Arrangements

O’Rourke, Chapter 6

Announcements

- Assignment 3 has been posted

Outline

- Review
- Duality
- Generalizing Voronoi Diagrams
- Ham-Sandwich Cuts

Review

The equation for a parabola in 2D is:

$$f(x) = x^2$$

⇒ The derivative of the parabola at α is:

$$f'(\alpha) = 2\alpha$$

⇒ The equation for the line tangent to the parabola at α is:

$$y_\alpha(x) = 2\alpha x - \alpha^2$$

Review

The equation for a parabola in 3D is:

$$f(x,y) = x^2 + y^2$$

⇒ The derivatives of the parabola at $p = (\alpha,\beta)$ is:

$$\frac{\partial f}{\partial x}(\alpha,\beta) = 2\alpha \qquad \frac{\partial f}{\partial y}(\alpha,\beta) = 2b$$

⇒ The equation for the tangent plane to the parabola at $p = (\alpha,\beta)$ is:

$$z_p(x,y) = 2\alpha x + 2\beta y - \alpha^2 - \beta^2$$

$$\Updownarrow$$

$$z_p(r) = 2\langle p, r\rangle - \|p\|^2$$

Outline

- Review
- Duality
- Generalizing Voronoi Diagrams
- Ham-Sandwich Cuts

Duality

Definition:

Given a point $p = (\alpha, \beta)$ in the plane, define the *dual line* to be the (non-vertical) line with equation:

$$D(p) = \{(x, y) | y = 2\alpha x - \beta\}$$

Note:

- The slope depends on the x-coordinate of p.
- The height depends on the y-coordinate of p. (Height decreases as the y-coordinate increases.)

Duality

Definition:

Given a point $p=(\alpha,\beta)$ in the plane, define the *dual line* to be the (non-vertical) line with equation:

$$D(p)=\{(x,y)|y=2\alpha x-\beta\}$$

Given a (non-vertical) line $L=\{(x,y)|y=mx+b\}$, define the *dual point* to be the point with coordinates:

$$D(L)=\left(\frac{m}{2},-b\right)$$

Duality

Claim (Inverse):

The two dual maps are inverses of each other.

Proof (Points):

$$p = (\alpha, \beta)$$
$$\Rightarrow D(p) = \{(x, y) | y = 2\alpha x - \beta\}$$
$$\Rightarrow D\big(D(p)\big) = \left(\left(\frac{2\alpha}{2}\right), \beta\right) = p$$

$$D((\alpha, \beta)) = \{(x, y) | y = 2\alpha x - \beta\} \qquad D(\{(x, y) | y = mx + b\}) = \left(\frac{m}{2}, -b\right)$$

Duality

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The two dual maps are inverses of each other.

Proof (Points):

$$p = (\alpha, \beta)$$
$$\Rightarrow D(p) = \{(x, y) | y = 2\alpha x - \beta\}$$
$$\Rightarrow D\big(D(p)\big) = \left(\left(\frac{2\alpha}{2}\right), \beta\right) = p$$

Since every line is the dual of some point, it follows that the maps are inverses.

$D((\alpha, \beta)) = \{(x, y) | y = 2\alpha x - \beta\}$ $\quad$ $D(\{(x, y) | y = mx + b\}) = \left(\frac{m}{2}, -b\right)$

Duality

Claim (Incidence):

Given $p = (\alpha, \beta)$ and $L = \{(x, y) | y = mx + b\}$:

$$p \in L \quad \Leftrightarrow \quad D(L) \in D(p).$$

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Duality

Claim (Incidence):

Given $p = (\alpha, \beta)$ and $L = \{(x, y) | y = mx + b\}$:

$$p \in L \quad \Leftrightarrow \quad D(L) \in D(p).$$

Proof:

$$p \in L$$
$$\Leftrightarrow \beta = m\alpha + b$$
$$\Leftrightarrow -b = 2\alpha\left(\frac{m}{2}\right) - \beta$$
$$\Leftrightarrow D(L) \in D(p)$$

$$D((\alpha, \beta)) = \{(x, y) | y = 2\alpha x - \beta\} \qquad D(\{(x, y) | y = mx + b\}) = \left(\frac{m}{2}, -b\right)$$

Duality

Claim (Incidence):

Given $p = (\alpha, \beta)$ and $L = \{(x, y) | y = mx + b\}$:

$$p \in L \quad \Leftrightarrow \quad D(L) \in D(p).$$

Corollary:

$p \in L_1 \cap L_2$ if and only if $D(L_1), D(L_2) \in D(p)$.

$D((\alpha, \beta)) = \{(x, y) | y = 2\alpha x - \beta\}$ $D(\{(x, y) | y = mx + b\}) = \left(\frac{m}{2}, -b\right)$

Duality

Claim (Ordering):

If line $L = \{(x, y) | y = mx + b\}$ is below/above point $p = (\alpha, \beta)$ then line $D(L)$ is above/below $D(p)$.

Duality

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If line $L = \{(x, y) | y = mx + b\}$ is below/above point $p = (\alpha, \beta)$ then line $D(L)$ is above/below $D(p)$.

Proof:

L is below p
$\Leftrightarrow \beta > m\alpha + b$
$\Leftrightarrow -b > 2\alpha\left(\frac{m}{2}\right) - \beta$
$\Leftrightarrow$ the point $\left(\frac{m}{2}, -b\right)$ is above the line $\{(x, y) | y = 2\alpha x - \beta\}$
$\Leftrightarrow D(L)$ is above $D(p)$

Duality

Claim (Parabola):

p is on the parabola if and only if $D(p)$ is the tangent to the parabola at p.

Duality

Claim (Parabola):

p is on the parabola if and only if $D(p)$ is the tangent to the parabola at p.

Proof:

$L = \{(x,y) | y = mx + b\}$ is tangent to the parabola at α

$$\Updownarrow$$

$$L = \{(x,y) \big| y = 2\alpha x - \alpha^2\}$$

$$\Updownarrow$$

$$D(L) = (\alpha, \alpha^2)$$

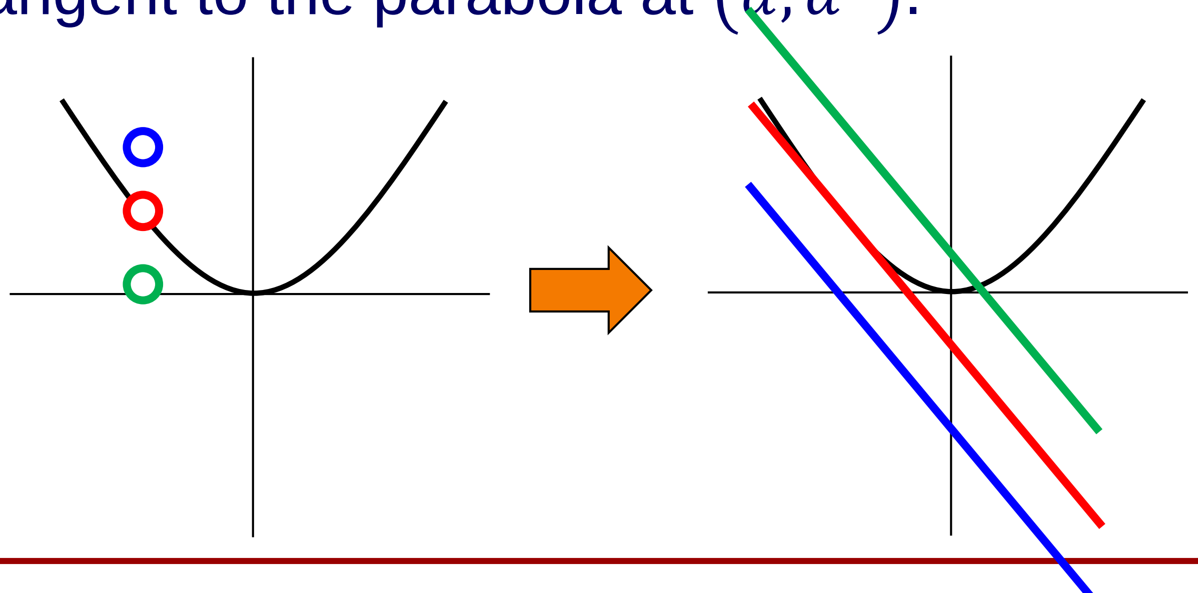
$$\Updownarrow$$

$D(L)$ is on the parabola

Duality

For a point $p = (\alpha, \beta)$:

- $\beta = \alpha^2$: If p is on the parabola, $D(p)$ is the tangent to the parabola at (α, α^2).
- $\beta < \alpha^2$: If p is below the parabola, $D(p)$ is parallel and above the tangent to the parabola at (α, α^2).
- $\beta > \alpha^2$: If p is above the parabola, $D(p)$ is parallel and below the tangent to the parabola at (α, α^2).



Duality

In a similar manner, we can define duality between points and (non-vertical) hyper-planes in d dimensions:

$$(\alpha_1, \ldots, \alpha_d)$$

$$\updownarrow$$

$$\left\{ (x_1, \ldots, x_d) \mid x_d = 2 \sum_{i=1}^{d-1} \alpha_i x_i - \alpha_d \right\}$$

Outline

- Review
- Duality
- Generalizing Voronoi Diagrams
- Ham-Sandwich Cuts

Generalizing Voronoi Diagrams

Recall:

- Given a point $P(p) = (p, \|p\|^2)$ on the paraboloid, the tangent plane is given by:
$$z_p(r) = 2\langle p, r\rangle - \|p\|^2$$
- For any point q the (vertical) distance between the points on the parabola and the tangent plane are:
$$P(q) - z_p(q) = \|p - q\|^2$$

⇒ Given points p and q, wherever the tangent plane at q is higher than the tangent plane at p, we are closer to q than to p.

Generalizing Voronoi Diagrams

Given a set of points in the plane $P = \{p_1, \ldots, p_n\}$ if we draw the tangents to the paraboloid at the points $\{(p_i, \|p_i\|^2)\}$ and view from above, we "see" the Voronoi diagram.

Generalizing Voronoi Diagrams

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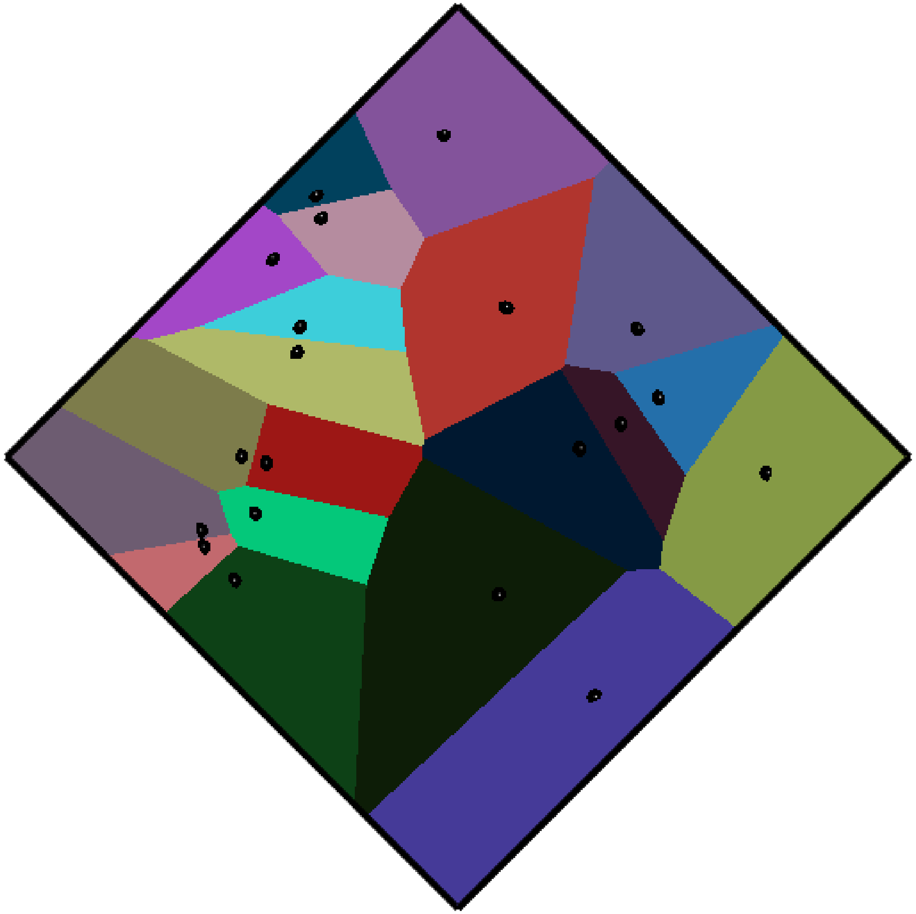
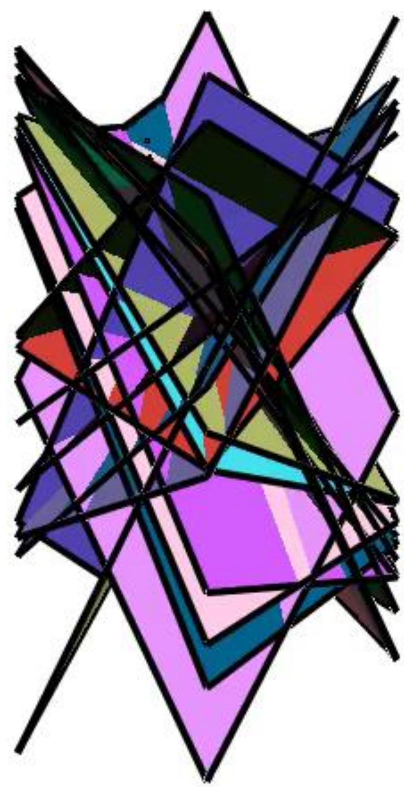


Generalizing Voronoi Diagrams

Given a set of points in the plane $P = \{p_1, \dots, p_n\}$ if we draw the tangents to the paraboloid at the points $\{(p_i, \|p_i\|^2)\}$ and view from below, we "see" the furthest-point Voronoi diagram.

Generalizing Voronoi Diagrams

Given a set of points in the plane $P = \{p_1, \dots, p_n\}$ if we draw the tangents to the paraboloid at the points $\{(p_i, \|p_i\|^2)\}$ and view <u>from below</u>, we "see" the furthest-point Voronoi diagram.



Generalizing Voronoi Diagrams

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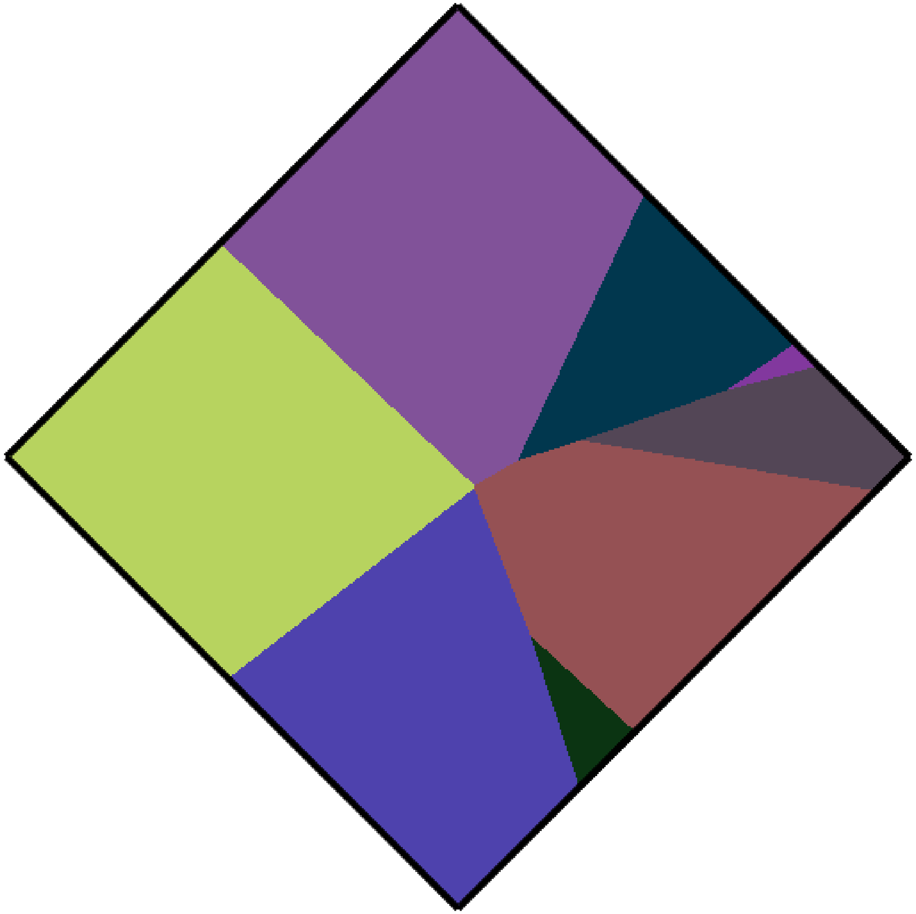
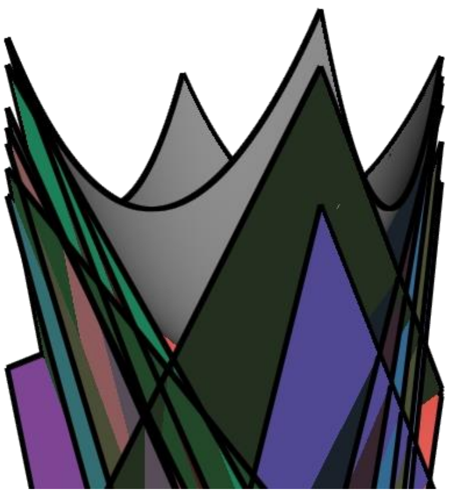


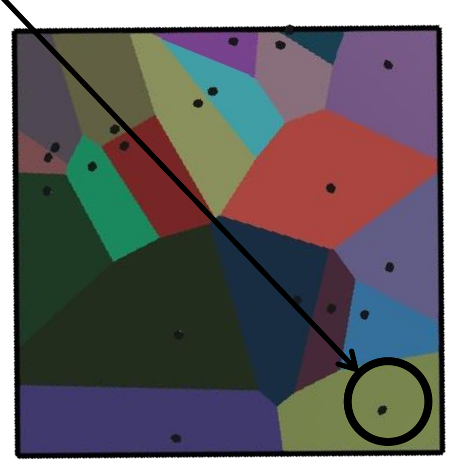
Generalizing Voronoi Diagrams

<u>From Above</u>: (Nearest-Point) Voronoi Diagram

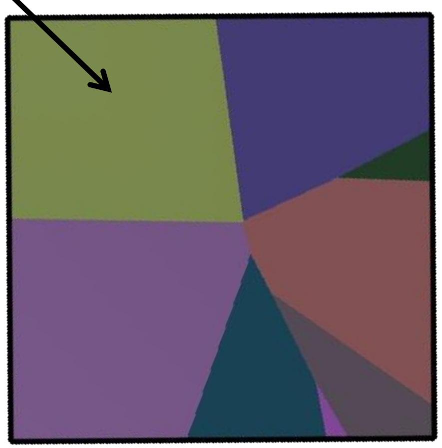
<u>From Below</u>: Furthest-Point Voronoi Diagram

The points here are further from this site than from any other site.





Nearest



Furthest

Generalizing Voronoi Diagrams

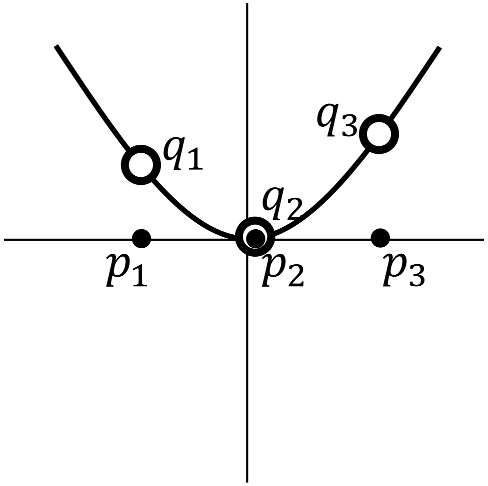
<u>Definition</u>:

The k*-th order Voronoi Diagram* is a partition of space into convex cells, indexed by k-tuples of points $(p_{i_1}, \ldots, p_{i_k})$, with $i_j < i_{j+1}$, such that a point q is in cell $(p_{i_1}, \ldots, p_{i_k})$ iff. the k nearest neighbors of q are $\{p_{i_1}, \ldots, p_{i_k}\}$.

Generalizing Voronoi Diagrams

Given sites in 2D, lift them to the paraboloid:

$$p_i \to q_i = \left(p_i, \|p_i\|^2\right)$$

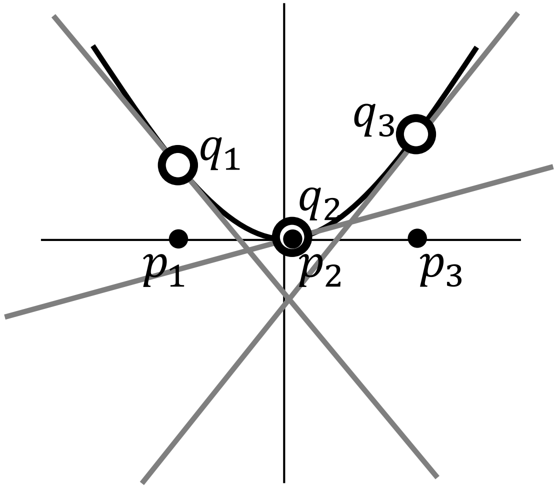


Generalizing Voronoi Diagrams

Given sites in 2D, lift them to the paraboloid:

$$p_i \to q_i = (p_i, \|p_i\|^2)$$

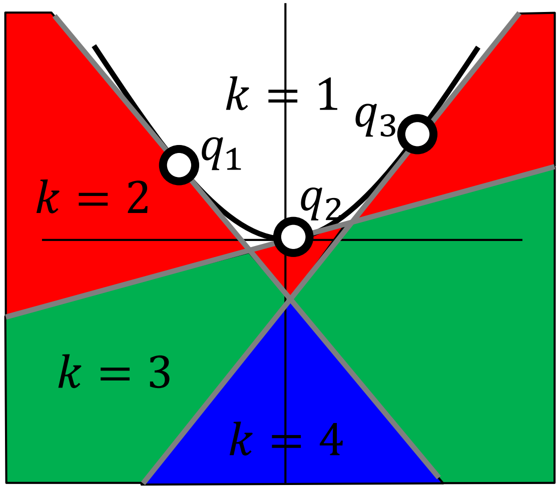
The set of tangent planes to the paraboloid at these points form an arrangement.



Generalizing Voronoi Diagrams

Definition:

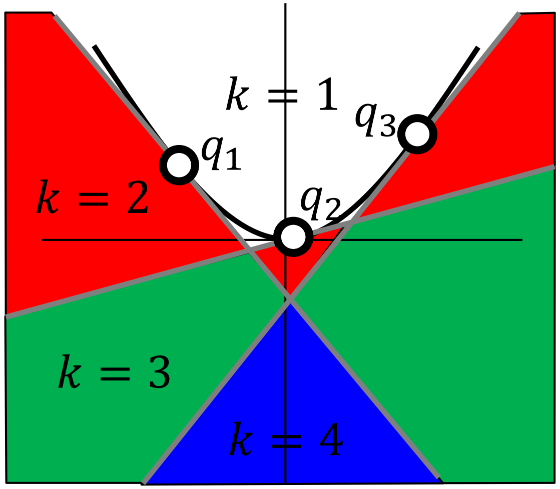
The k*-th level of the arrangement* is the set of cells in the arrangement which have exactly $k-1$ planes above them.



Generalizing Voronoi Diagrams

Note:

The set of faces in the arrangement between k-th level and the $(k+1)$-st level project uniquely onto the xy-plane.



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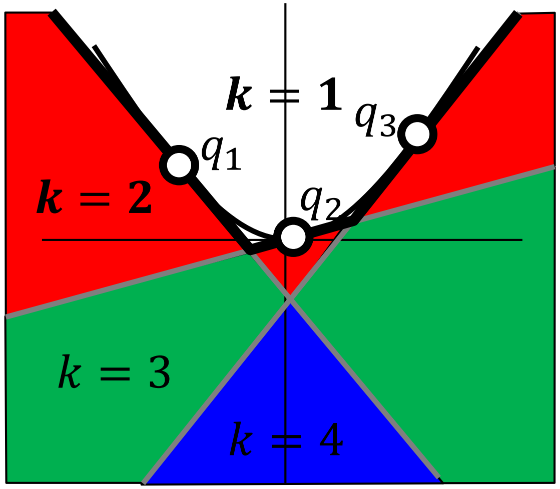

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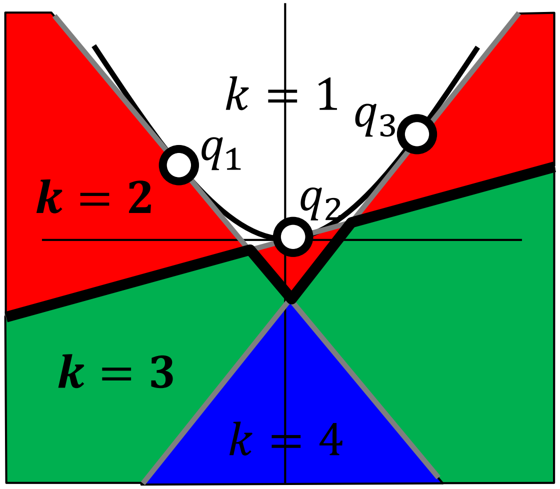

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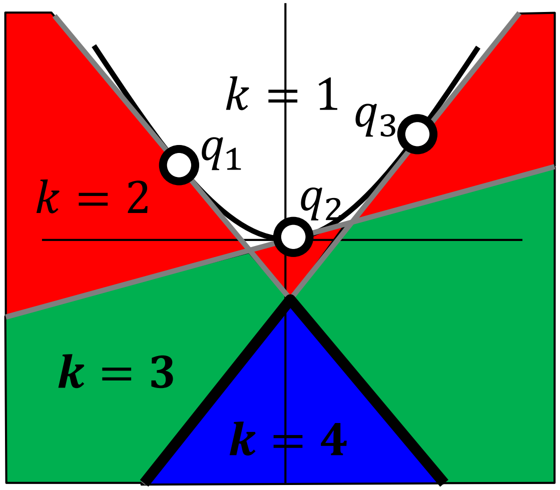
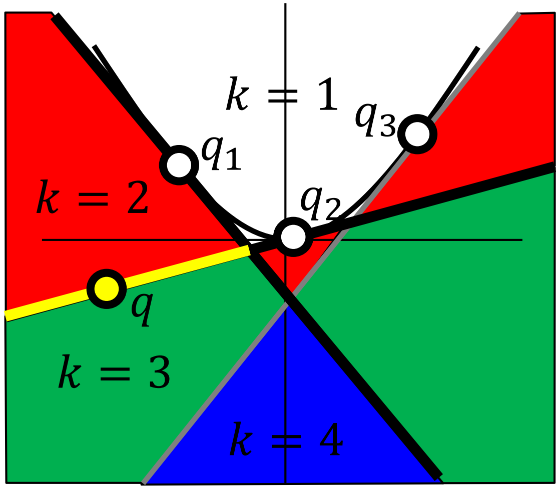


Generalizing Voronoi Diagrams

Note:

Every point q on a face of the intersection of the k-th and $(k+1)$-st levels of the arrangement has the same set of k planes on or above it.



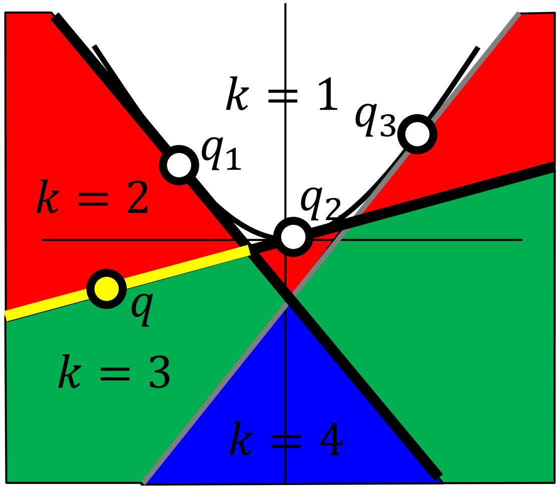
Generalizing Voronoi Diagrams

Note:

Every point q on a face of the intersection of the k-th and $(k+1)$-st levels of the arrangement has the same set of k planes on or above it.

Recall:

- The planes of the arrangement are the tangent planes to the paraboloid at the lifted sites, q_i.



Generalizing Voronoi Diagrams

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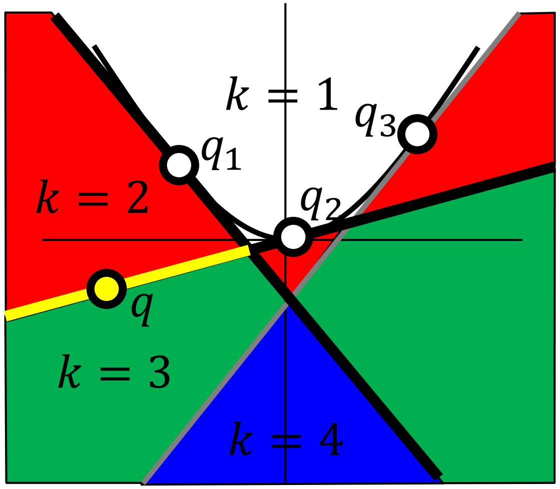
Every point q on a face of the intersection of the k-th and $(k+1)$-st levels of the arrangement has the same set of k planes on or above it.

Recall:

- The planes of the arrangement are the tangent planes to the paraboloid at the lifted sites, q_i.
- Their duals are the lifted sites.



Generalizing Voronoi Diagrams

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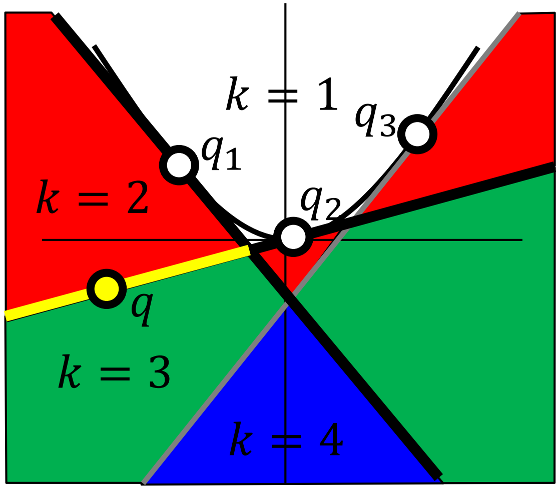
<u>Recall</u>:

- The planes of the arrangement are the tangent planes to the paraboloid at the lifted sites, q_i.
- Their duals are the lifted sites.
- The projection of the planes' duals onto the xy-plane are the sites p_i.

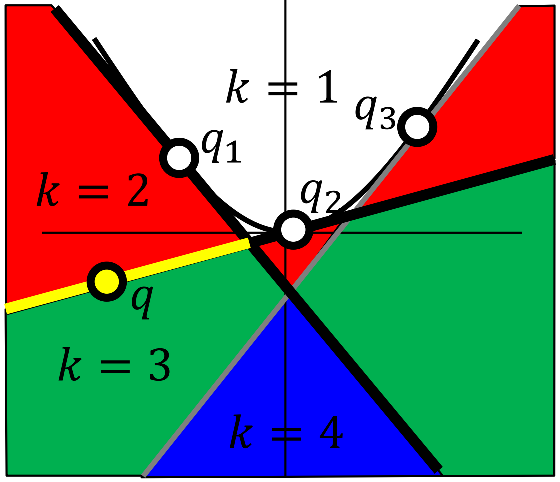


Generalizing Voronoi Diagrams

Note:

Every point q on a face of the intersection of the k-th and $(k+1)$-st levels of the arrangement has the same set of k planes on or above it.

⇒ The projection of the duals of those k planes are the sites closest to the projection of q onto the xy-plane.



Generalizing Voronoi Diagrams

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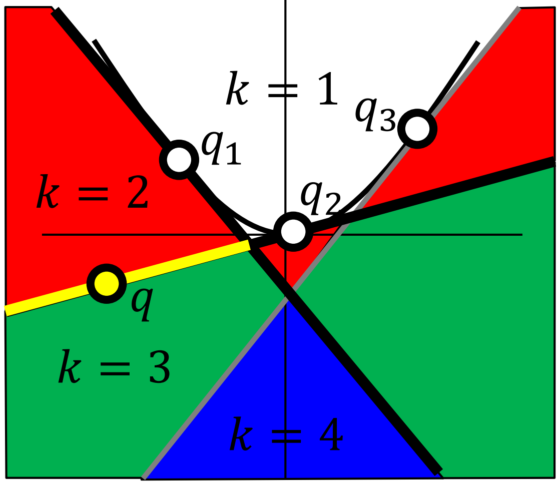
⇒ The projection of the duals of those k planes are the sites closest to the projection of q onto the xy-plane.

⇒ The projection of the faces is a connected component of the k-th level Voronoi diagram.



Generalizing Voronoi Diagrams

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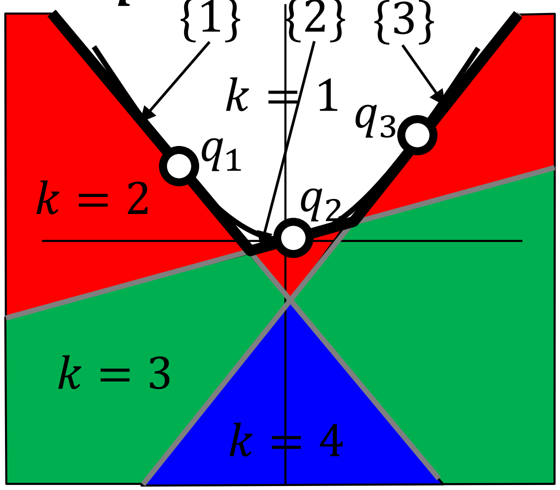

Generalizing Voronoi Diagrams

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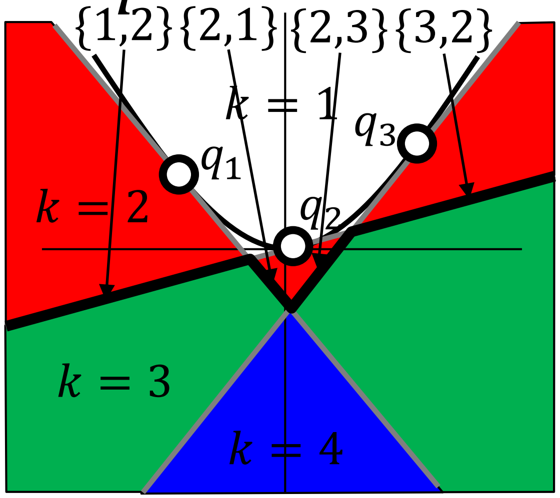

Generalizing Voronoi Diagrams

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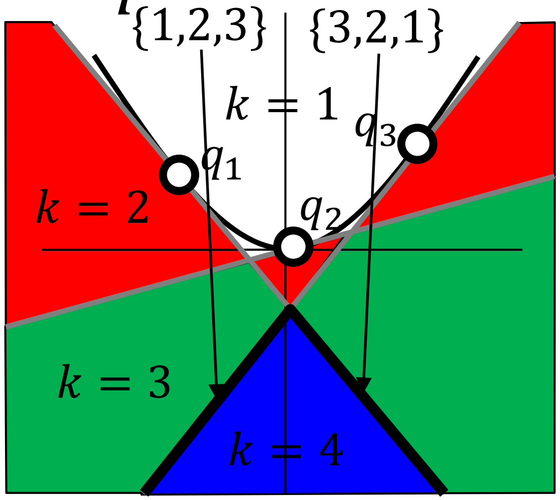
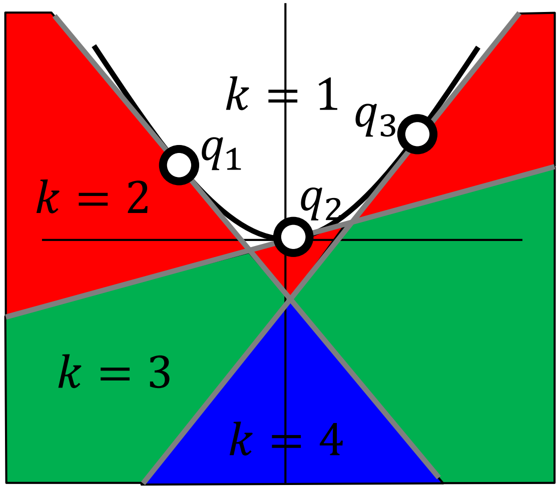


[Edelsbrunner 1987]

Theorem:

The points of intersection of the k-th and $(k+1)$-th levels in the arrangement project to the k-th order Voronoi diagram.



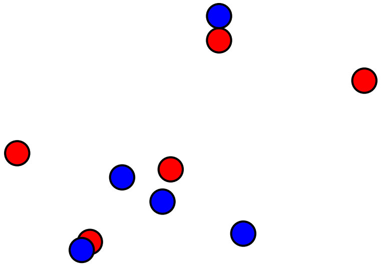
Outline

- Review
- Duality
- Generalizing Voronoi Diagrams
- Ham-Sandwich Cuts
  - Red-Blue Matching

Ham-Sandwich Cuts

Claim:

Given two sets of points, P_1 and P_2, in the plane, there is a line that simultaneously bisects both sets.



Ham-Sandwich Cuts

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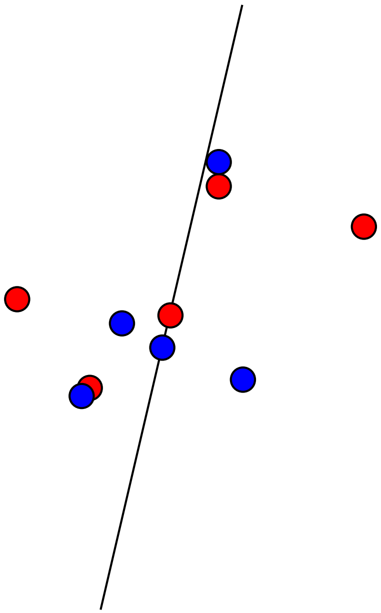
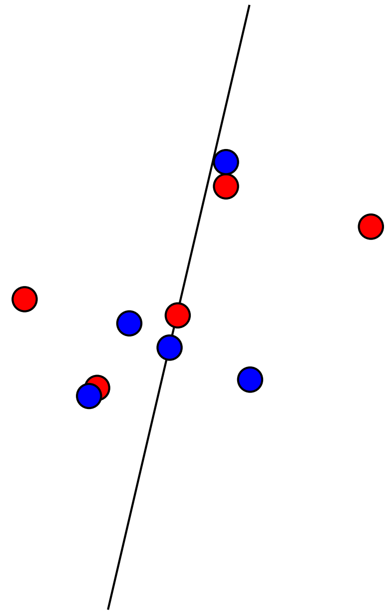


Ham-Sandwich Cuts

Proof:

Assume general position and, with some loss of generality, that the two point-sets each have an odd number of points.

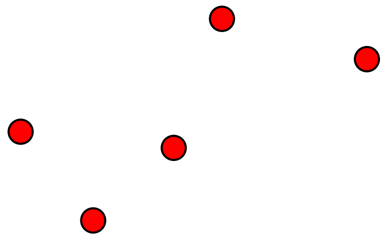


Ham-Sandwich Cuts

Note:

A line splits the point set P_1 in two if:

1. It passes through one of the points, and

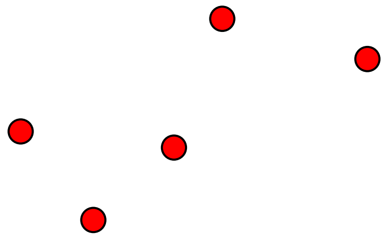


Ham-Sandwich Cuts

Note:

A line splits the point set P_1 in two if:

1. It passes through one of the points, and
2. It has the same number of points above and below.



Ham-Sandwich Cuts

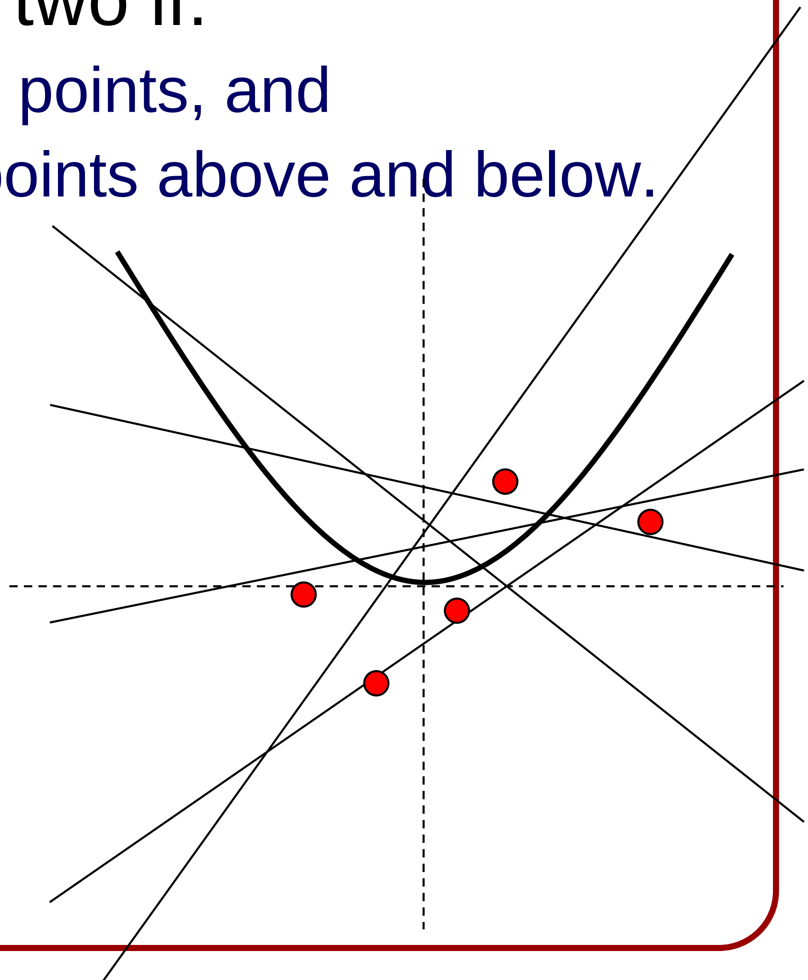
Note:

A line splits the point set P_1 in two if:

1. It passes through one of the points, and
2. It has the same number of points above and below.

Equivalently:

1. Its dual lies on the dual of one of the points, and



Ham-Sandwich Cuts

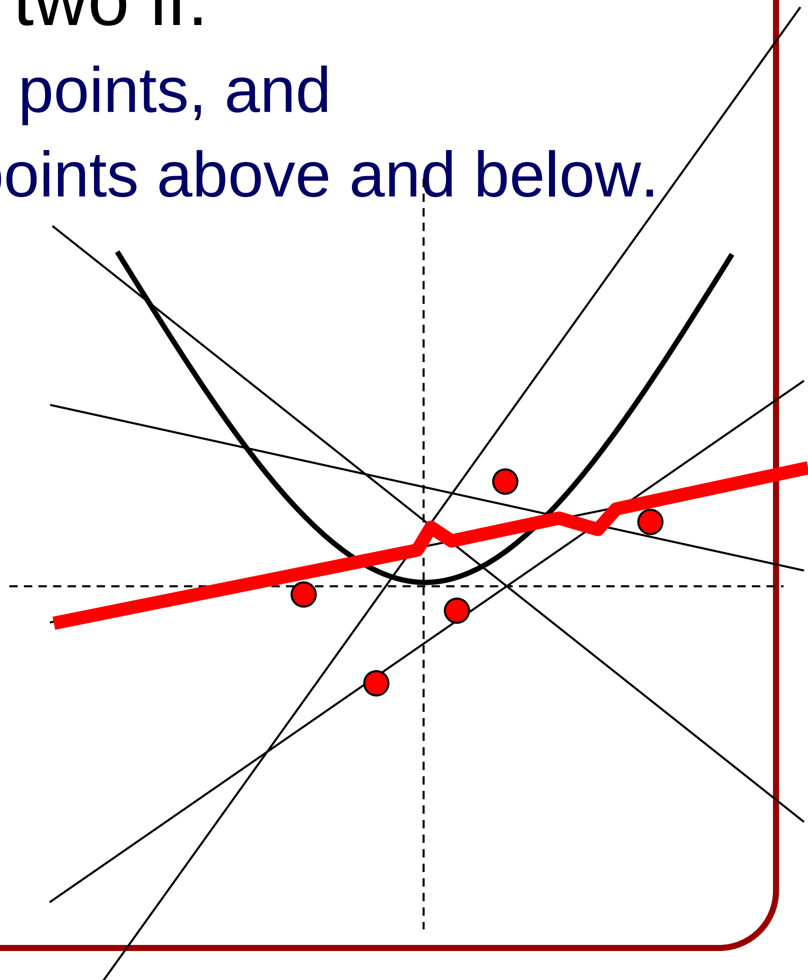
Note:

A line splits the point set P_1 in two if:

1. It passes through one of the points, and
2. It has the same number of points above and below.

Equivalently:

1. Its dual lies on the dual of one of the points, and
2. Its dual is on the median level of the dual arrangement.



Ham-Sandwich Cuts

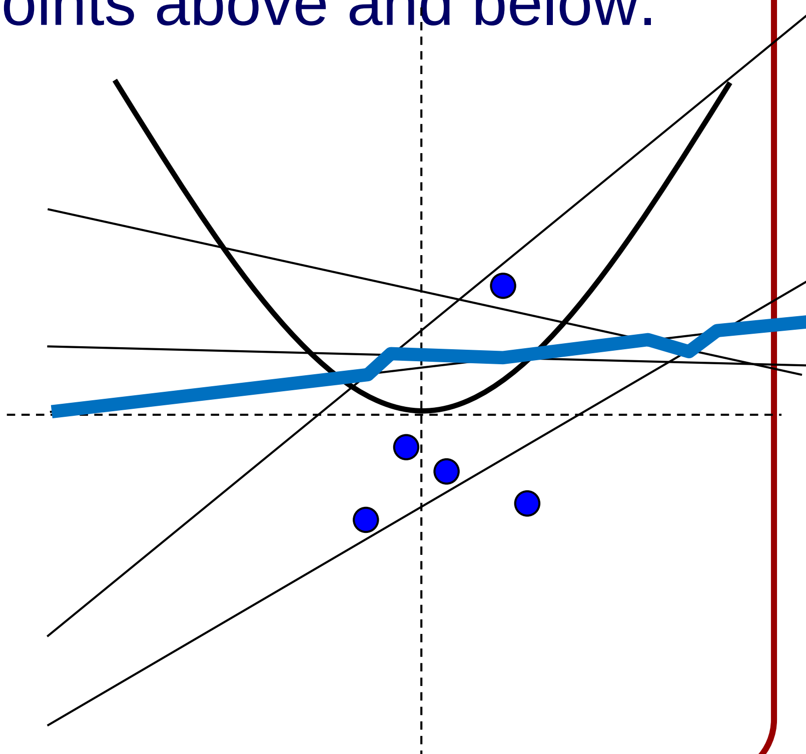
Note:

A line splits the point set P_2 in two if:

1. It passes through one of the points, and
2. It has the same number of points above and below.

Equivalently:

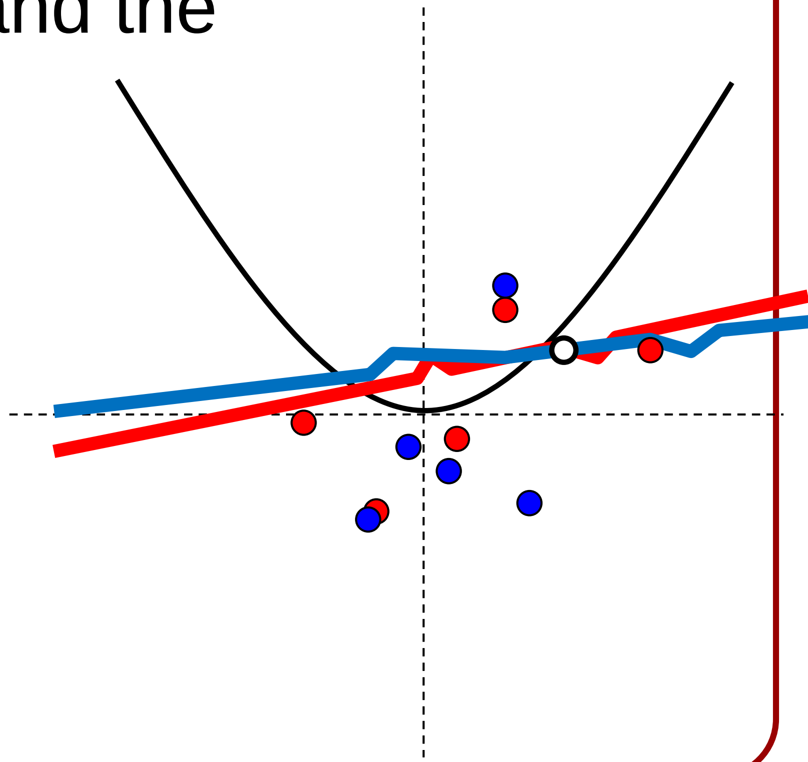
1. Its dual lies on the dual of one of the points, and
2. Its dual is on the median level of the dual arrangement.



Ham-Sandwich Cuts

<u>Note</u>:

A line splits both P_1 and P_2 in two if it passes through a point in P_1 and a point in P_2 and has the same number of points in P_1 and the same number of points in P_2 above and below.

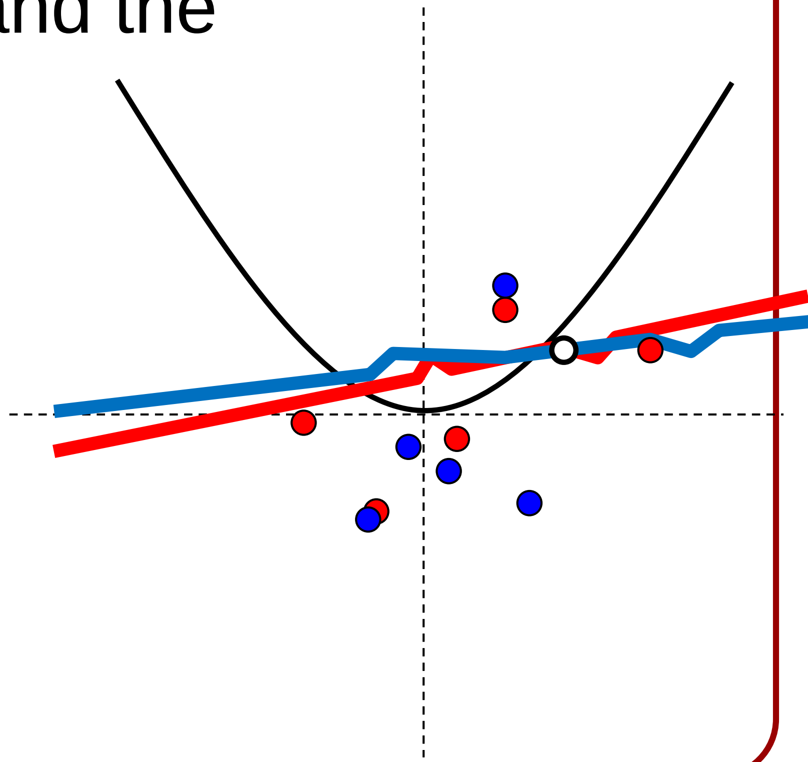


Ham-Sandwich Cuts

Note:

A line splits both P_1 and P_2 in two if it passes through a point in P_1 and a point in P_2 and has the same number of points in P_1 and the same number of points in P_2 above and below.

⇒ To find the cut, we need to find the intersection of the median levels of the two arrangements.



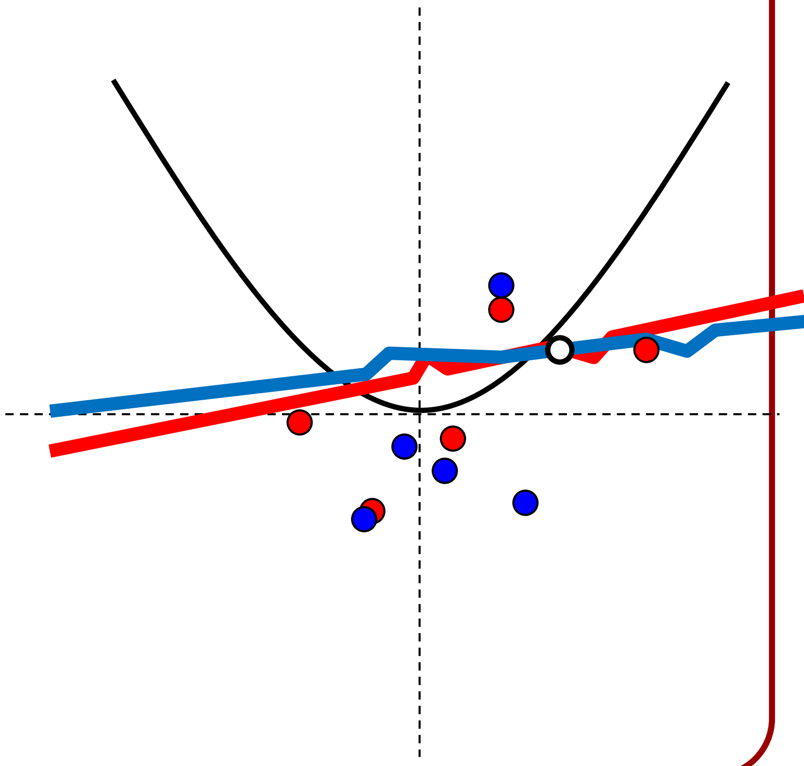
Ham-Sandwich Cuts

Claim:

The median levels of two arrangements must intersect (an odd number of times).

Sub-Claim:

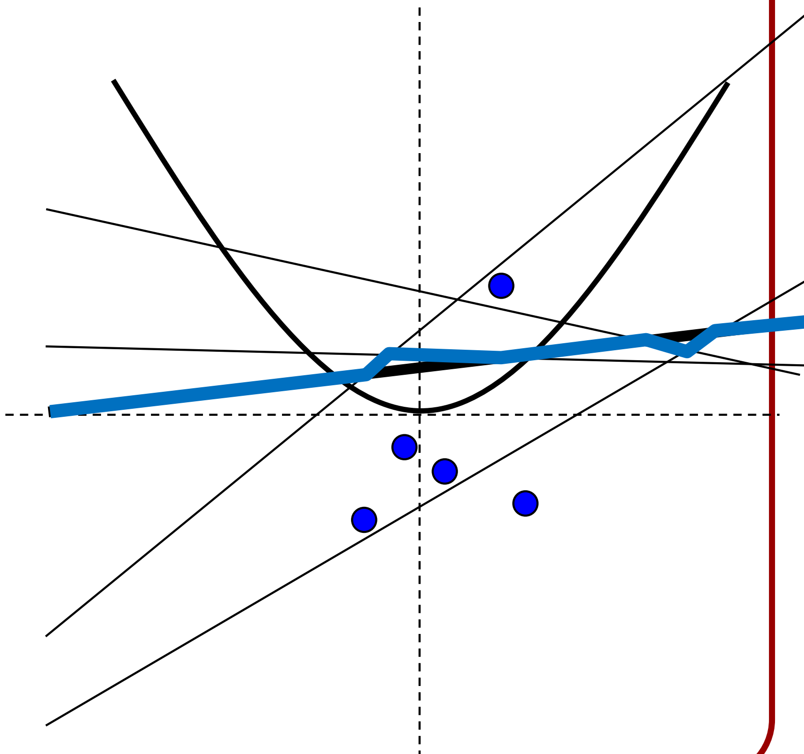
The two infinite edges of the median level are defined by the same line.



Ham-Sandwich Cuts

Proof (Sub-Claim):

Let L be the line giving the left edge of the median level.

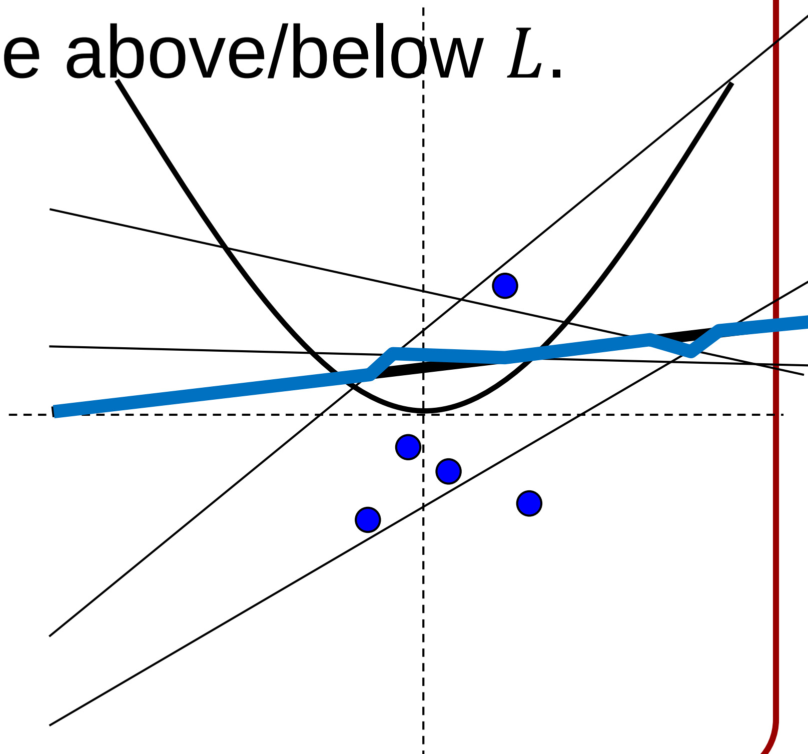


Ham-Sandwich Cuts

Proof (Sub-Claim):

Let L be the line giving the left edge of the median level.

$\Rightarrow$ As $x \to -\infty$ half the lines are above/below L.



Ham-Sandwich Cuts

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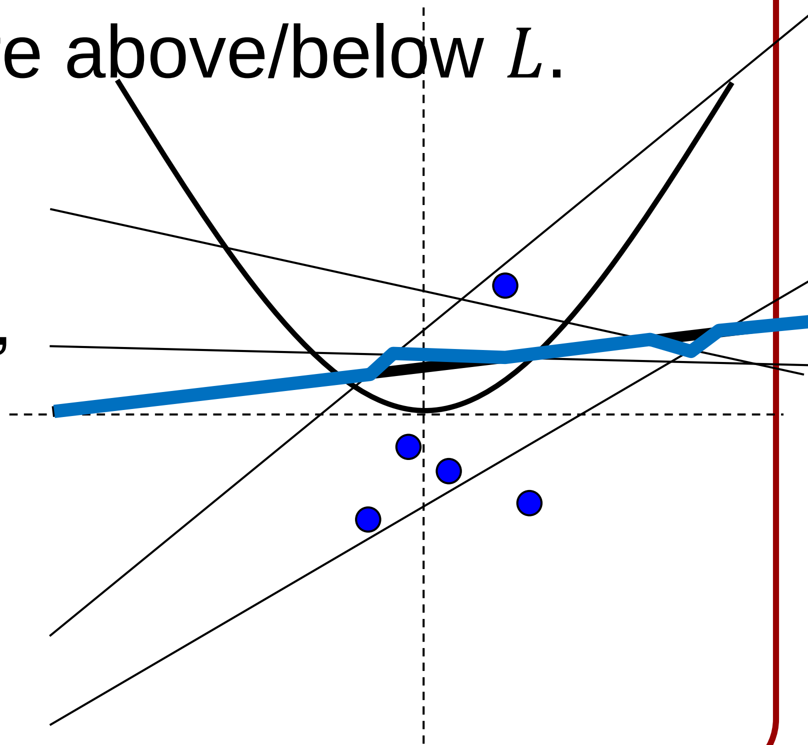
Let L be the line giving the left edge of the median level.

⇒ As $x \to -\infty$ half the lines are above/below L.

⇒ Assuming general position, at $x = \infty$ the "above" lines are "below" and the "below" lines are "above".



Ham-Sandwich Cuts

Proof (Sub-Claim):

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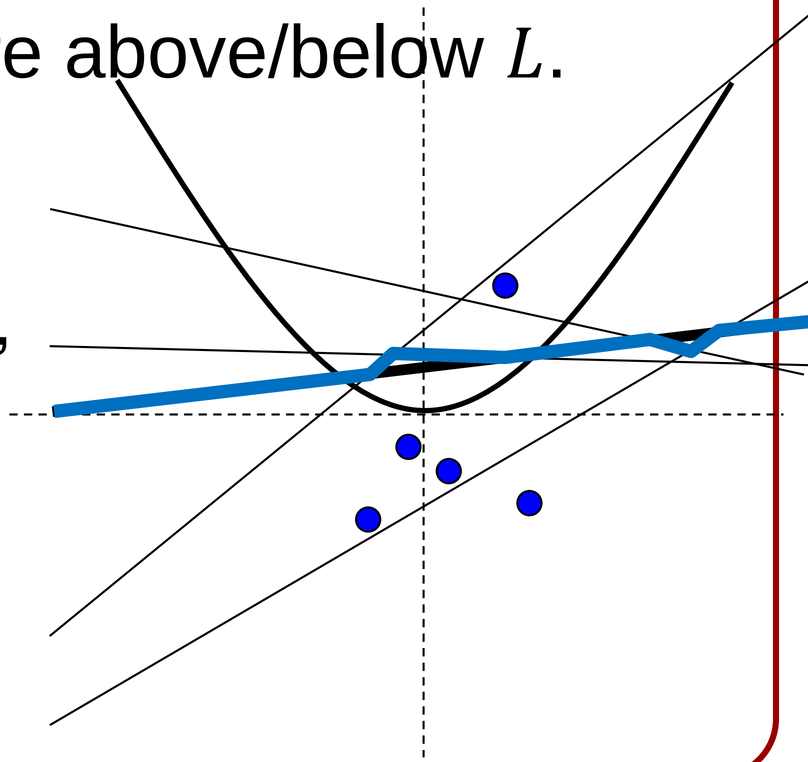
⇒ As $x \to -\infty$ half the lines are above/below L.

⇒ Assuming general position, at $x = \infty$ the "above" lines are "below" and the "below" lines are "above".

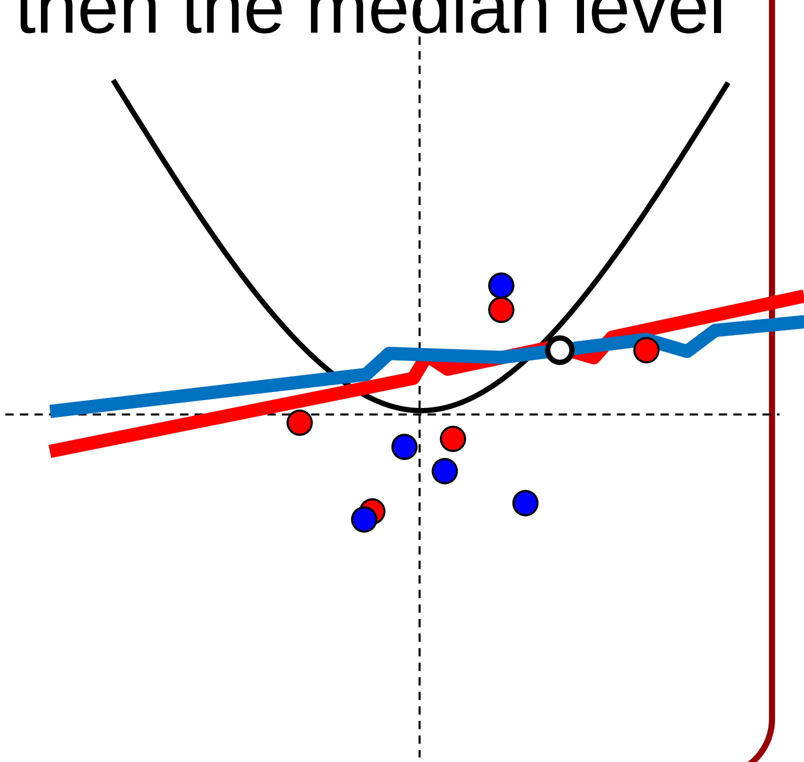
⇒ L also defines the right edge of the median level.



Ham-Sandwich Cuts

Proof (Claim):

Since the left/right-most edges lie on the same line, if the median level of P_1 is above (resp. below) the median level of P_2 as $x \to -\infty$ then the median level of P_1 is below (resp. above) the median level of P_2 as $x \to \infty$.



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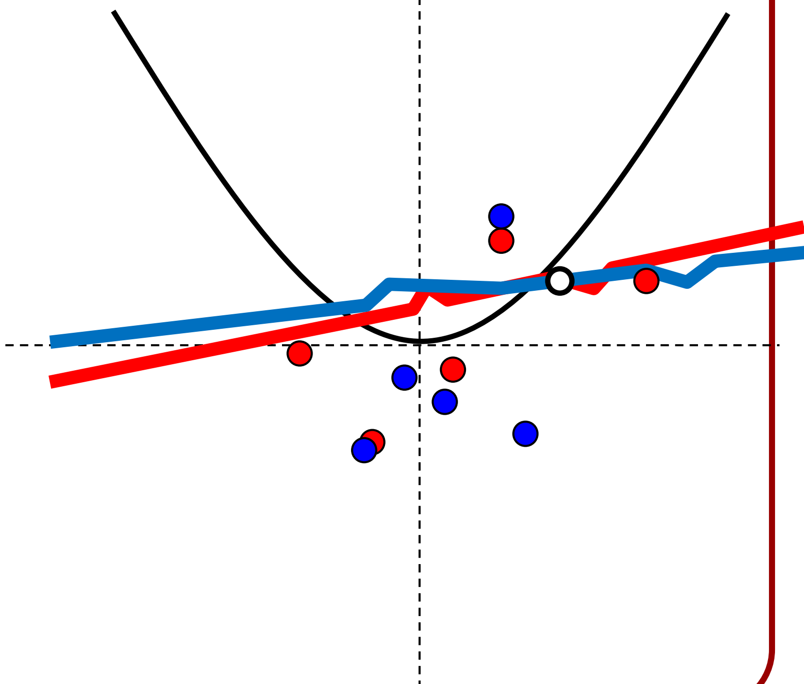

$\Rightarrow$ The median levels cross (an odd number of times).

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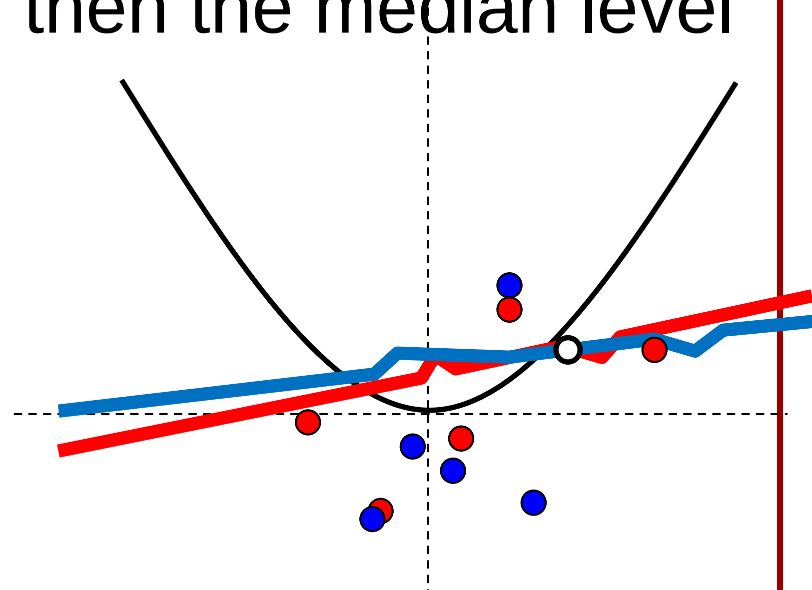


⇒ The median levels cross (an odd number of times).

[Lo, Maoutsek, and Steiger, 1994]:
The intersection can be found in $O(|P_1| + |P_2|)$ time.

Ham-Sandwich Cuts

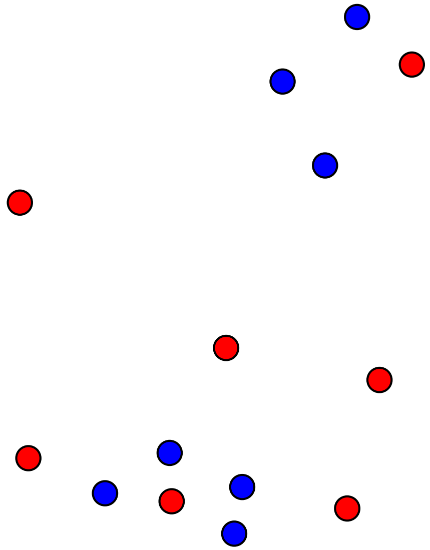
A similar argument holds in d dimensions:

Given d sets of points, $P_1, P_2, \ldots, P_d$, in d-dimensions, there is a hyperplane that simultaneously bisects each of the sets.

Red-Blue Matching

Claim:

Given n red and n blue points in the plan, we can pair them up using non-intersecting line segments.



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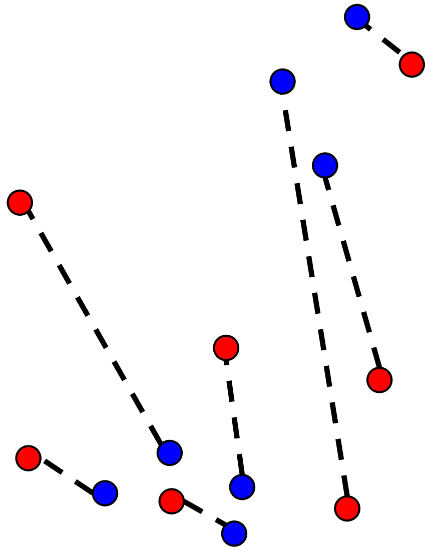

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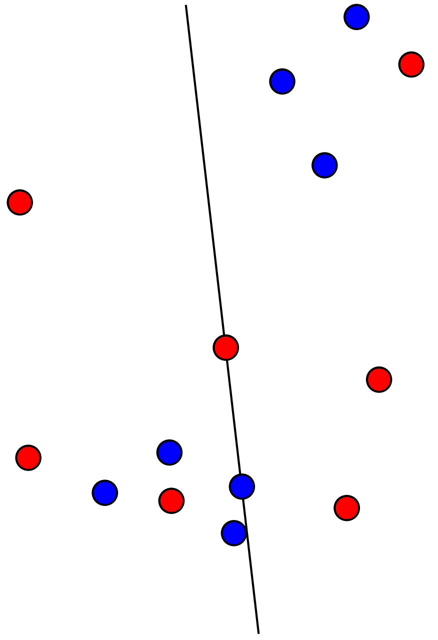
Proof (by Algorithm):

- Compute a ham-sandwich cut



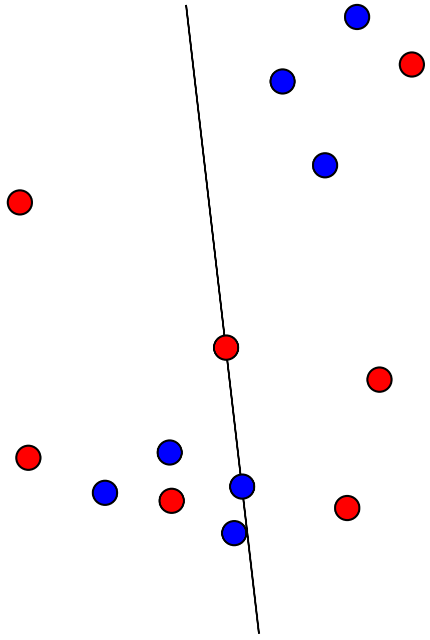
Red-Blue Matching

Claim:

Given n red and n blue points in the plan, we can pair them up using non-intersecting line segments.

Proof (by Algorithm):

- Compute a ham-sandwich cut
- (Recursively) compute a matching.



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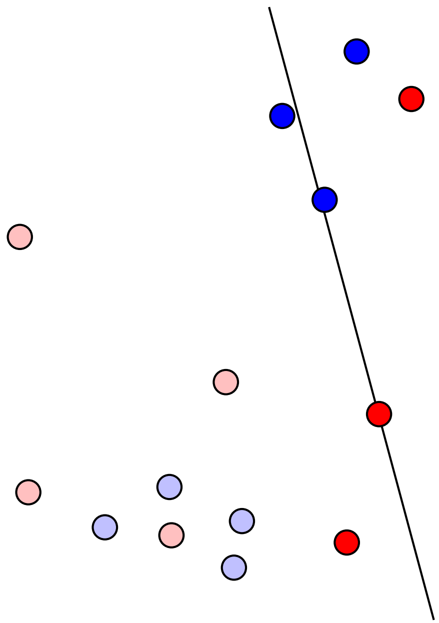

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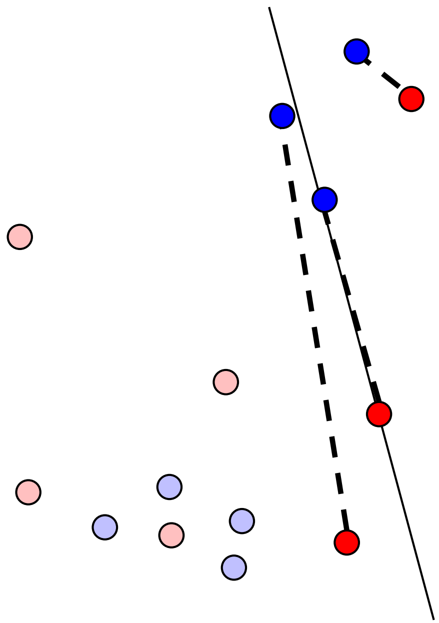

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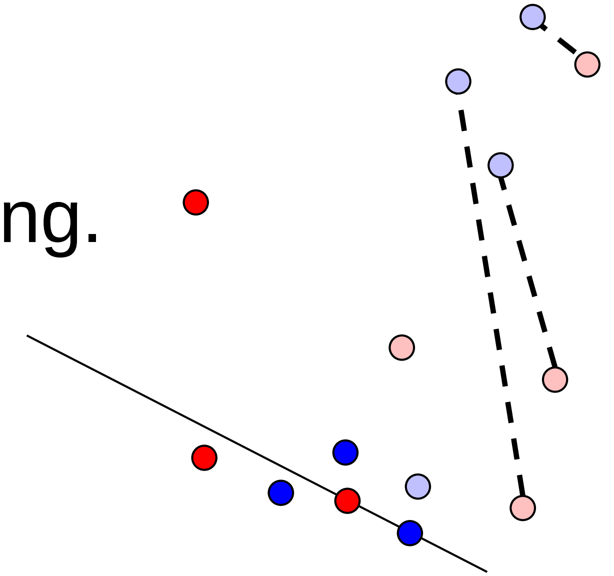

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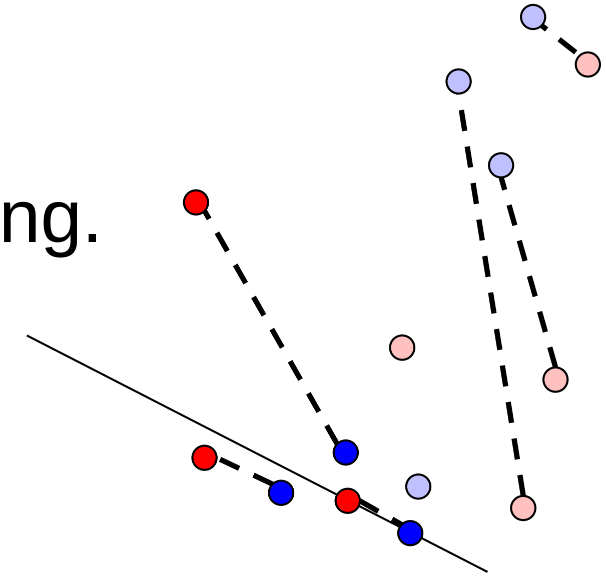

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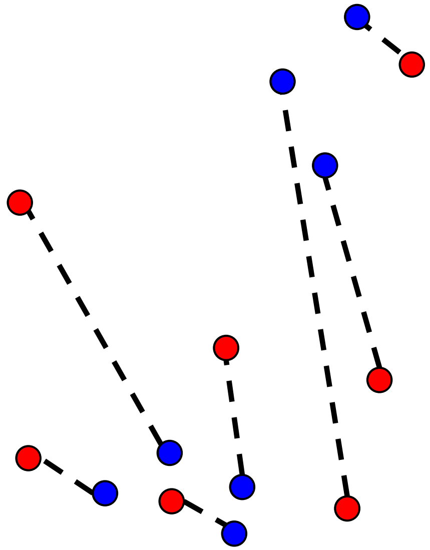

Since the line-segments for each sub-problem are on one side of the cut, the segments from the two sub-problems do not intersect.

Red-Blue Matching

Claim:

Given n red and n blue points in the plan, we can pair them up using non-intersecting line segments.

Proof (by Algorithm):

- Compute a ham-sandwich cut
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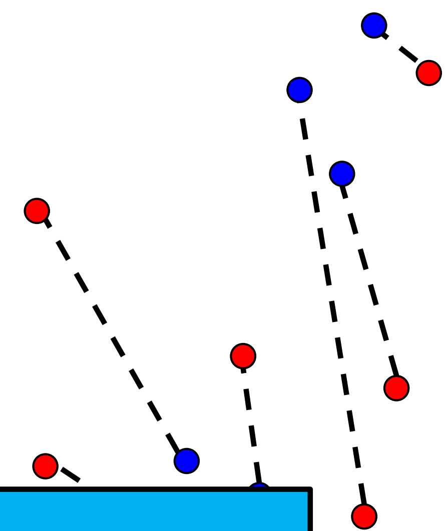

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sub

[Lo, Maoutsek, and Steiger, 1994]:
The matching can be found in $O(n \log n)$ time.