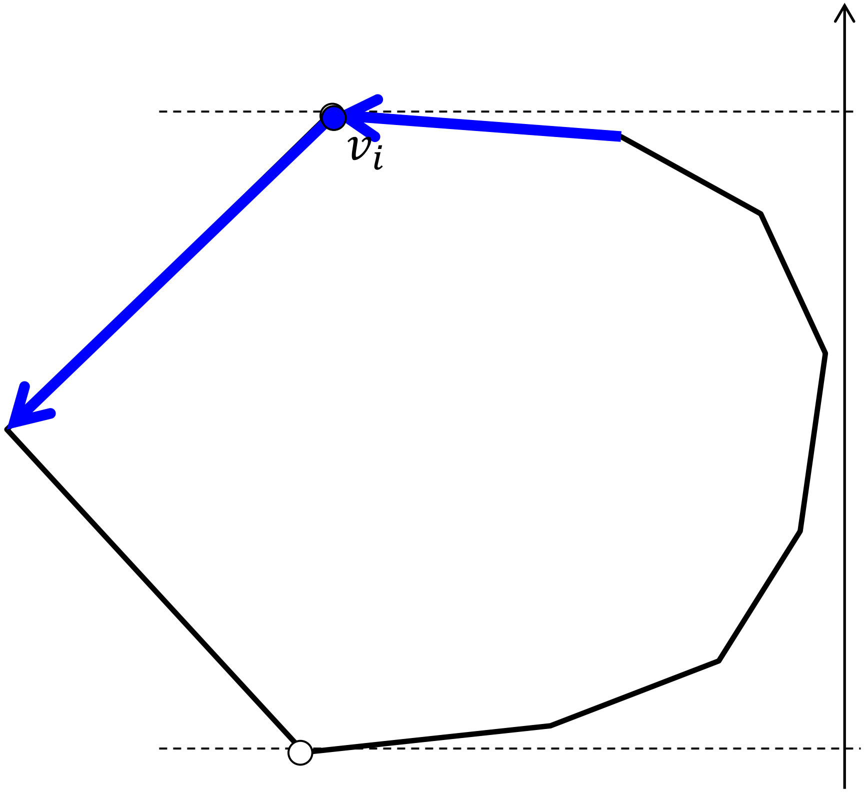
- Otherwise v_i is extremal



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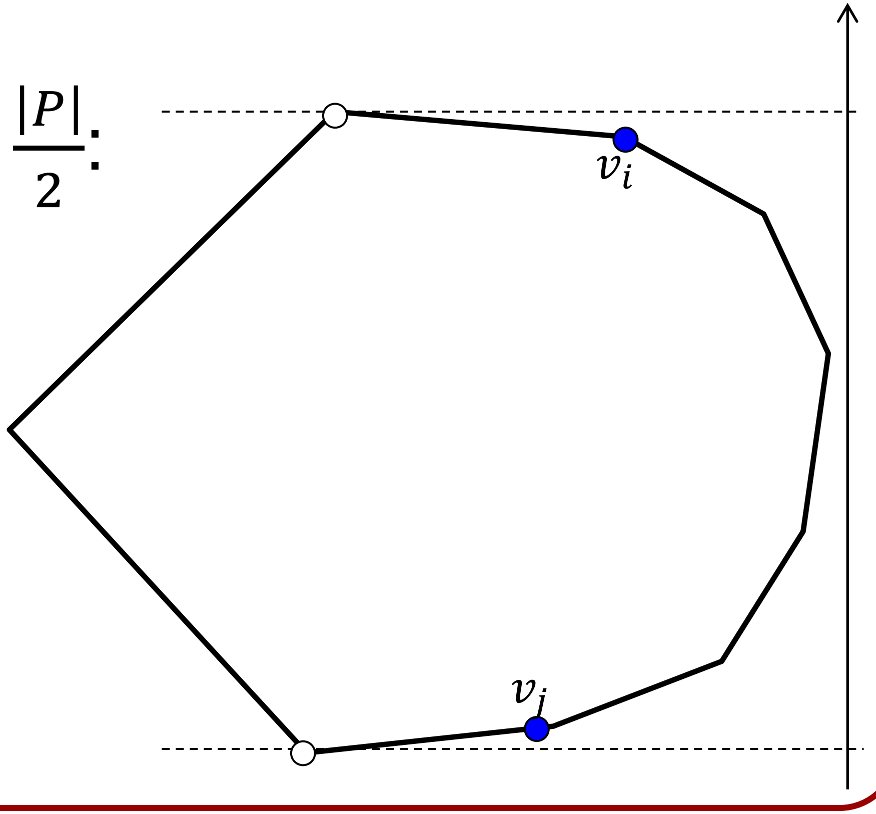
Consider vertex v_j, $j = i + \frac{|P|}{2}$:



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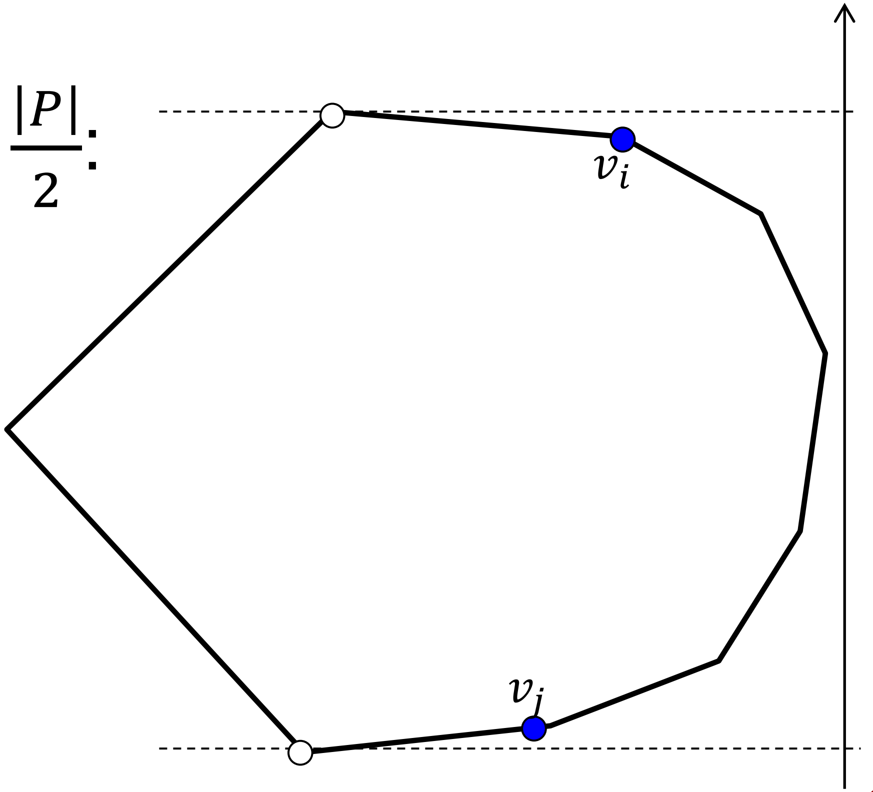
- If v_i is left and v_j right:
  ⇒ max ∈ $[j, i]$ min ∈ $[i, j]$



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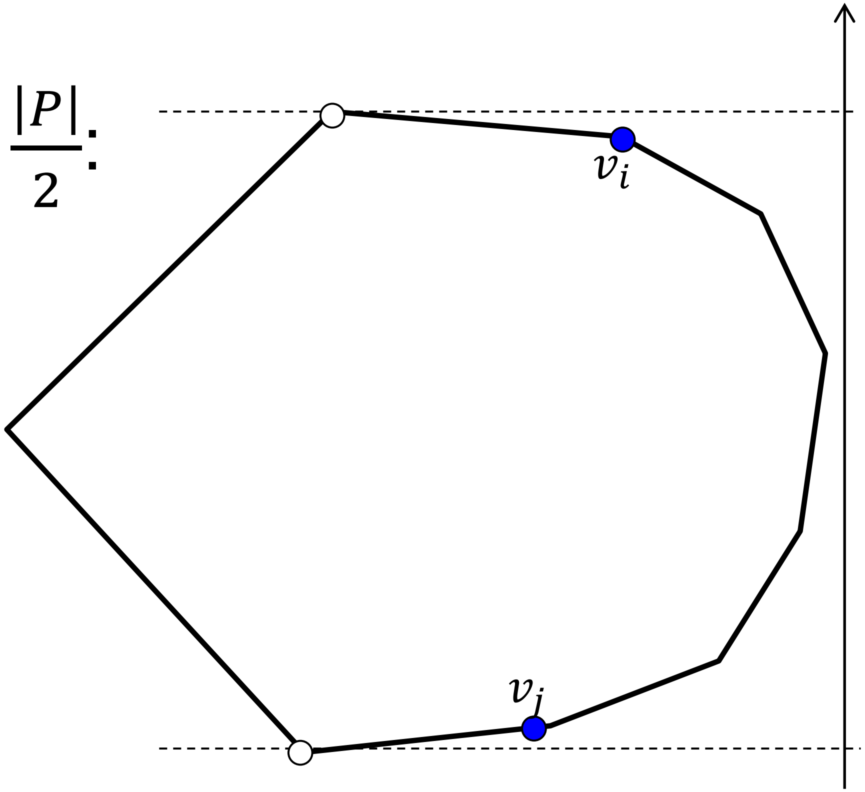

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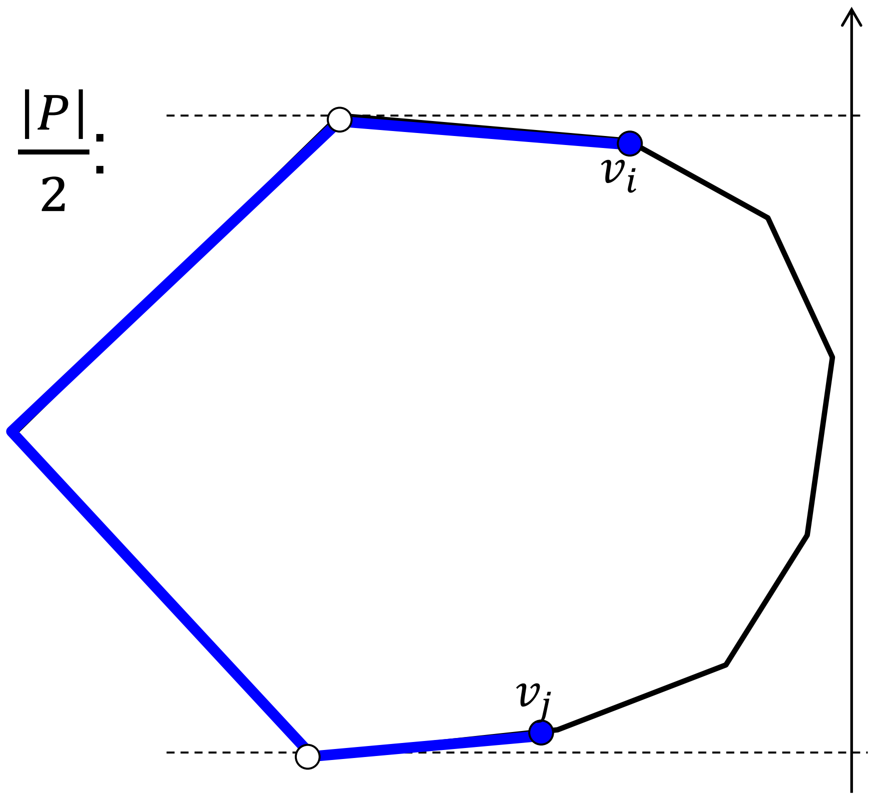

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Consider vertex v_j, $j = i + \frac{|P|}{2}$:

- If v_i is left and v_j right:
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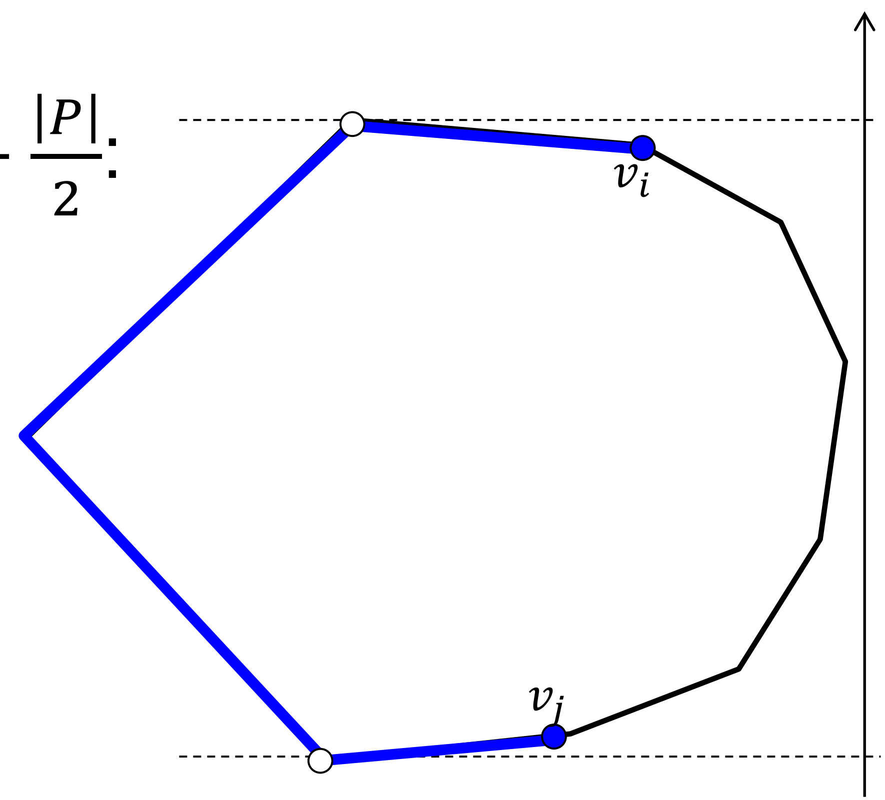
- If v_i right and v_j left: …
- If v_i and v_j right:
  - » If v_i above v_j:
    - – Both extrema in $[i, j]$



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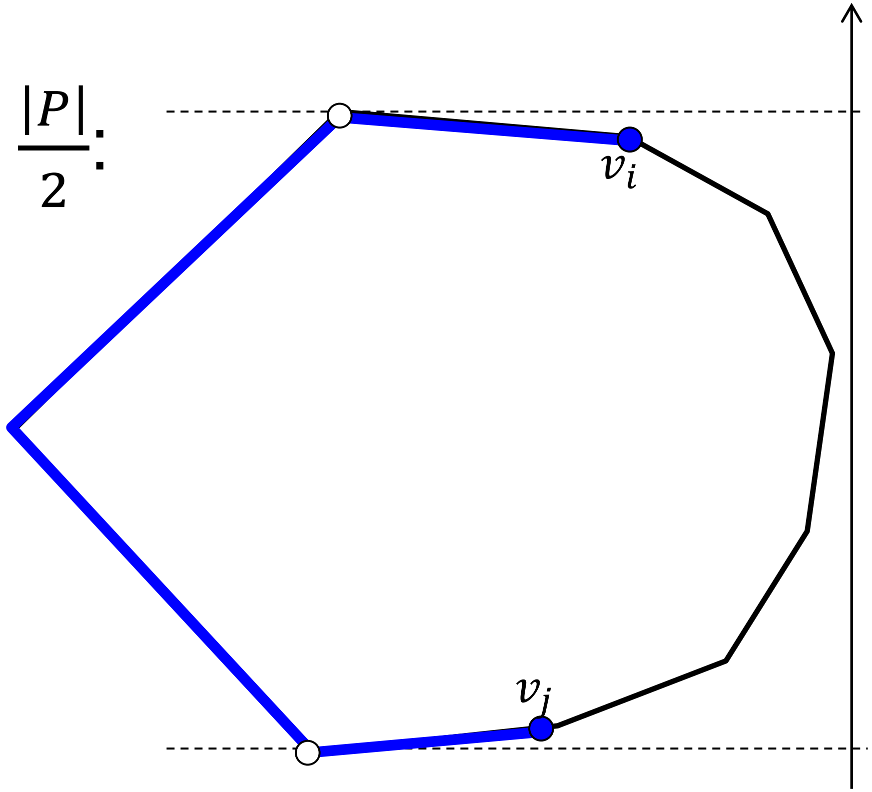

Extreme Points (2D)

Pick a vertex v_i at random:

- Incoming and outgoing edge rising ⇒ Right
- Inc
- Oth

With repeated bisection, we can find the two extrema in $O(\log|P|)$ time.

Consider vertex v_j, $j = i + \frac{|P|}{2}$:

- If v_i is left and v_j right:
  ⇒ max ∈ $[j, i]$ min ∈ $[i, j]$
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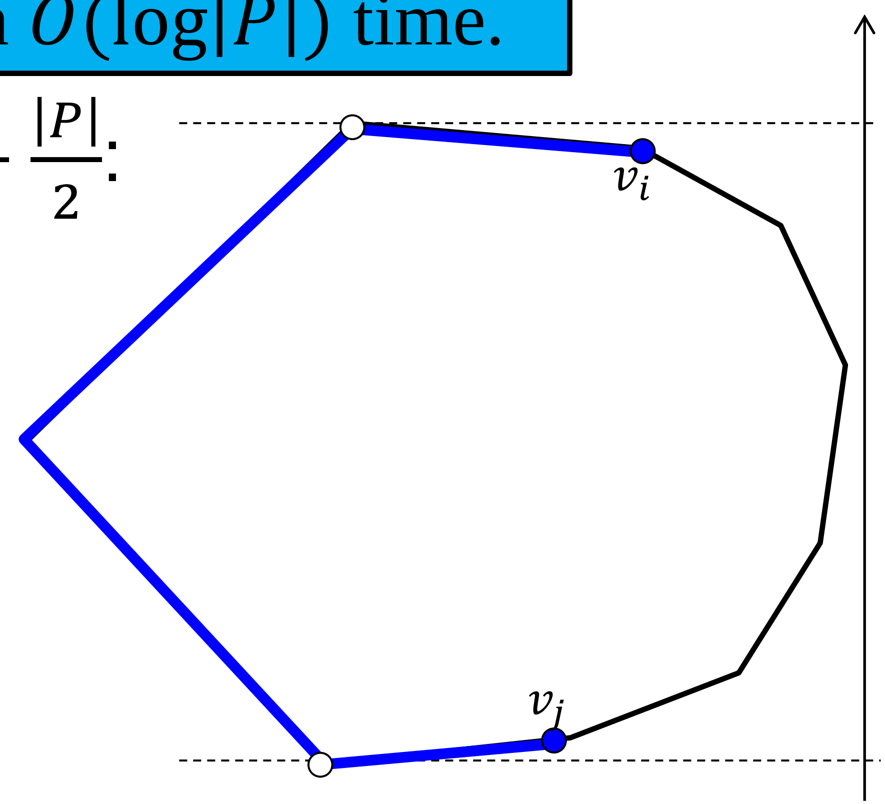

Outline

- Trapezoidal Decomposition
- Extreme Points (2D)
- Extreme Points (3D)

Extreme Points (3D)

Given a convex polyhedron P, we would like to find the (without loss of generality) highest point.

Extreme Points (3D)

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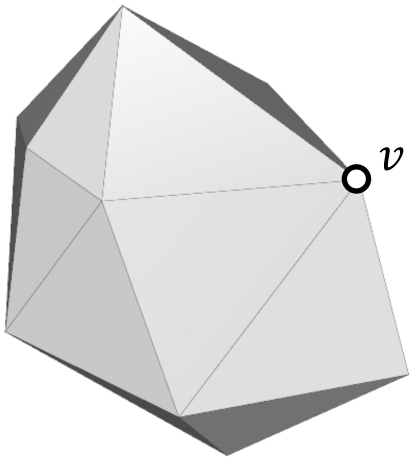
Challenge:

We can no longer order points on the convex hull.

Extreme Points (3D)

Observation 1:

Given a convex polyhedron and a vertex $v \in P$, the cone apexed at v and going through the neighbors of v must contain all of P.



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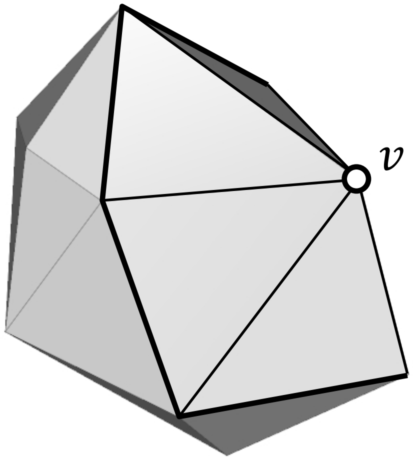
Given a convex polyhedron and a vertex $v \in P$, the cone apexed at v and going through the neighbors of v must contain all of P.

Proof:

- The cone is the intersection of the half spaces defined by the triangles incident on v.



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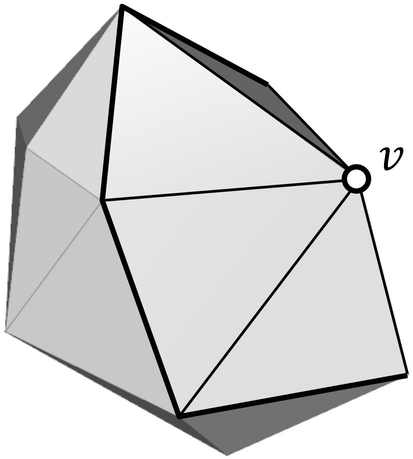
Proof:

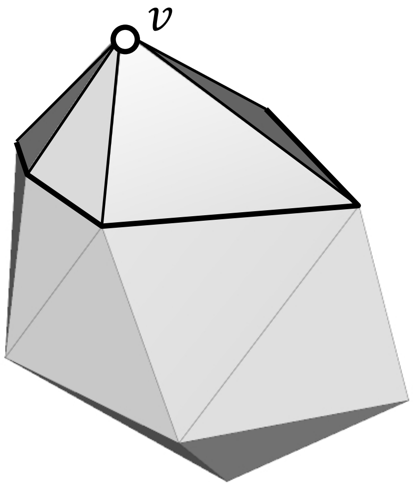
- The cone is the intersection of the half spaces defined by the triangles incident on v.
- Since each triangle incident to v is on the hull, the set P is entirely to one side.



Extreme Points (3D)

Observation 2:

Given a convex polyhedron and a vertex $v \in P$, if v is higher than its neighbors, then v is the highest point on P.



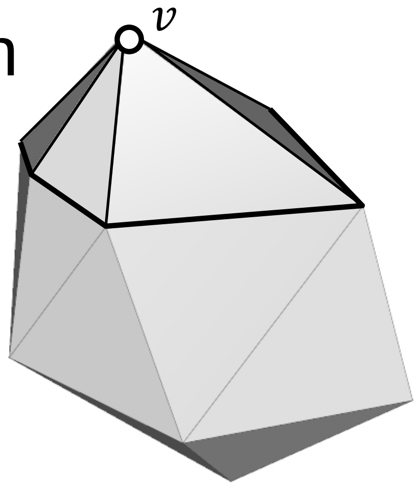
Extreme Points (3D)

Observation 2:

Given a convex polyhedron and a vertex $v \in P$, if v is higher than its neighbors, then v is the highest point on P.

Proof:

- The cone apexed at v and going through the neighbors of v contains all of P and is below v.



Extreme Points (3D)

Given a convex polyhedron P, we would like to find the (without loss of generality) highest point.

<u>Challenge</u>:

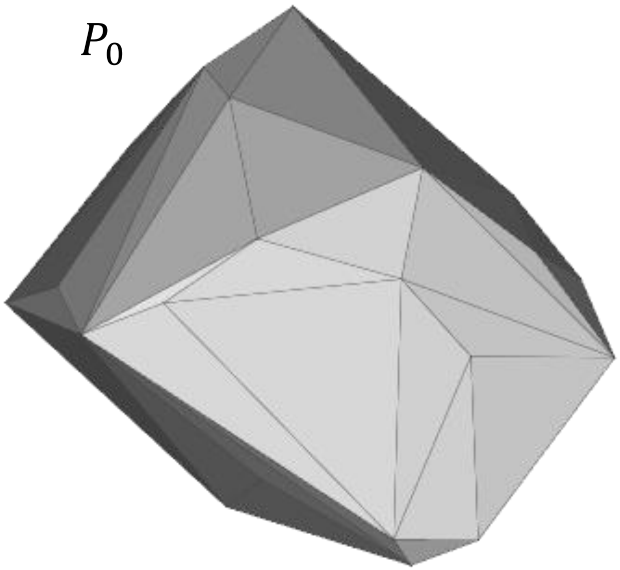
We can no longer order points on the convex hull.

<u>[Kirkpatrick, 1983]</u>

Compute a hierarchy of nested polytopes, compute the highest point on the coarsest polytopes and use that to efficiently compute the highest point on the next polytope.

Extreme Points (3D)

Compute a hierarchy of nested polytopes, $\{P_0 = P \supset P_1 \supset \cdots \supset P_l\}$, use extremals on P_{k+1} to find extremals on P_k.



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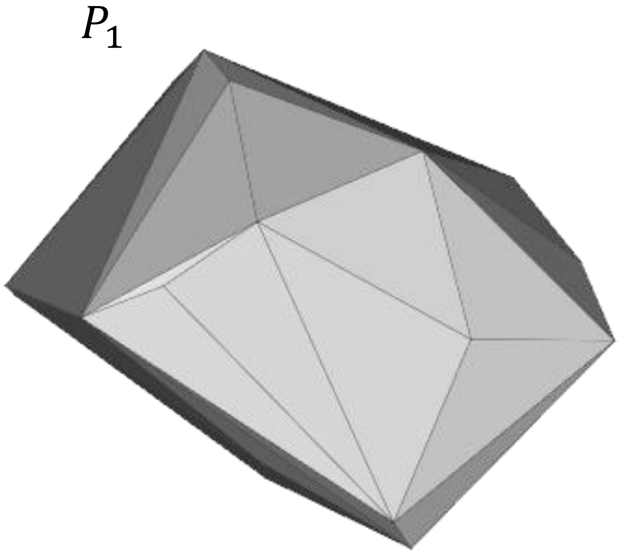

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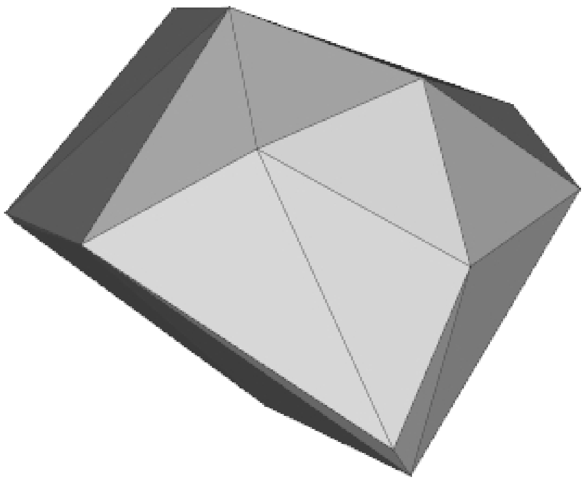
P_2



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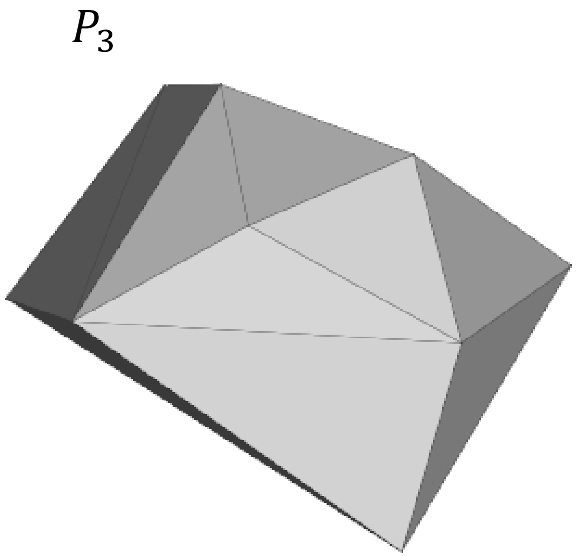

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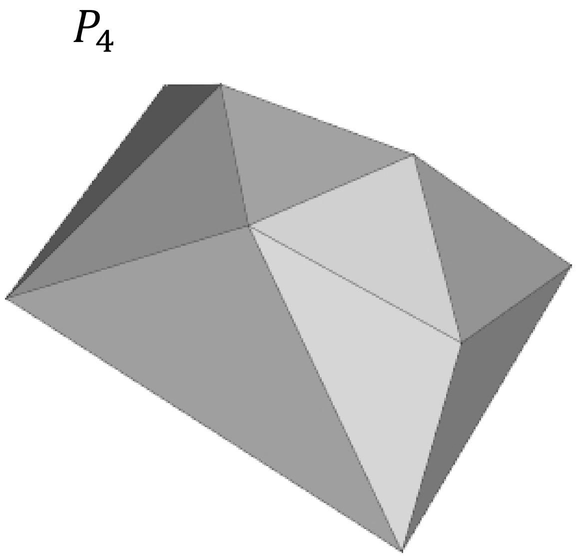

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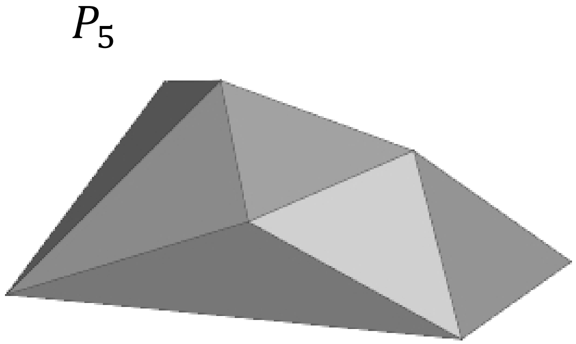

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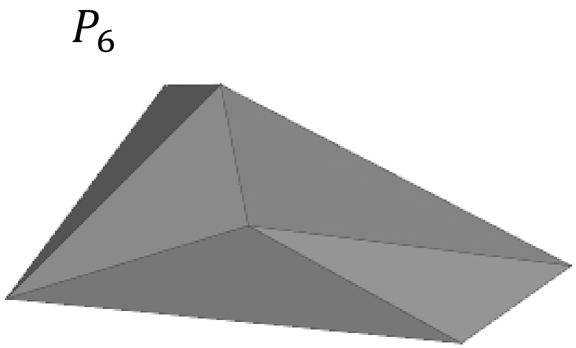

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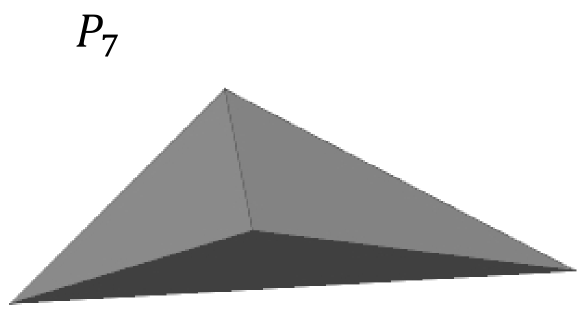

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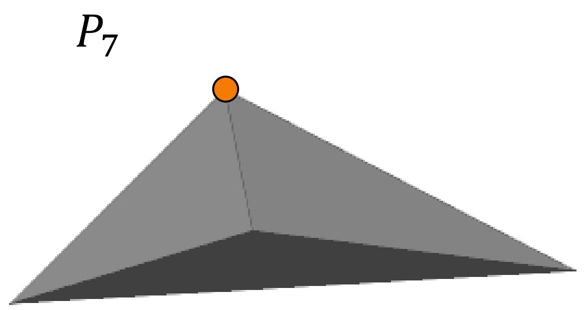

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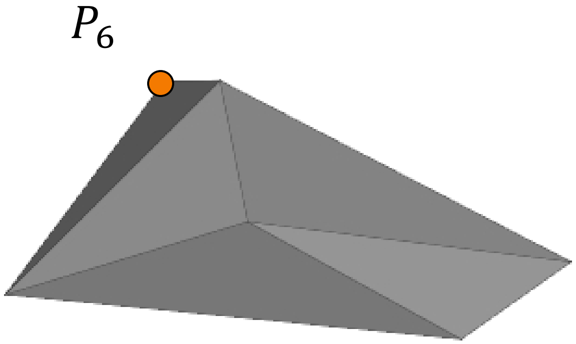

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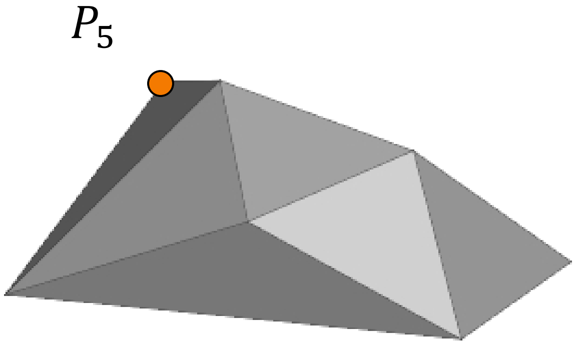

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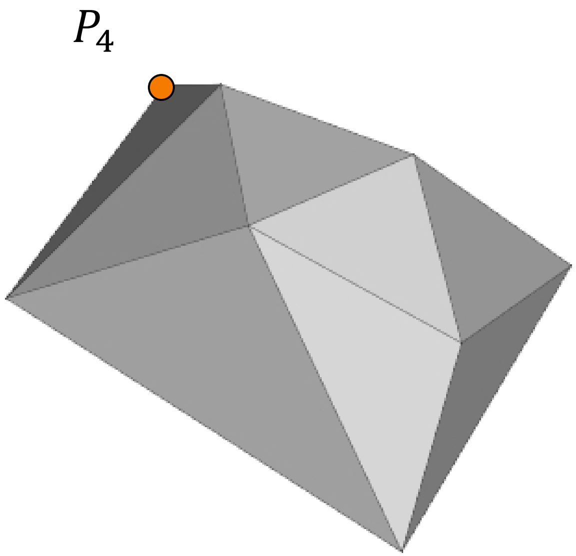

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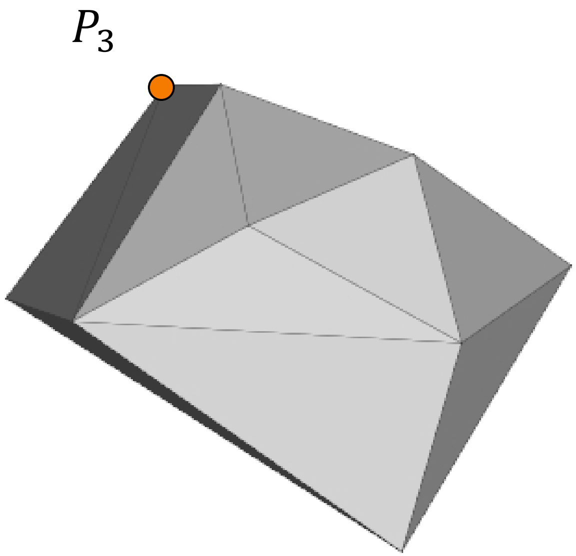

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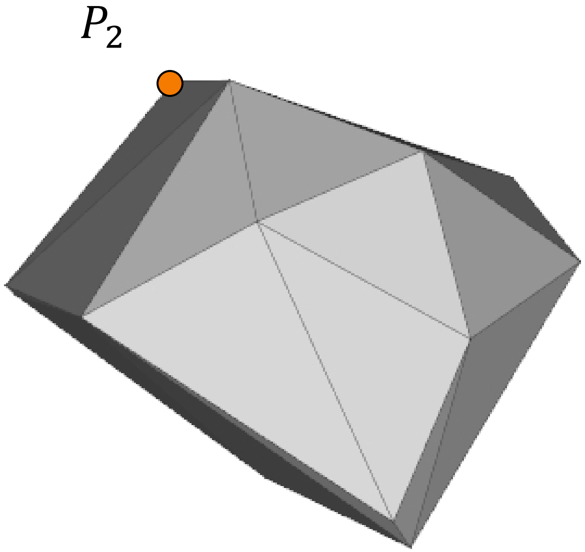

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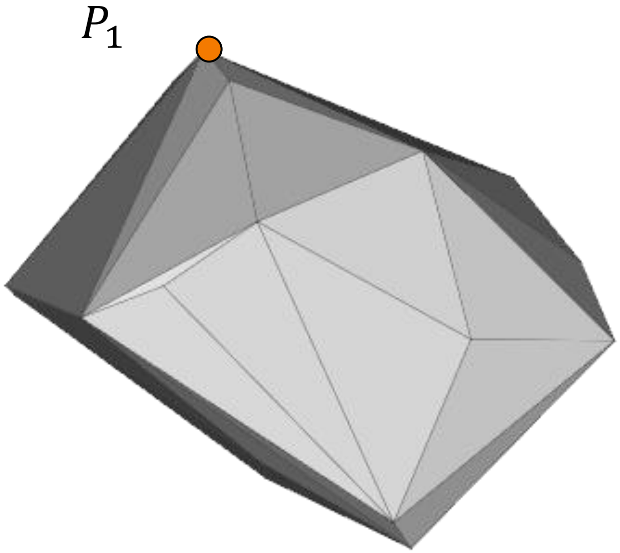
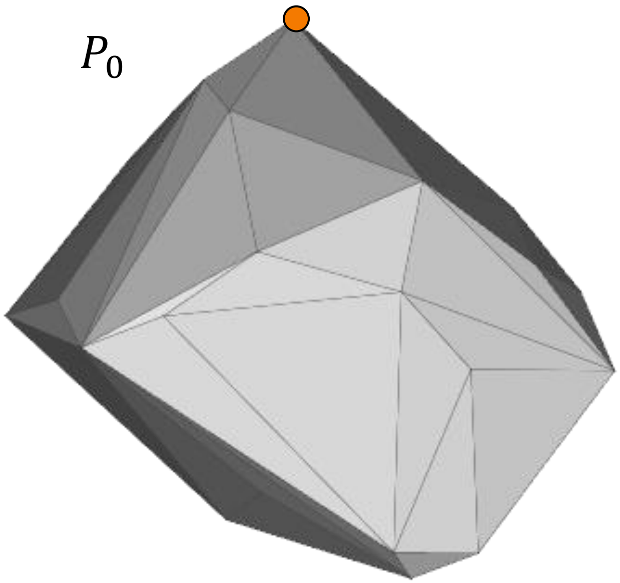


Extreme Points (3D)

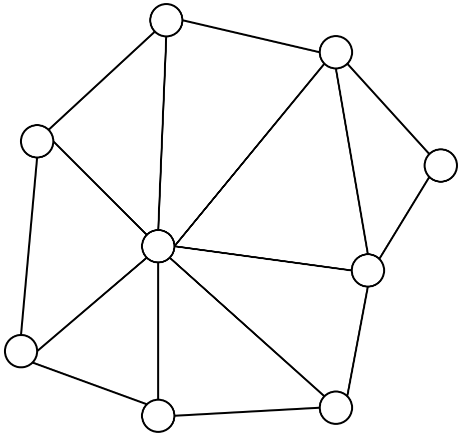
Compute a hierarchy of nested polytopes, $\{P_0 = P \supset P_1 \supset \cdots \supset P_l\}$, use extremals on P_{k+1} to find extremals on P_k.



Constructing the Hierarchy

Definition:

Given a graph, a subset of vertices is said to be *independent* if there is no edge in the graph that connects vertices in the subset.



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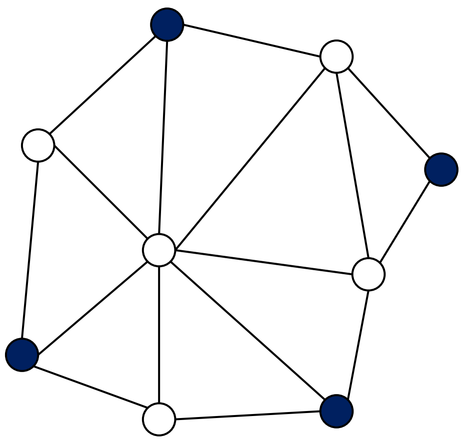

Constructing the Hierarchy

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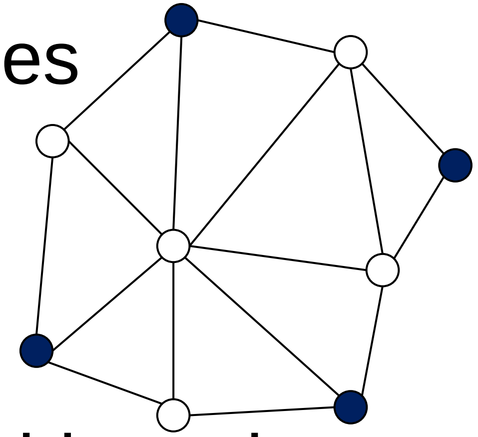
Given a graph, a subset of vertices is said to be *independent* if there is no edge in the graph that connects vertices in the subset.

Key Idea:

Identify an independent set of vertices on P_k with low degree, remove those, and set P_{k+1} to the convex hull of what's left.



Repeat for subsequent levels of the hierarchy.

Constructing the Hierarchy

Greedy Algorithm:

- While not done
  - Find a vertex with degree ≤ 8.
  - If none of its neighbors have been marked as independent, mark it as independent.

Constructing the Hierarchy

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Need to show this generates a sufficiently large subset of independent vertices.

Claim:

This algorithm will mark at least 1/18 of the vertices as independent.

Constructing the Hierarchy

Proof:

By Euler's formula, for a triangulated polyhedron:

$$E = 3V - 6.$$

Constructing the Hierarchy

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$$E = 3V - 6.$$

⇒ The sum of the degrees of the vertices, Σ, is equal to twice the number of edges, and hence:

$$\Sigma = 6V - 12.$$

Constructing the Hierarchy

Proof:

$$\Sigma = 6V - 12.$$

$\Rightarrow$ There are at least $V/2$ vertices with degree ≤ 8.

Otherwise, there are at least $V/2$ vertices with degree ≥ 9 and the rest have degree at least 3:

$$\begin{aligned}\Sigma &\geq \frac{9V}{2} + \frac{3V}{2} \\ &= 6V \\ &> 6V - 12 \\ &= \Sigma\end{aligned}$$

Constructing the Hierarchy

Proof:

There are at least $V/2$ vertices with degree ≤ 8.

$\Rightarrow$ Marking a low-degree vertex as independent, we invalidate (at most) 8 other vertices.

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$\Rightarrow$ At least 1/18 of all vertices will be independent.

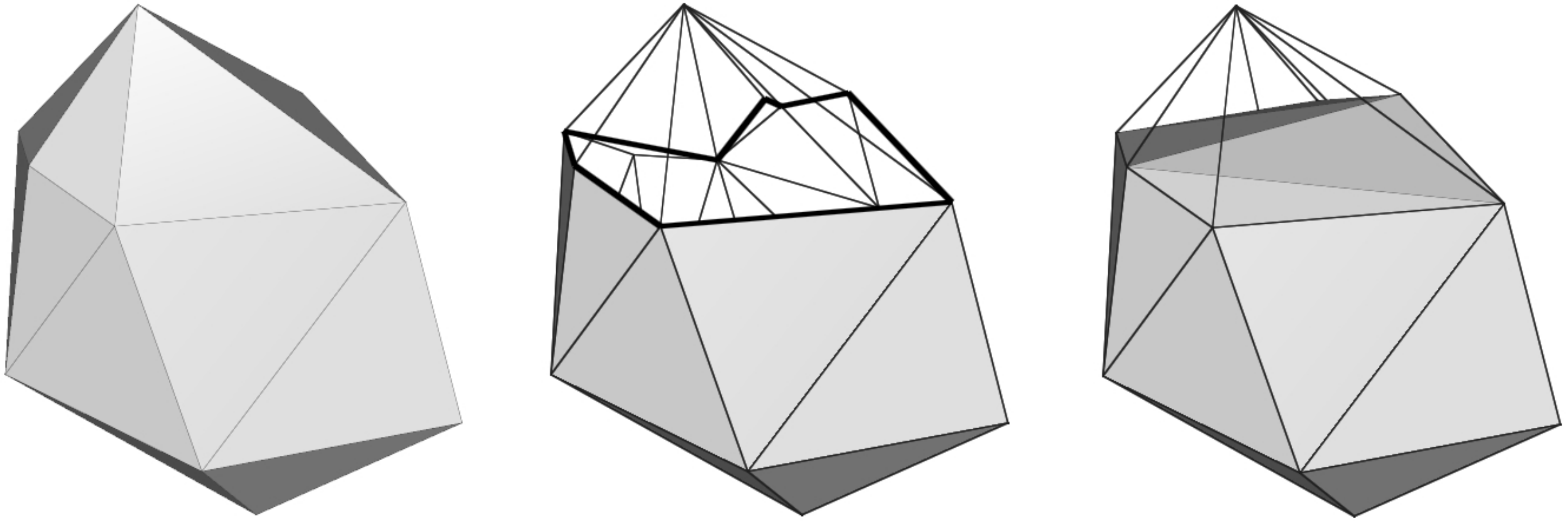
Using this to construct our polytope hierarchy:

- We will have $O(\log|P|)$ levels.
- We will require $O(|P|)$ storage.

Constructing the Hierarchy

Claim:

If we remove a point on a polytope, the convex hull of the remaining points can be obtained by computing the convex hull of the points on the boundary and using the “outer” half of the triangles.



Constructing the Hierarchy

Claim:

If we remove a point on a polytope, the convex hull of the remaining points can be obtained by computing the convex hull of the points on the boundary and using the “outer” half of the triangles.

Proof:

The remaining triangles must still be on the hull.

Any new triangles can’t connect to non-boundary vertices because then the hull would be non-manifold.

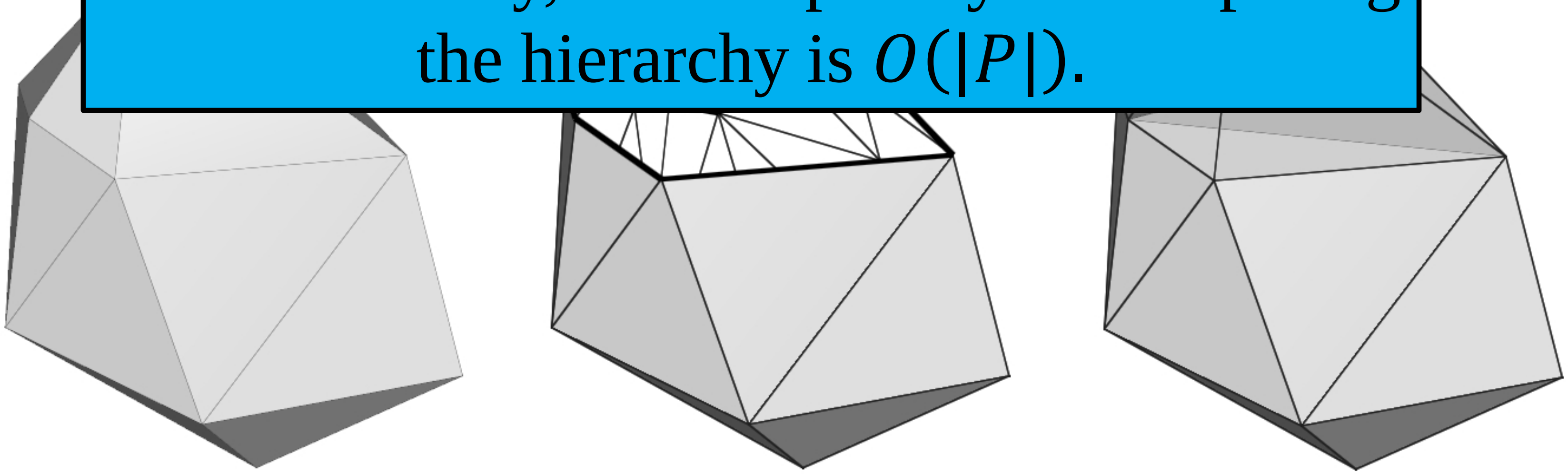
Constructing the Hierarchy

Claim:

Since the removed vertices are independent and have degree ≤ 8, the coarser convex hull can be computed in time proportional to the number of removed points.

bou … gles.

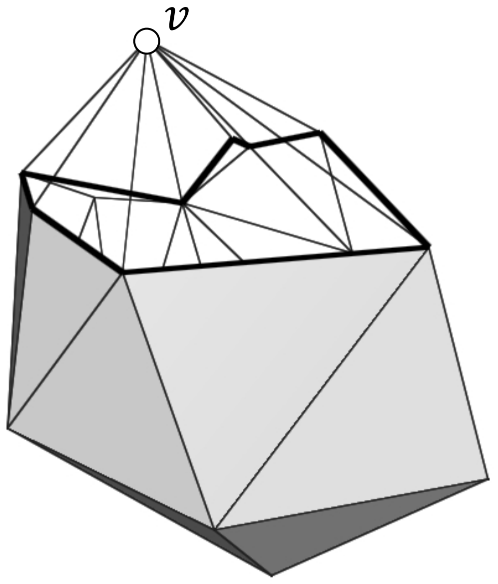
Since a removed vertex does not appear later in the hierarchy, the complexity of computing the hierarchy is $O(|P|)$.



Constructing the Hierarchy

Claim:

After removing the highest vertex $v \in P$, the next highest vertex w has to be in the one-ring of v.



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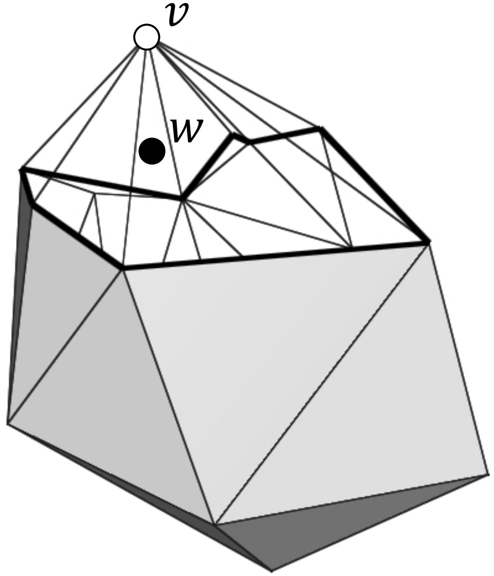
Claim:

After removing the highest vertex $v \in P$, the next highest vertex w has to be in the one-ring of v.

Proof: (by contradiction)

Assume w is interior.



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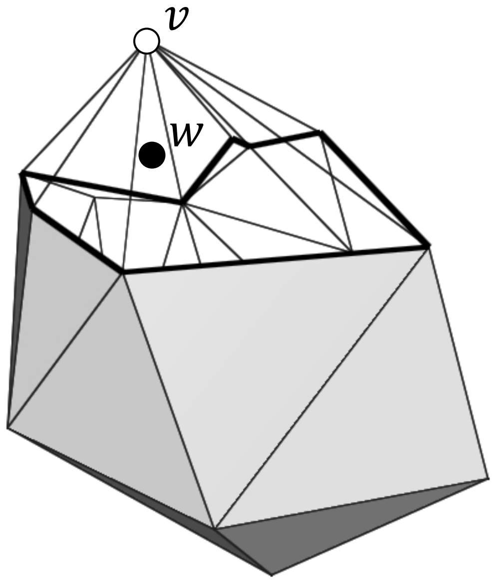

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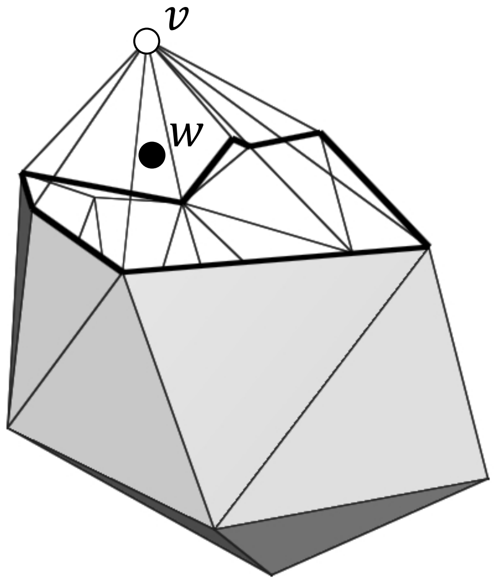

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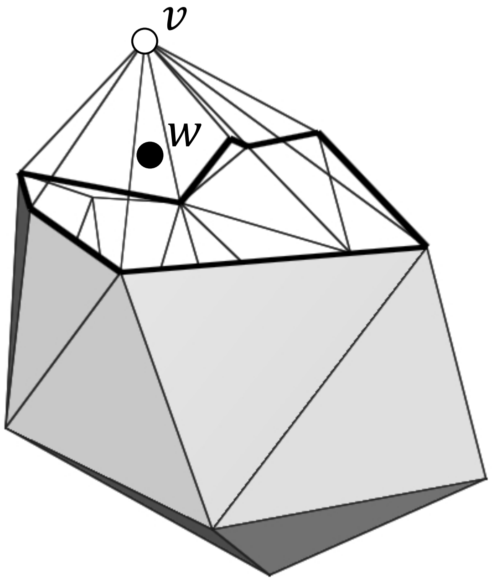


Assume w is interior.

$\Rightarrow$ The closed loop of neighbors of w are below w.

$\Rightarrow$ w is the highest vertex.

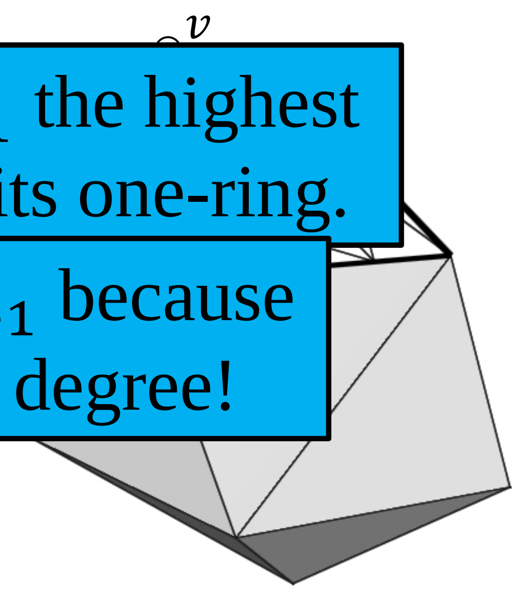
$\Rightarrow$ v was not highest.

Constructing the Hierarchy

Claim:

After removing the highest vertex $v \in P$, the next highest vertex w has to be in the one-ring of v.

Proof: (by contradiction)



Given the highest vertex, $v_{k+1} \in P_{k+1}$ the highest vertex $v_k \in P_k$ is either v_{k+1} or is in its one-ring.

⇒ Th

of

We can't test all neighbors of v_{k+1} because v_{k+1} may have arbitrarily large degree!

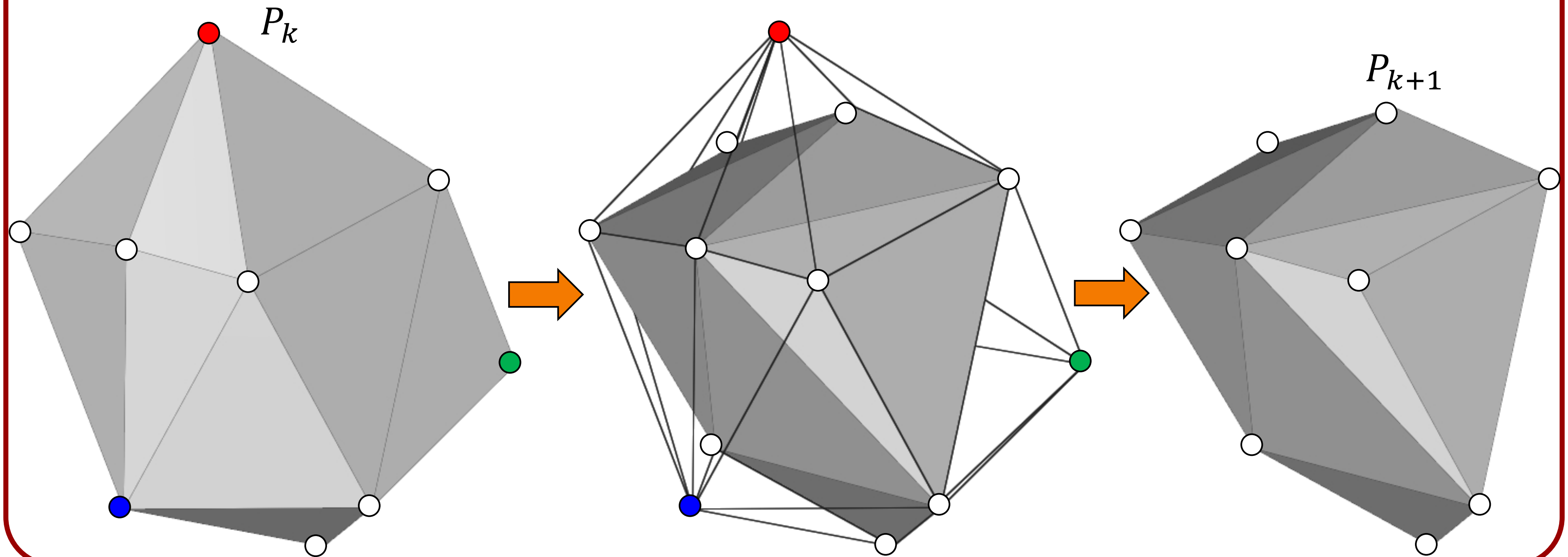
⇒ w is the highest vertex.

⇒ v was not highest.

Constructing the Hierarchy

Definition:

An edge e on P_{k+1} is *exposed* by a vertex $p \in P_k$, if e is in the triangulation of the hole resulting from the removal of p.



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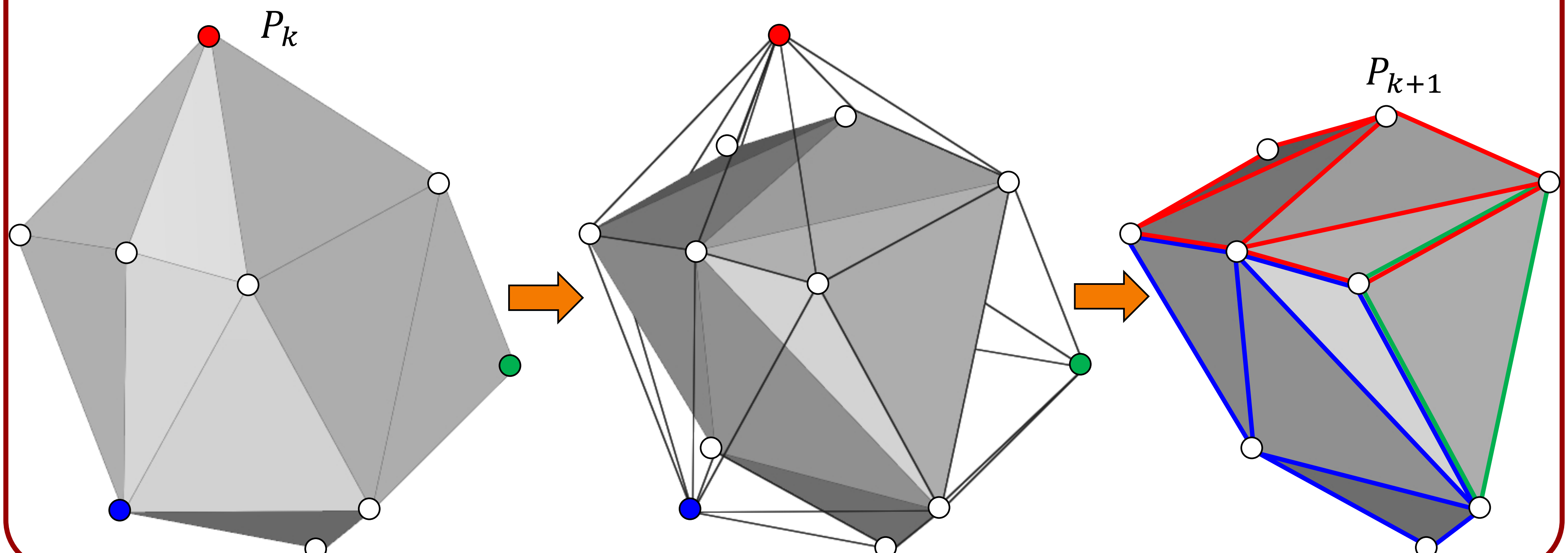


Constructing the Hierarchy

Definition:

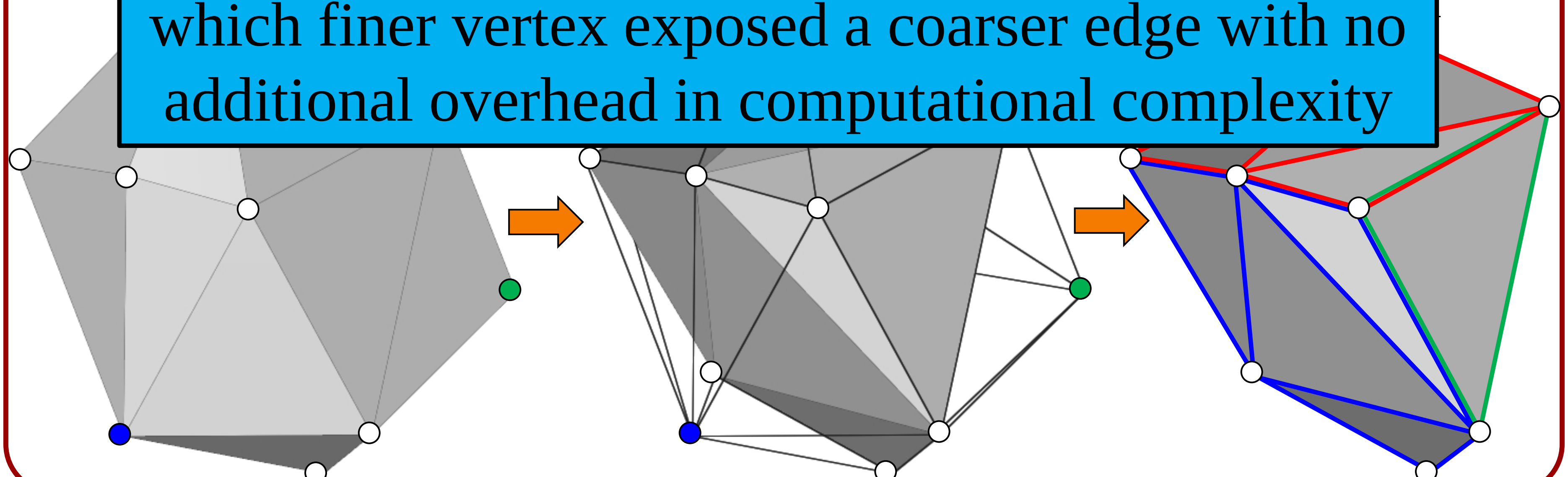
An edg[e … $\in P_k$], if

e is in [… fr]om the

An edge $e \subset P_{k+1}$ is exposed by $p \in P_k$ if e is visible to p when viewing P_{k+1}.

An edge in P_{k+1} can be exposed by at most two vertices.

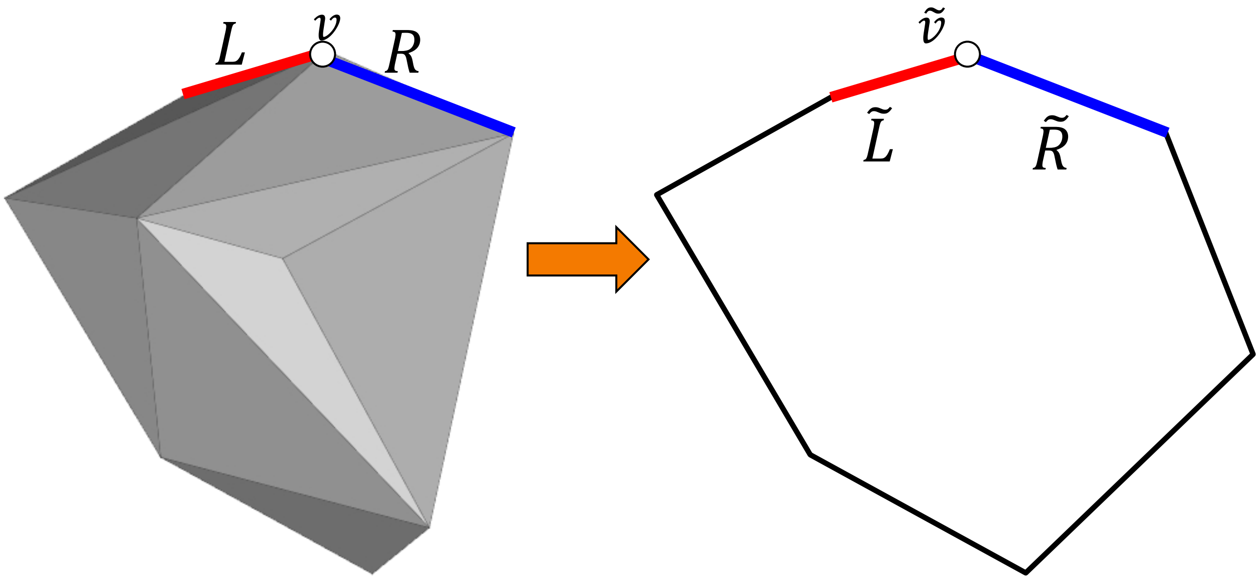
In constructing the hierarchy, we can also track which finer vertex exposed a coarser edge with no additional overhead in computational complexity



Constructing the Hierarchy

Notation:

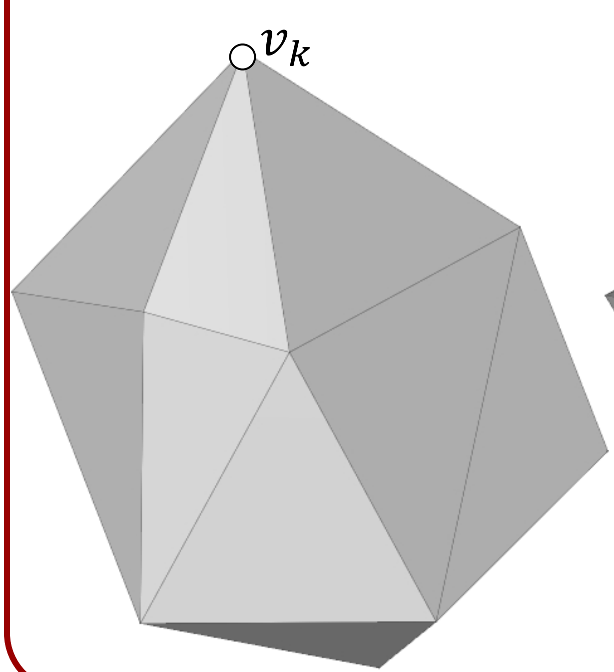
Given a polytope P, we can project it onto the yz-plane. We denote by L and R the edges that project on to the left-most and right-most edges coming out of the projected highest vertex.

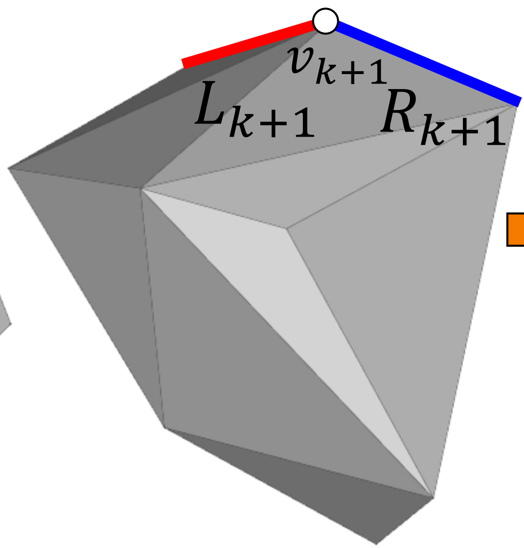


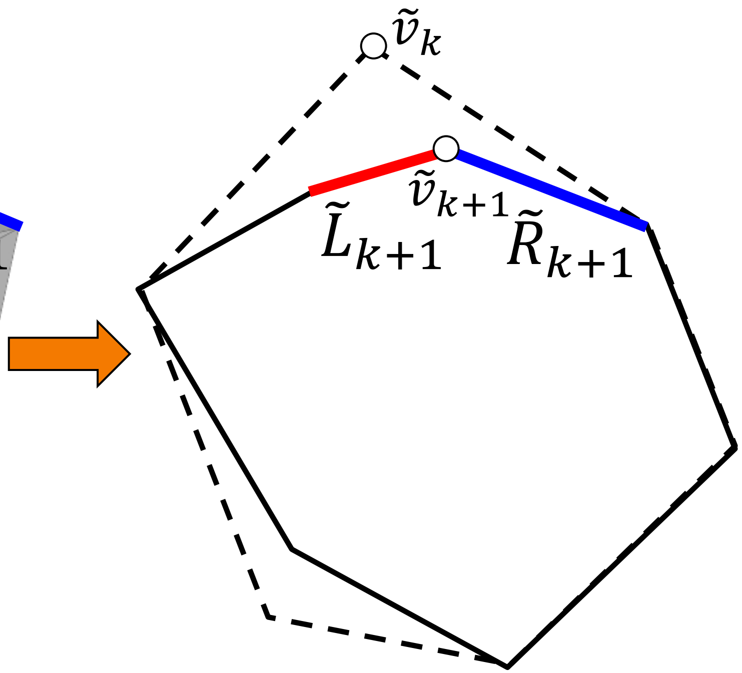
Constructing the Hierarchy

<u>Claim</u>:

Given the highest vertex, $v_{k+1} \in P_{k+1}$ the highest vertex $v_k \in P_k$ is either v_{k+1}, or a vertex that exposes one of L_{k+1} and R_{k+1}.



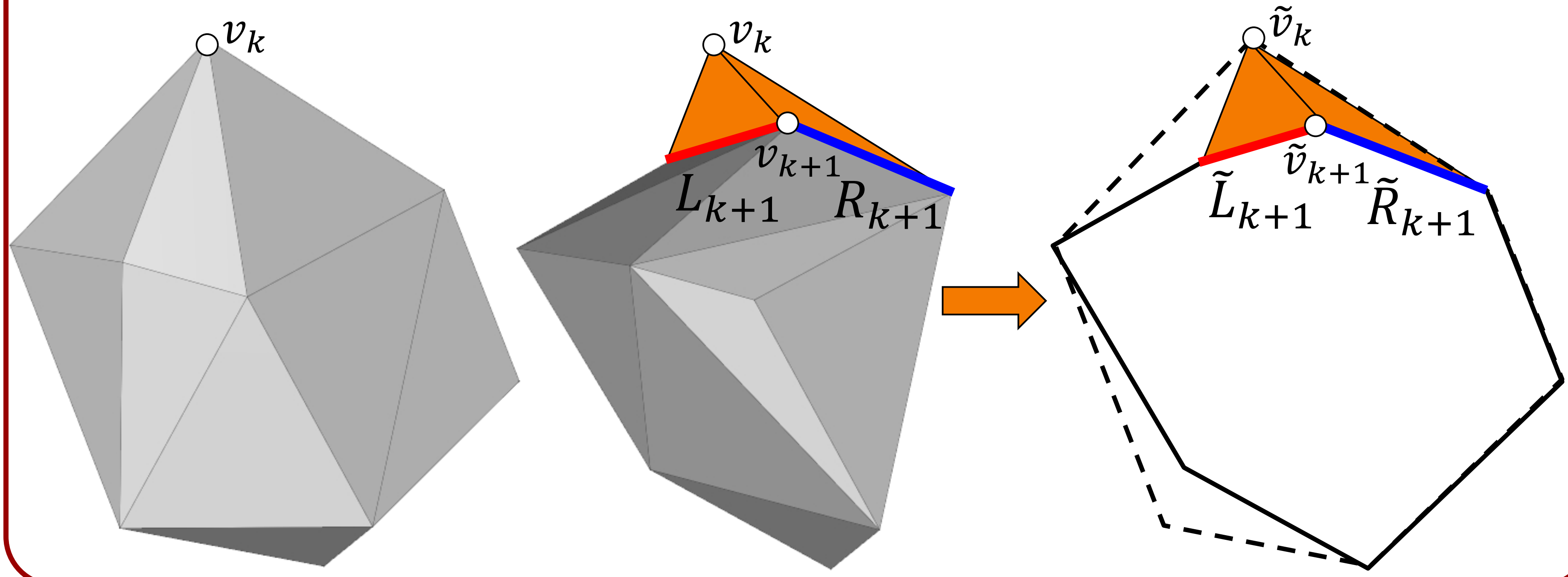




Constructing the Hierarchy

Proof:

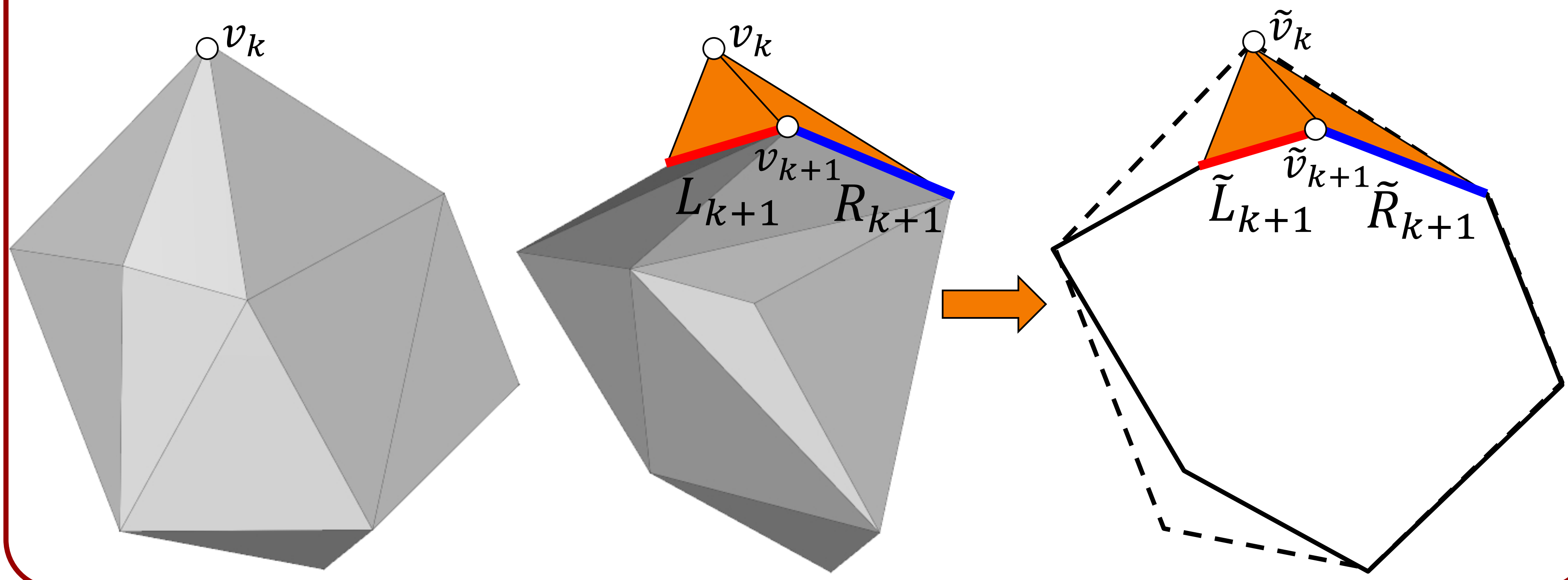
- Draw triangles (v_k, L_{k+1}) and (v_k, R_{k+1}) and project.



Constructing the Hierarchy

Proof:

- Draw triangles (v_k, L_{k+1}) and (v_k, R_{k+1}) and project.
- One of these does not intersect the projection of P_{k+1}.

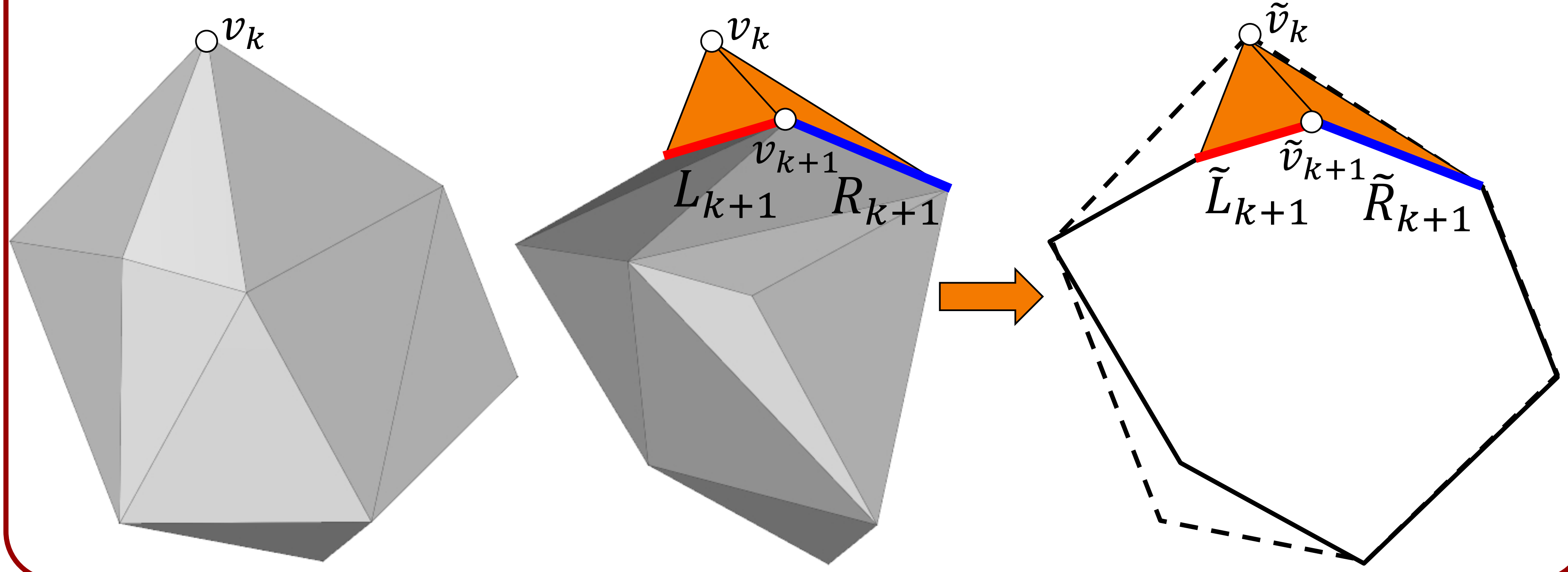


Constructing the Hierarchy

Proof:

- Draw triangles (v_k, L_{k+1}) and (v_k, R_{k+1}) and project.
- One of these does not intersect the projection of P_{k+1}.

$\Rightarrow$ One of the original triangles does not intersect P_{k+1}



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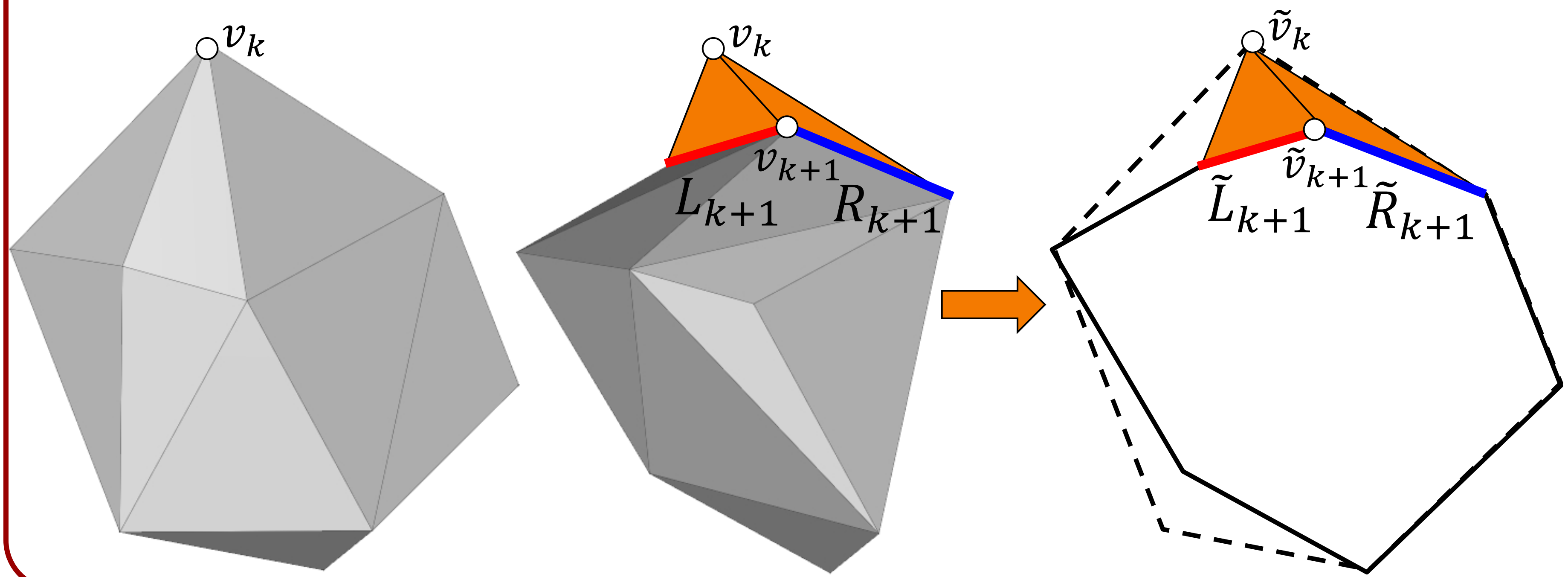
$\Rightarrow$ One of the original triangles does not intersect P_{k+1}

$\Rightarrow$ One of L_{k+1} or R_{k+1} is exposed by v_k.



Constructing the Hierarchy

Proof:

If we know the highest vertex $v_{k+1} \in P_{k+1}$ and we know L_{k+1} and R_{k+1}, then we get the highest vertex $v_k \in P_k$ in $O(1)$.

If v_{k+1} is removed at level $k+2$:
$\Rightarrow$ v_{k+1} has degree ≤ 8 in P_{k+1}
$\Rightarrow$ We can find L_{k+1} and R_{k+1} with exhaustive search.
Otherwise, we can use L_{k+2} and R_{k+2} to compute L_{k+1} and R_{k+1} in time $O(1)$.

- We can construct the polytope hierarchy in $O(|P|)$ time.
- We can find the extreme point, with respect to an arbitrary direction, in $O(\log|P|)$.