Arrangements

O’Rourke, Chapter 6

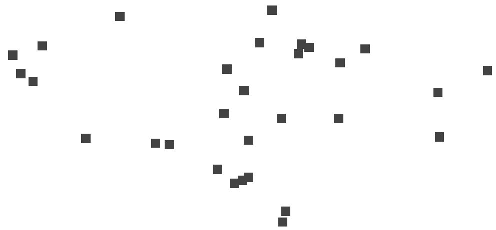
Outline

- Voronoi Diagrams
- Arrangements

Voronoi Diagrams

Recall:

We can compute the Delaunay Triangulation by raising the points to a paraboloid and computing the projection of the lower hull.



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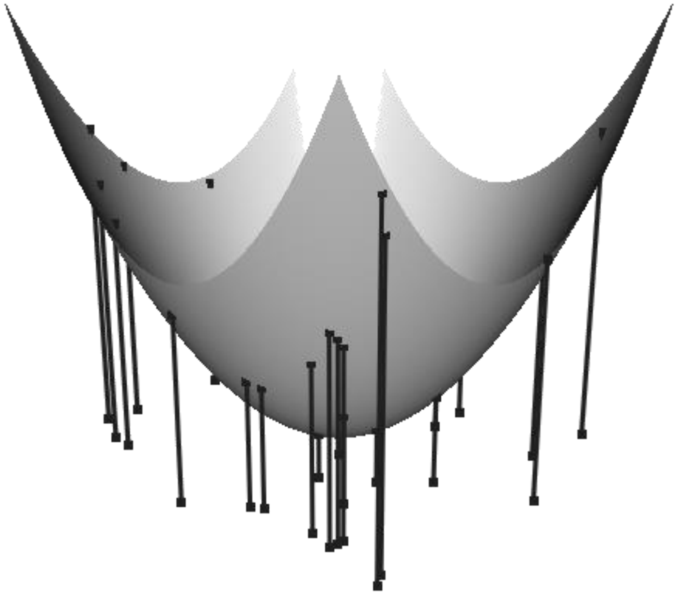

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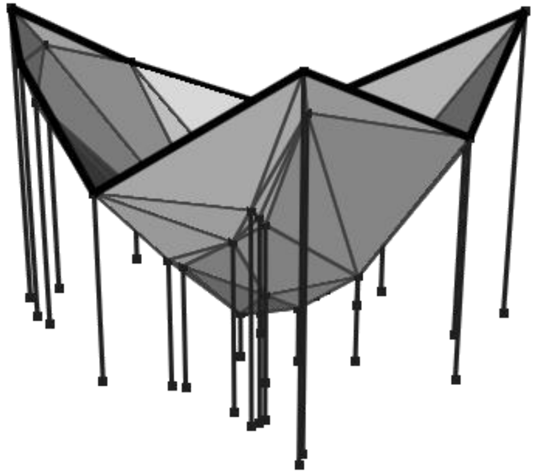

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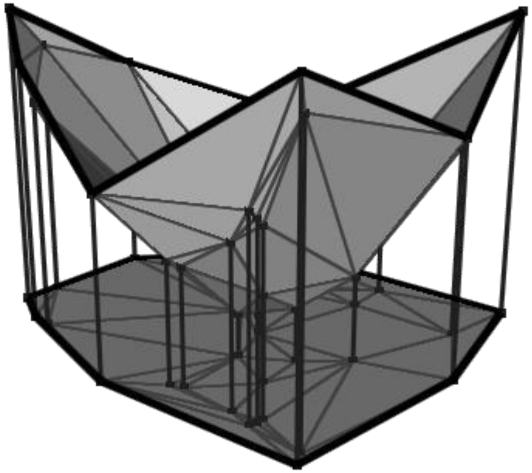

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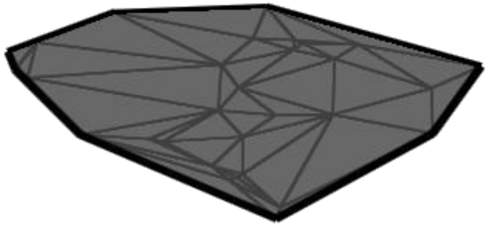
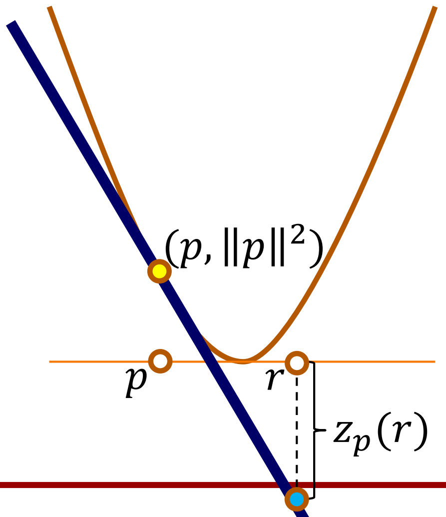


Voronoi Diagrams

Recall:

Given a point $P(p) = (p, \|p\|^2)$ on the paraboloid, the tangent plane is given by:

$$z_p(r) = 2\langle p, r\rangle - \|p\|^2$$



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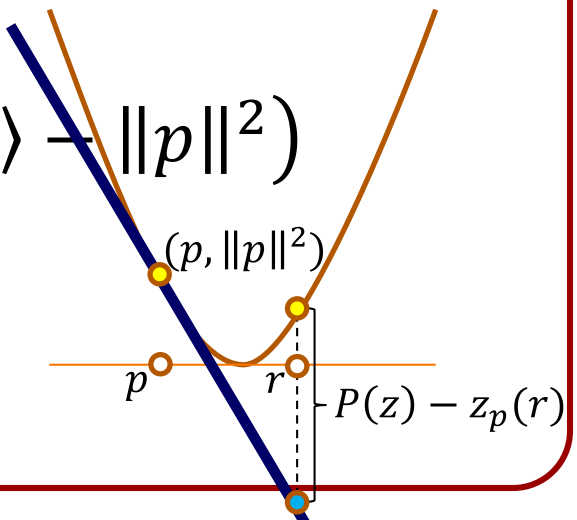
For any point r the (vertical) distance between its position on the parabola and its position on the tangent plane at p is:

$$P(r) - z_p(r) = \|r\|^2 - \left(2\langle r, p\rangle - \|p\|^2\right)$$



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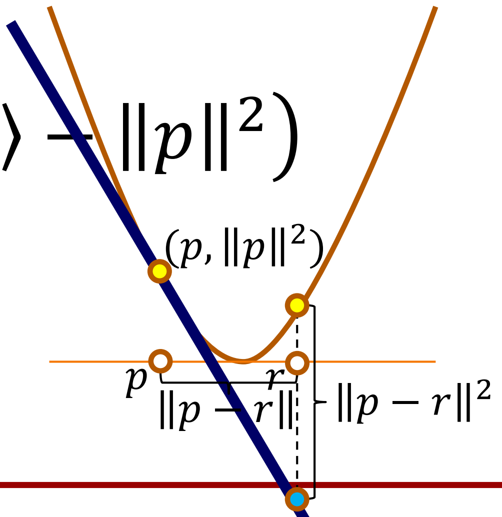
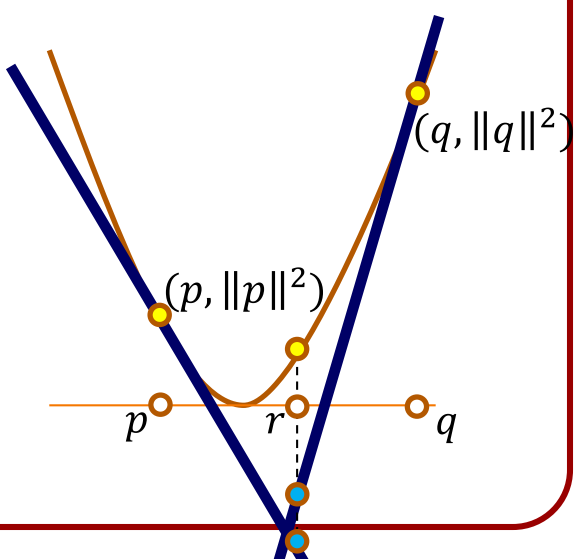
$$P(r) - z_p(r) = \|r\|^2 - \left(2\langle r, p\rangle - \|p\|^2\right)$$
$$= \|p - r\|^2$$



Voronoi Diagrams

⇒ Given points p and q, wherever the tangent plane at q is higher than the tangent plane at p, we are closer to q than to p.

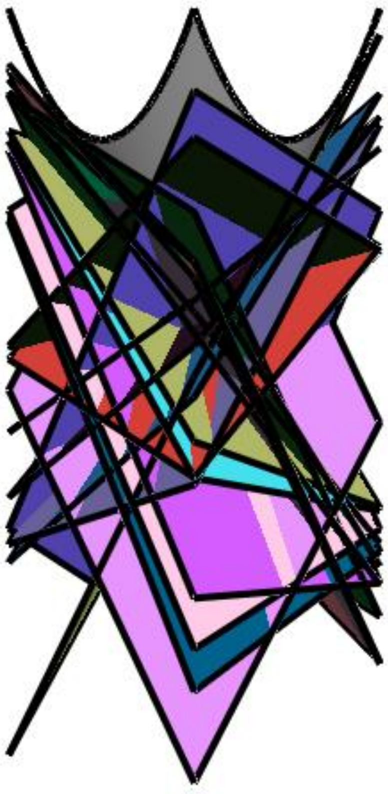
$$z_p(r) \leq z_q(r)$$
$$\Updownarrow$$
$$P(r) - z_p(r) \geq P(r) - z_q(r)$$
$$\Updownarrow$$
$$\|p - r\|^2 \geq \|q - r\|^2$$



Voronoi Diagrams

⇒ Given points p and q, wherever the tangent plane at p is higher than the tangent plane at q, we are closer to p than to q.

⇒ We can visualize the Voronoi diagram by drawing the tangent planes at the sites and looking down the z-axis.



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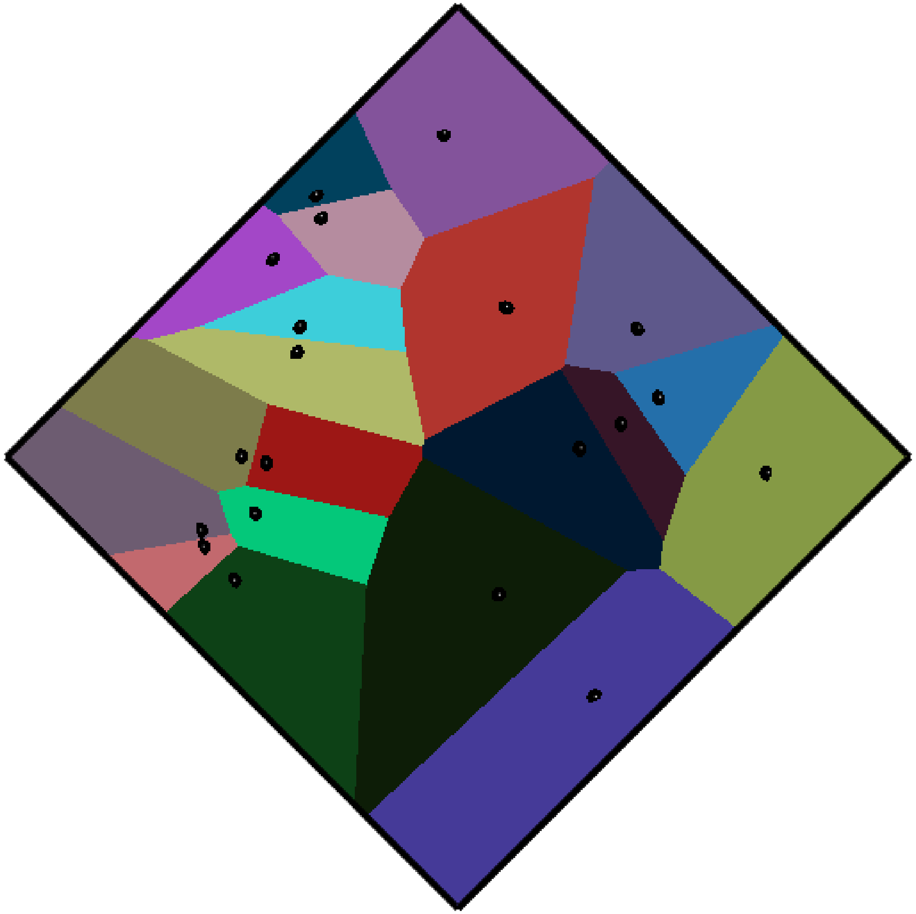

Outline

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- Arrangements

Arrangements

Definition:

An *arrangement of lines* is a set of lines in the plane, inducing a partition of the domain into (convex) faces, edges, and vertices.

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An *arrangement of lines* is a set of lines in the plane, inducing a partition of the domain into (convex) faces, edges, and vertices.

An arrangement is *simple* if all pairs of lines intersect, and no three lines intersect at the same point.

Combinatorics

Claim:

A simple arrangement of *n* lines has

- $\binom{n}{2}$ vertices,
- n^2 edges, and
- $\binom{n}{2} + n + 1$ faces.

Combinatorics

Proof (Vertices):

Since each pair of lines intersects exactly once, the total number of vertices is the number of distinct line pairs, $\binom{n}{2}$.

Combinatorics

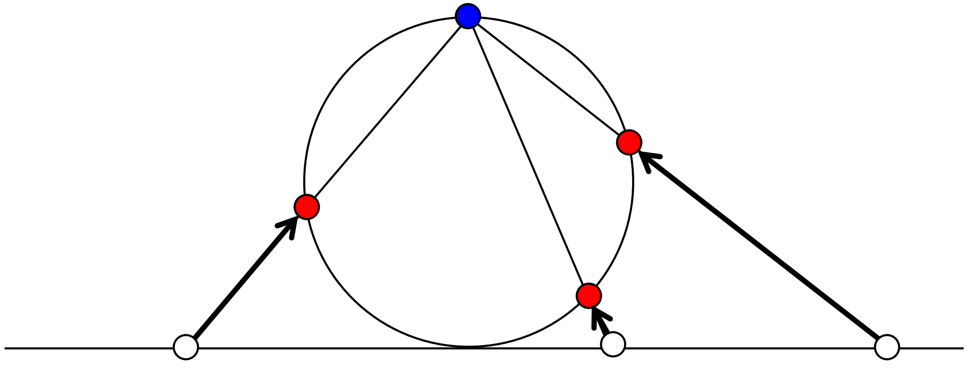
Proof (Edges):

Since each line is intersected by $n-1$ other lines, partitioning the lines into n edges, the total number of edges is n^2.

Combinatorics

Proof (Faces):

Using stereographic mapping, arrangements of lines in the plane can be thought of as polygonizations of the sphere.



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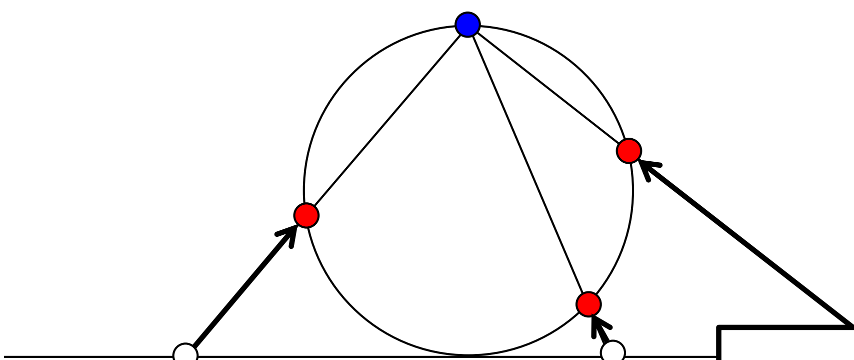
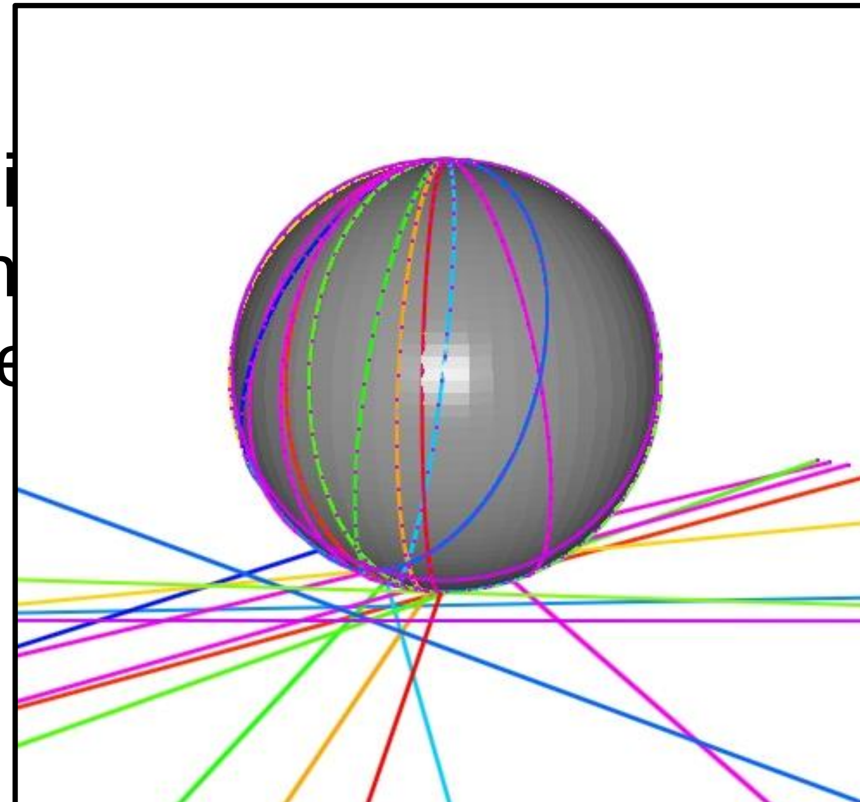




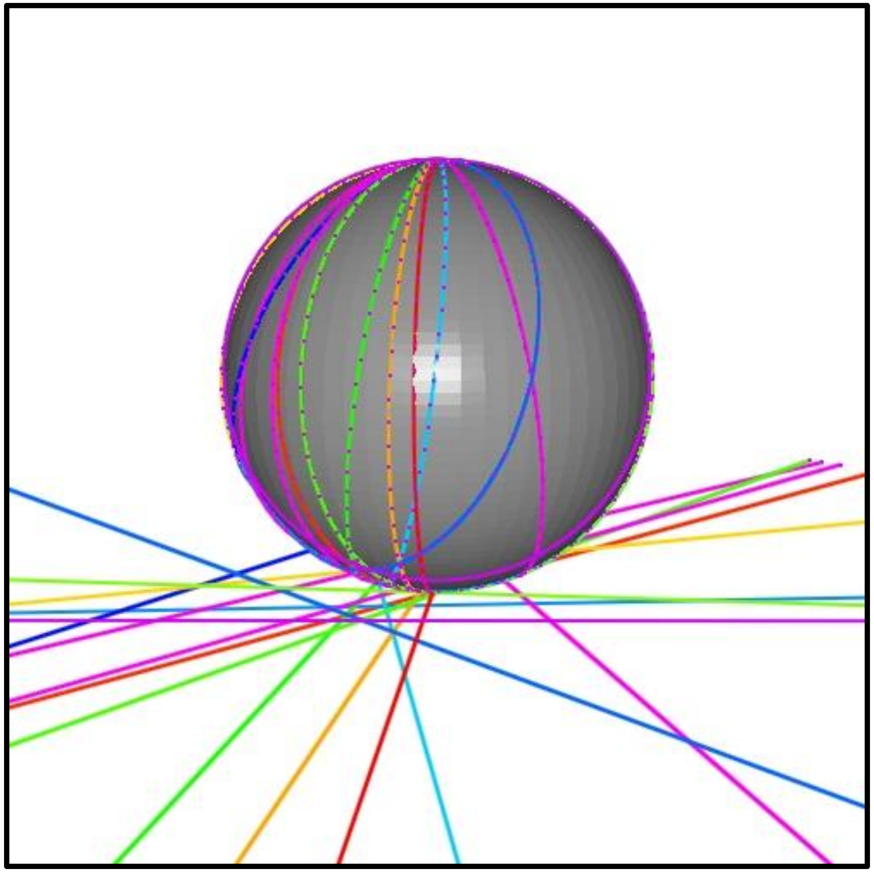
Note:
The stereographic mapping of the lines intersect at the North Pole.

Combinatorics

<u>Proof (Faces)</u>:

By Euler's theorem the number of faces is:

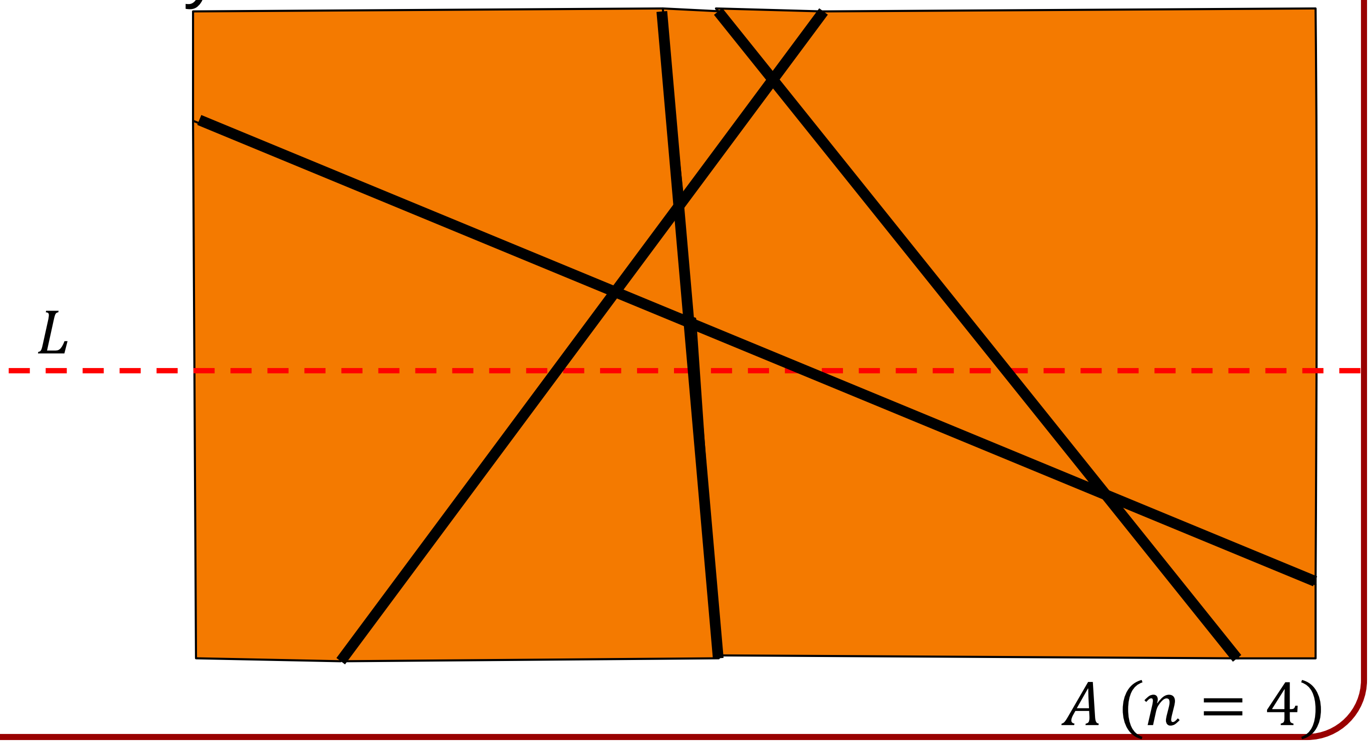
$$F = 2 - (V + 1) + E$$
$$= 2 - \binom{n}{2} - 1 + n^2$$
$$= \binom{n}{2} + n + 1$$



Zone Theorem

Definition:

Given an arrangement A and a line L (s.t. $A \cup \{L\}$ is simple) the *zone* of L in A, $Z(L)$, is the set of faces of A intersected by L.



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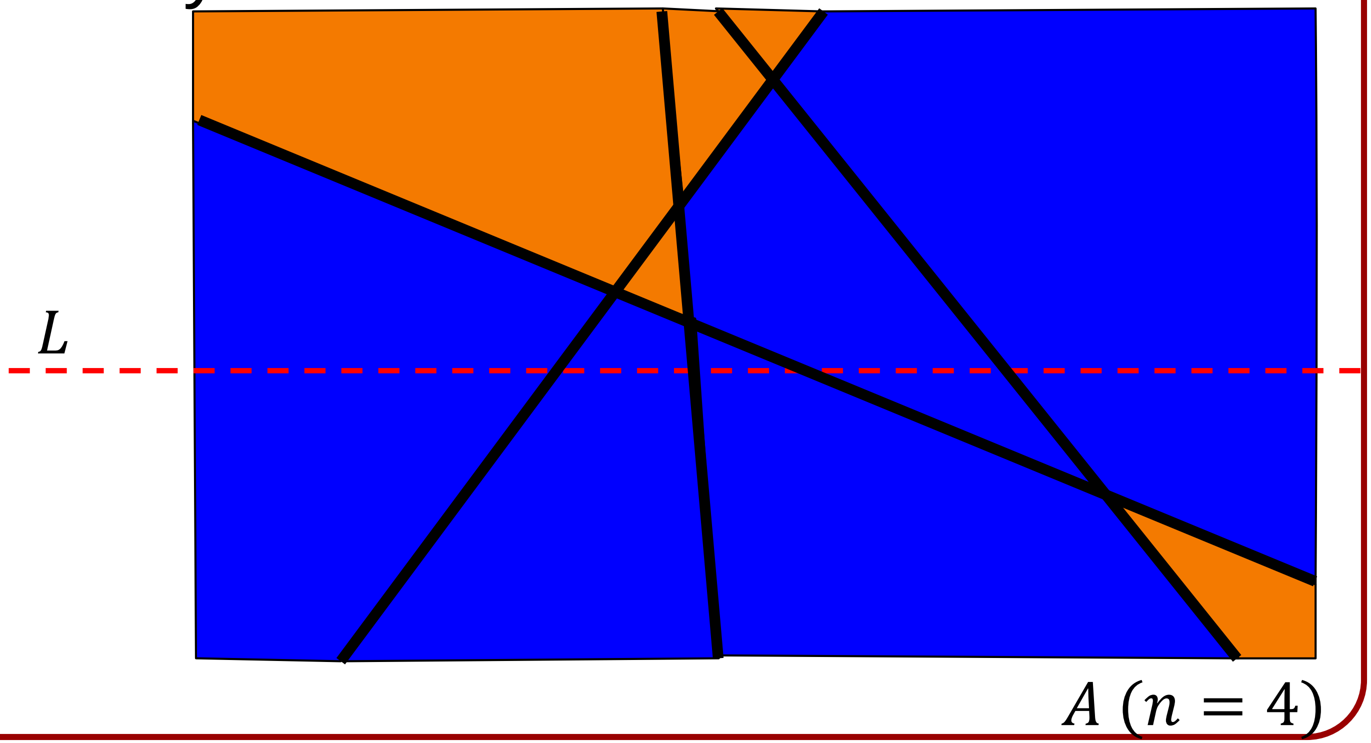
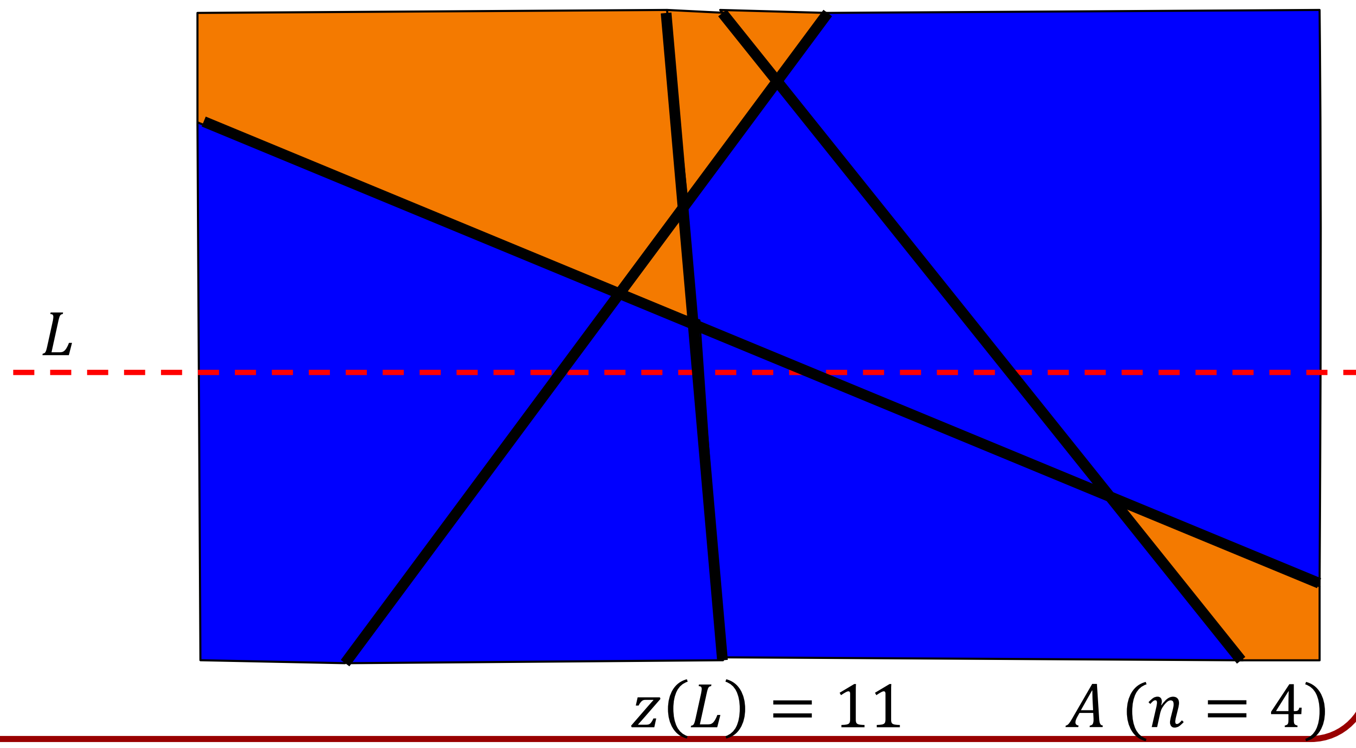


Zone Theorem

Notation:

The number of edges in $Z(L)$ is denoted $z(L)$.

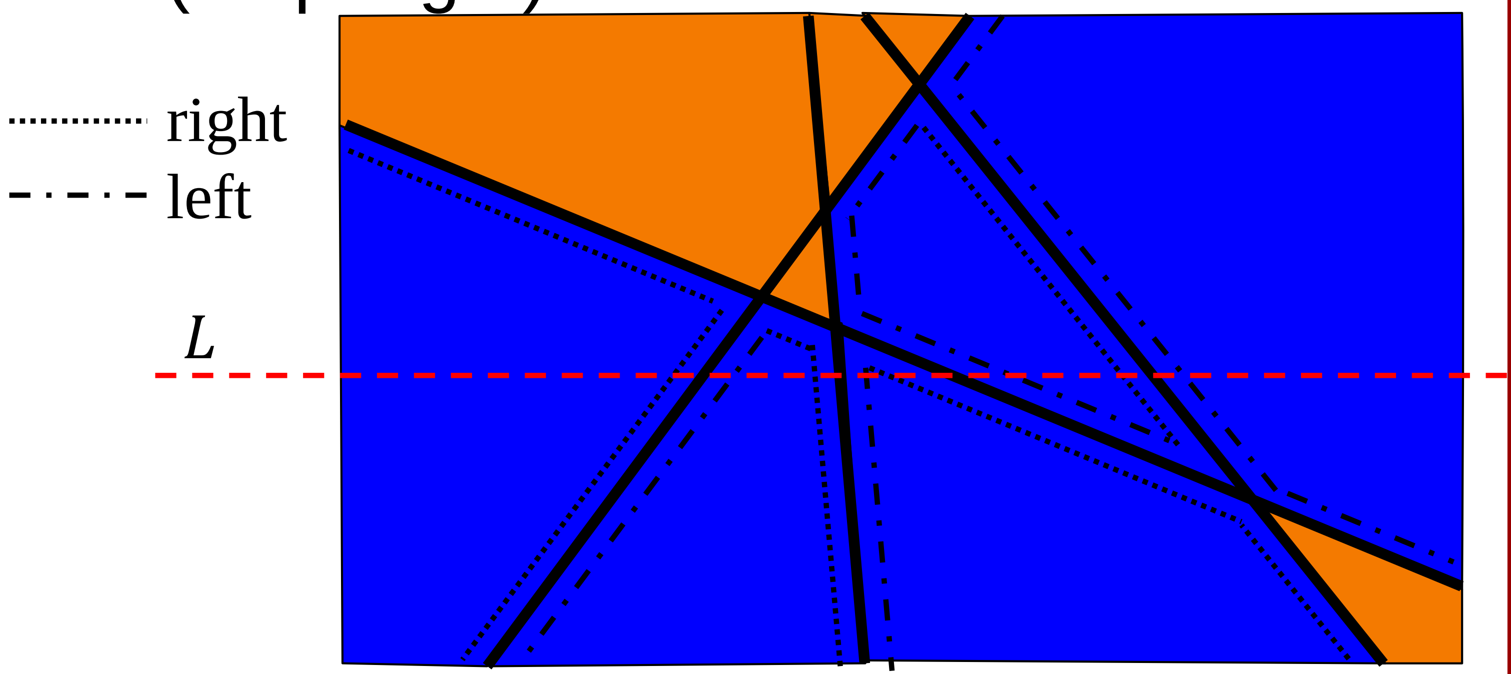
The max size of $z(L)$ over all lines is denoted z_n.



Zone Theorem

Note:

Assuming that no line in A is horizontal, we mark an edge as *left* (resp. *right*) if it bounds a face of $Z(L)$ from the left (resp. right).*



$z(L) = 11$ $A\ (n = 4)$

*Note that an edge can be marked both *left* and *right*.

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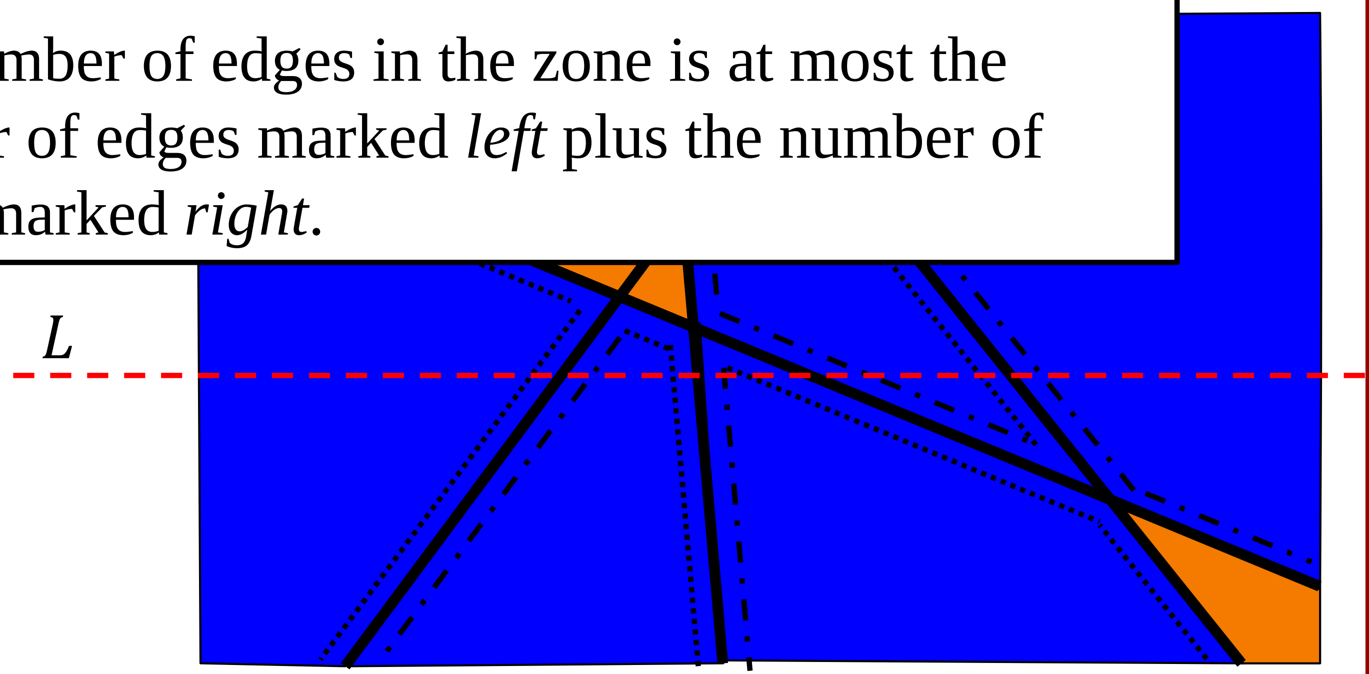
Note:
The number of edges in the zone is at most the number of edges marked *left* plus the number of edges marked *right*.



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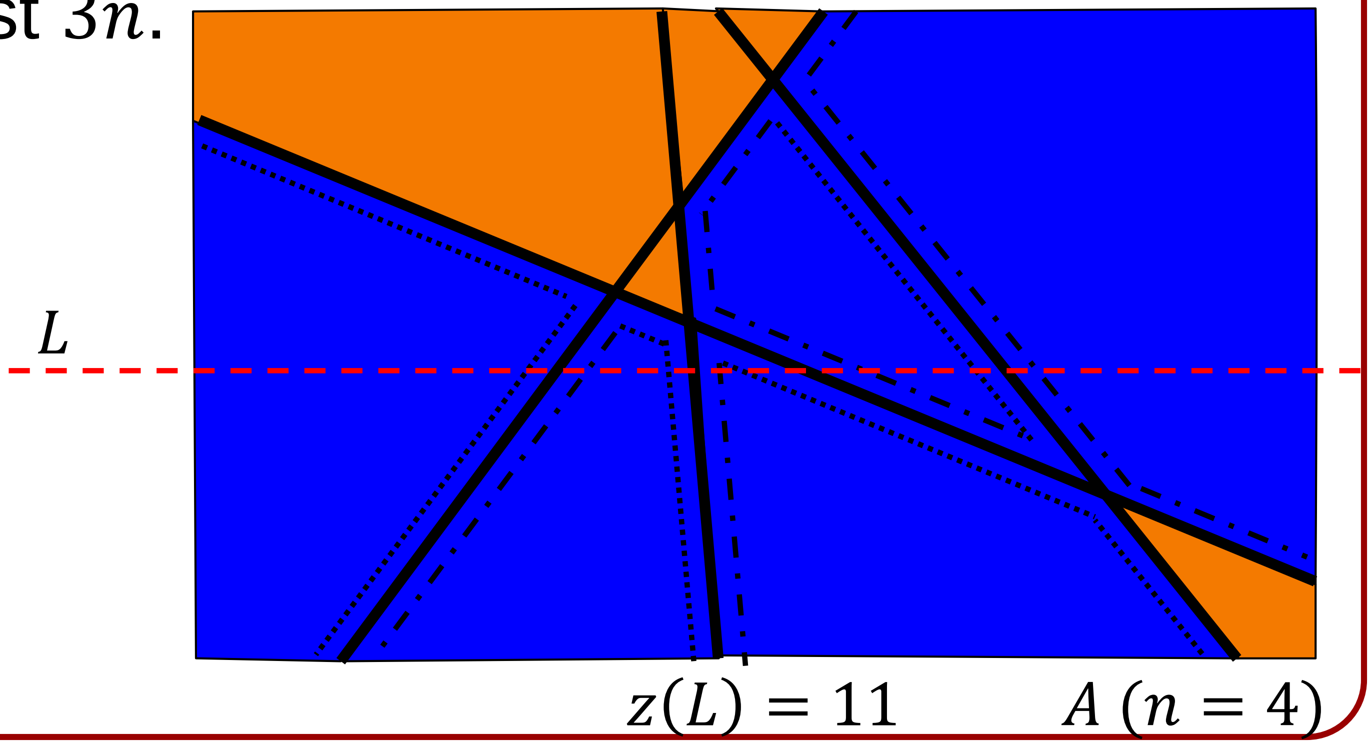
Zone Theorem

Theorem:

For an arrangement of n lines, $z_n \le 6n$.

In particular, the number of edges marked left (resp. right) at most $3n$.



Zone Theorem

<u>Proof</u>:

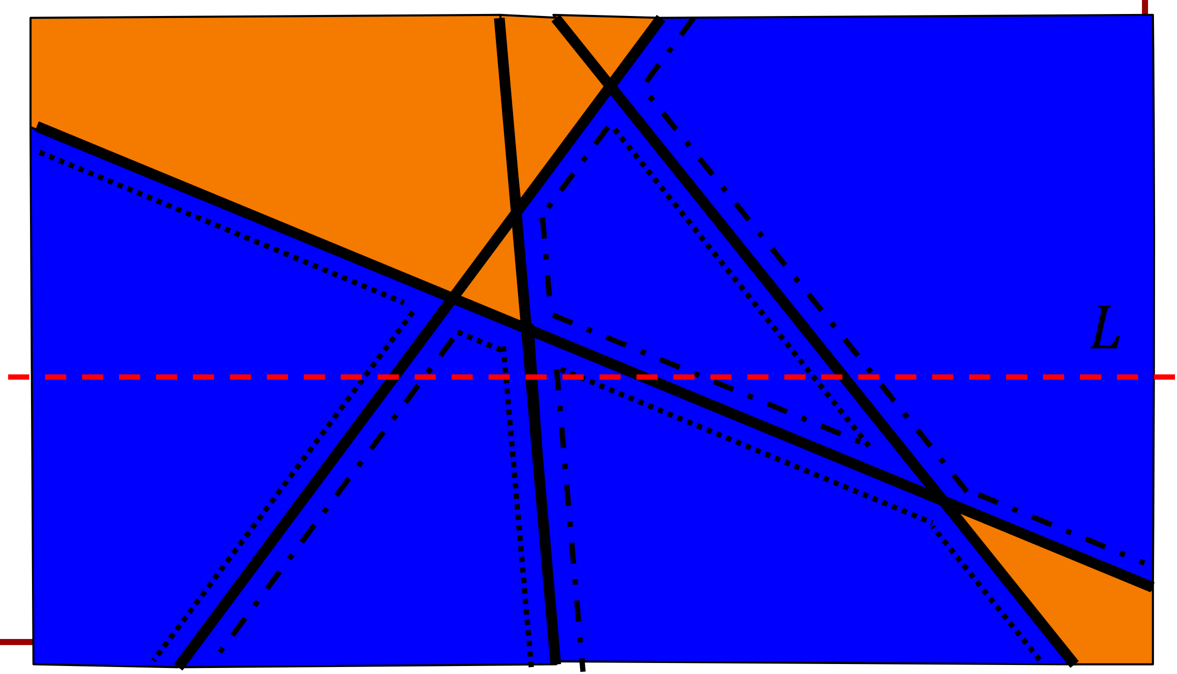
Without loss of generality, assume that the line L is horizontal.

Proceed by induction.

Zone Theorem

Proof (base case):

Trivially true when $n = 0$.



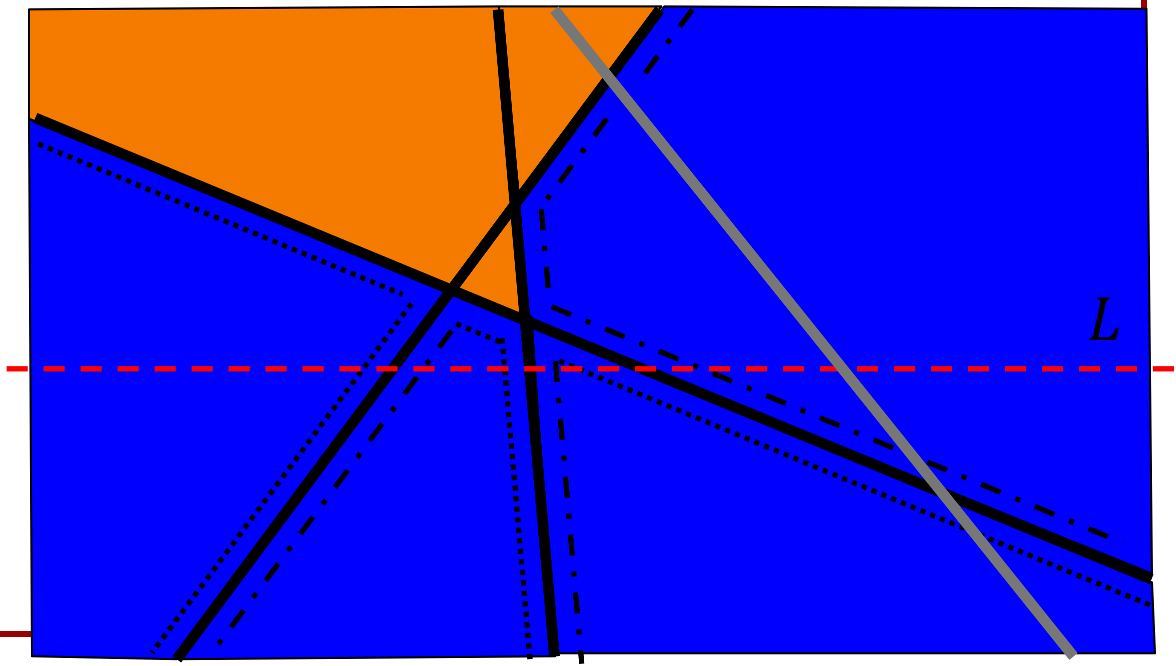
Zone Theorem

Proof (inductive case):

Remove the right-most line on L.

By induction, the number of left edges crossed is at most $3(n-1)$.

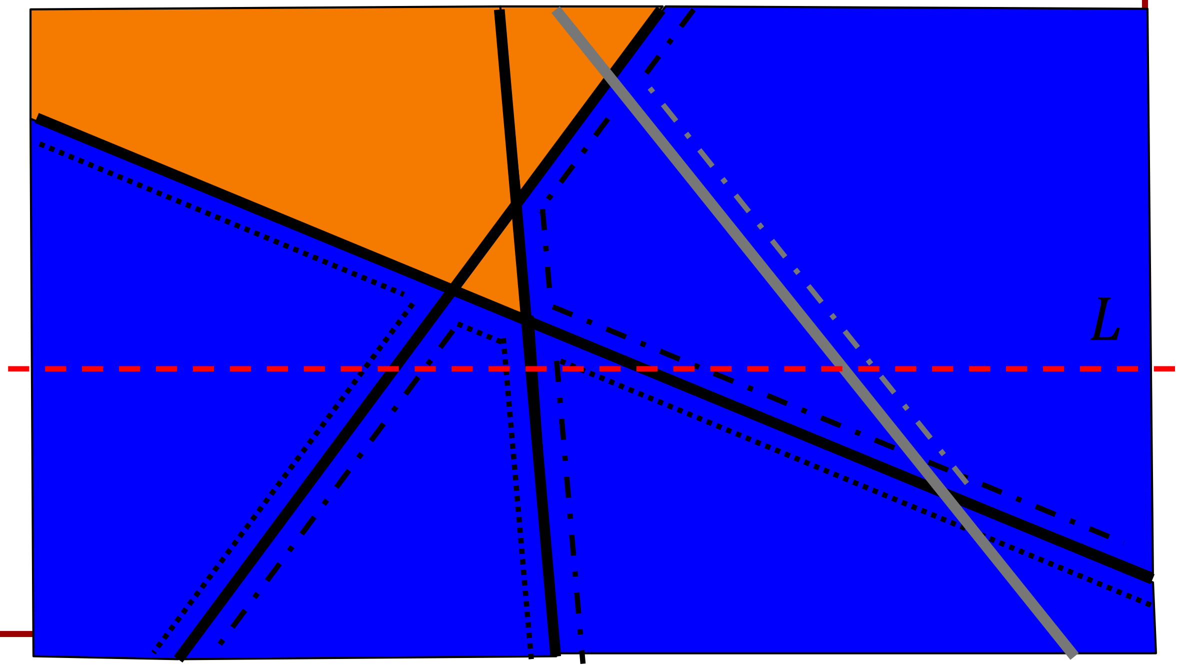
Need to show that adding the line back generates at most 3 additional left edges.



Zone Theorem

Claim 1:

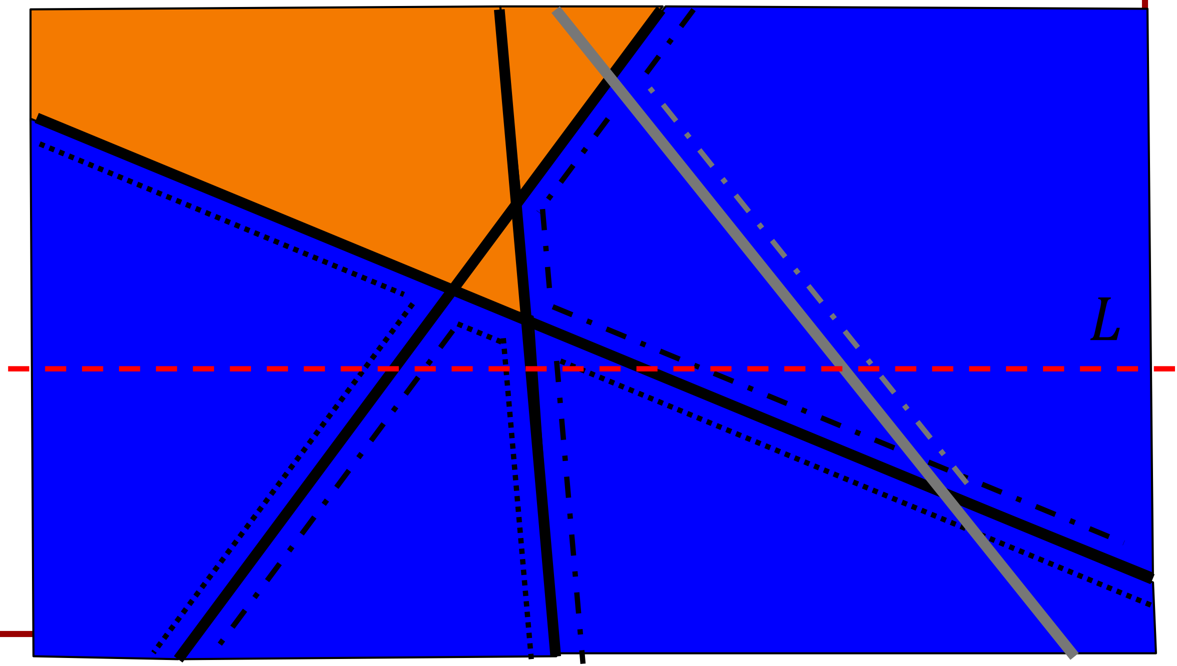
Adding the right-most line introduces exactly one new left edge.



Zone Theorem

Proof of Claim 1:

It introduces one because this will be a left edge of the right-most face.



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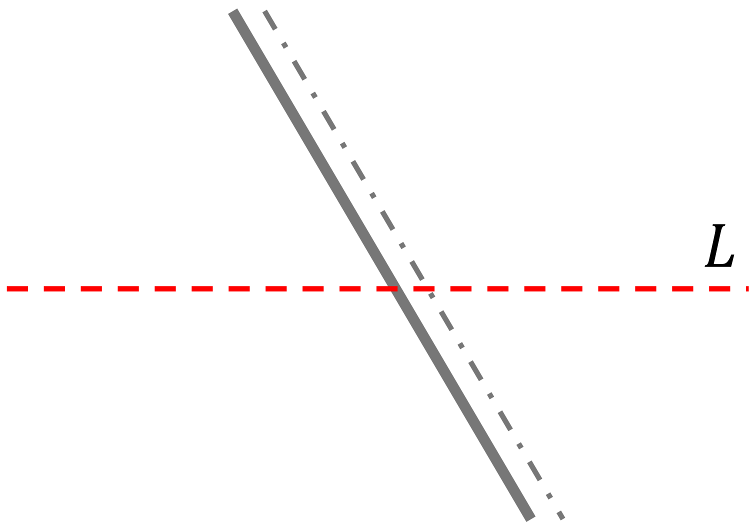
Proof of Claim 1:

It introduces one because this will be a left edge of the right-most face.

It introduces exactly one because a right-most line cannot contribute more than one left edge.



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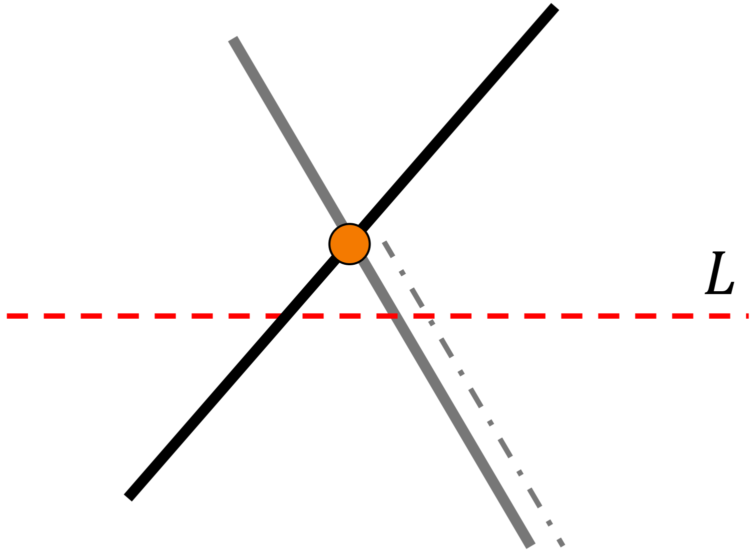

If it is split by a line from the left, only one of the two segments will be in the zone, (the one containing *L*.)

Zone Theorem

Proof of Claim 1:

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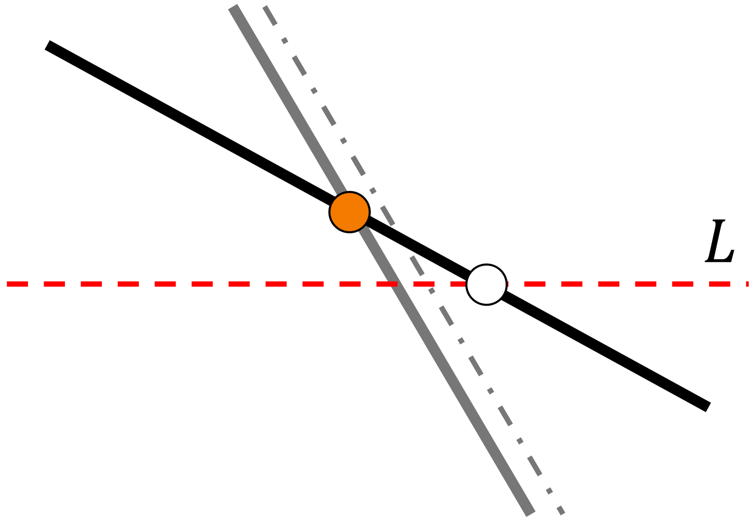
It introduces exactly one because a right-most line cannot contribute more than one left edge.

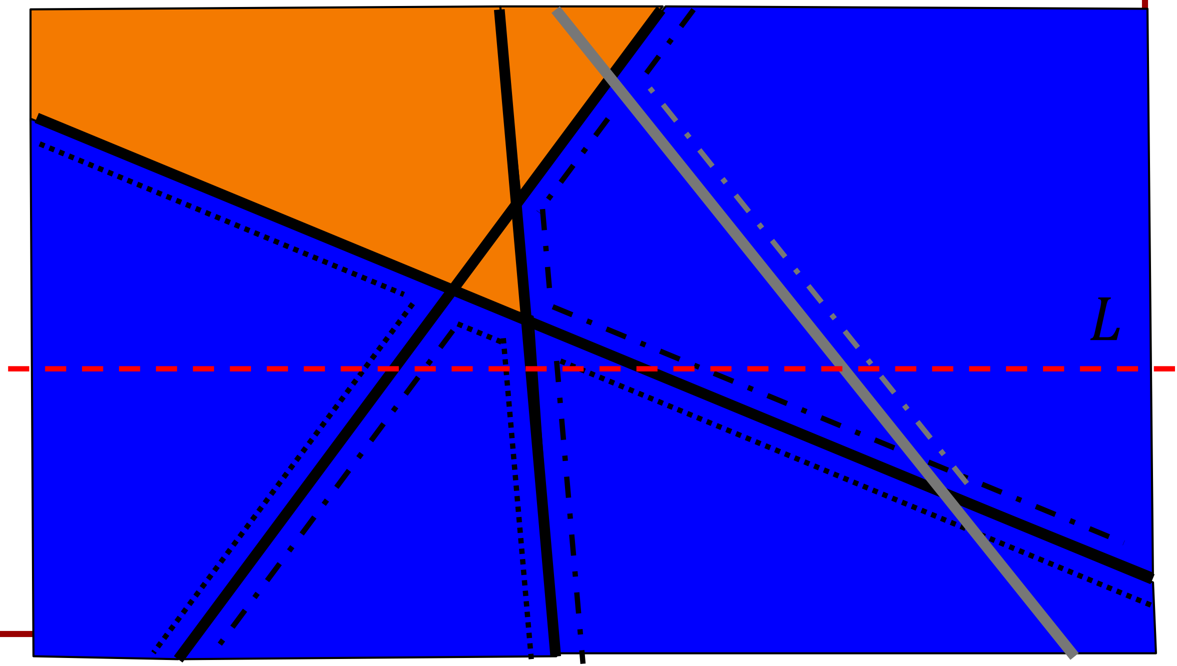


If it is split by a line from the right, then it wasn't right-most.

Zone Theorem

Claim 2:

Adding the right-most line splits at most two existing left edges.



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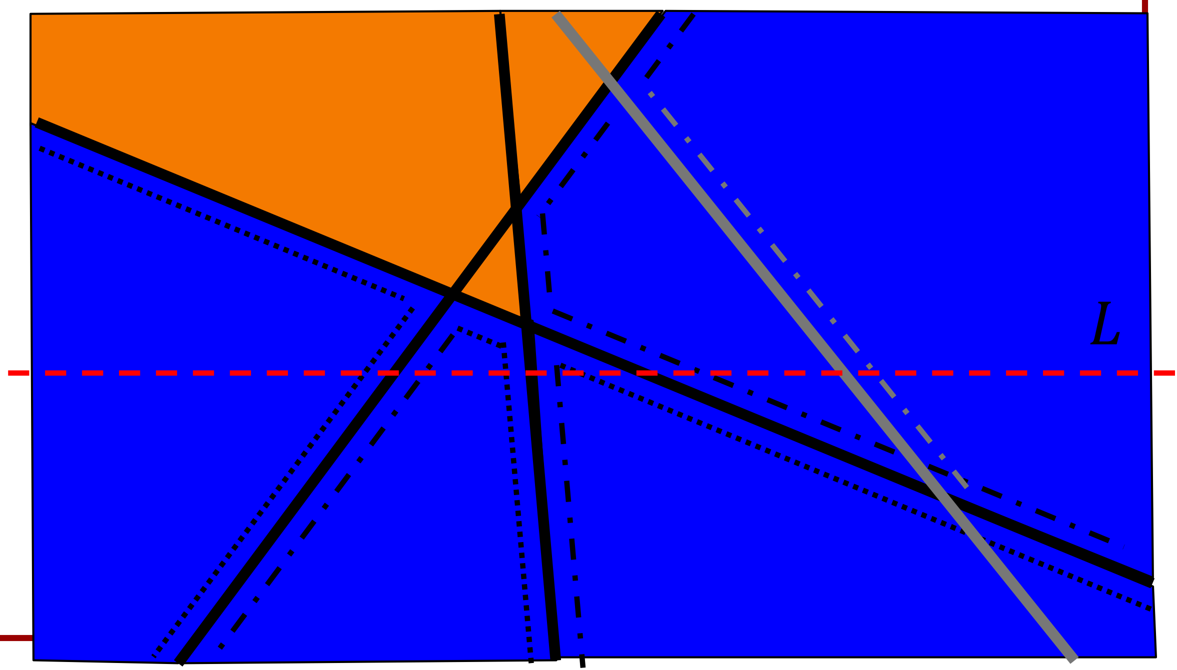
Proof of Claim 2:

If the right-most line splits a left edge in two, the edge must be on the right-most face.



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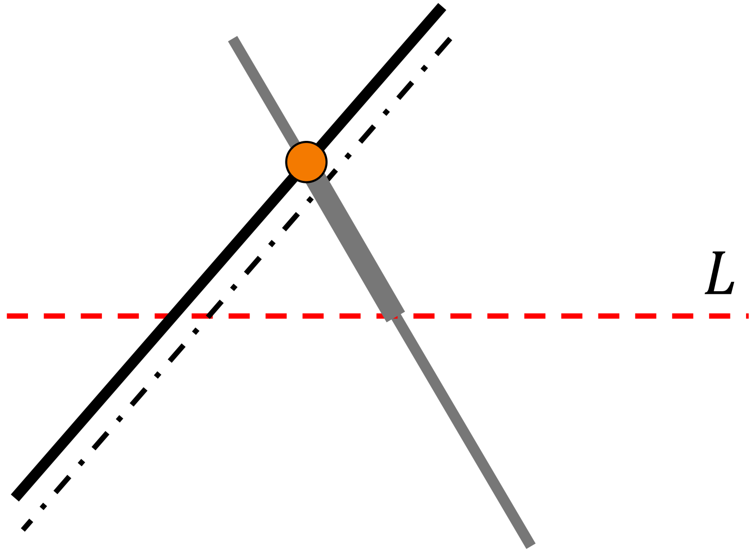
Consider the segment of the right-most line from L to the left edge.



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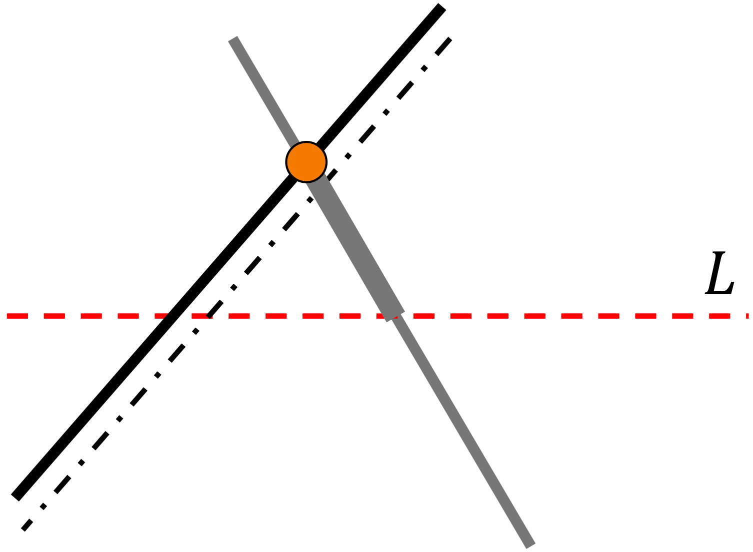

If it is not split by another line, the left edge must have been on the right-most face.

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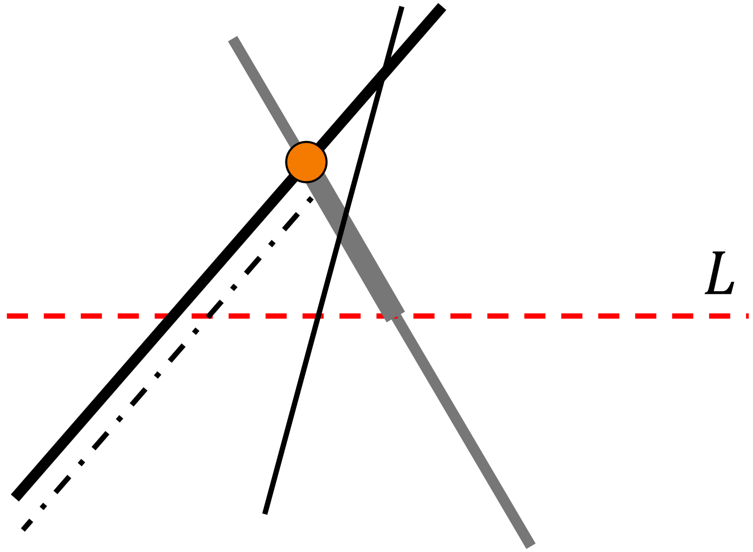


If it is split by another line, only one of the two sides of the left edge will be in the zone.

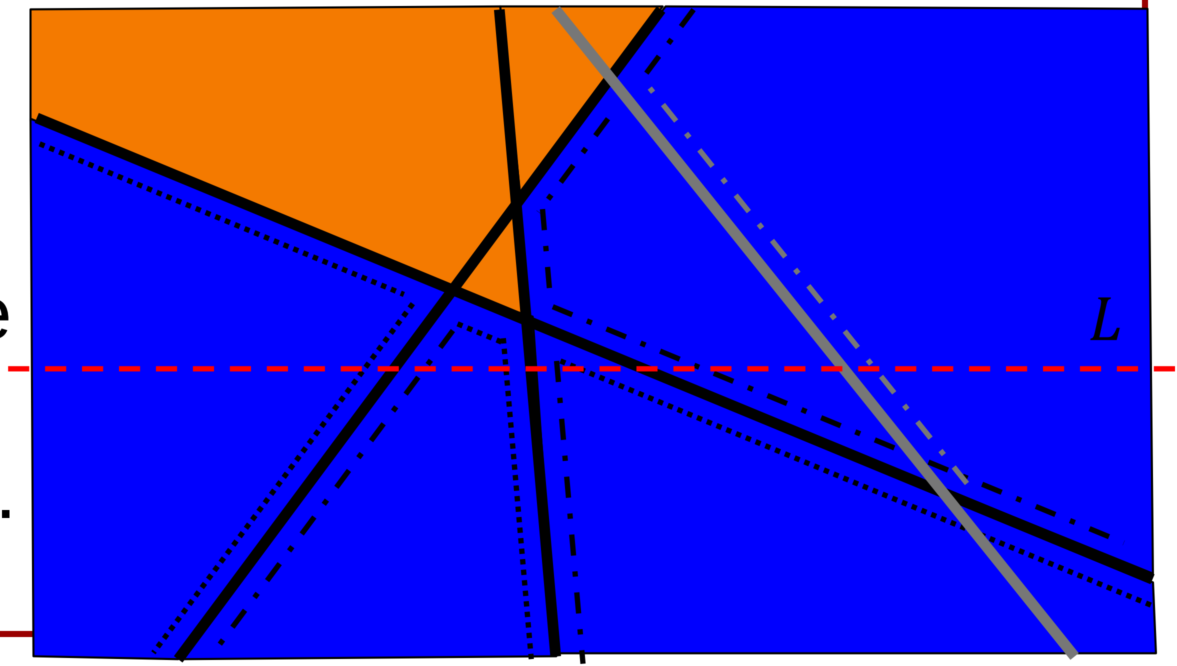
Zone Theorem

Proof of Claim 2:

If the right-most line splits a left edge in two, the edge must be on the right-most face.

Since faces are convex, the line splits at most two edges on the right-most face.

These must be left edges because otherwise the line was not right-most.



Zone Theorem

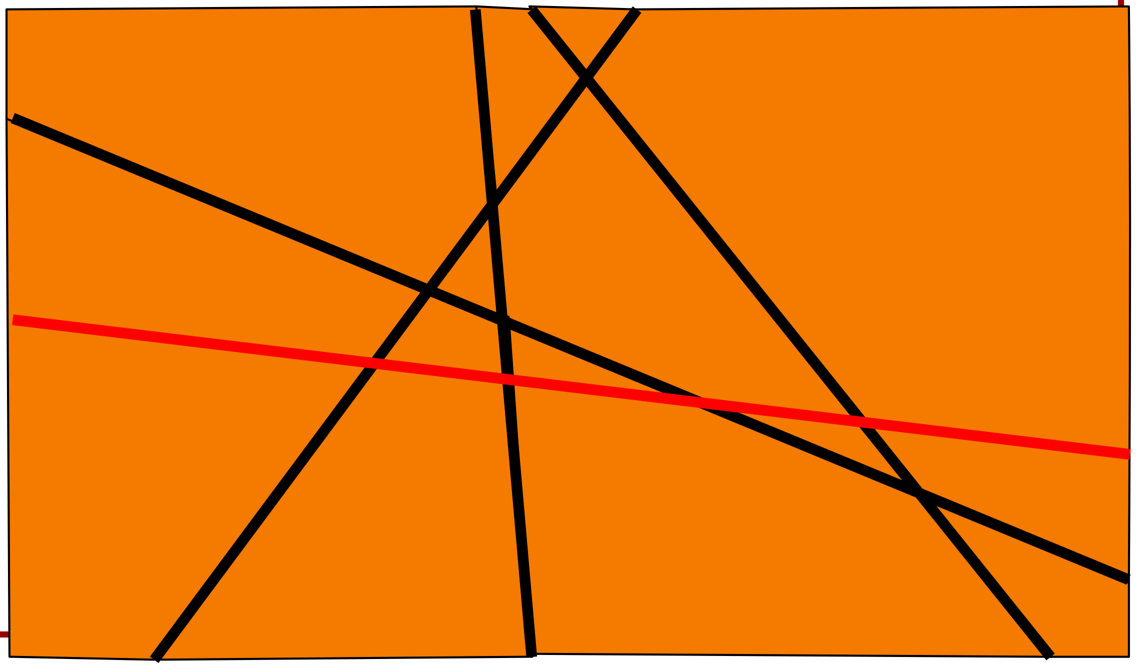
Corollary:

We can construct a (simple) arrangement of n lines in $O(n^2)$ time.

Zone Theorem

Proof:

Iteratively add the k-th line.

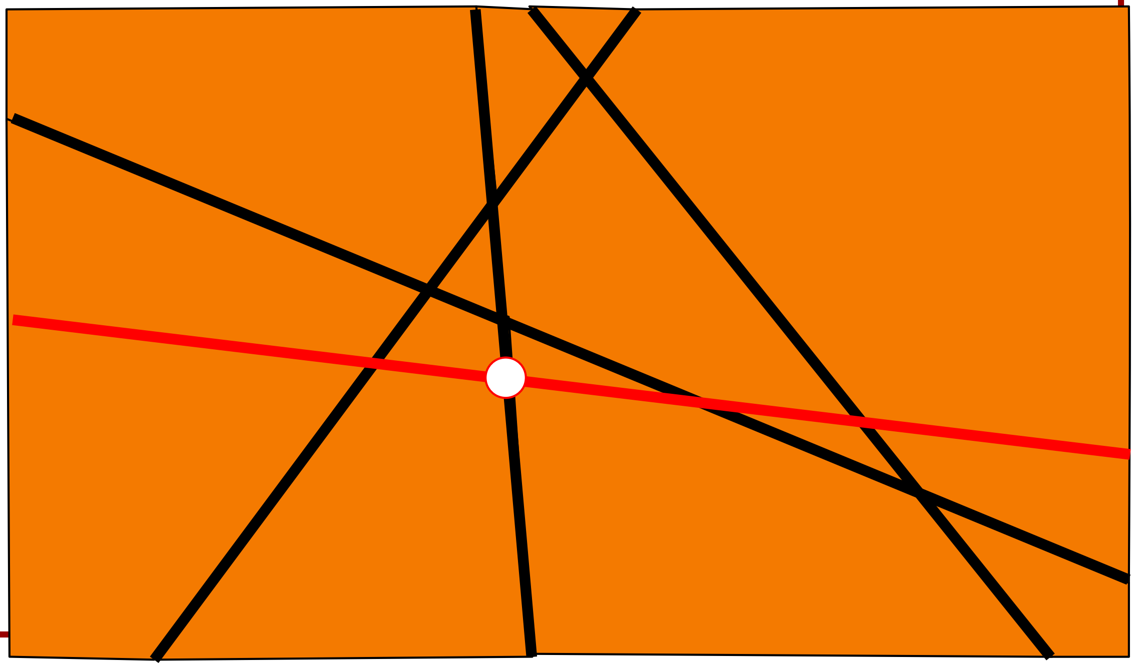


Zone Theorem

Proof:

Iteratively add the k-th line.

- Find an intersection with an existing edge.

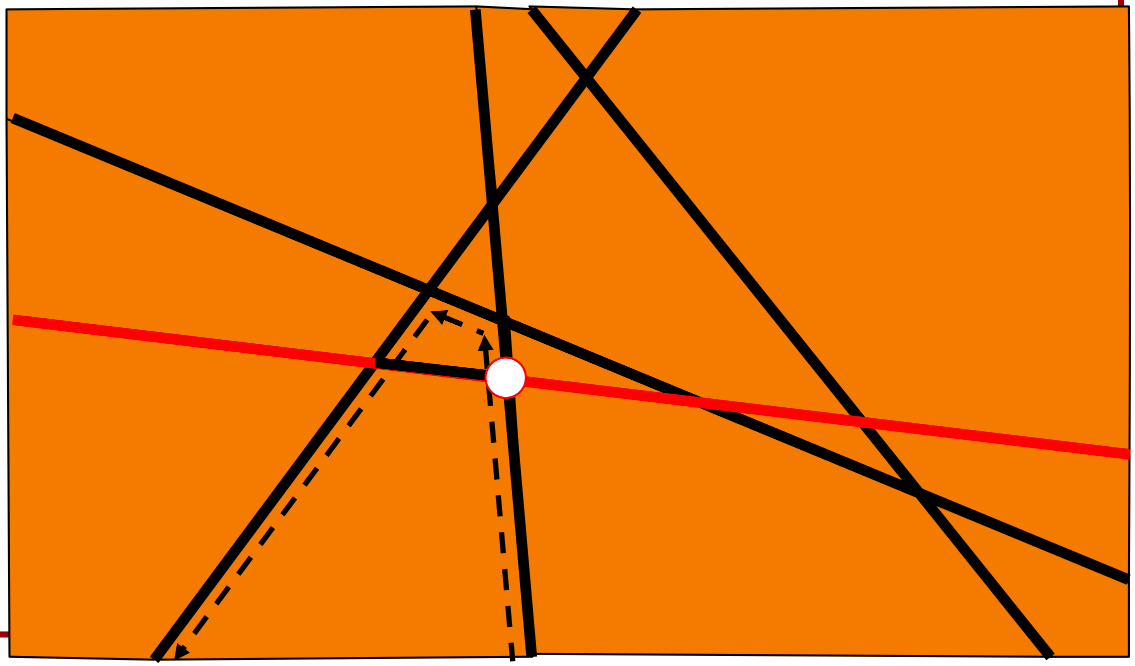


Zone Theorem

Proof:

Iteratively add the k-th line.

- Find an intersection with an existing edge.
- Cycle around faces to the left



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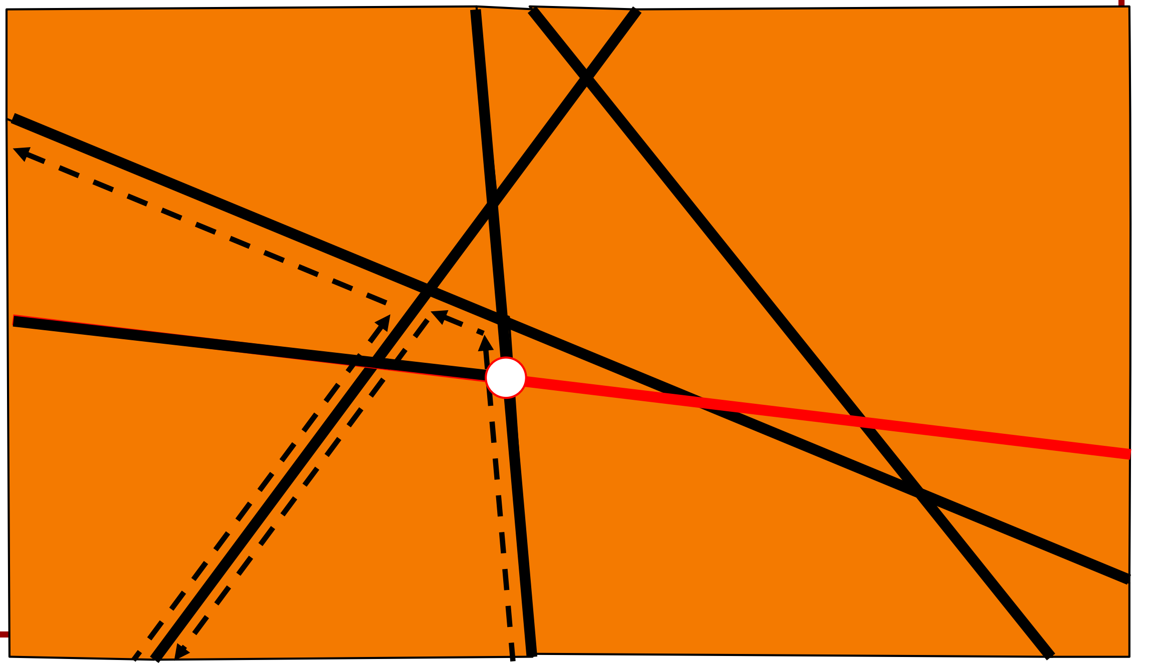

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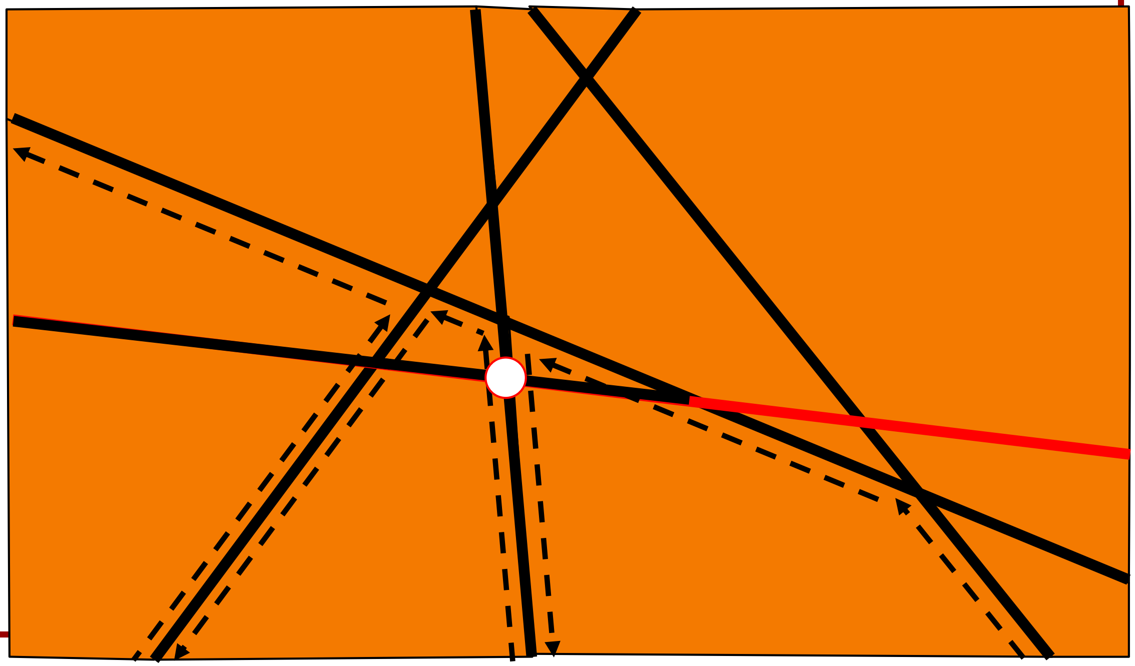

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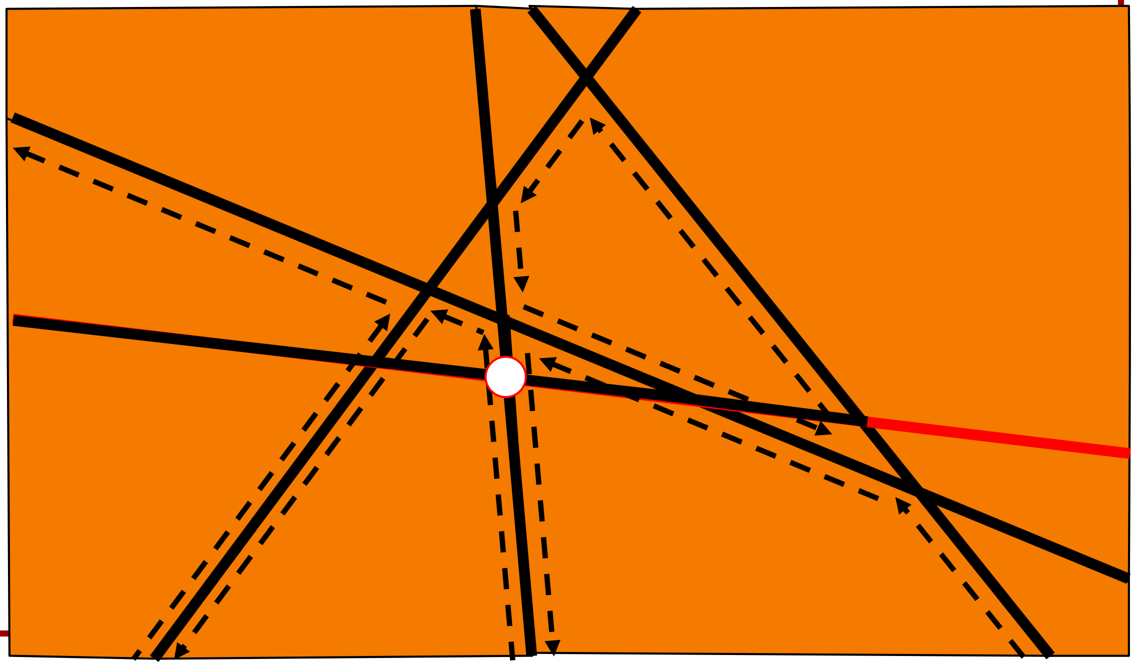

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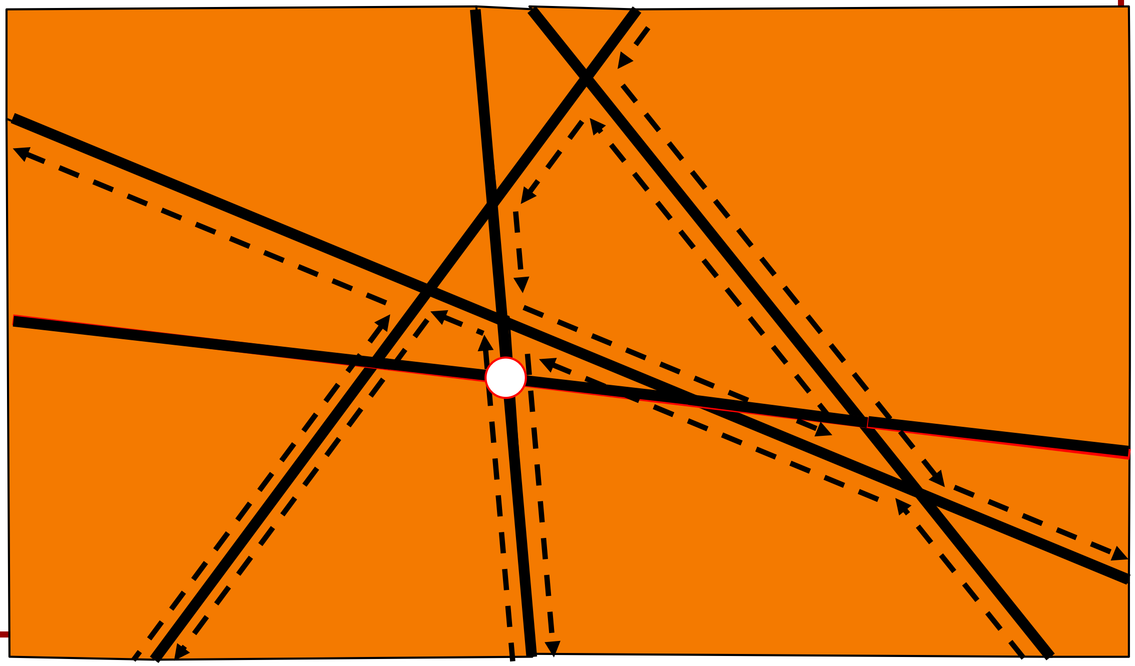

Zone Theorem

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Iteratively add the k-th line. } $O(n)$ iterations

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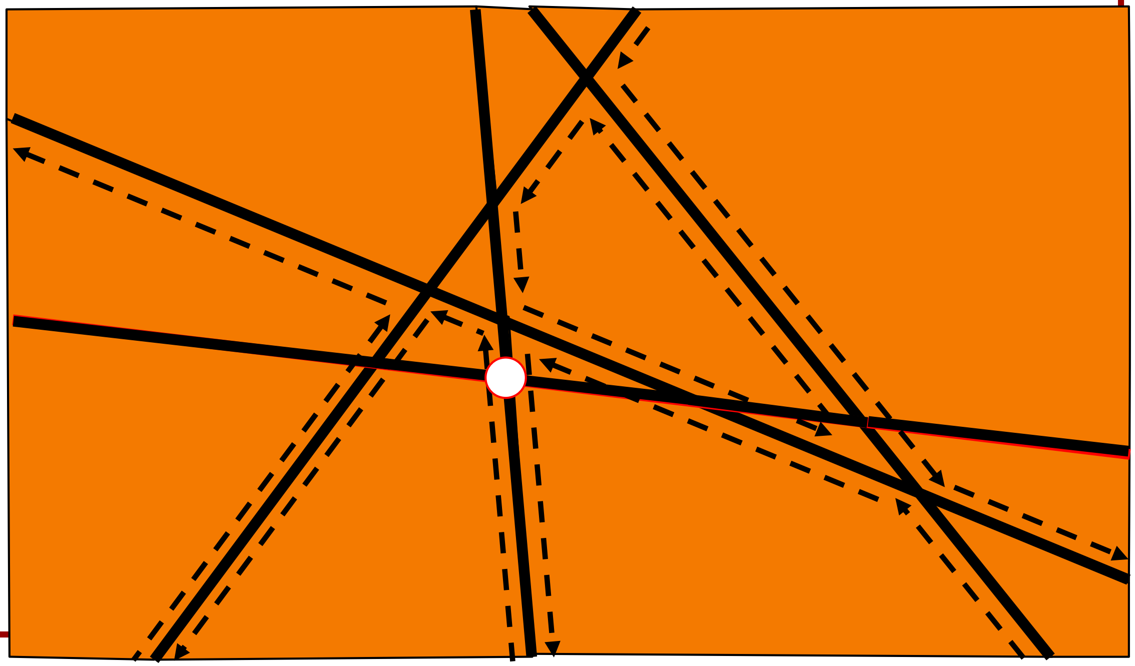

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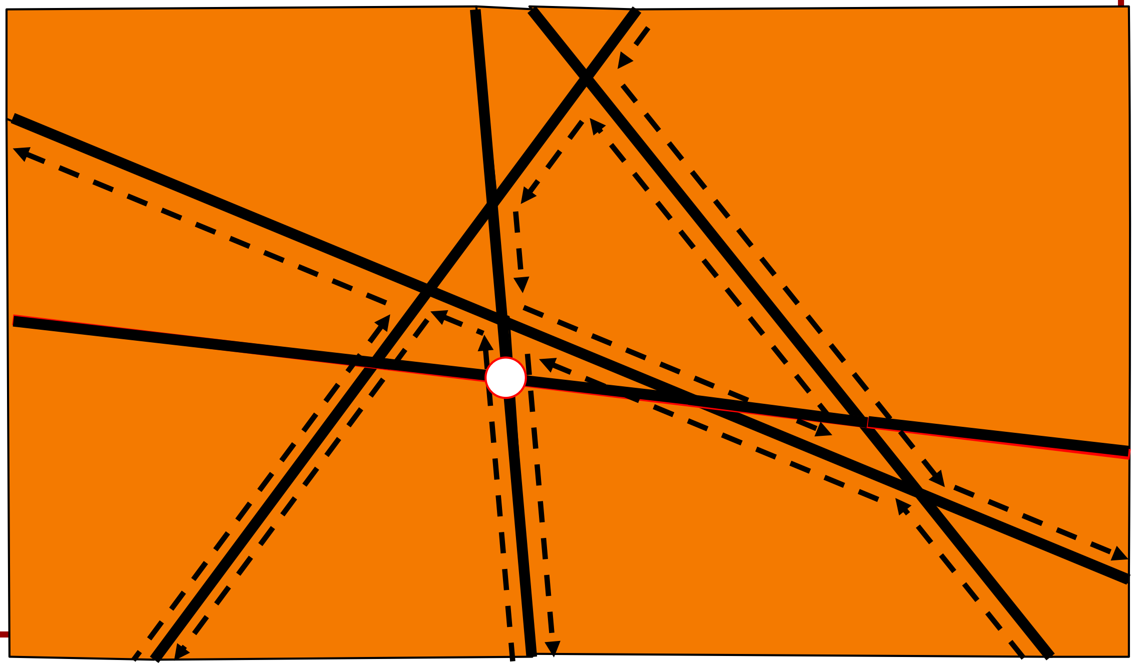

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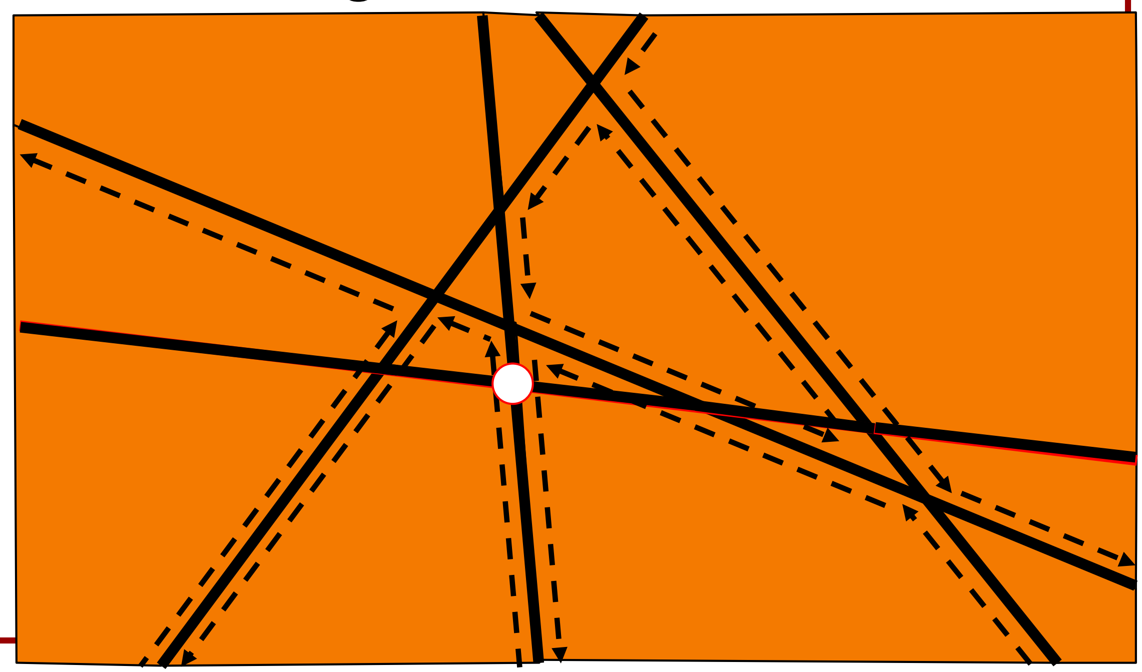

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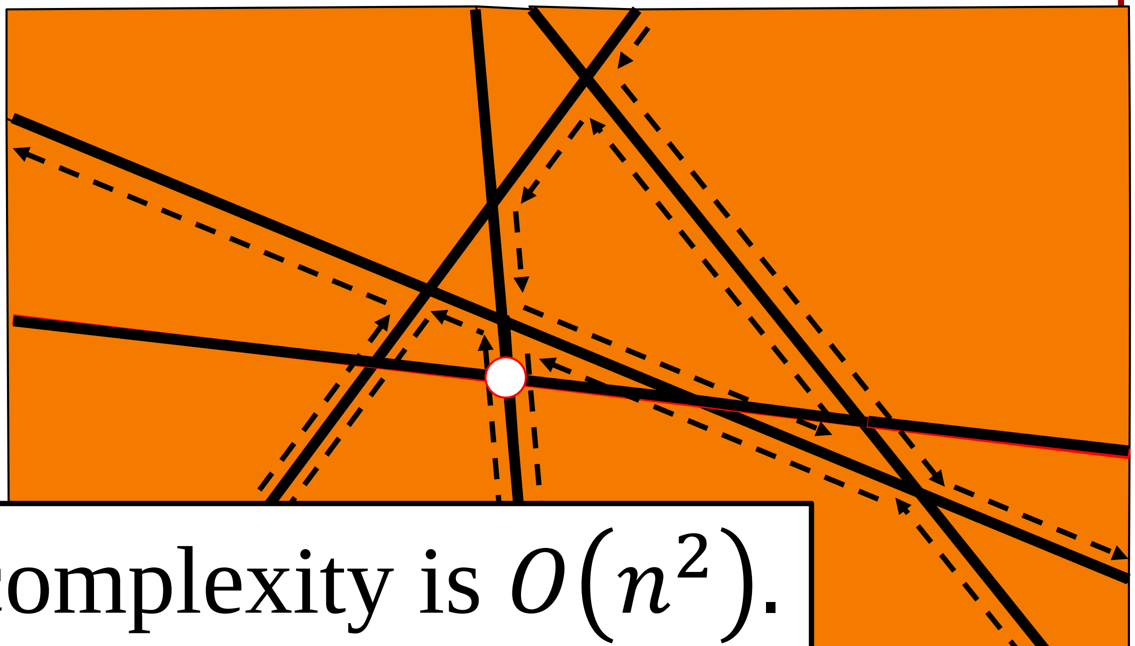

The total complexity is $O(n^2)$.

Zone Theorem

Generalizations:

In d-dimensional space:

- The number of faces of any dimension of an arrangement is $O\left(n^d\right)$.
- The number of faces in the zone of a hyper-plane is bounded by $O\left(n^{d-1}\right)$.
- The arrangement can be computed in $O\left(n^d\right)$ time.