Convex Hulls (3D)

O'Rourke, Chapter 4

Announcements

- For assignment 1:
  I have posted additional polygon files for testing.

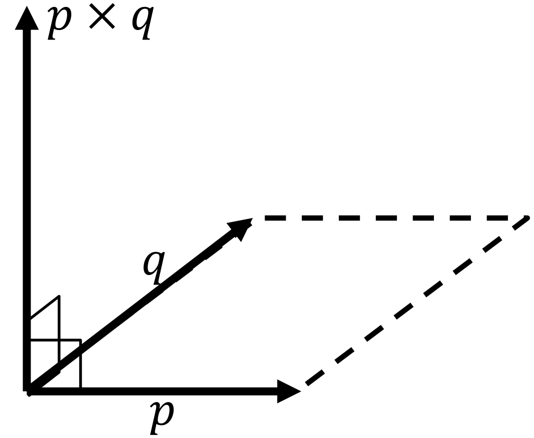
Outline

- Review
- Gift-Wrapping
- Divide-and-Conquer

Recall

Given points $p, q \in \mathbb{R}^3$, the *cross-product* $p \times q \in \mathbb{R}^3$ is the vector:

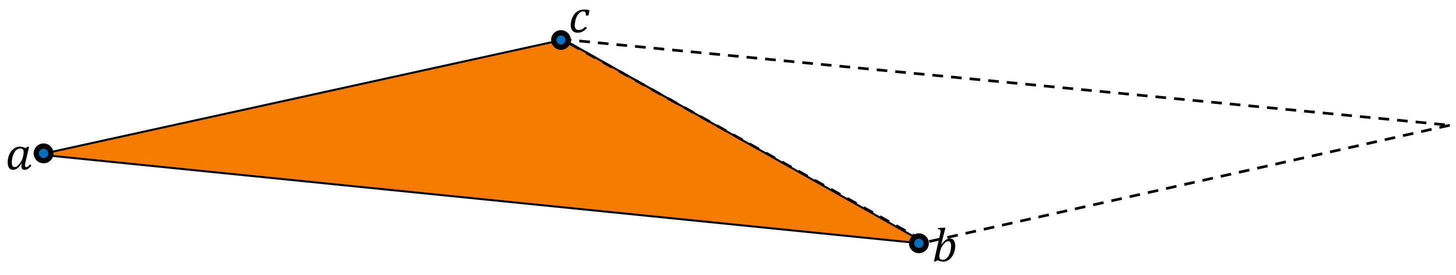
- perpendicular to both p and q,
- oriented according to the right-hand-rule,
- with length equal to the area of the parallelogram defined by p and q.



Recall

Given a triangle T with vertices $(a, b, c) \in \mathbb{R}^3$, the area of the triangle is:

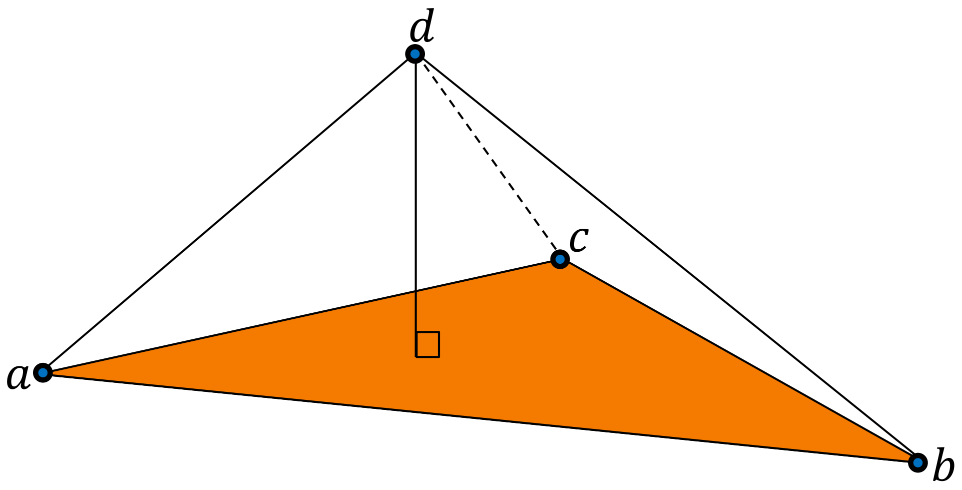
$$\text{Area}(T) = \frac{1}{2} \times \|(b - a) \times (c - a)\|$$



Recall

Given a tetrahedron T with vertices $(a, b, c, d) \in \mathbb{R}^3$, the volume of the tetrahedron is:

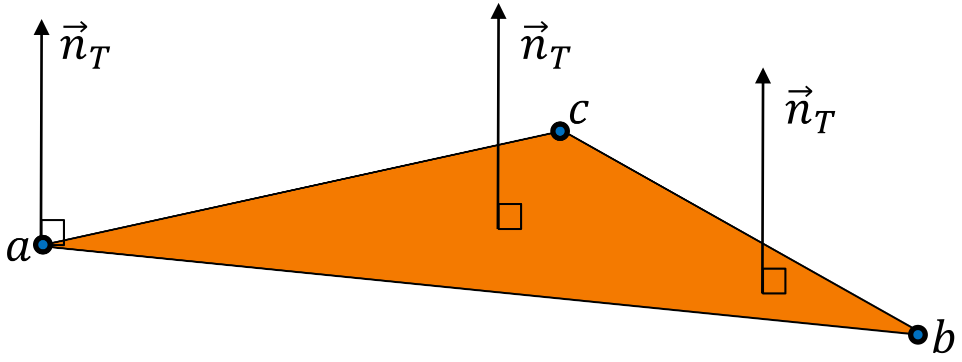
$$\text{Volume}(T) = \frac{1}{3} \times \text{base} \times \text{height}$$



Recall

Given a triangle T with vertices $(a, b, c) \in \mathbb{R}^3$, the triangle normal is:

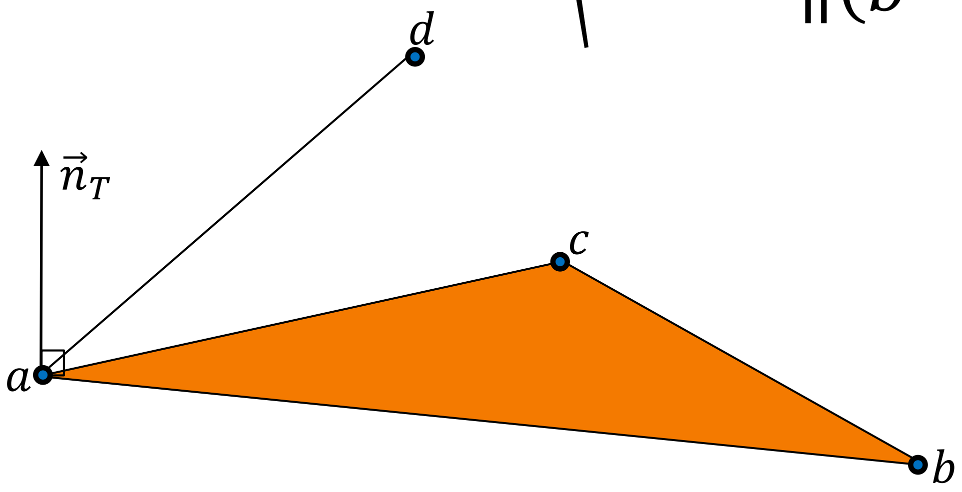
$$\vec{n}_T = \frac{(b-a)\times(c-a)}{\|(b-a)\times(c-a)\|}$$



Recall

Given a triangle T with vertices $(a, b, c) \in \mathbb{R}^3$ and given a point $d \in \mathbb{R}^3$, the signed perpendicular height of d from the plane containing (a, b, c) is:

$$\begin{aligned}\text{Height}(T, d) &= \langle d - a, \vec{n}_T \rangle \\ &= \left\langle d - a, \frac{(b-a) \times (c-a)}{\|(b-a) \times (c-a)\|} \right\rangle\end{aligned}$$



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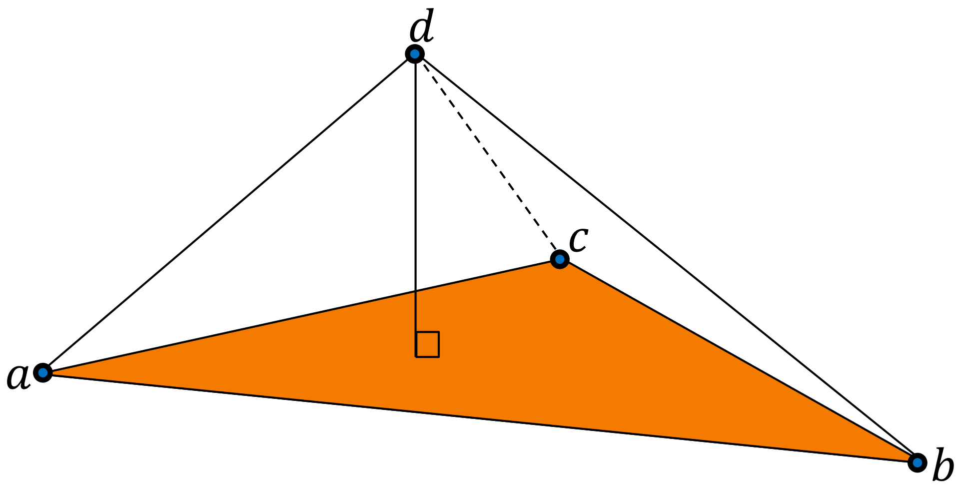
$$\text{Volume}(T) = \frac{1}{3} \times \text{base} \times \text{height}$$
$$= \frac{1}{6} \times \langle d - a, (b - a) \times (c - a) \rangle$$



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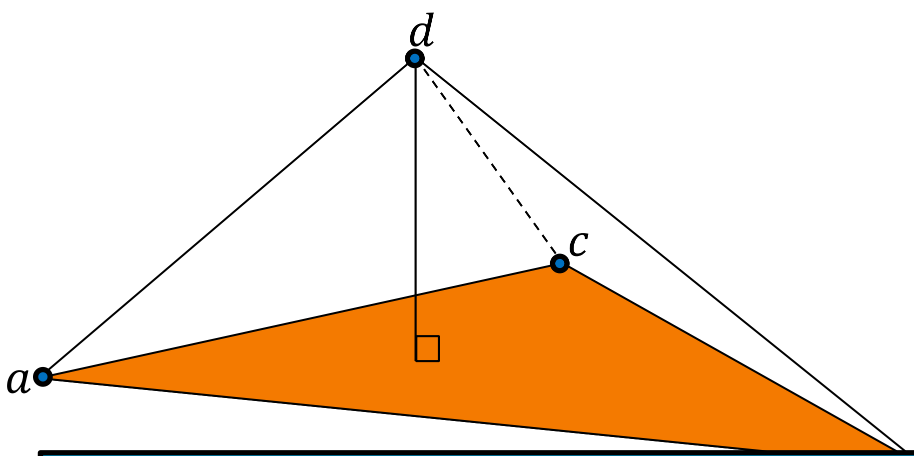


The volume is positive if d is to the left of the plane defined by the triangle (a, b, c).

Recall

If we have a graph G, we can identify the connected component containing a node v by performing a flood-fill.

```
FloodFill( v , G )
    if( NotMarked( v ) )
        Mark( v )
        for w ∈ Neighbors( v )
            FloodFill( w , G )
```

Complexity: $O(|E|)$

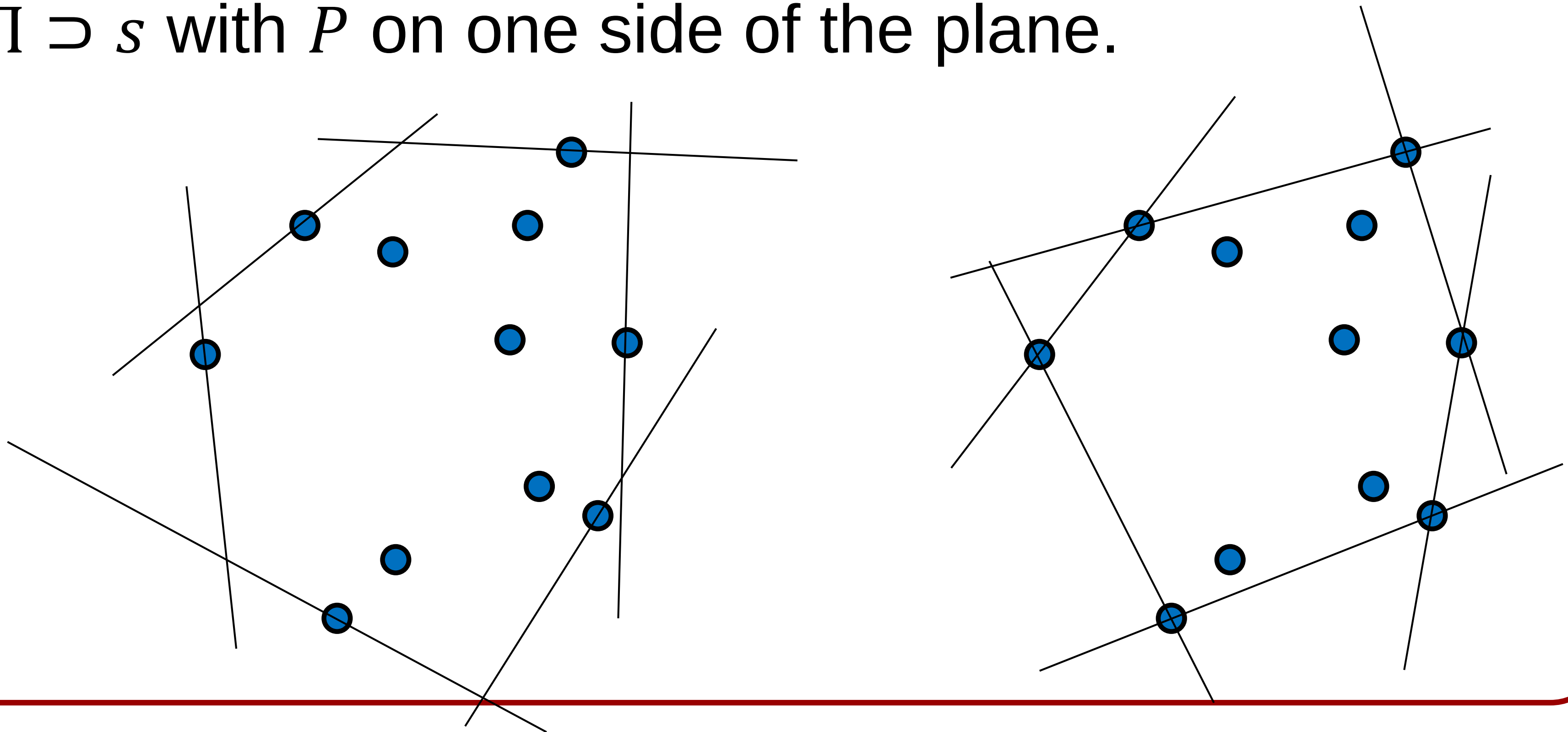
Recall

If we have a graph G, we can identify the connected component containing a node v by performing a flood-fill.

In particular, given a winged-edge representation of a triangle mesh and given a face in the mesh, we can compute the connected component of the face in linear time.

Supporting Simplices

Given a set of points $P \subset \mathbb{R}^d$, and given a simplex $s = \{p_1, \cdots, p_k\}$ (vertex, edge, triangle, etc.) formed by $k \leq d$ vertices, we say that the P is <u>supported</u> on s if there exists a $(d-1)$-dimensional hyperplane, $\Pi \supset s$ with P on one side of the plane.



Supporting Simplices

Note:

If we project P' is the projection of P onto $\mathbb{R}^{d'}$ and if P' is supported on a simplex $s' = \{p'_1, \dots, p'_k\}$ then P is supported on the simplex $s = \{p_1, \dots, p_k\}$.

Proof:

Extrude the $(d'-1)$-dimensional Π' along the direction of projection.

- The $(d-1)$-dimensional hyperplane Π has P on one side.
- The vertices of s lie on Π.

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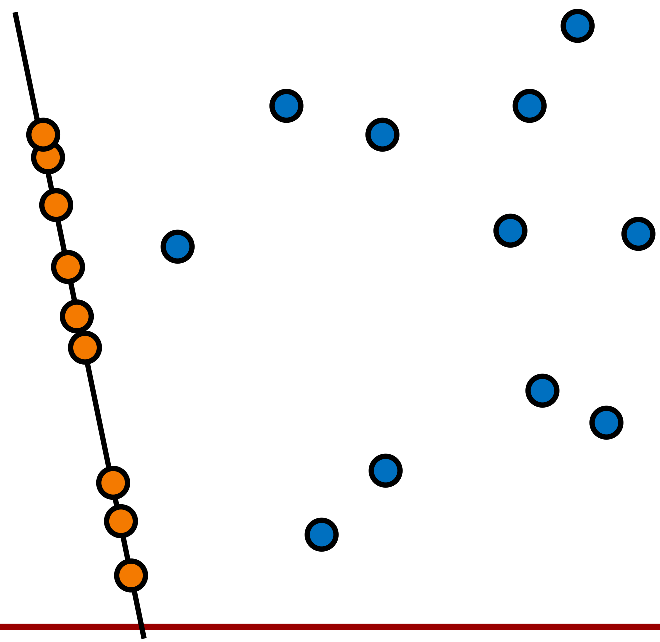

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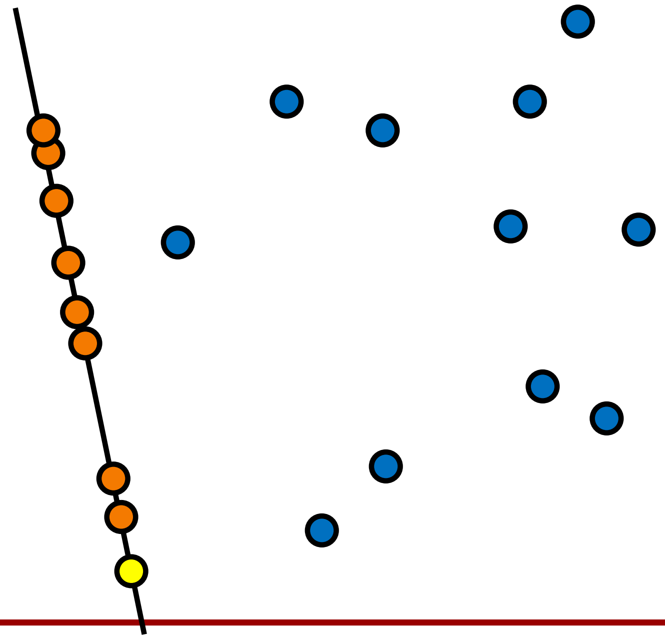

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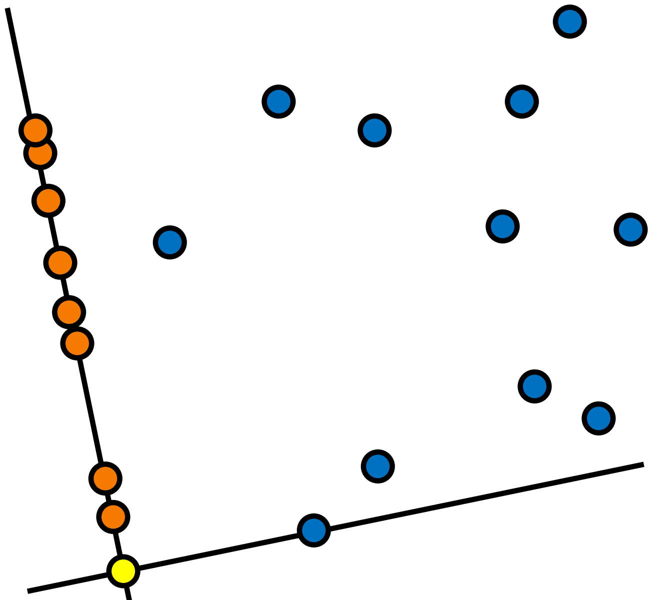
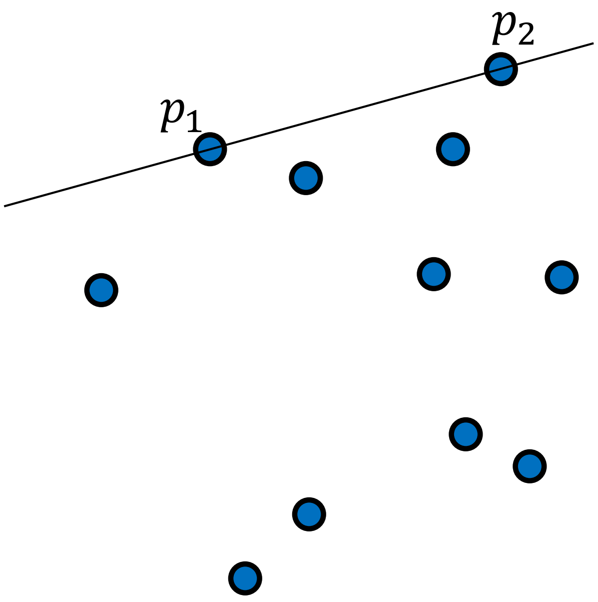


Supporting Simplices

Note:

If P is supported by the simplex $s = \{p_1, \ldots, p_k\}$ then the point-set $P' = P - \{p_1\}$ is supported by the simplex $s' = \{p_2, \ldots, p_k\}$.



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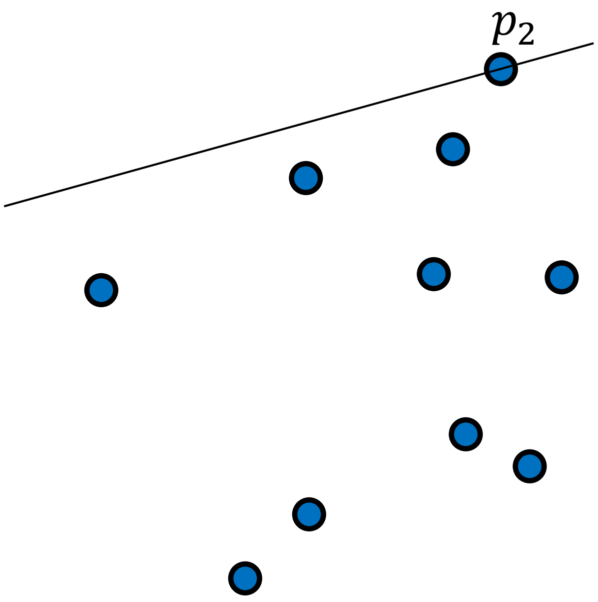

Outline

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- Gift-Wrapping
- Divide-and-Conquer

Gift-Wrapping

Initialization:

Find a triangle on the hull.

Iteratively:

Until the hull closes, pivot around a boundary edge.

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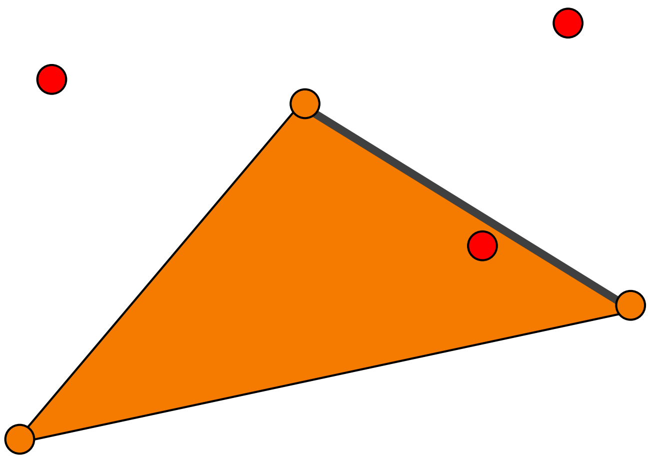

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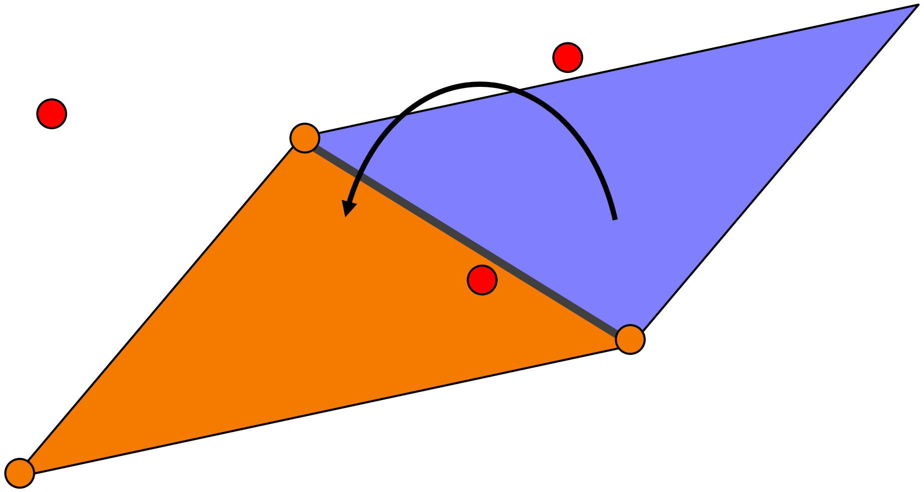

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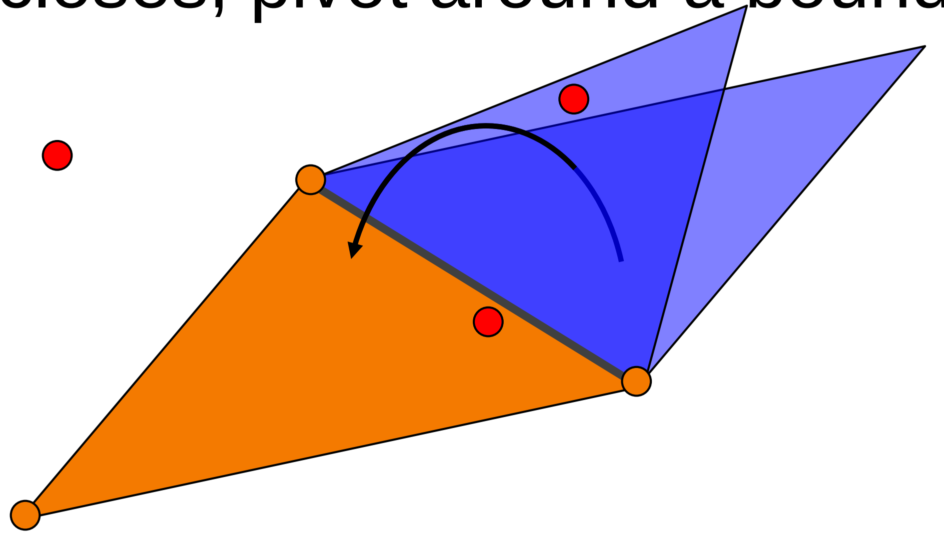

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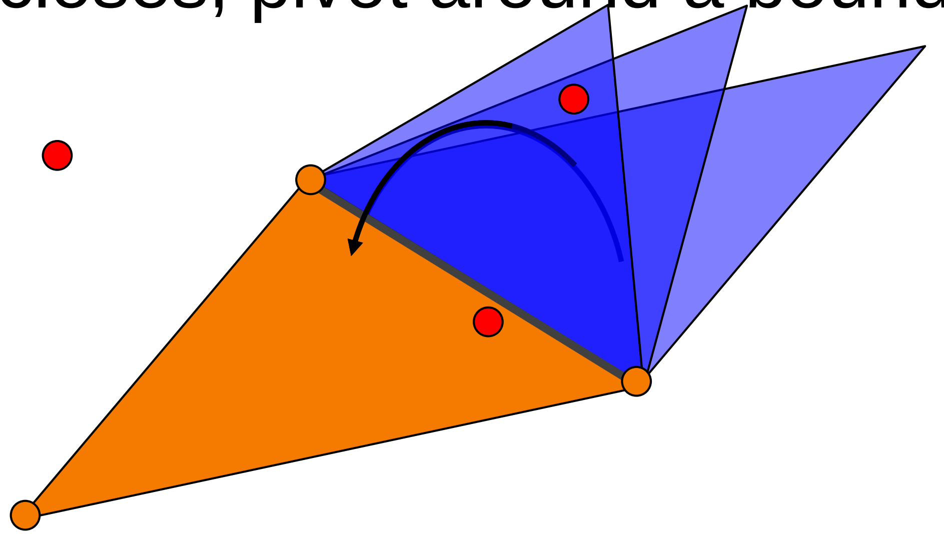

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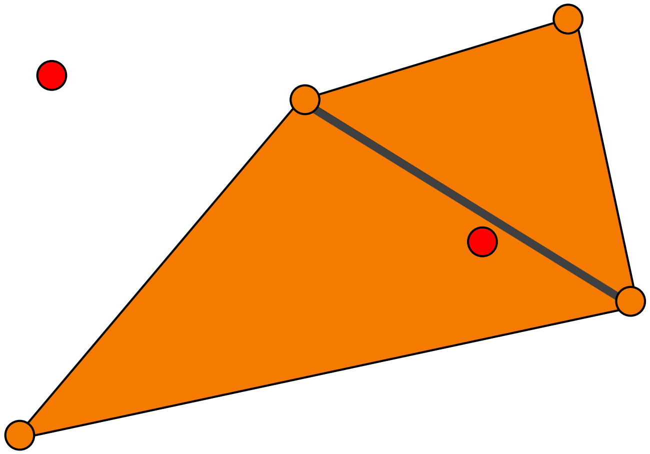

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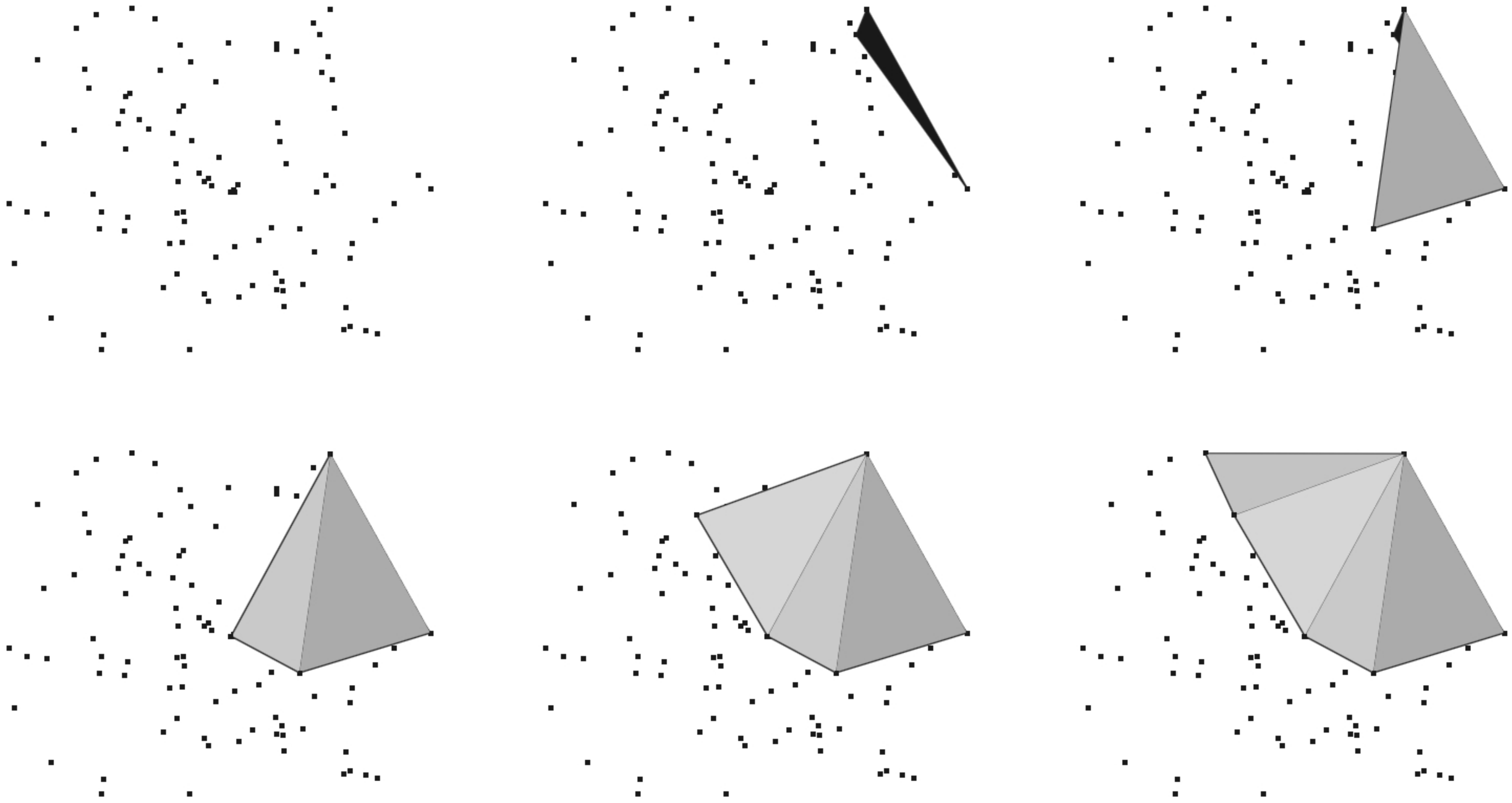

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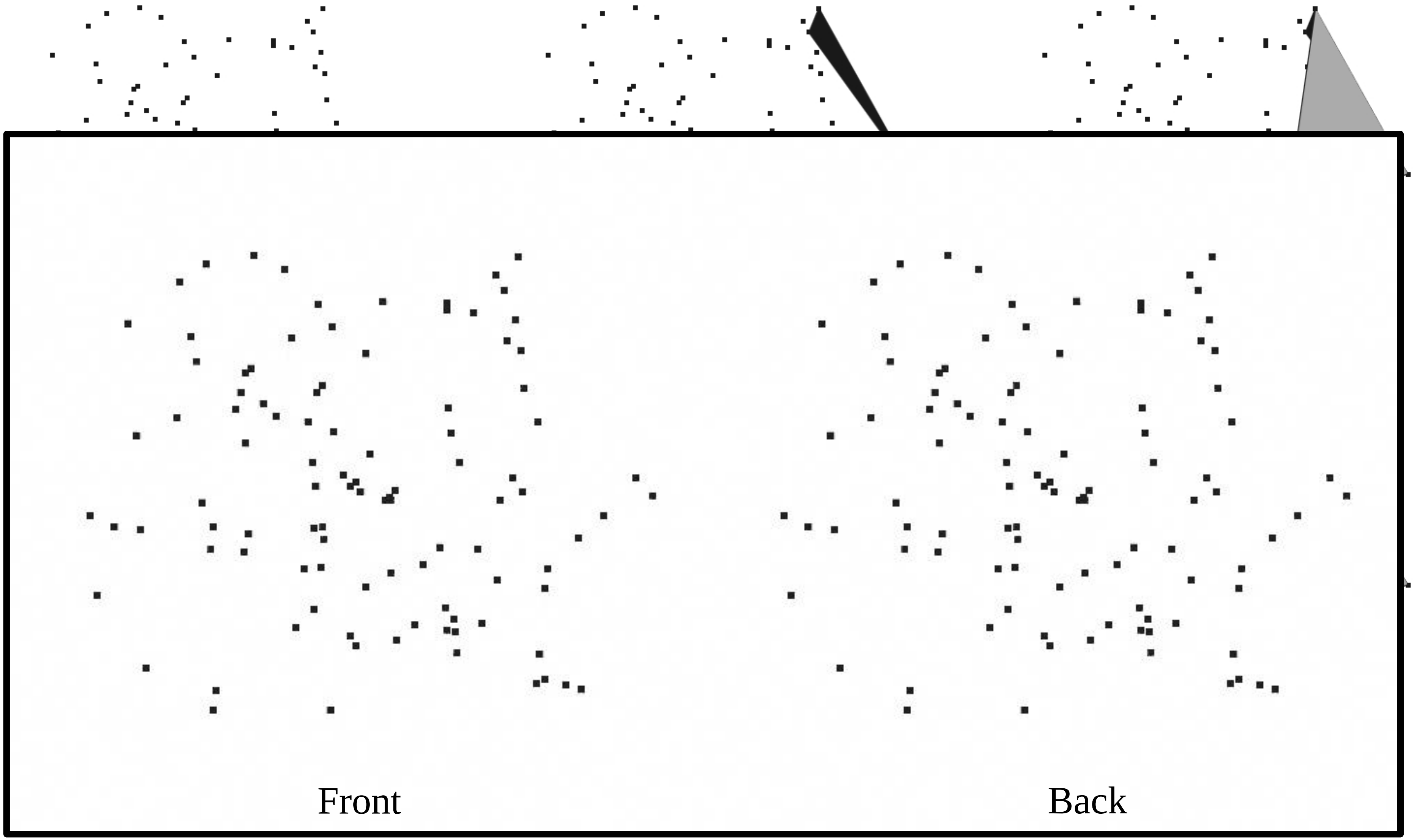


Gift-Wrapping

PivotAroundEdge($e = \{q_0, q_1\}$, $P = \{p_0, \dots, p_{n-1}\}$)

```
p ← p_0
area2 ← SquaredArea( q_0 , q_1 , p )
for p' ∈ {p_1, …, p_{n-1}}:
    volume ← SignedVolume( q_0 , q_1 , p , p' )
    if( volume<0 )
      p ← p'
    else if( volume==0 )
      _area2 ← SquaredArea( q_0 , q_1 , p' )
      if( _area2>area2 )
        p ← p'
        area2 ← _area2
return p
```

Complexity: $O(n)$

Gift-Wrapping

FindTriangleOnHull($P = \{p_0, \dots, p_{n-1}\}$)

$\{p, q\} \leftarrow$ FindEdgeOnHull(P)

$r \leftarrow$ PivotAroundEdge($\{p, q\}$, P)

return $\{p, q, r\}$

Complexity: $O(n)$ + Complexity of FindEdgeOnHull

Gift-Wrapping

FindEdgeOnHull($P = \{p_0, \dots, p_{n-1}\}$)

$p \leftarrow$ BottomMostLeftMostBackMost(P)

$q \leftarrow$ PivotOnEdge($\{p, p + (1,0,0)\}$, P)

return $\{p, q\}$

Complexity: $O(n)$

Gift-Wrapping

GiftWrap(P):

- $t \leftarrow$ FindTriangleOnHull(P)
- $Q \leftarrow \{(t_1, t_0), (t_2, t_1), (t_0, t_2)\}$ // hull boundary edges (?)
- $H \leftarrow \{t\}$ // the hull
- while($Q \neq \emptyset$)
  - $e \leftarrow Q$.pop_back()
  - if(NotProcessed(e))
    - $q \leftarrow$ PivotOnEdge(e)
    - $t \leftarrow$ Triangle(e , q)
    - $H \leftarrow H \cup \{t\}$
    - $Q \leftarrow Q \cup \{(t_1, t_0), (t_2, t_1), (t_0, t_2)\}$
    - MarkProcessedEdges(e)

Complexity: $O(n^2)$

Outline

- Review
- Gift-Wrapping
- Divide-and-Conquer

Divide And Conquer

DivideAndConquer(P):

$P \leftarrow$ SortByX(P)

return _DivideAndConquer(P)

_DivideAndConquer(P)

if($|P| < 8$) return Incremental(P)

$(P_1, P_2) \leftarrow$ SplitInHalf(P)

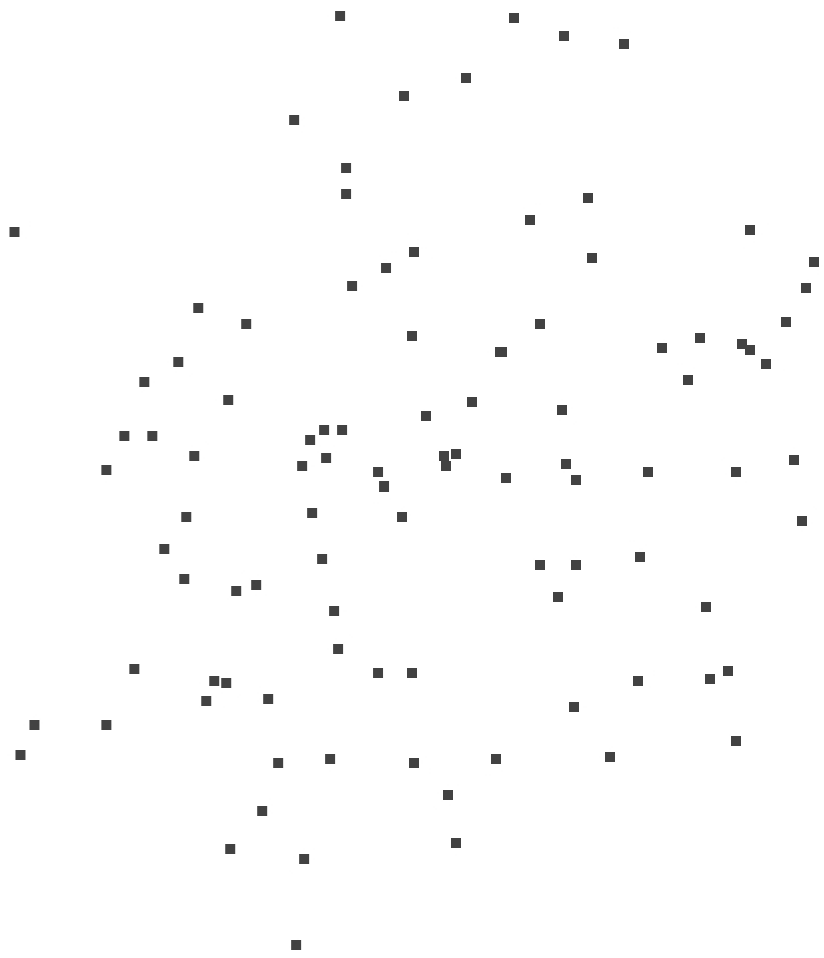
$H_1 \leftarrow$ _DivideAndConquer(P_1)

$H_2 \leftarrow$ _DivideAndConquer(P_2)

return Merge(H_1 , H_2)

Complexity: $O(n \log n)$

Divide And Conquer



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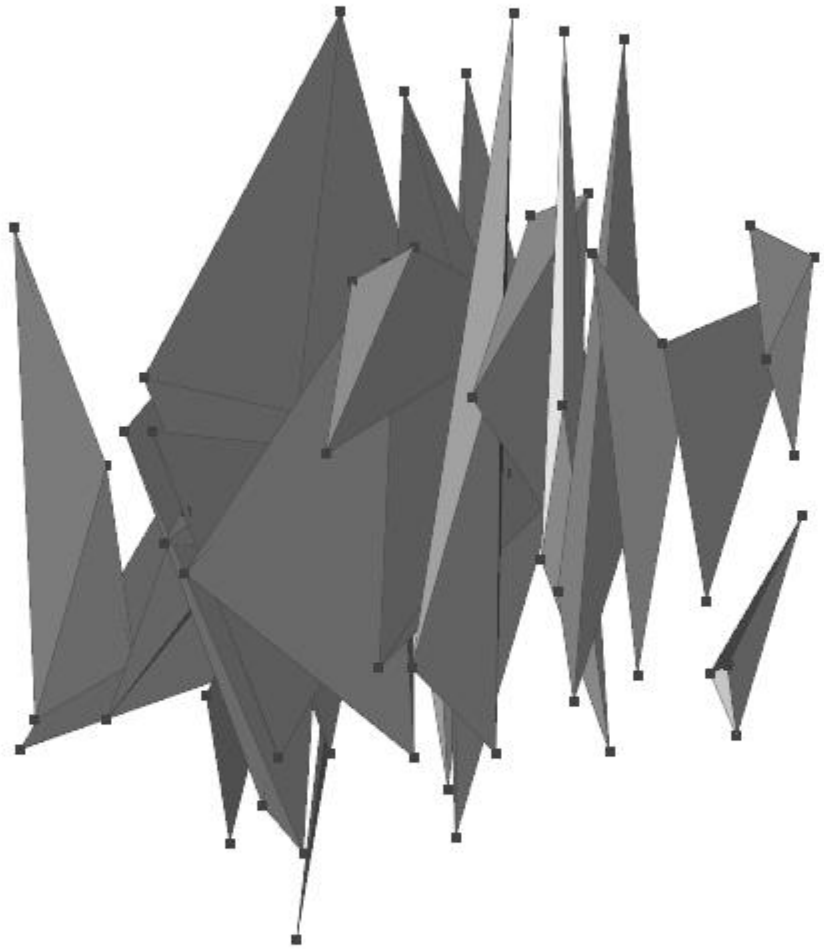

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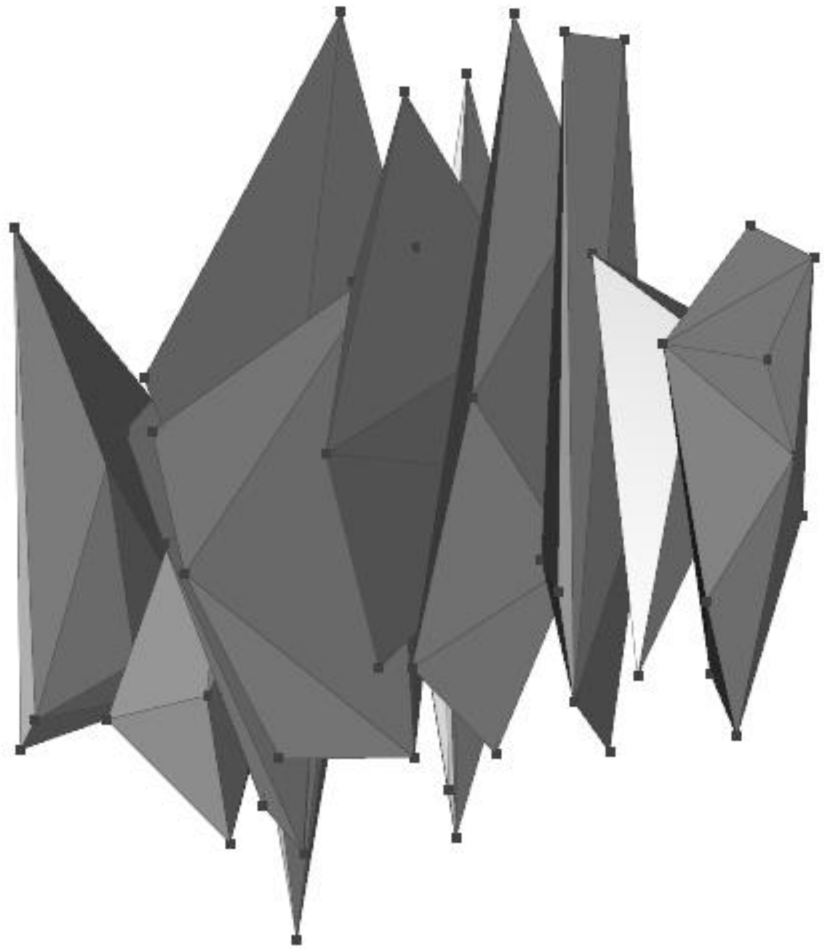

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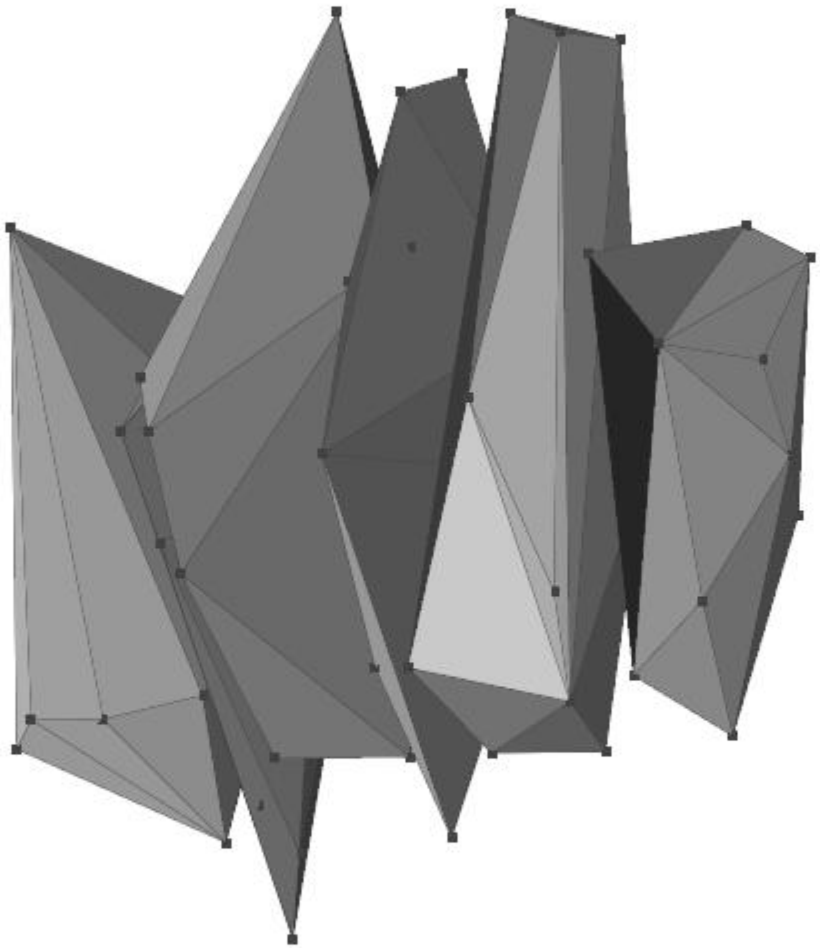

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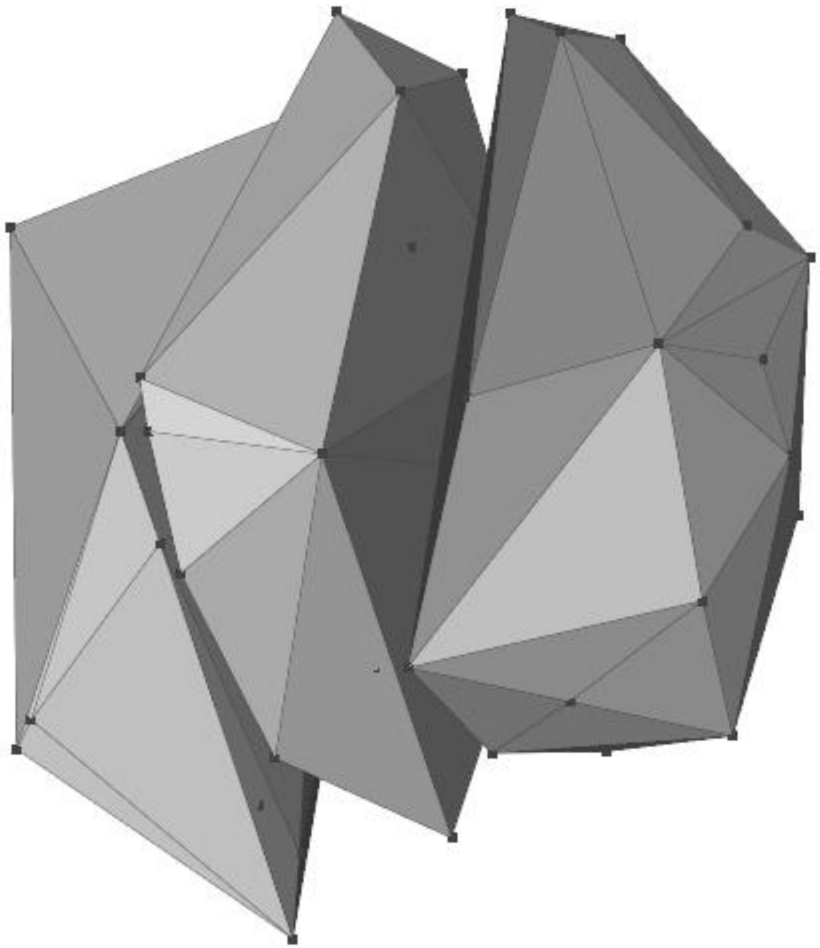

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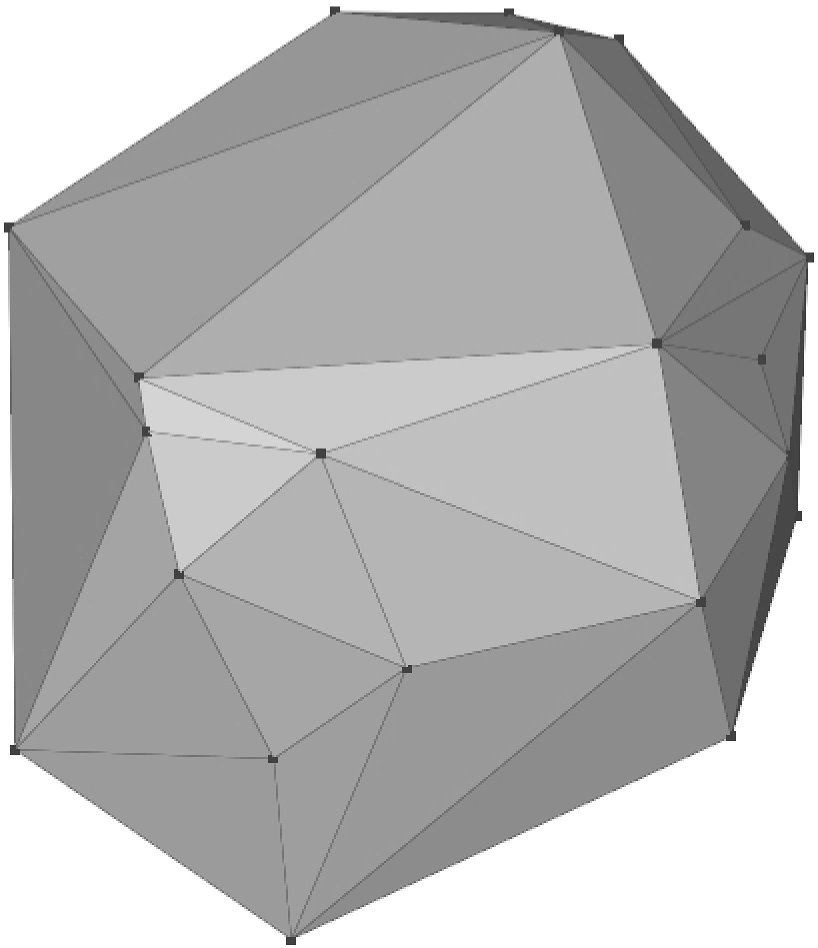
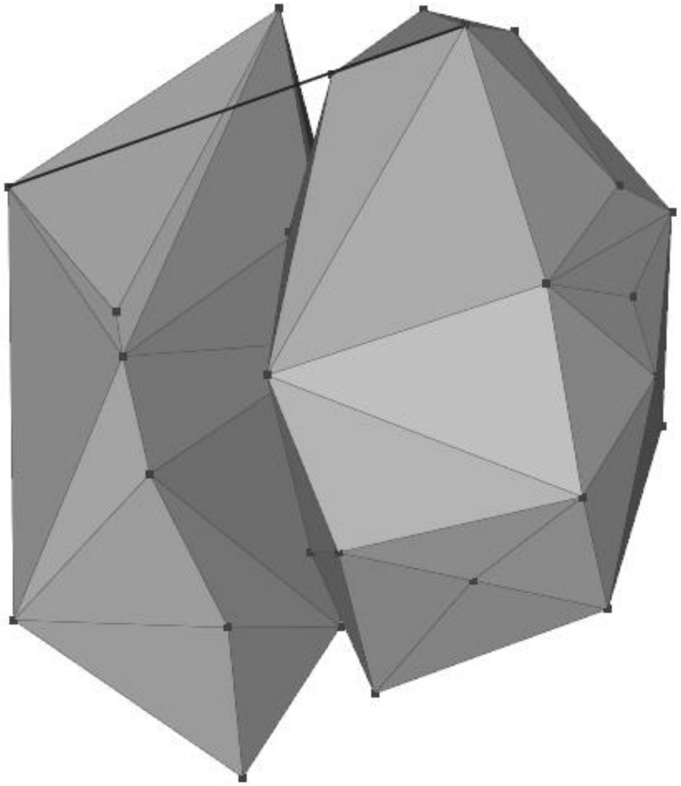


Divide And Conquer

Merge:

- Construct the fillet that merges the two hulls
- Remove the triangles that are no longer visible



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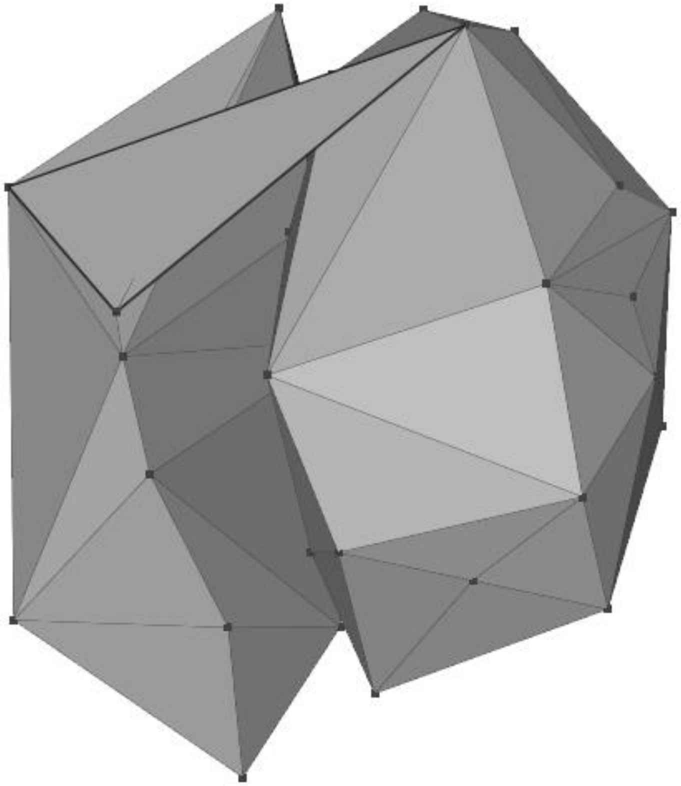

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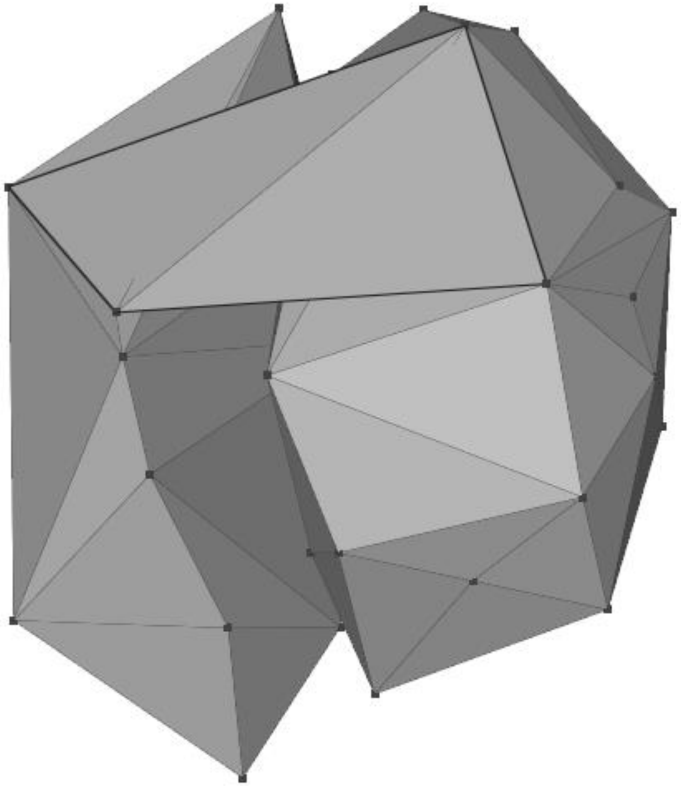

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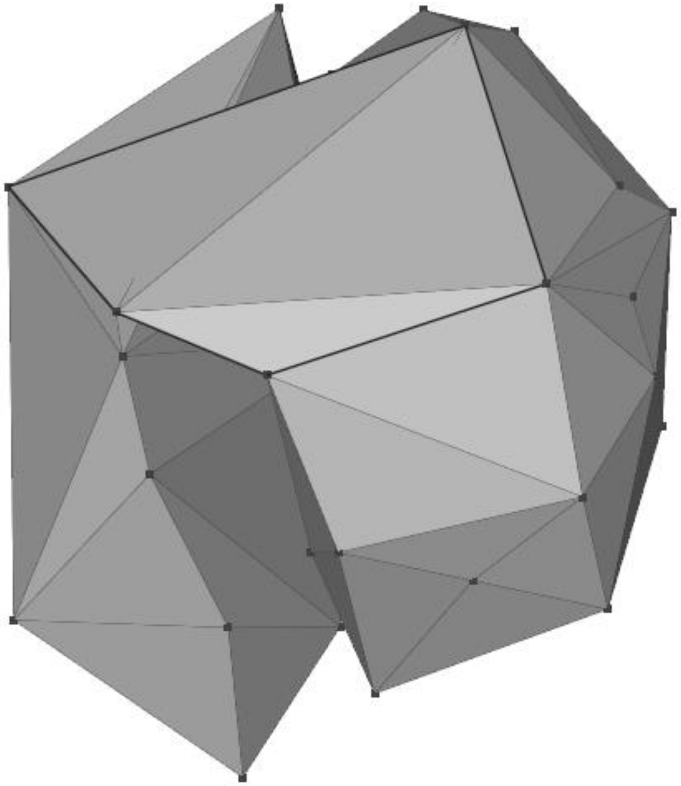

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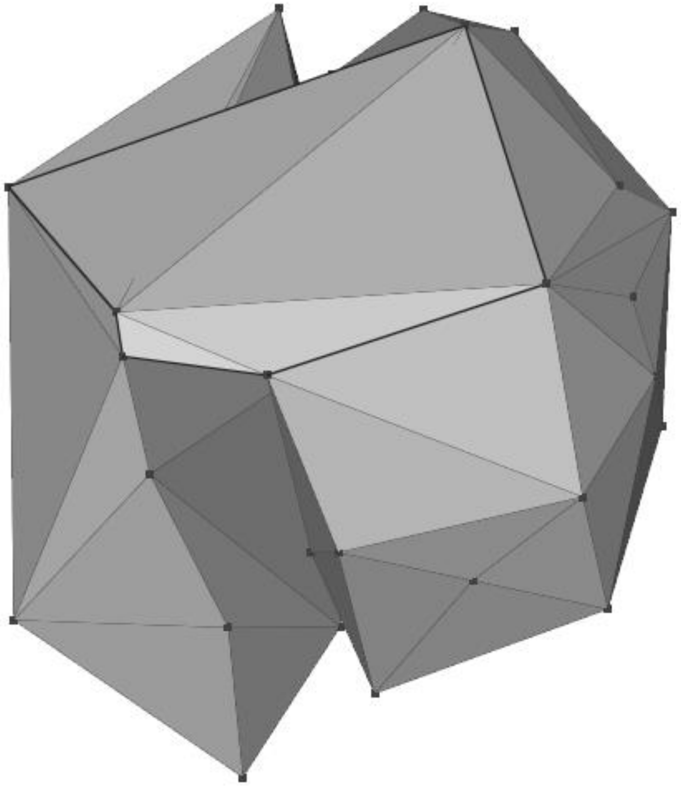

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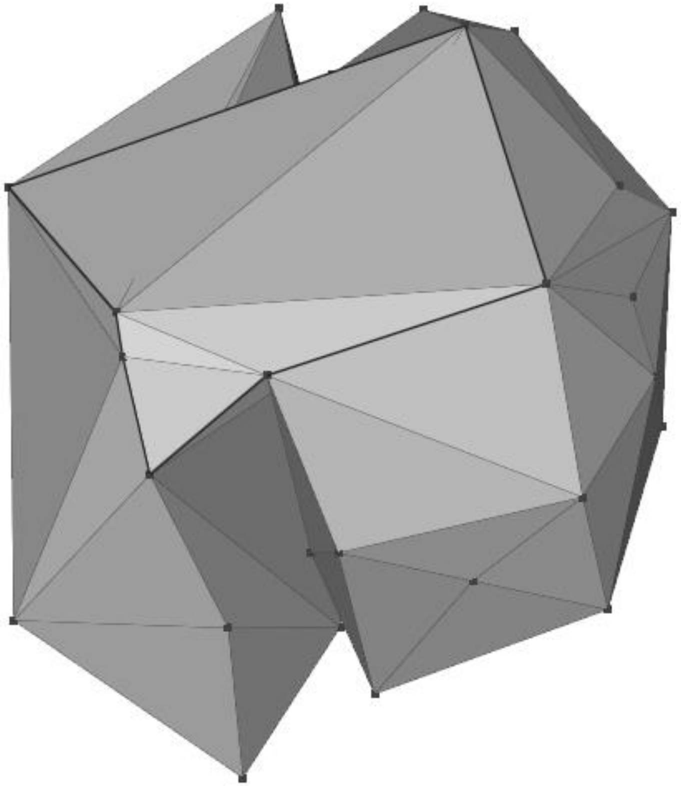

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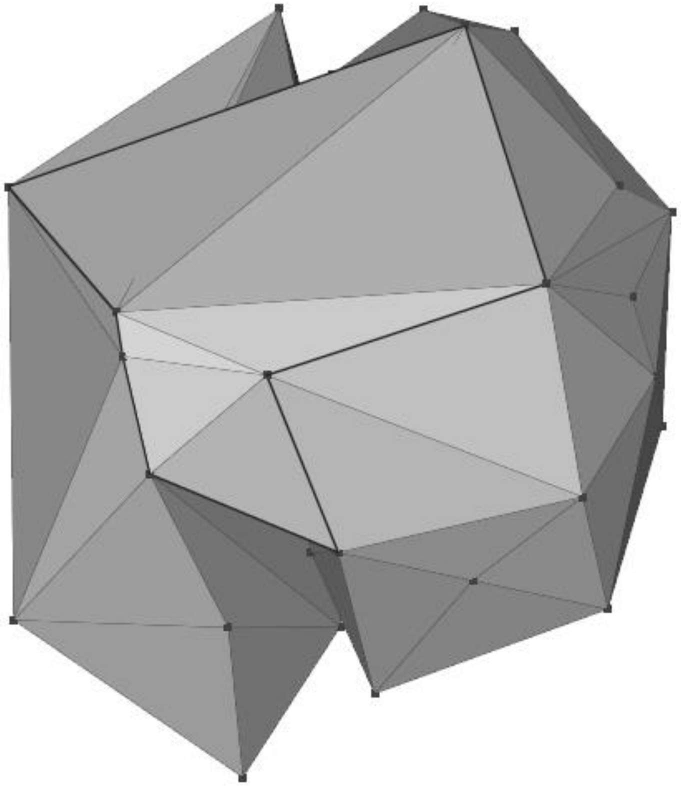

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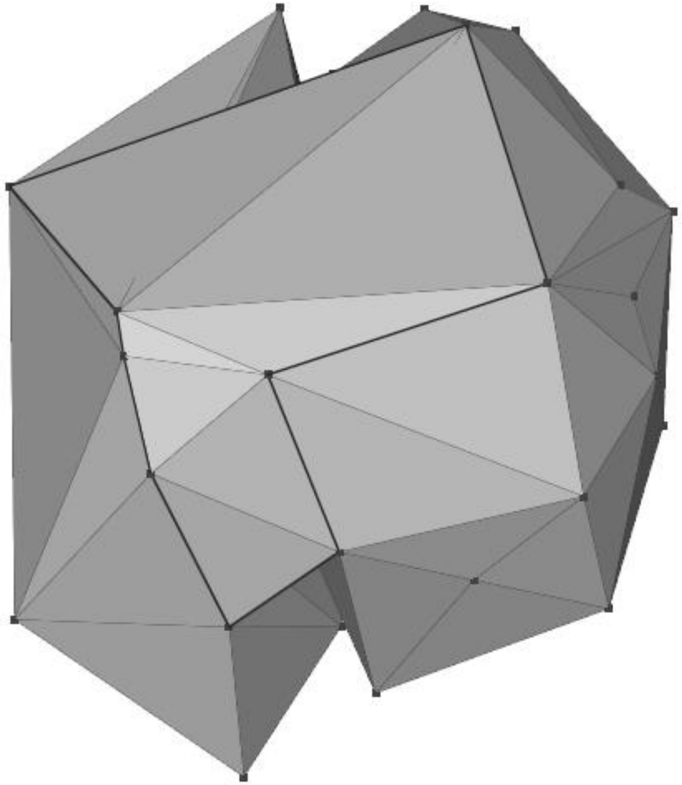

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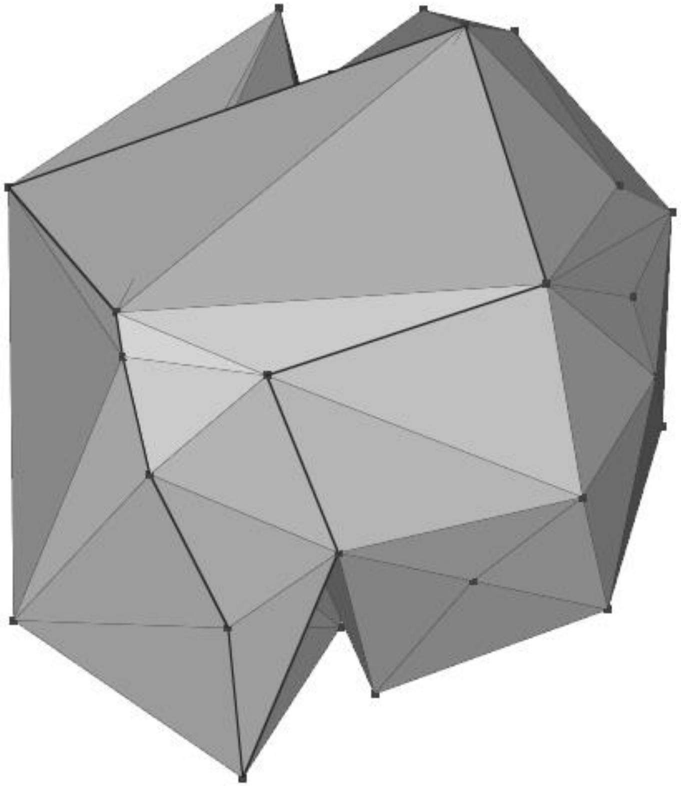

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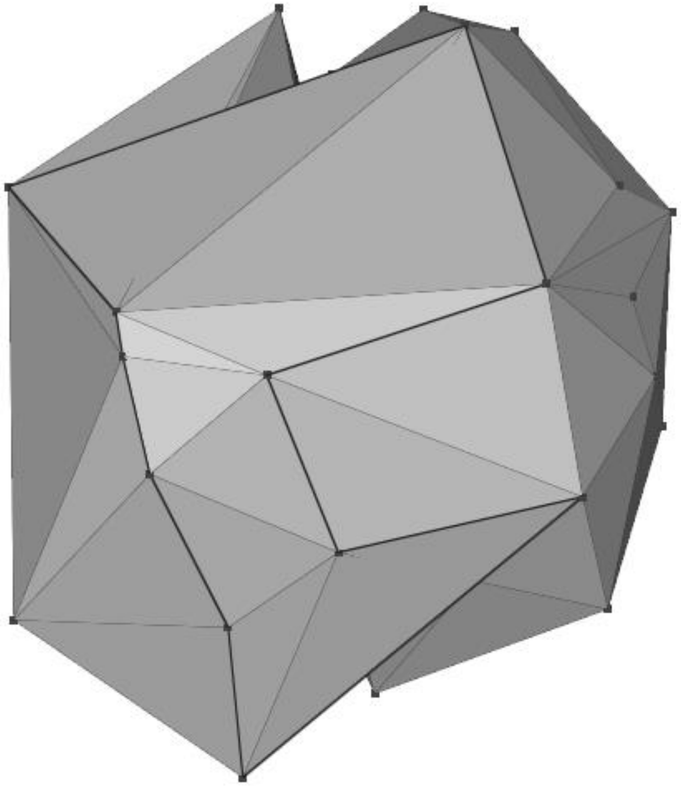

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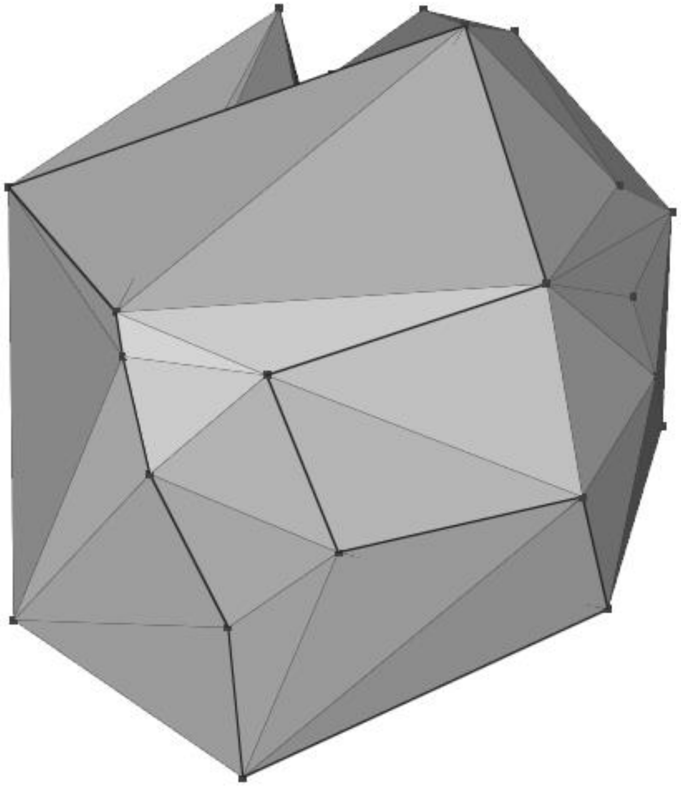

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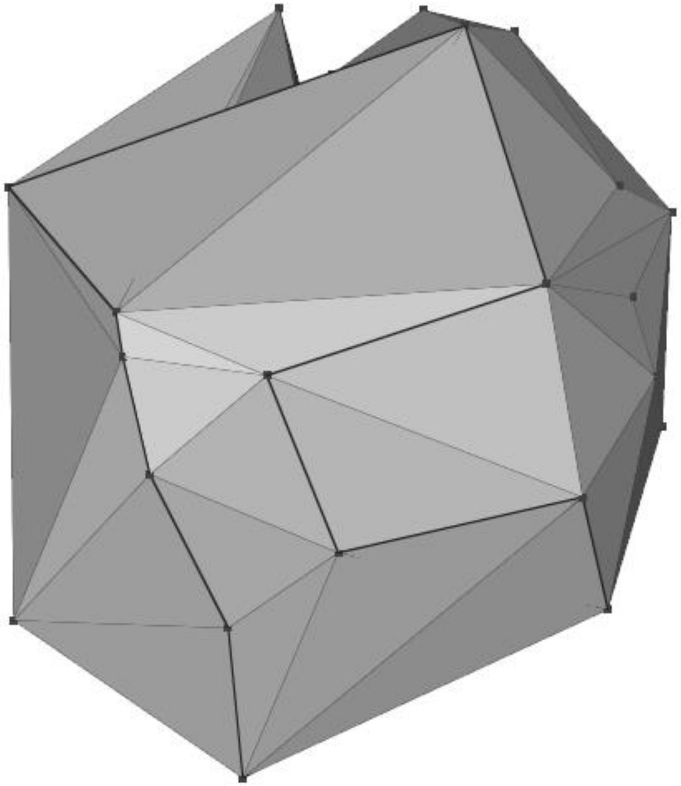

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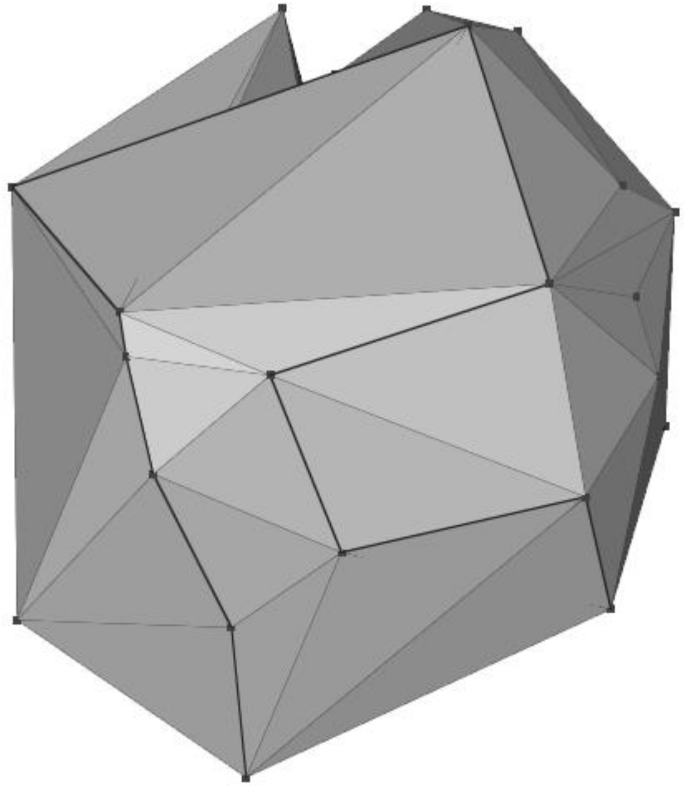

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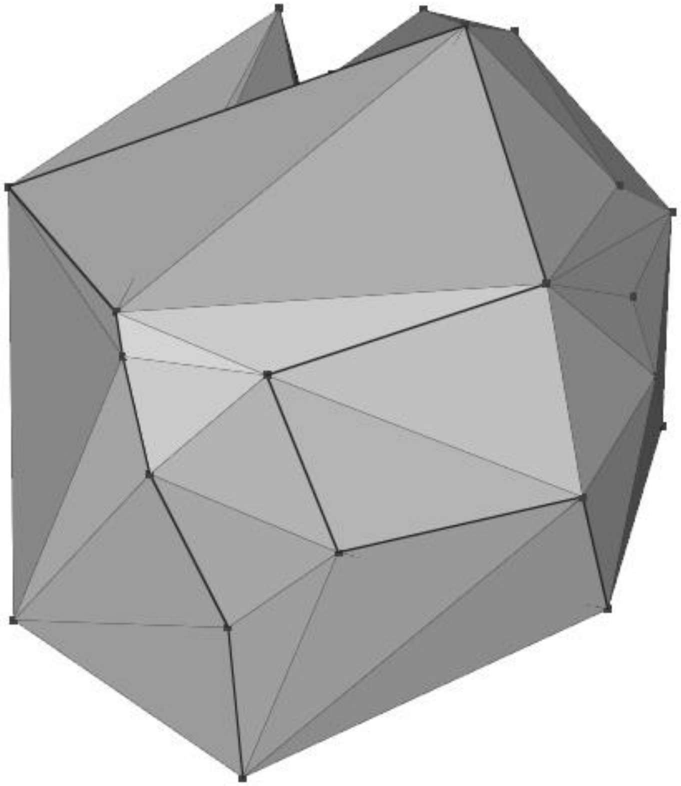

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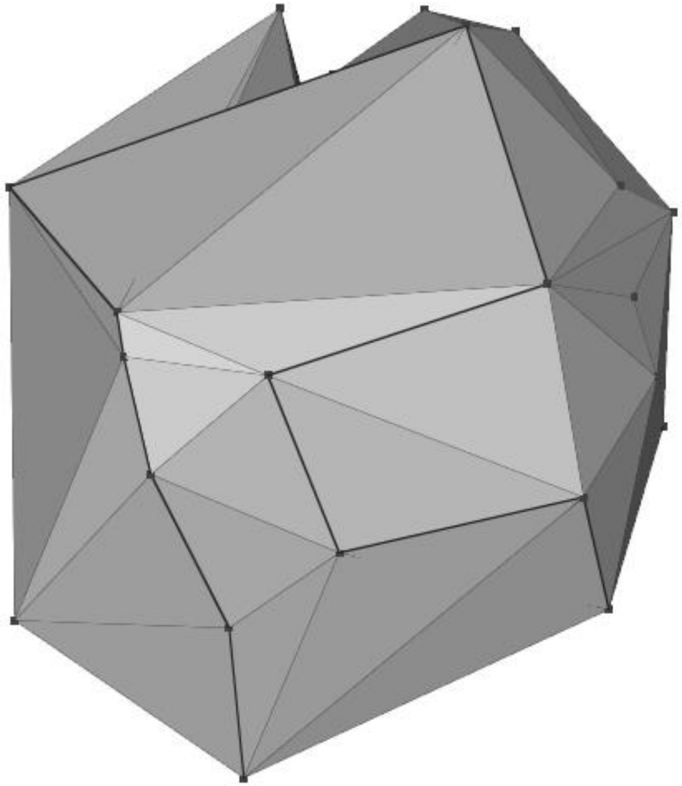

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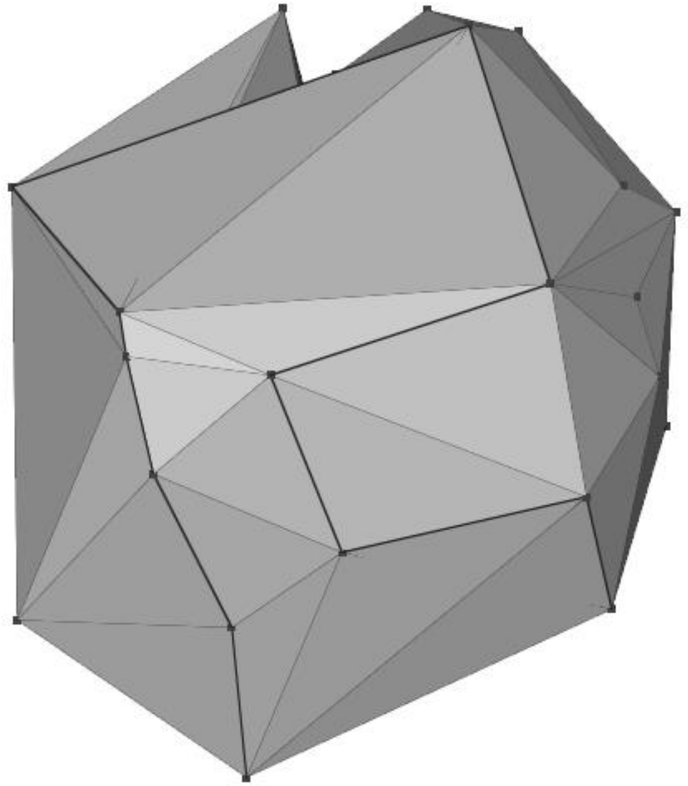

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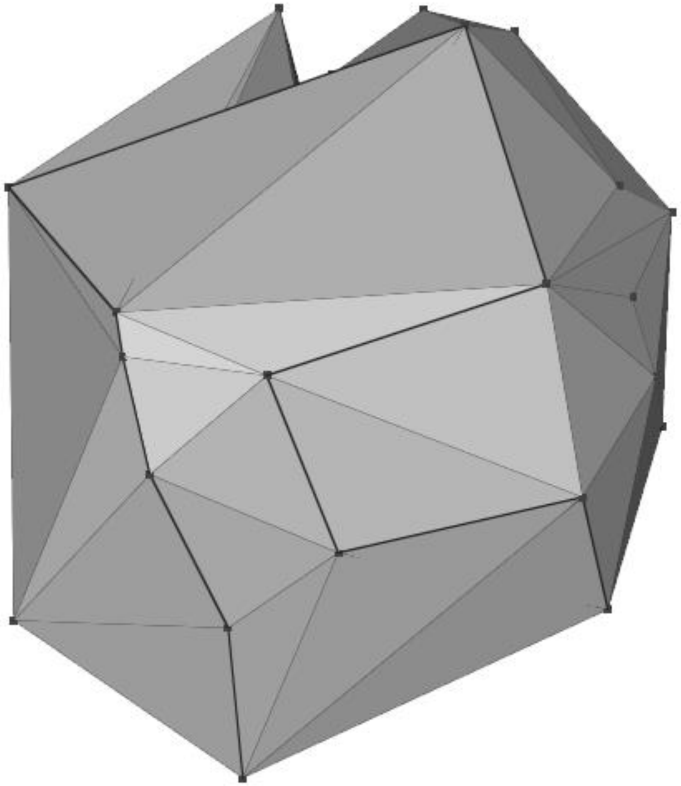

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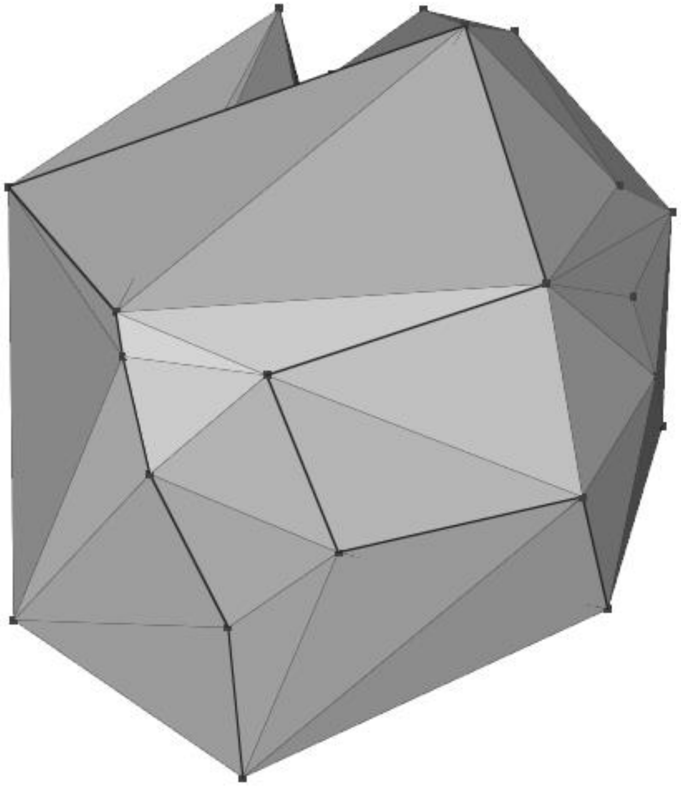

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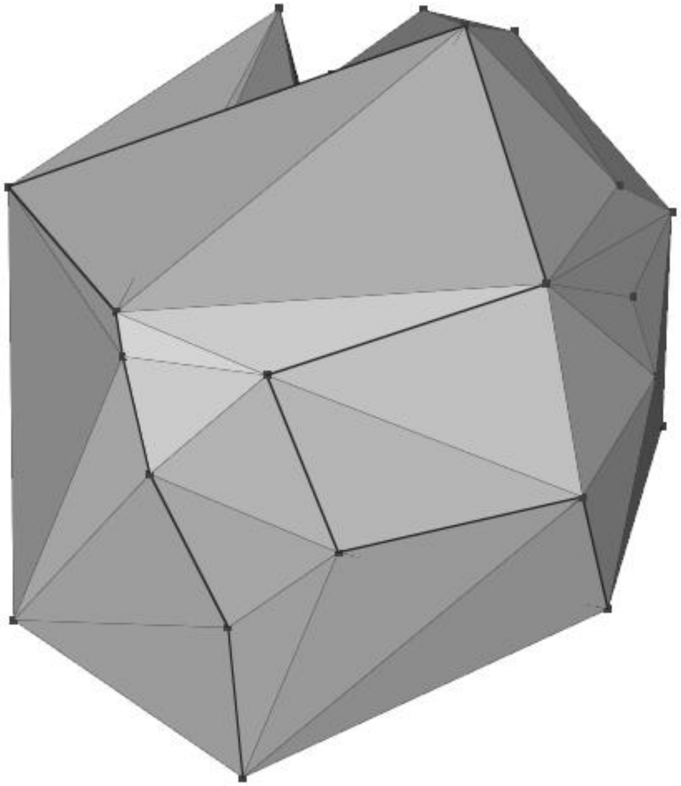

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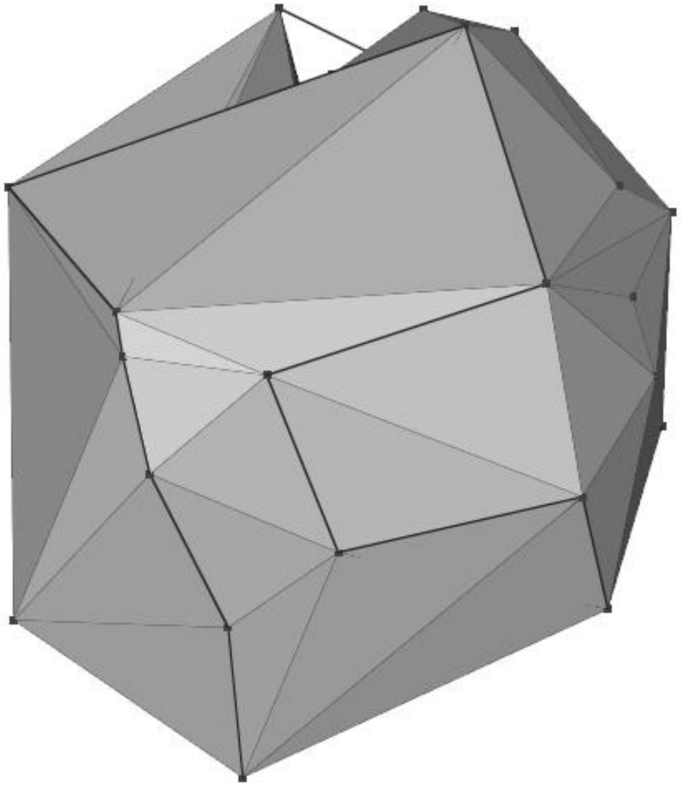

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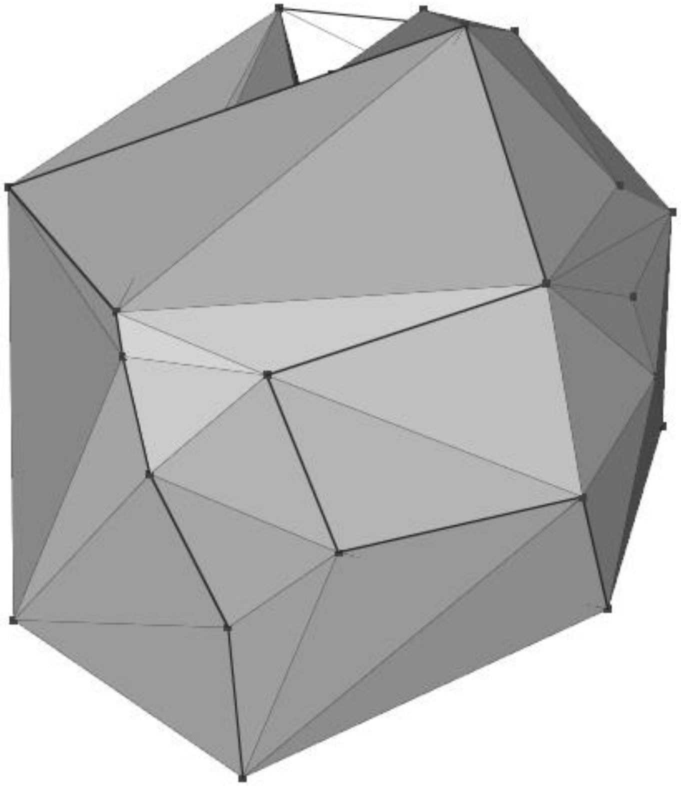

Divide And Conquer

Merge:

- Construct the fillet that merges the two hulls
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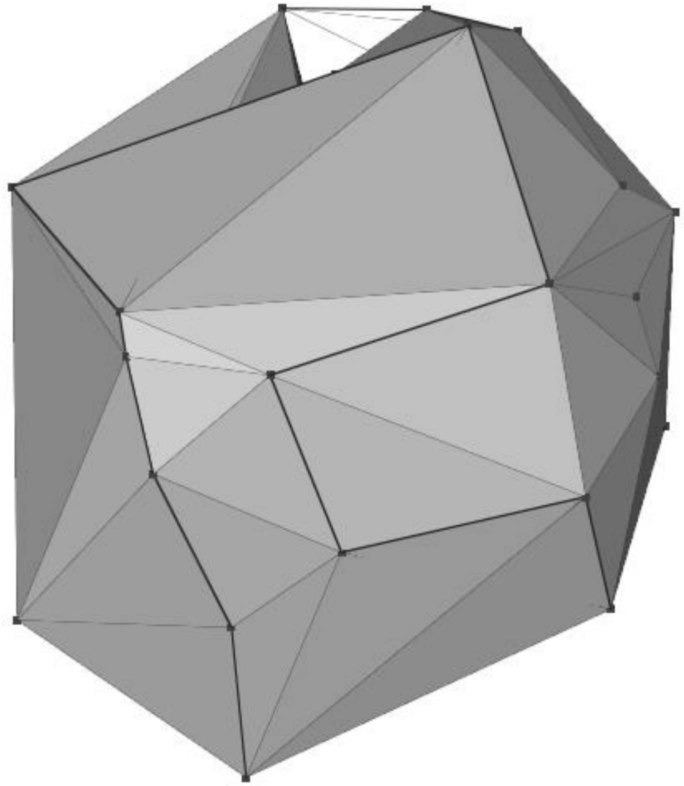

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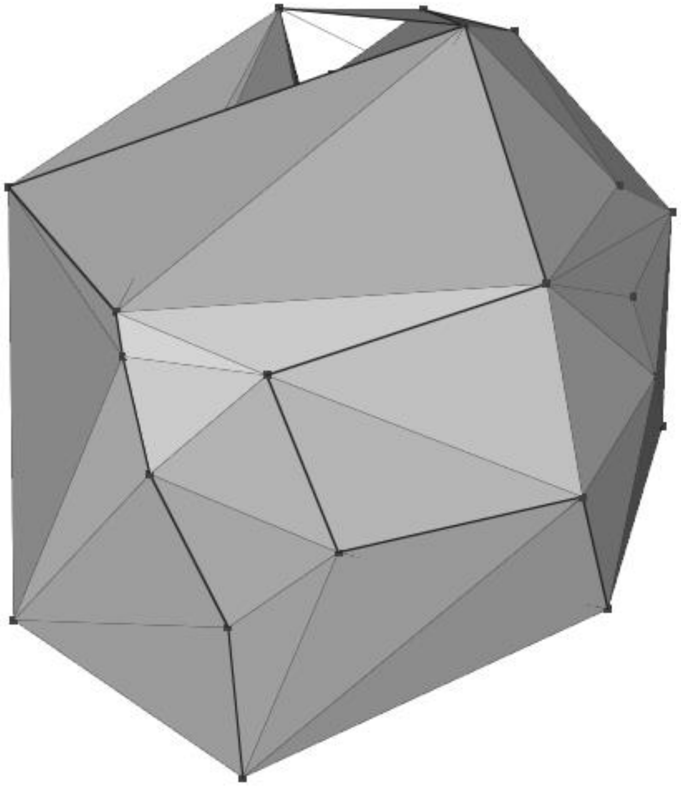

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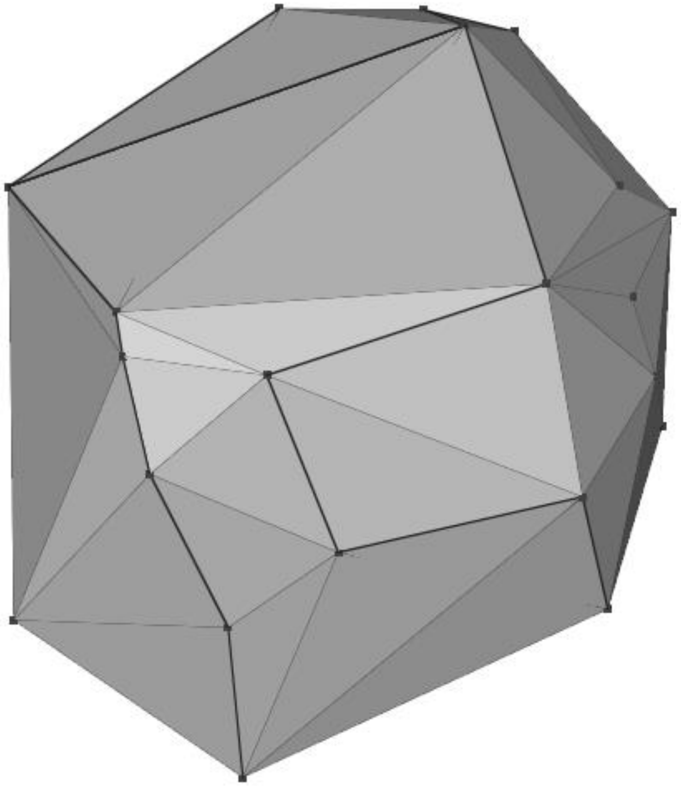

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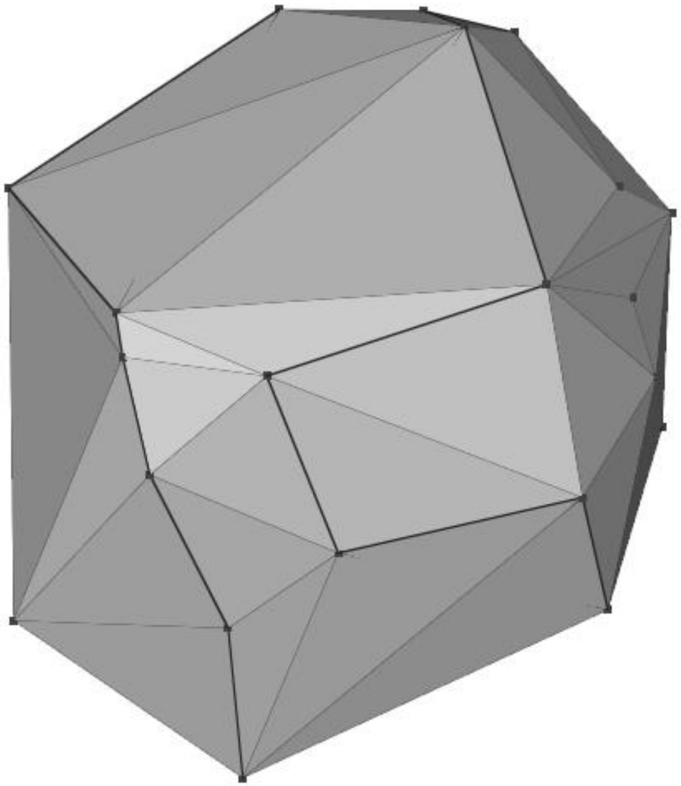

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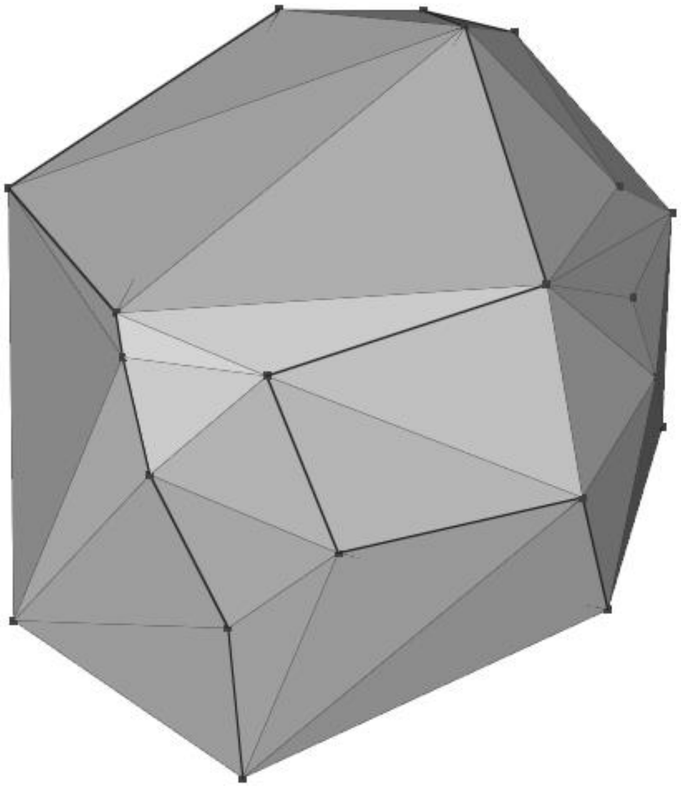


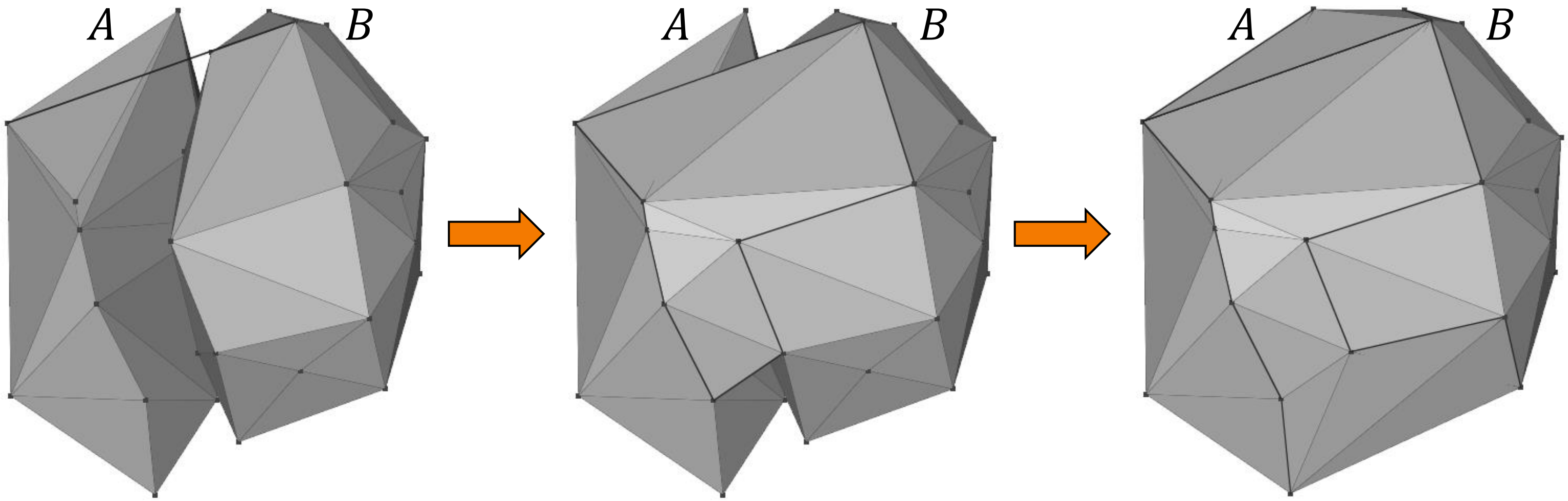
Note:

The fillet has linear complexity since each triangle on the fillet uses an edge from one of the two hulls.

Divide And Conquer

Constructing the Fillet:

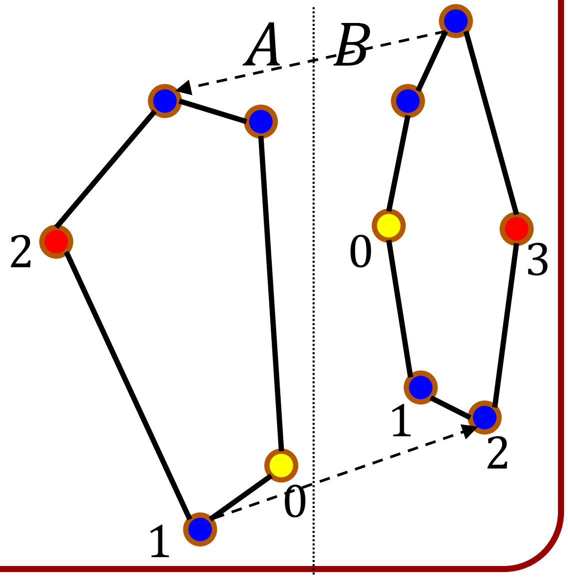
- Find a supporting line
- Pivot around the supporting line



Divide And Conquer

Finding a Supporting Line:

- While computing the 3D hull (recursively), simultaneously compute the 2D hull of the projection of the points onto the xy-plane.
- The supporting lines in 2D correspond to supporting lines in 3D.



Divide And Conquer

Pivot Around the Supporting Line:

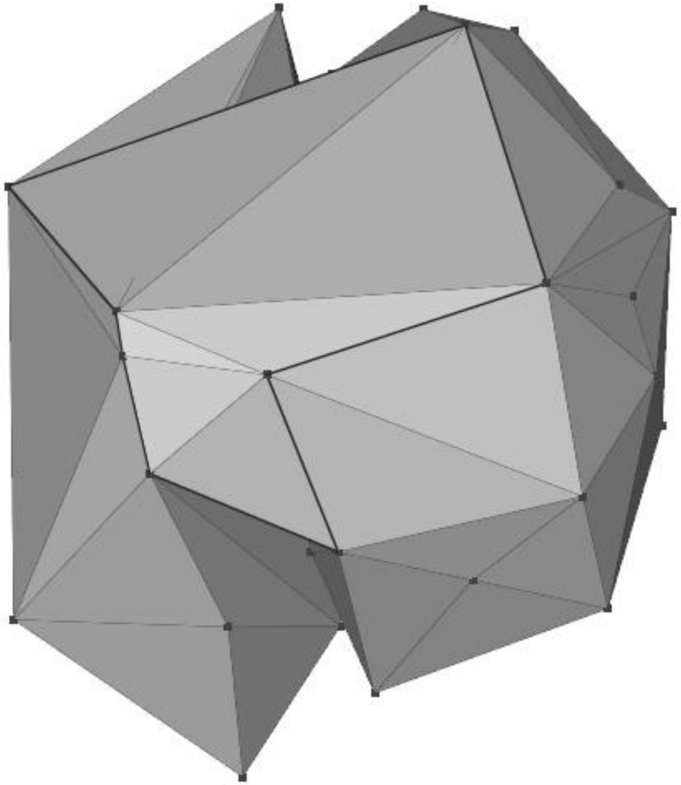
- Proceed as in the gift-wrap algorithm.

Challenge:

- To run in linear time, we can't try all points.

Observation:

- When we pivot, the first point we hit is one of the neighbors of the line's end-points.



Divide And Conquer

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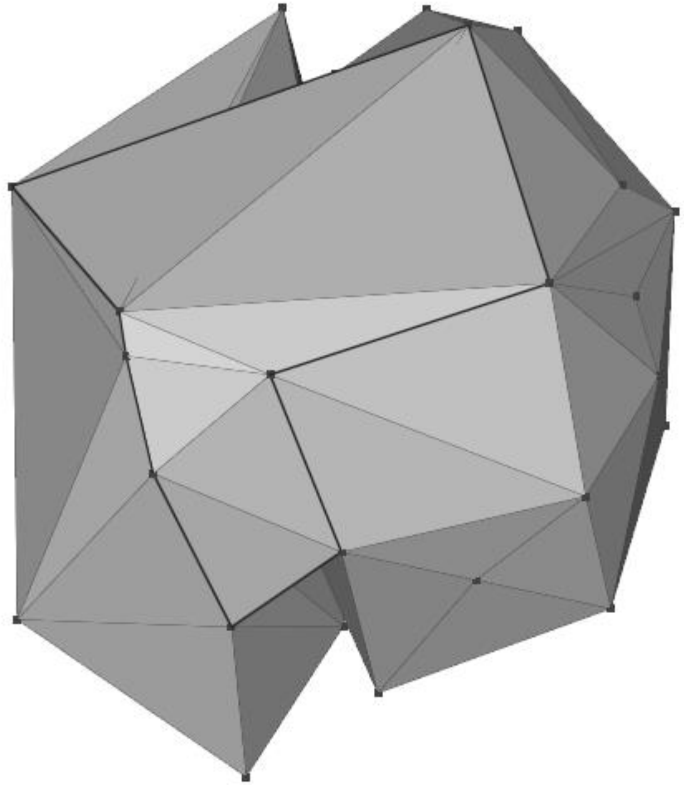

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Divide And Conquer

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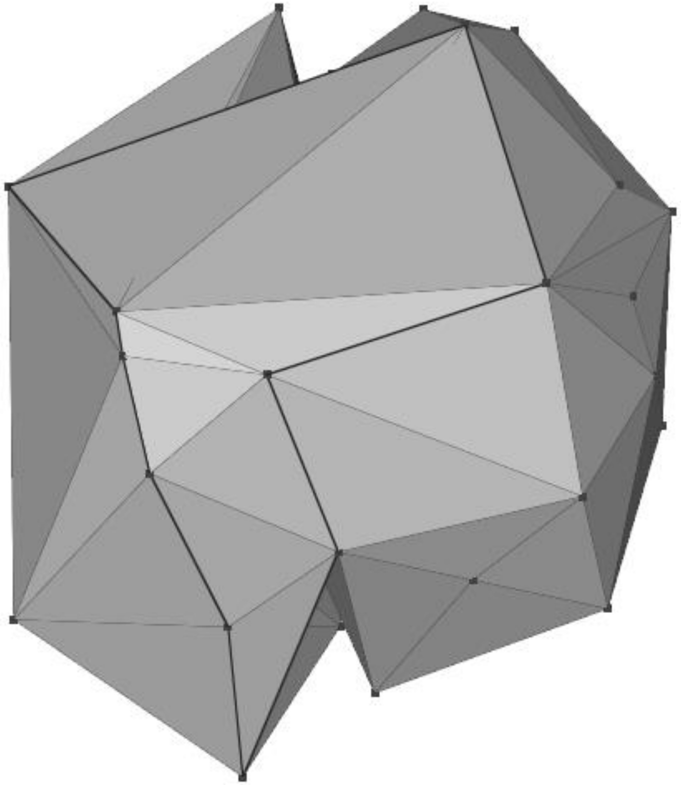

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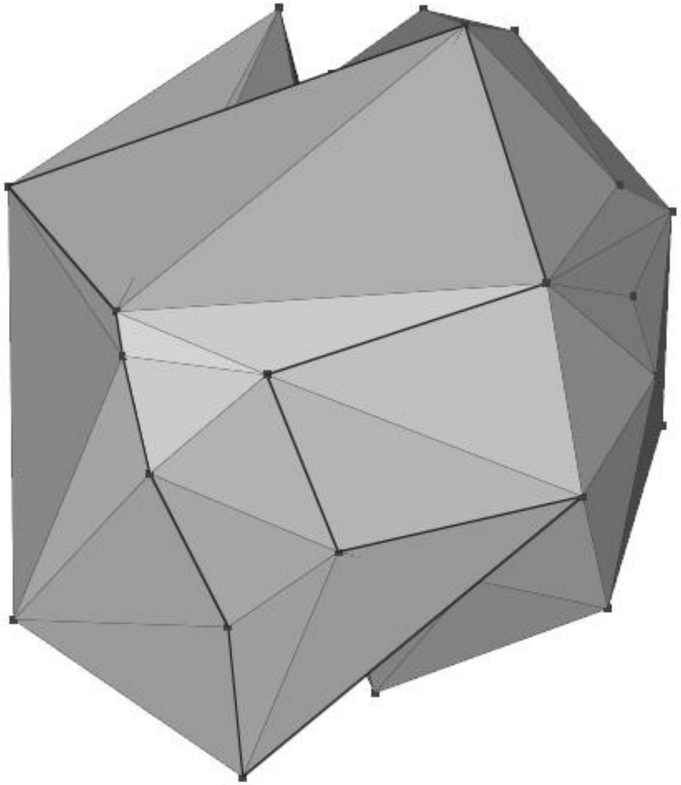

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Divide And Conquer

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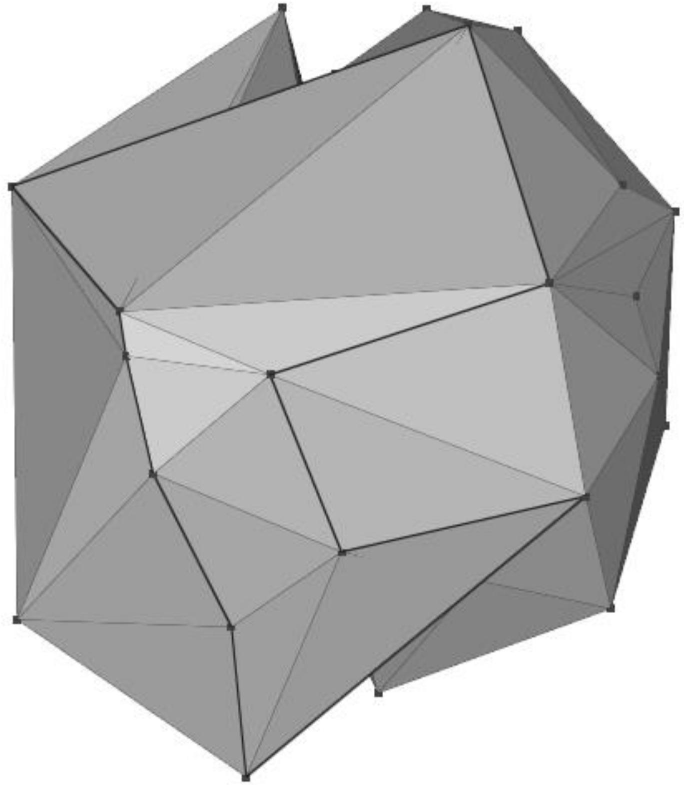


- This could still be costly since a vertex can have many neighbors.
  (e.g. If the right endpoint has many neighbors but the pivot keeps hitting a vertex on the left.)

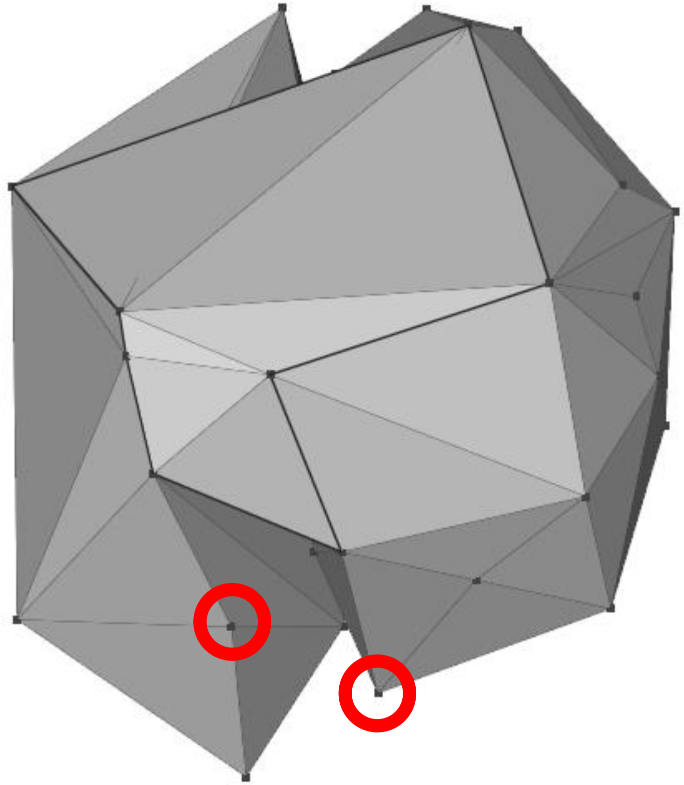
Divide And Conquer

<u>Pivot Around the Supporting Line</u>:

- Proceed as in the gift-wrap algorithm.

<u>Challenge</u>:

- This could still be costly since a vertex can have many neighbors.



<u>Observation</u>:

- We can use the previous estimated (failed) hit to constrain the next one.

Divide And Conquer

Pivot Around the Supporting Line:

- Proceed as in the gift-wrap algorithm.

Challenge:

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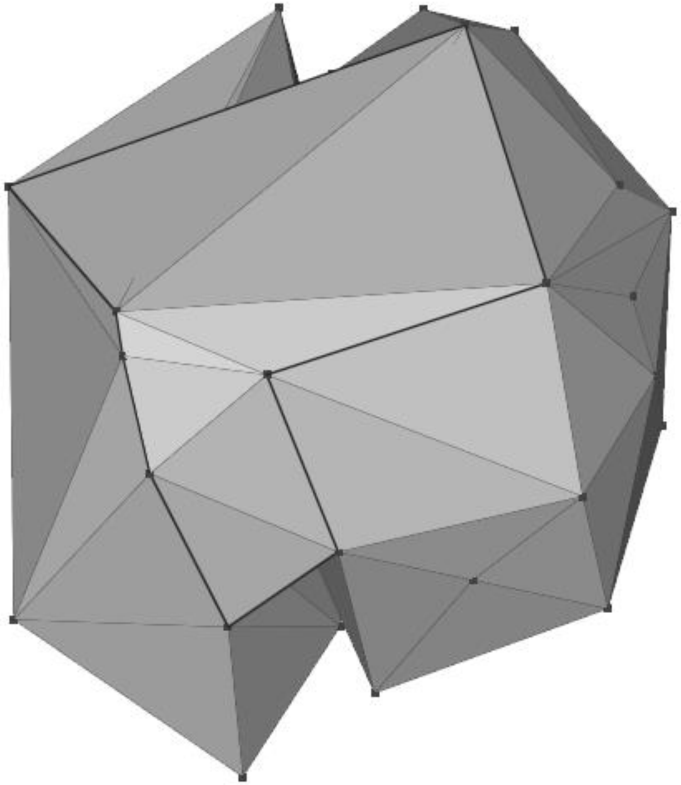

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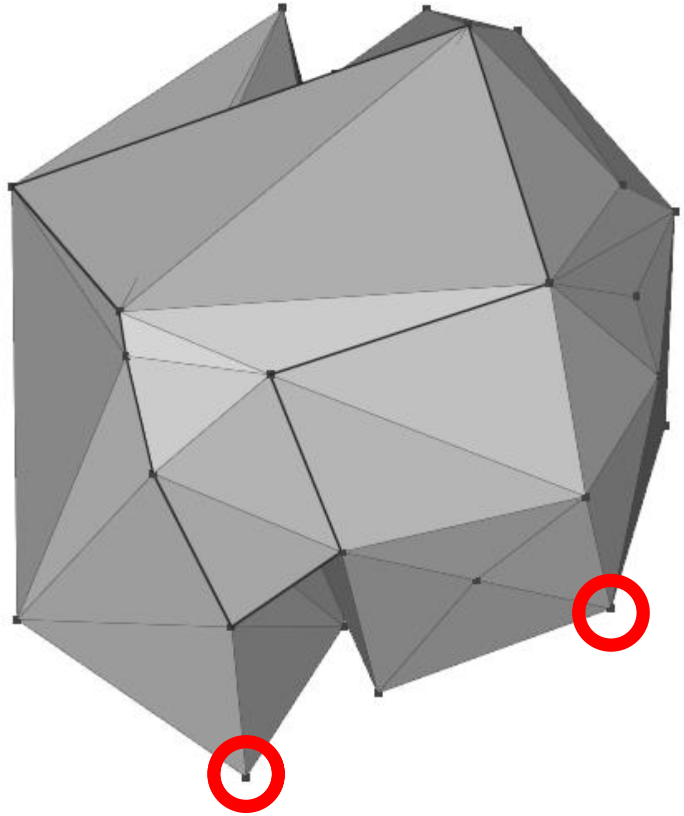

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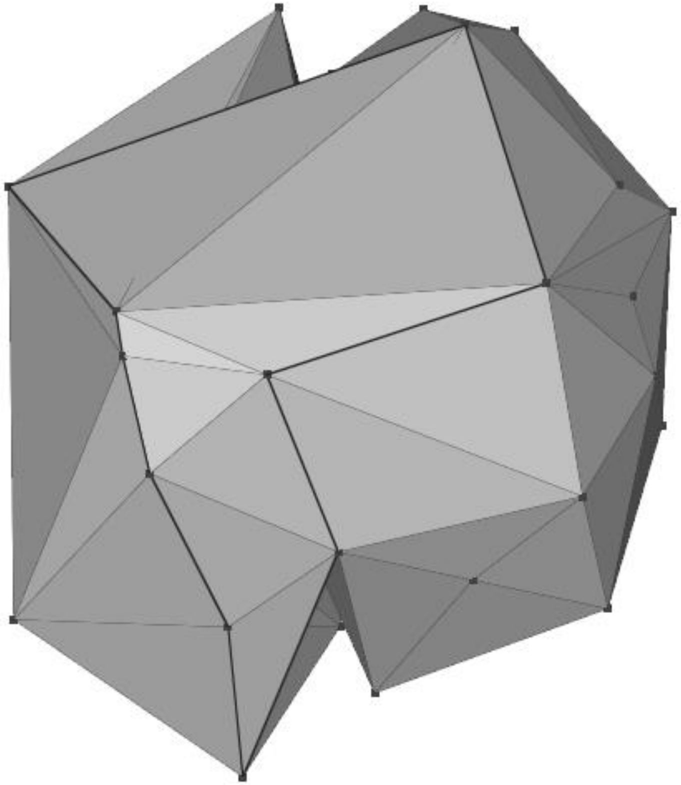

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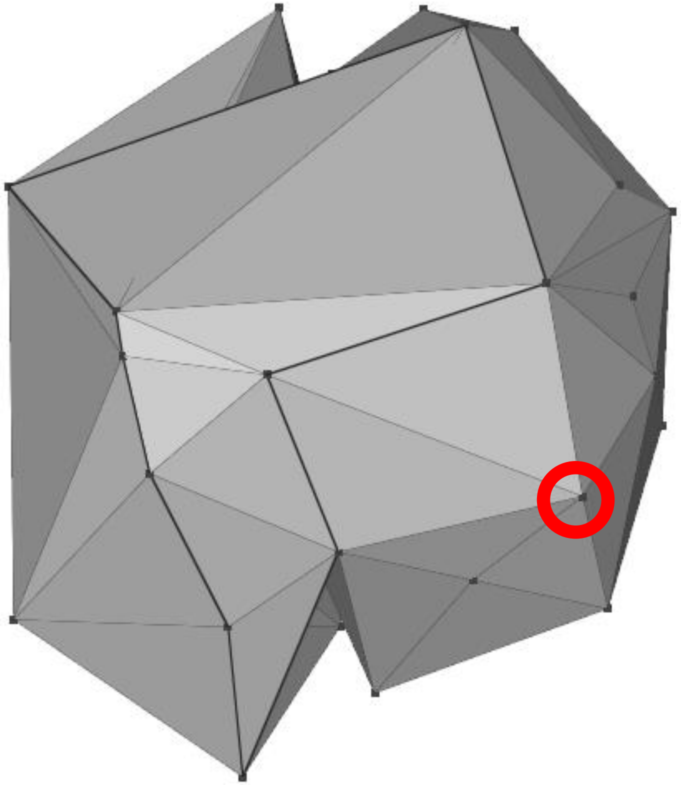

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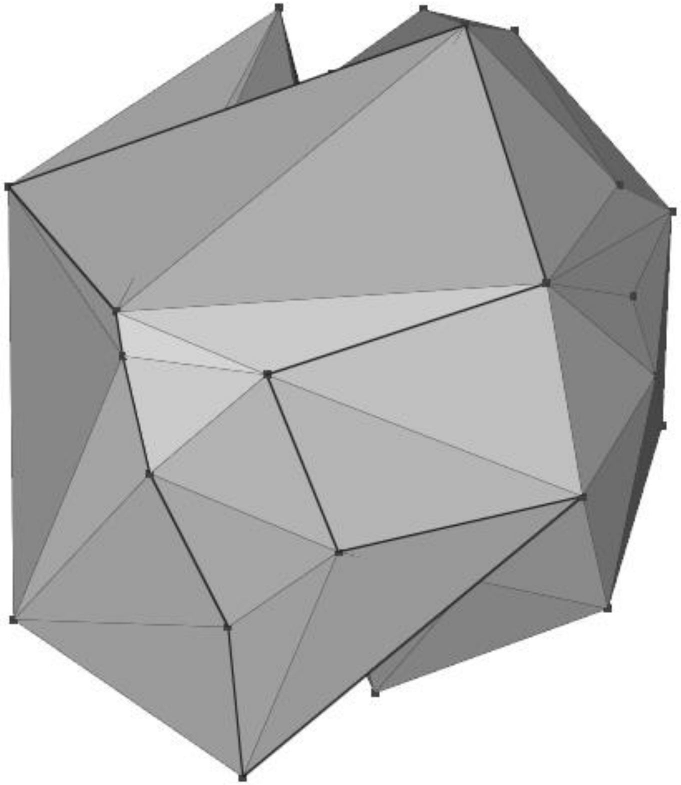

Challenge:

- This could still be costly since a vertex can have many neighbors.

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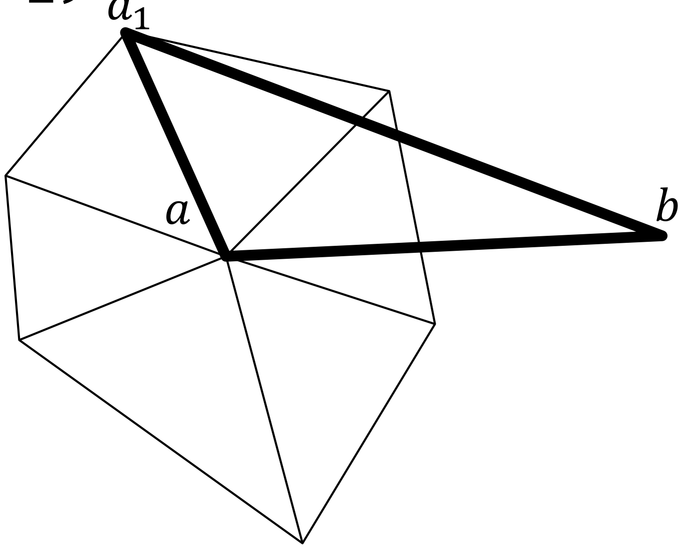
Divide And Conquer

More Specifically:

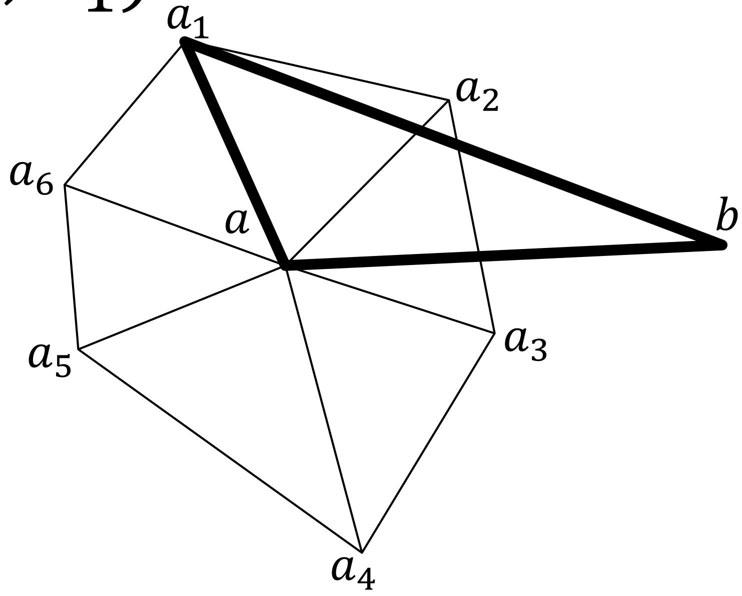
- Assume the fillet is at edge (a, b) having just added triangle (a, b, a_1).



Divide And Conquer

More Specifically:

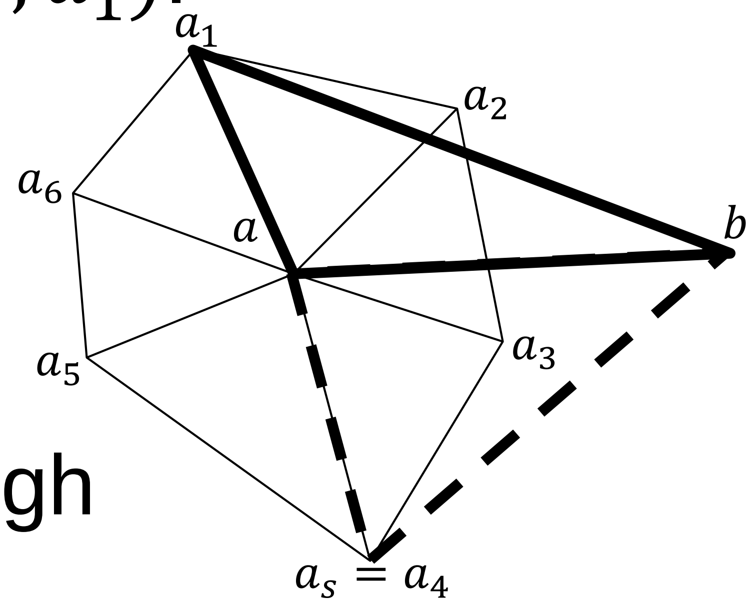
- Assume the fillet is at edge (a, b) having just added triangle (a, b, a_1).
- Sort the neighbors of a CW starting from a_1.



Divide And Conquer

<u>More Specifically</u>:

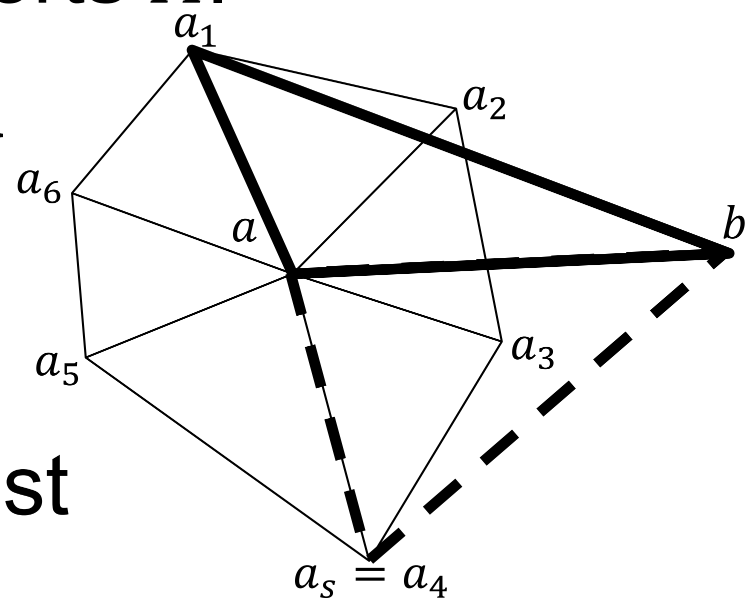
- Assume the fillet is at edge (a, b) having just added triangle (a, b, a_1).
- Sort the neighbors of a CW starting from a_1.
- Let a_s be the neighbor of a s.t. the plane through (b, a, a_s) supports A.



Divide And Conquer

More Specifically:

- Let a_s be the neighbor s.t. the plane through (b, a, a_s) supports A.
- The points $\{a_2, \ldots, a_{s-1}\}$ must be inside the hull.
- Even if we advance on b we won't need to retest these points.



Divide And Conquer

Merge(H_1 , H_2):

$(v_1, v_2) \leftarrow$ FindSupportingLine(H_1 , H_2)

$Q \leftarrow \{(v_1, v_2)\}$

$F \leftarrow \emptyset$

While ($Q \neq \emptyset$)

  $e \leftarrow Q$.pop_back()

  if($e \neq \{v_2, v_1\}$)

    $t \leftarrow$ SupportingTriangle(H_1 , H_2 , e)

    $F \leftarrow F \cup \{t\}$

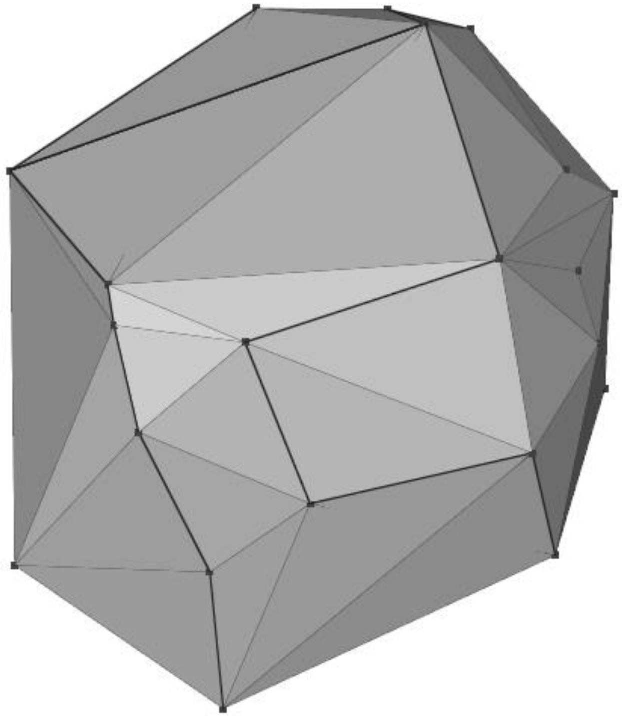
    $Q \leftarrow Q \cup$ CrossingEdges(t) / $\{e\}$

CleanUp

Divide And Conquer

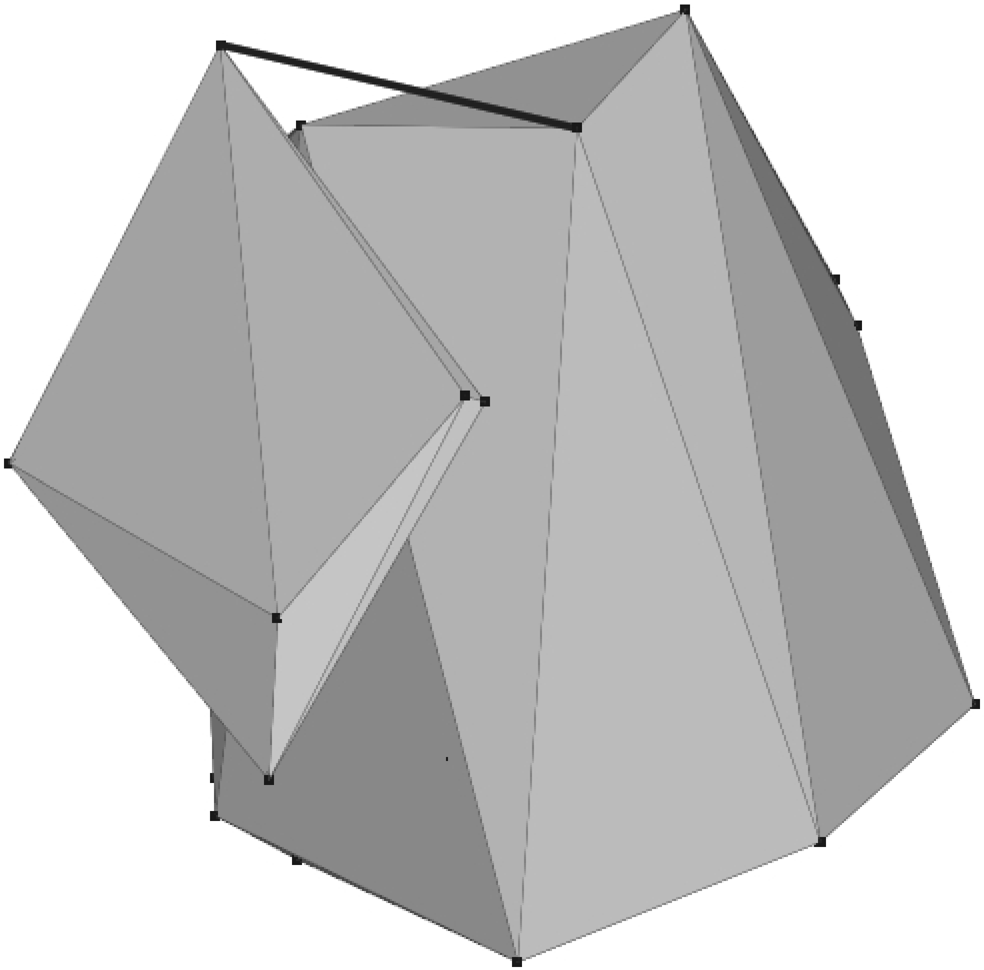
Clean-Up:

- Represent the two hulls with a winged-edge data structure.
- Replace the opposite edges of the silhouette with the edges of the new triangles.



- Flood-fill to find interior triangles.

Divide And Conquer



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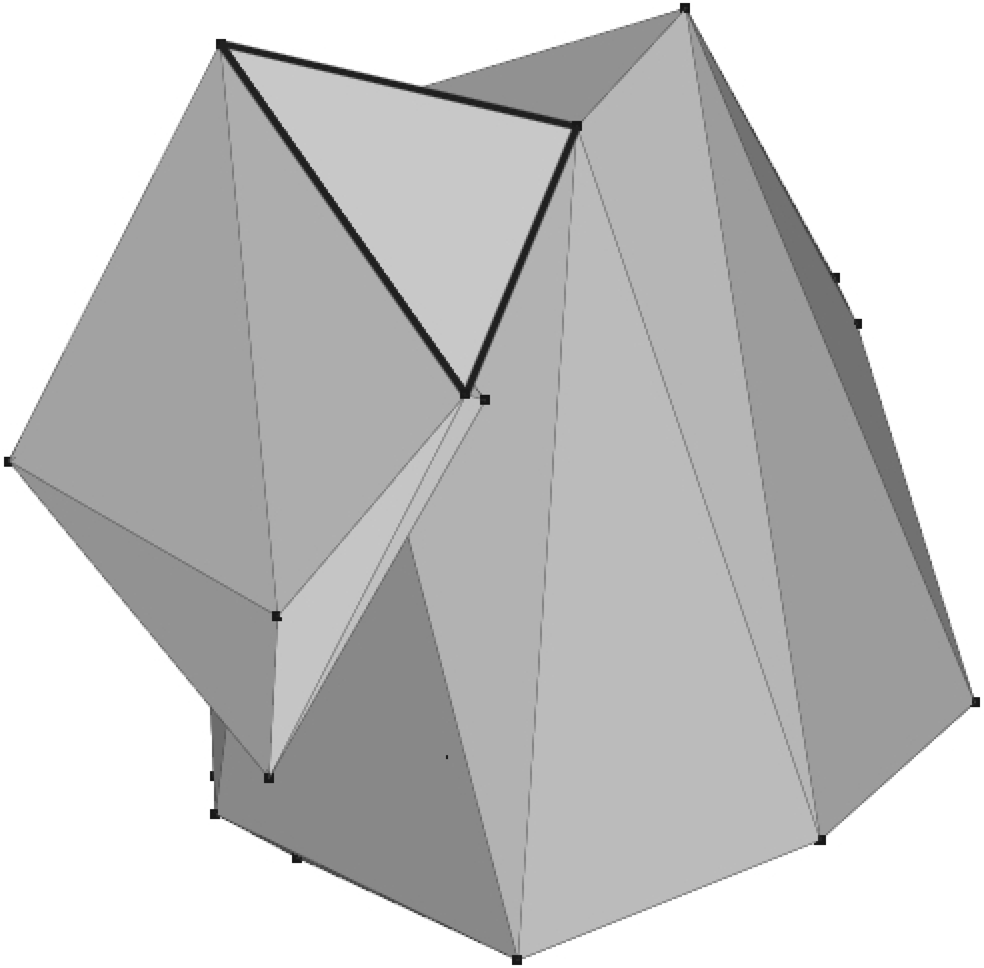

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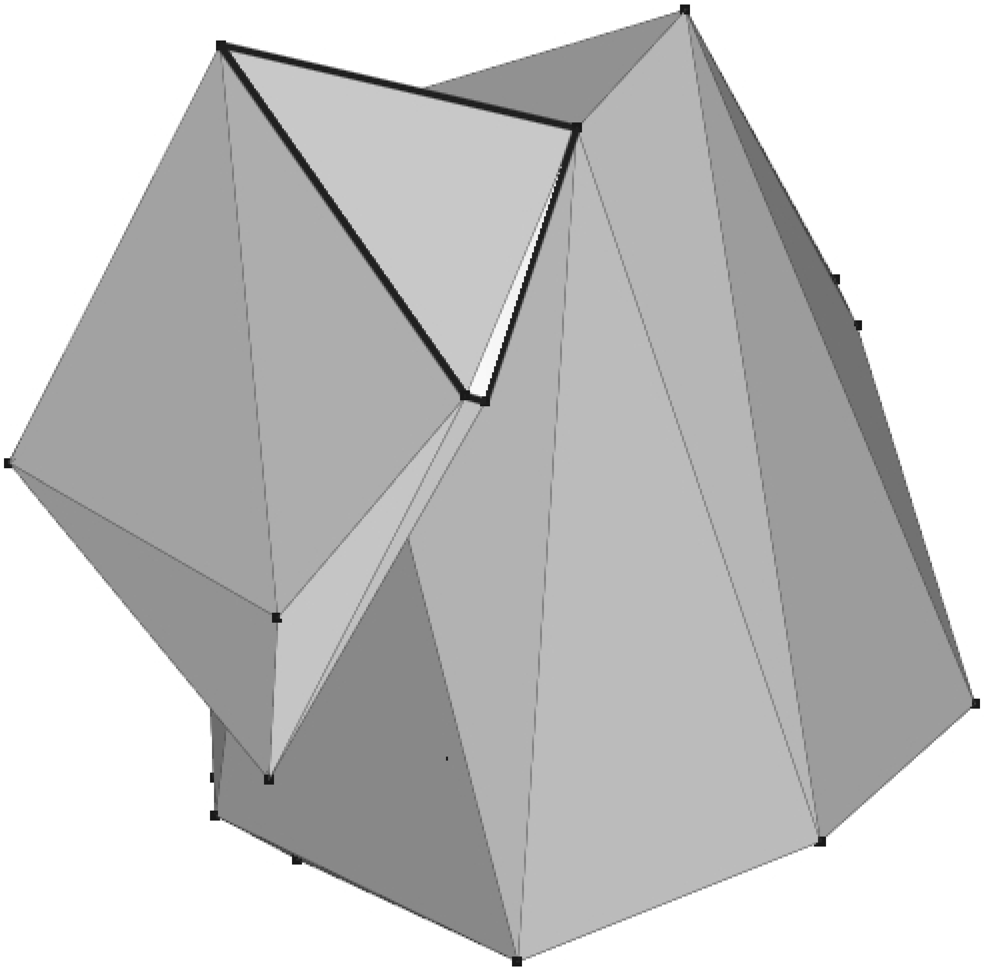

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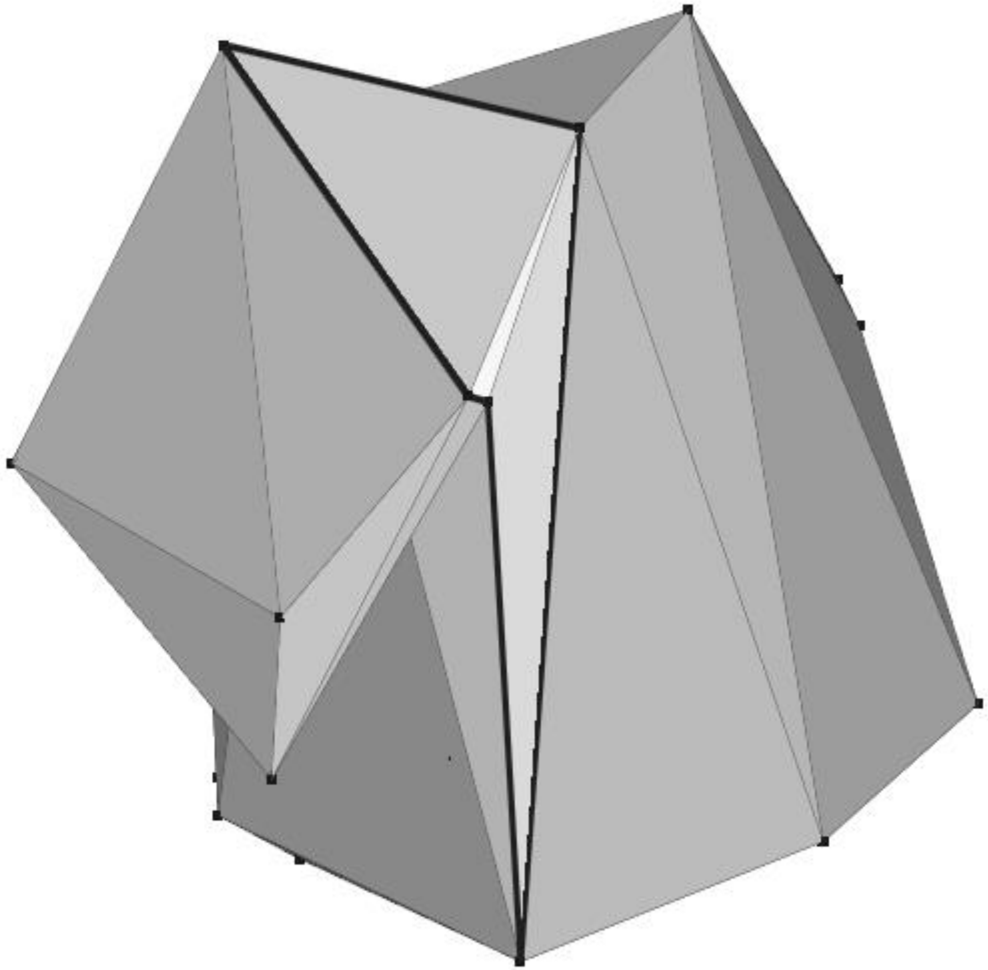

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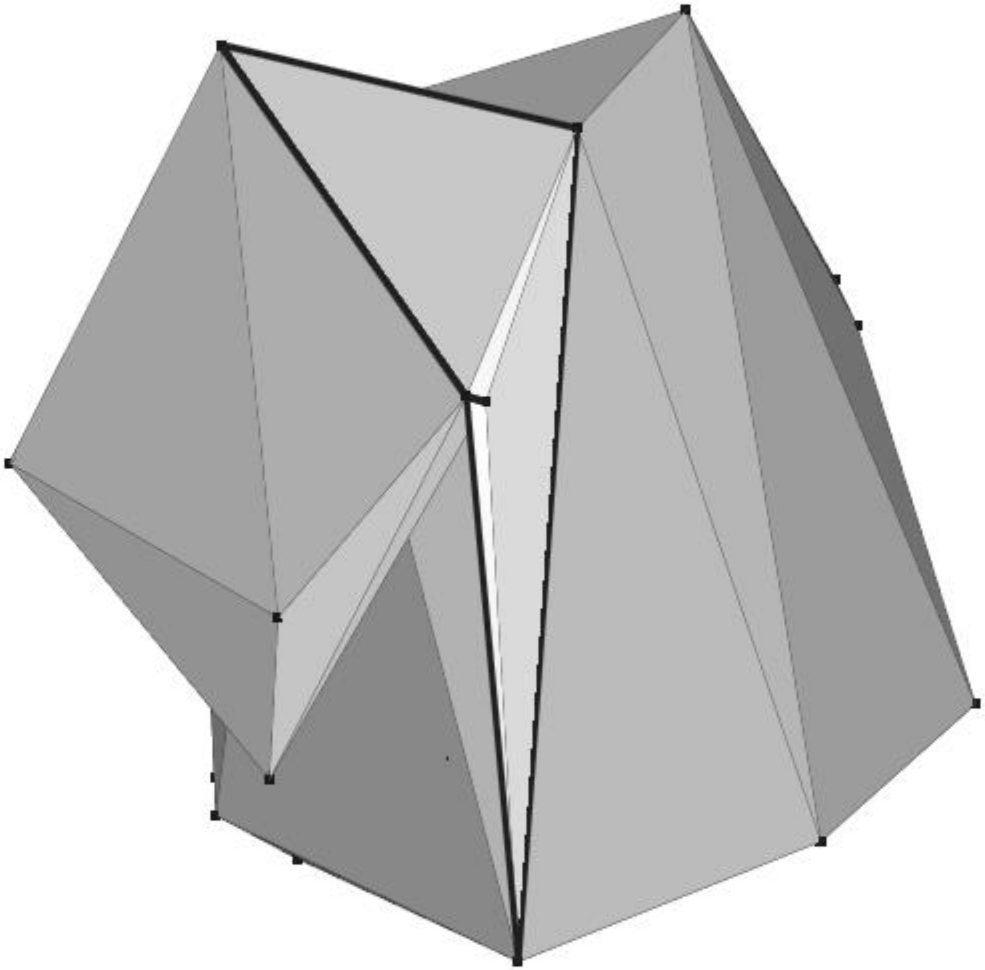

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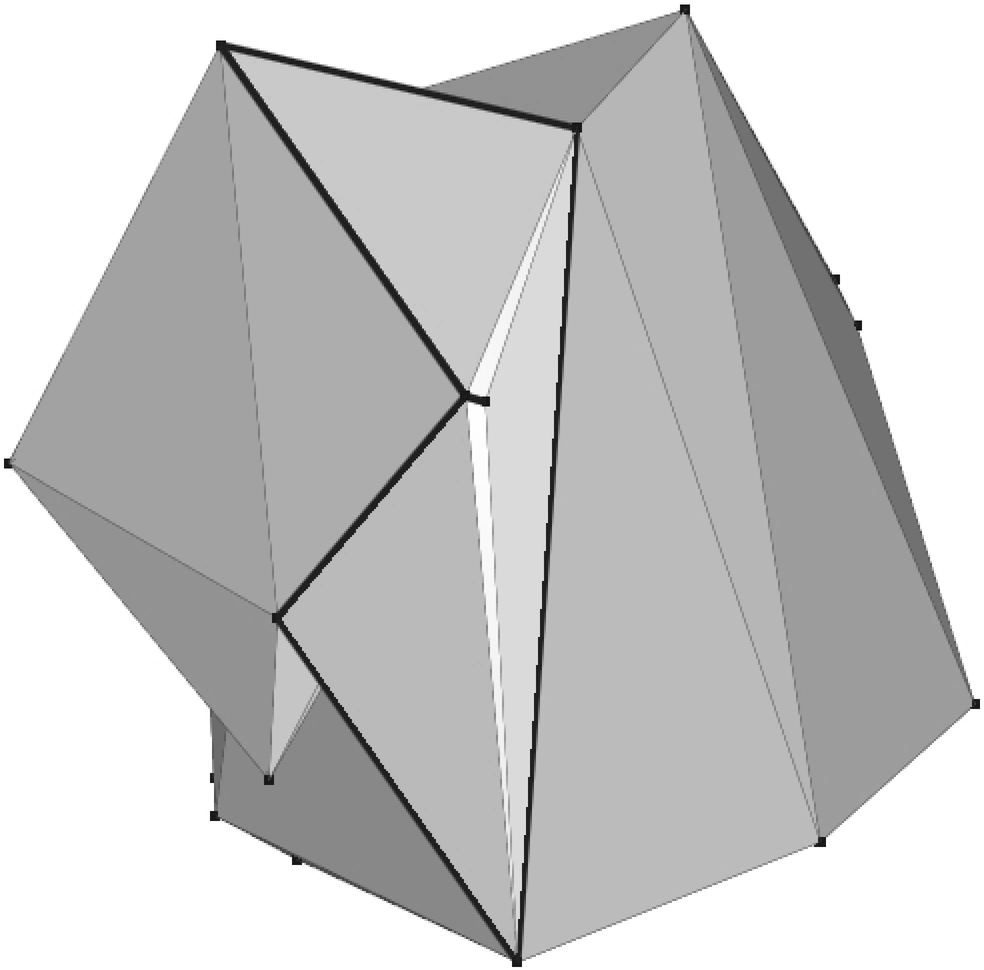

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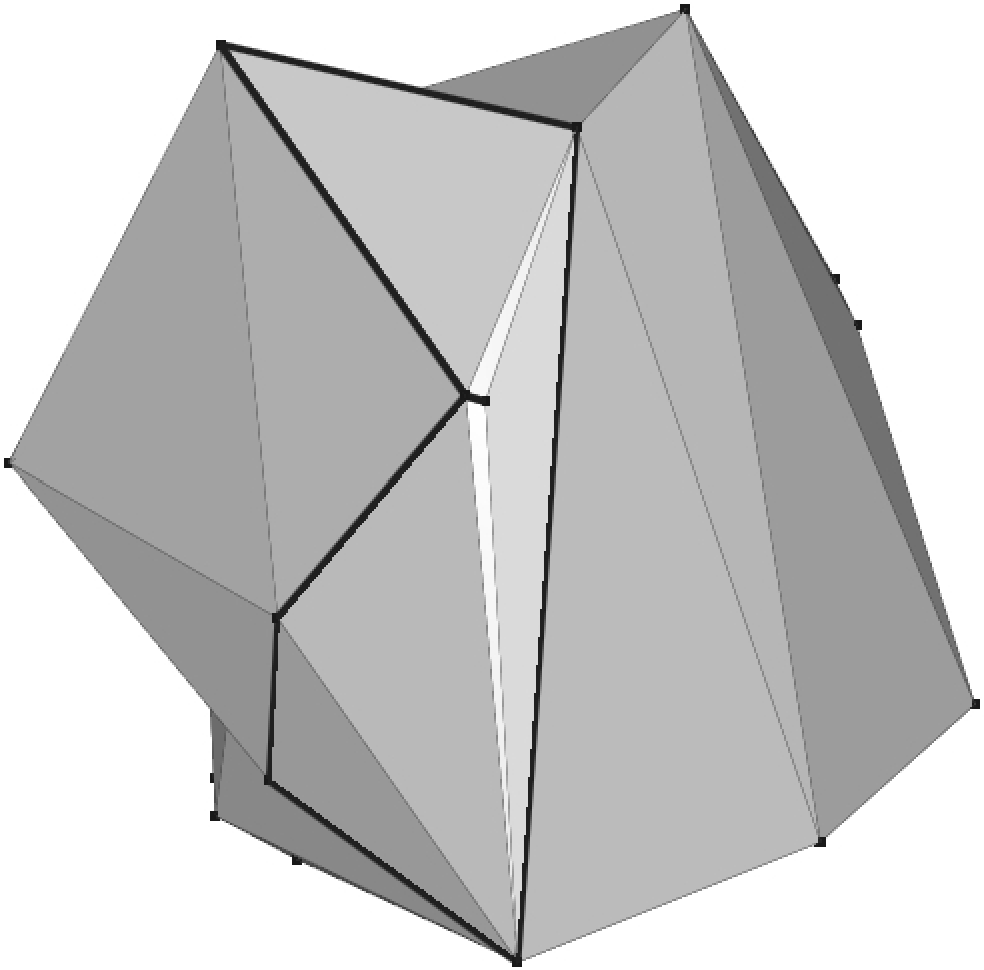

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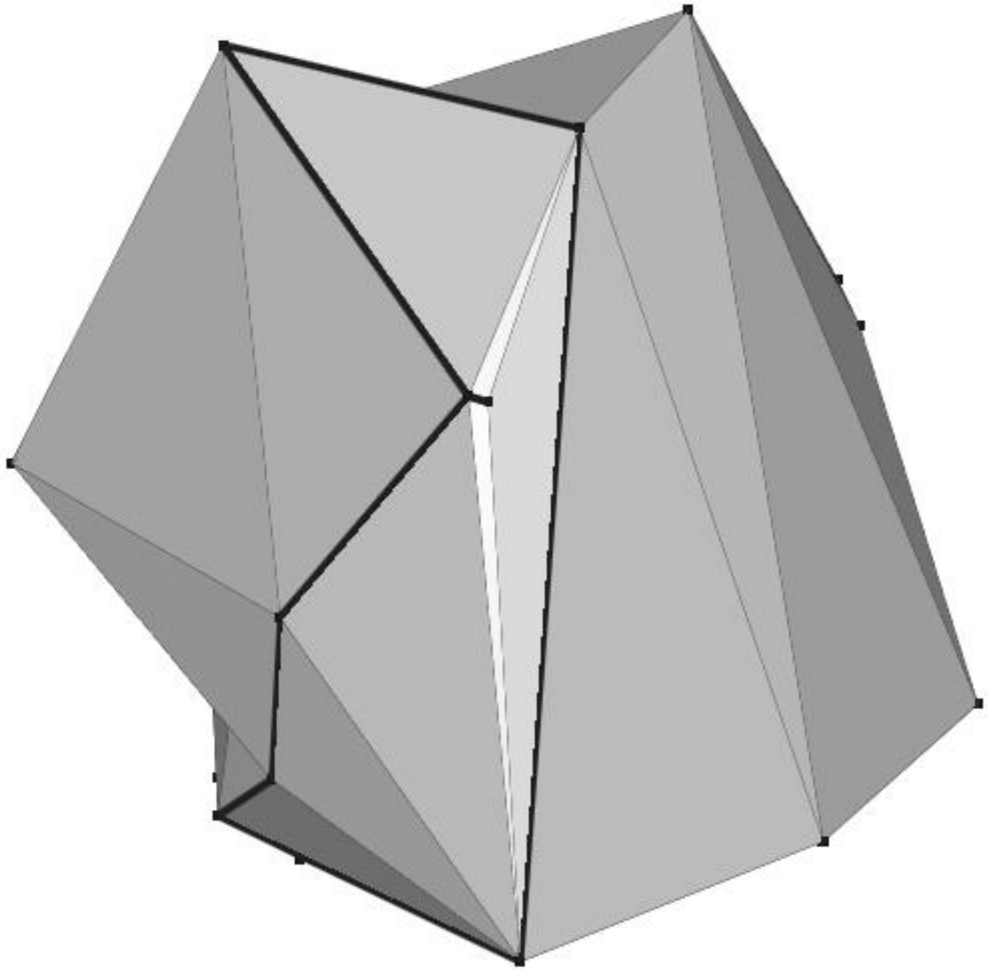

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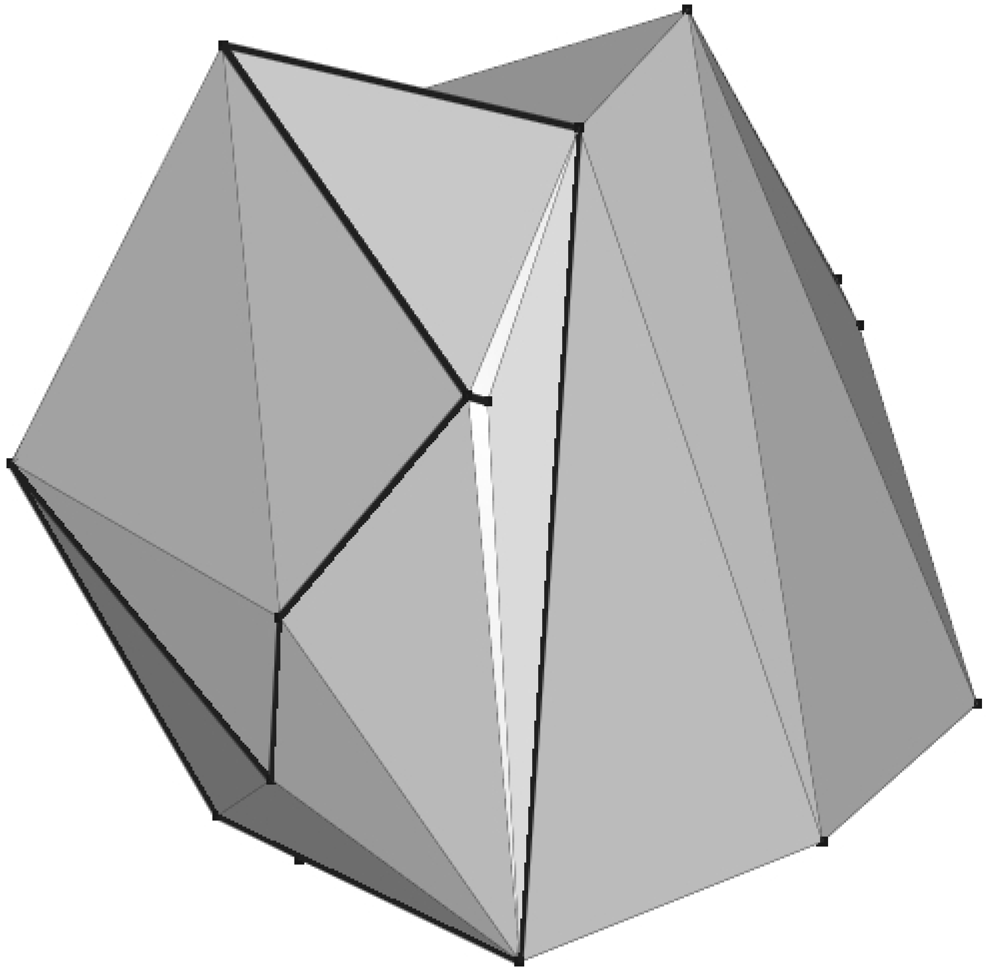

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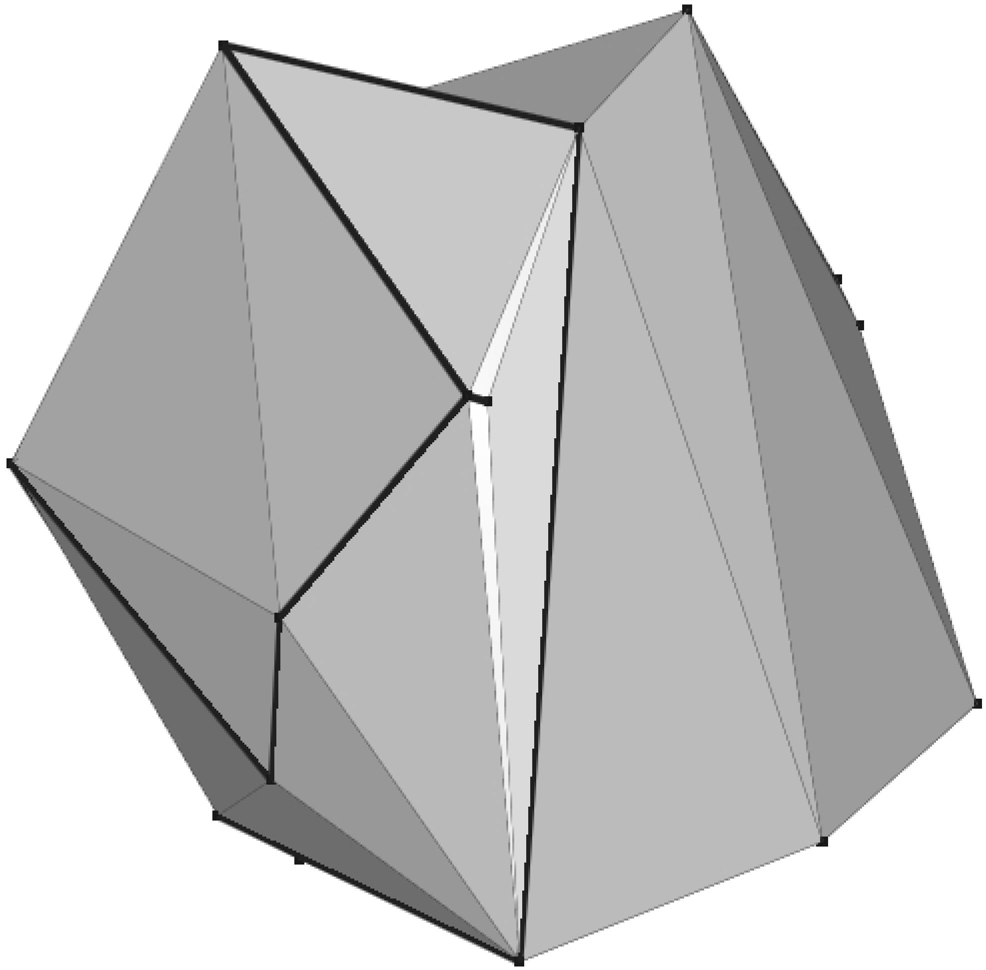

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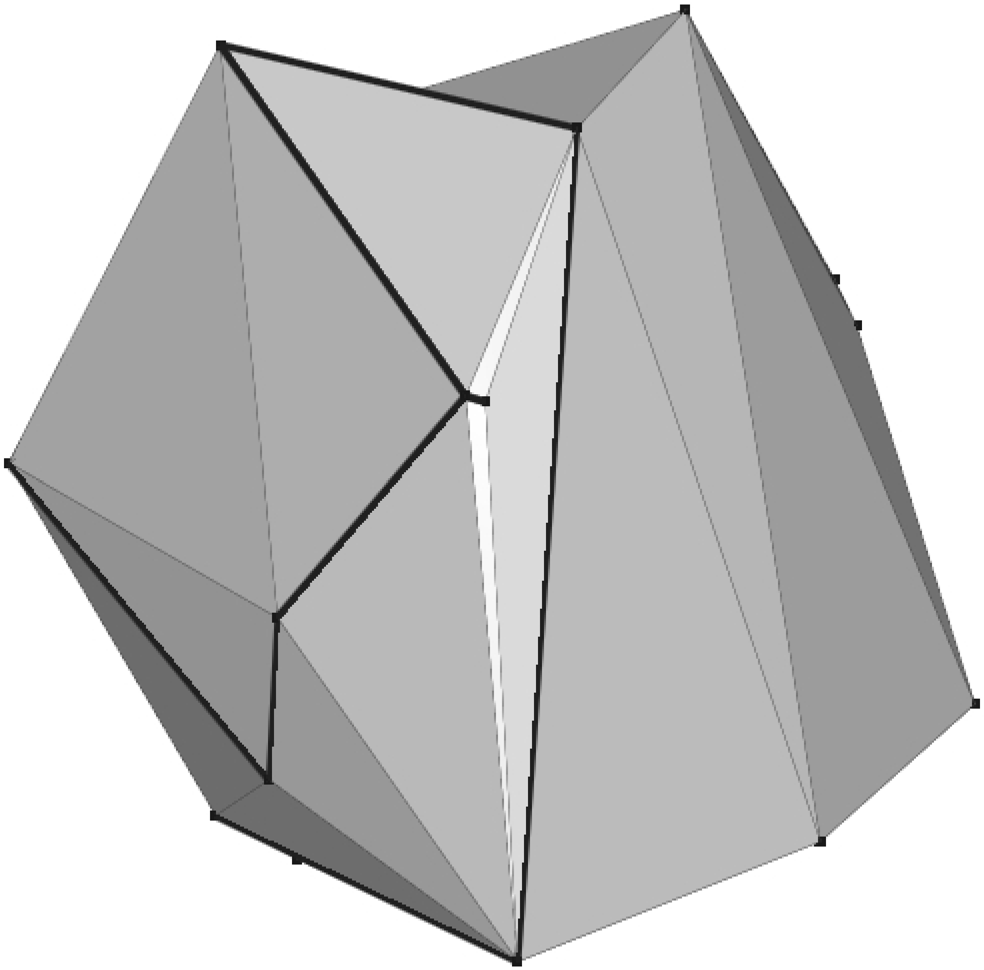

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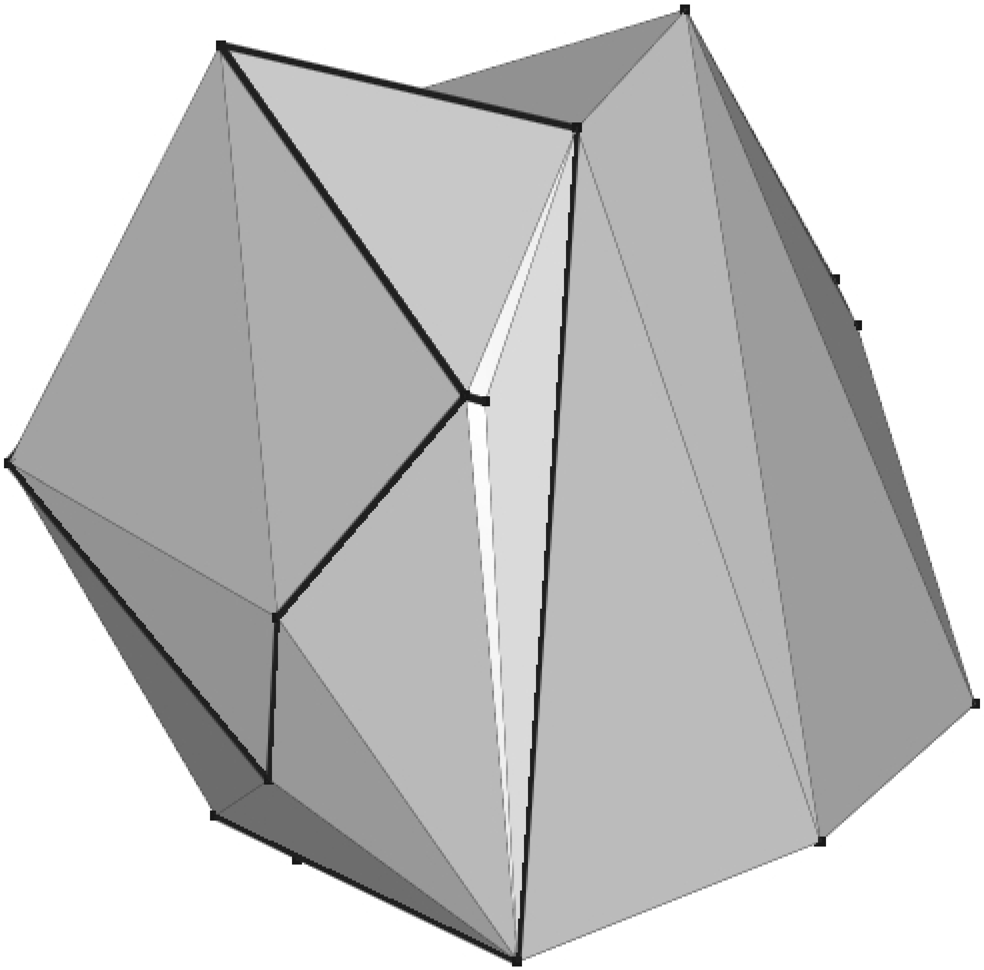

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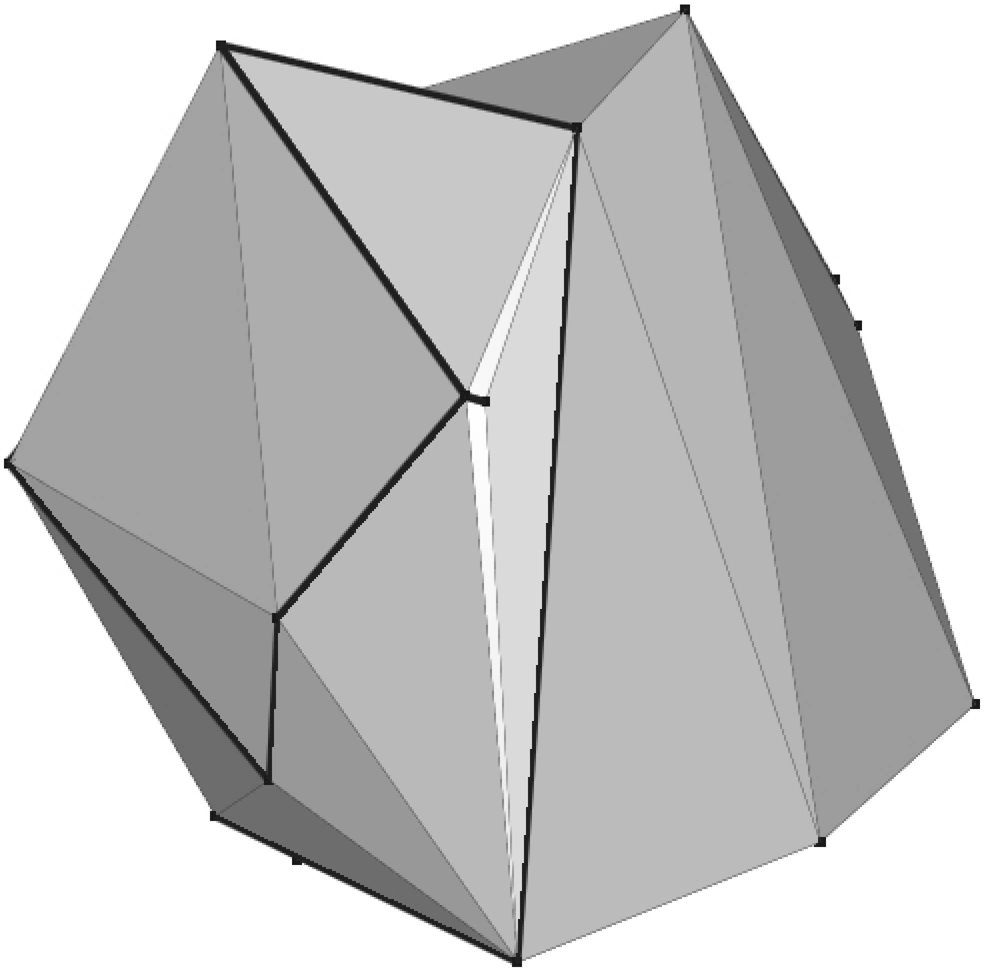

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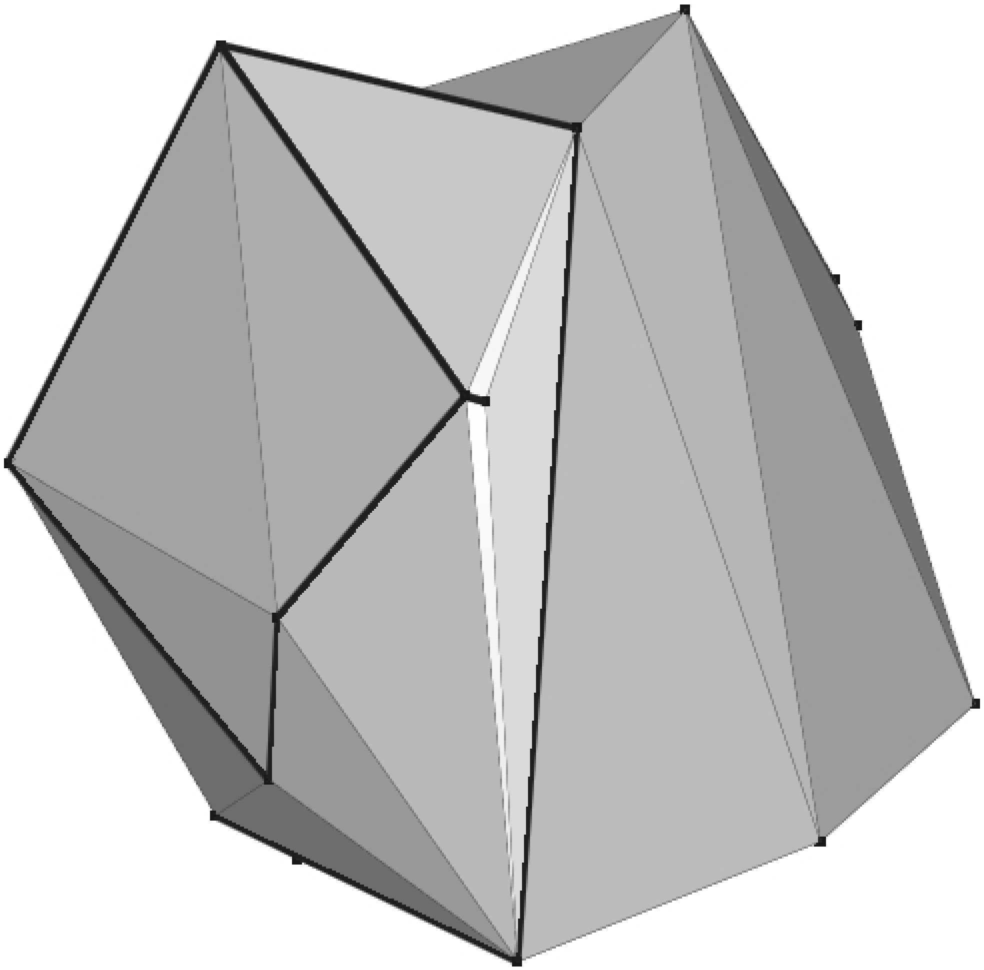

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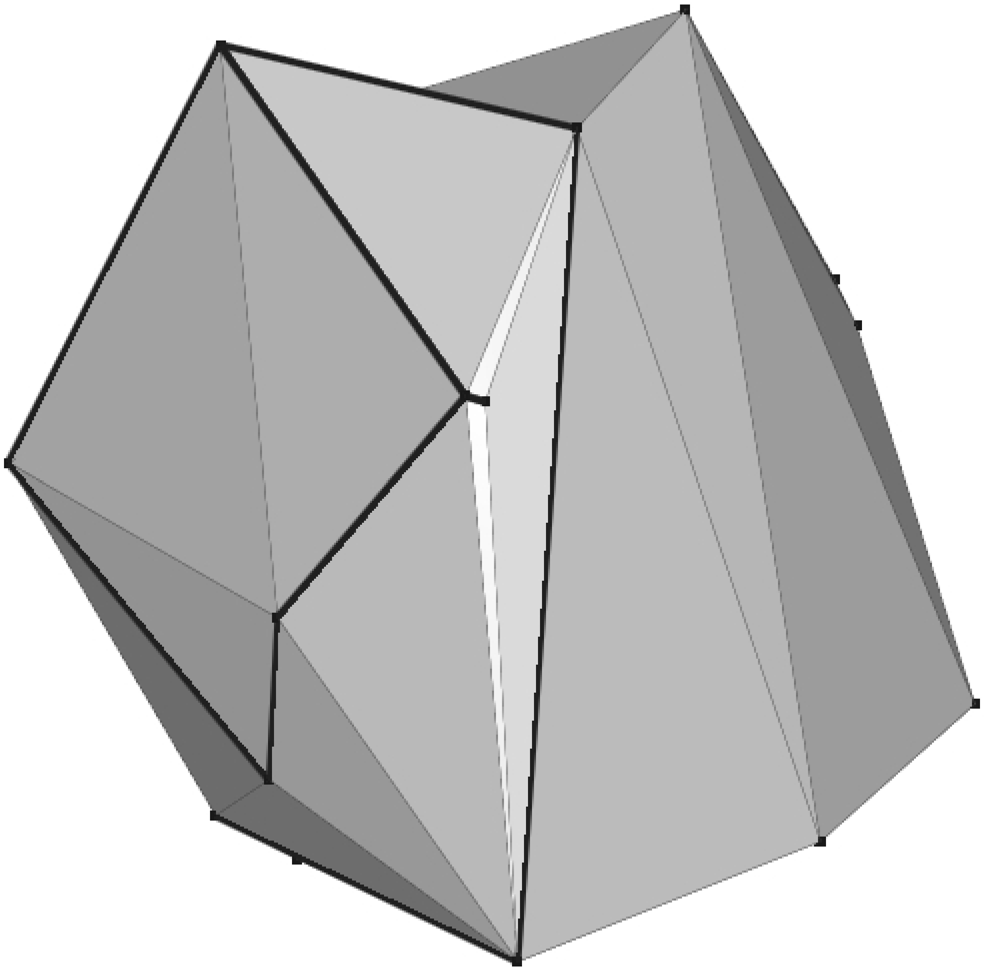

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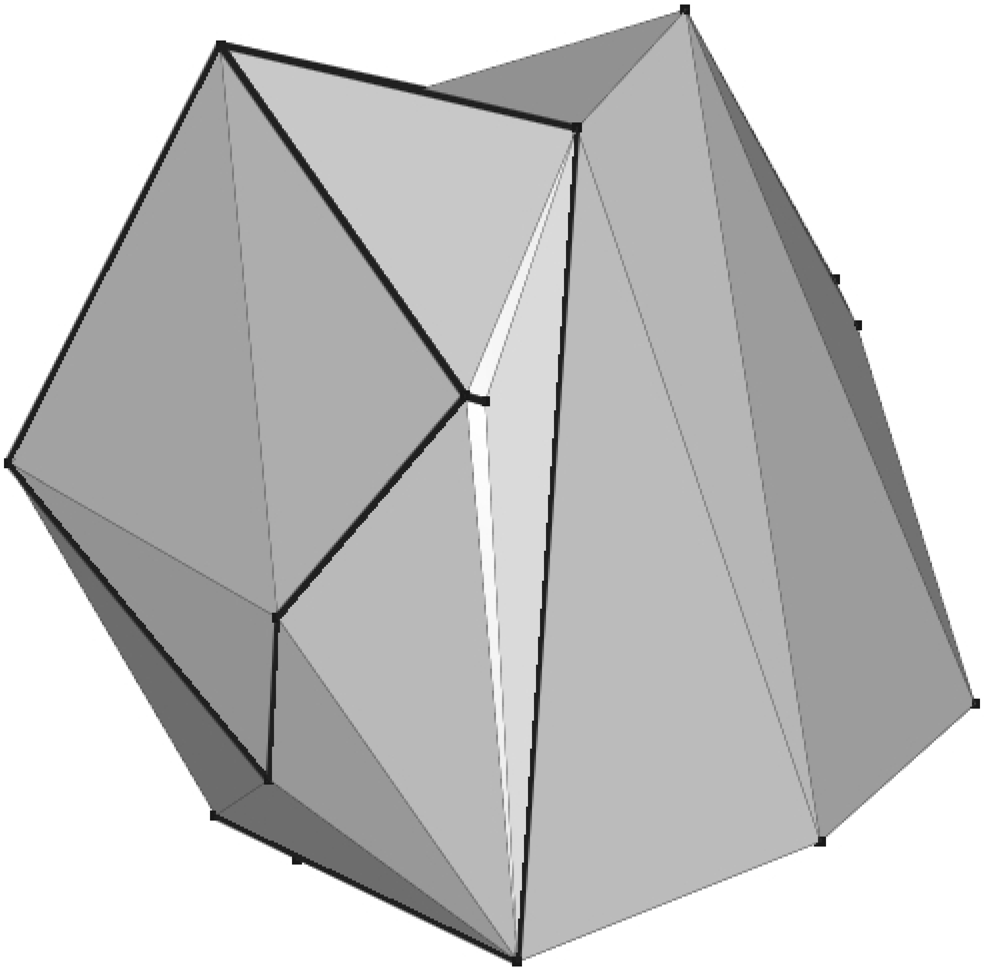

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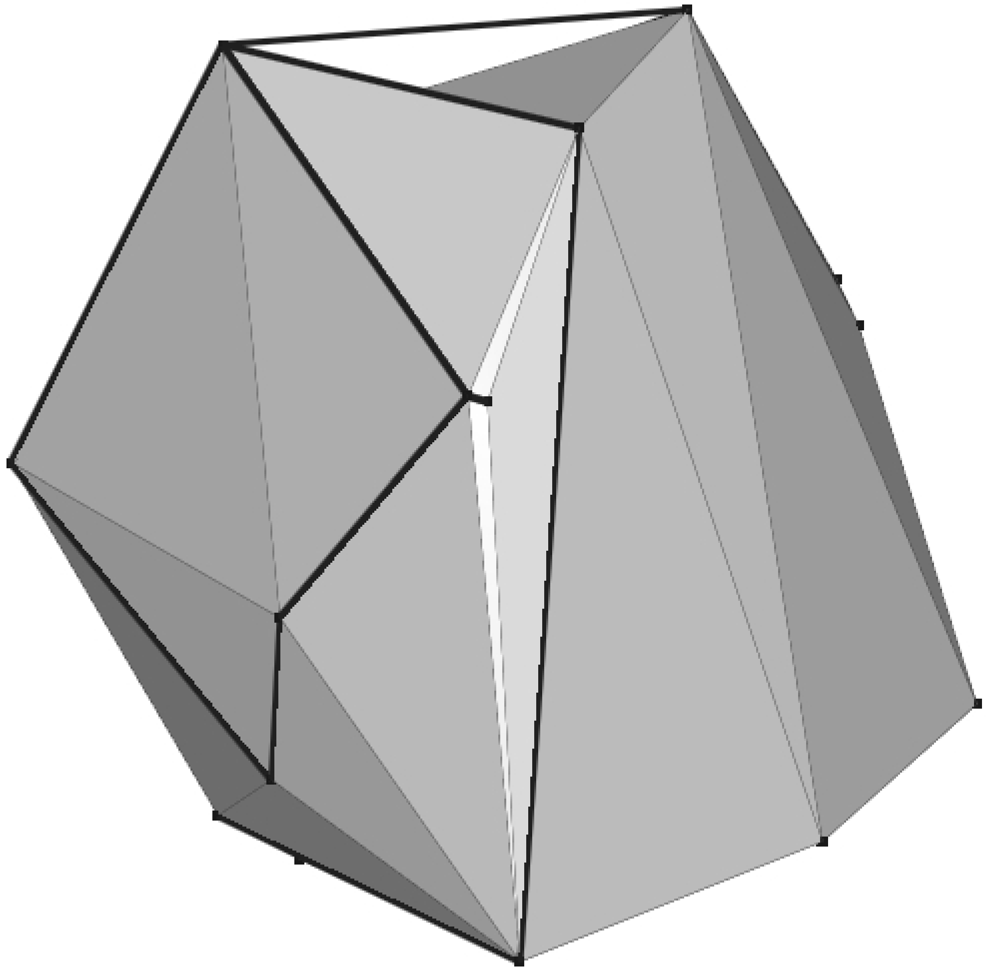

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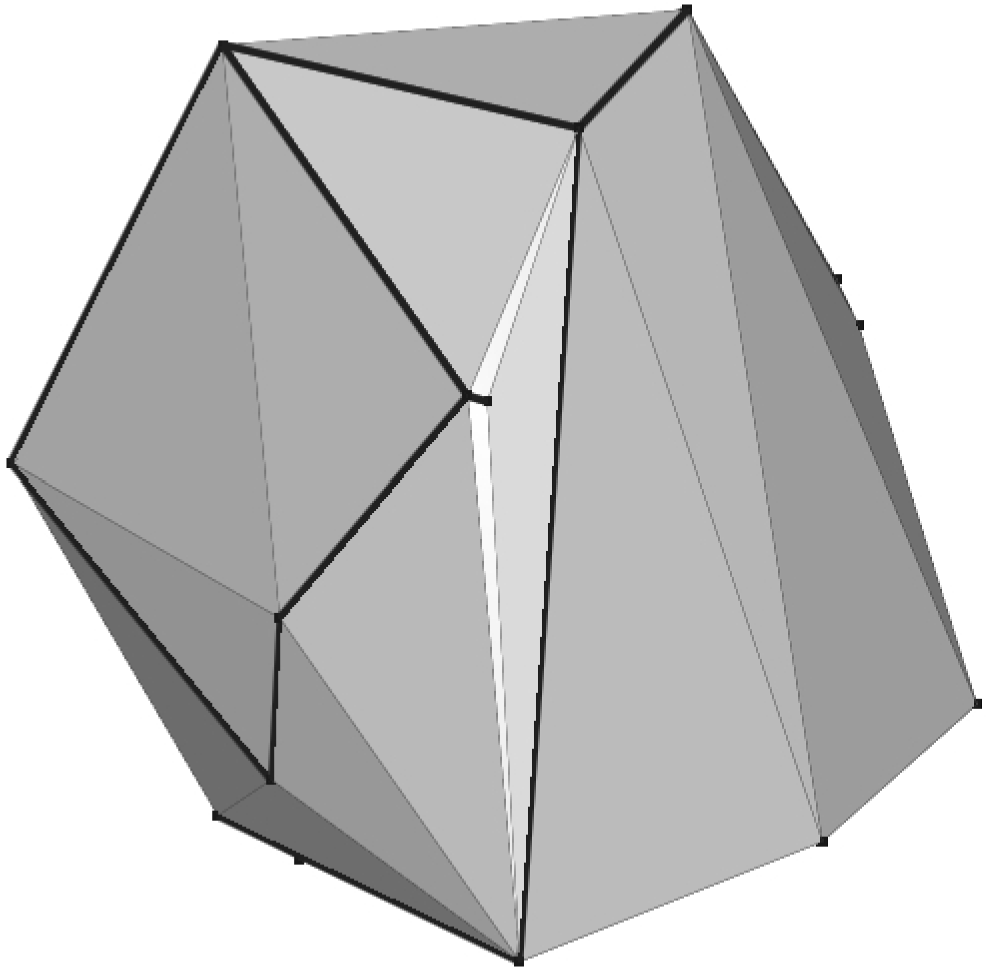

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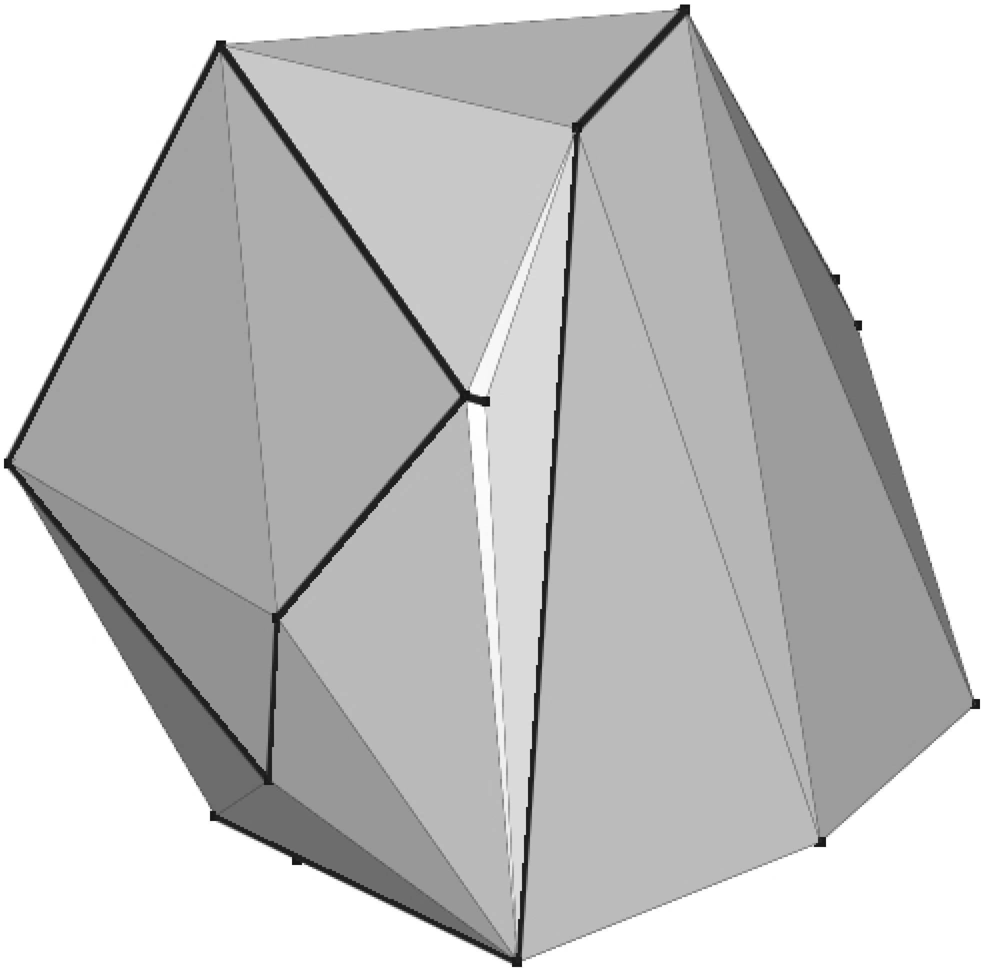

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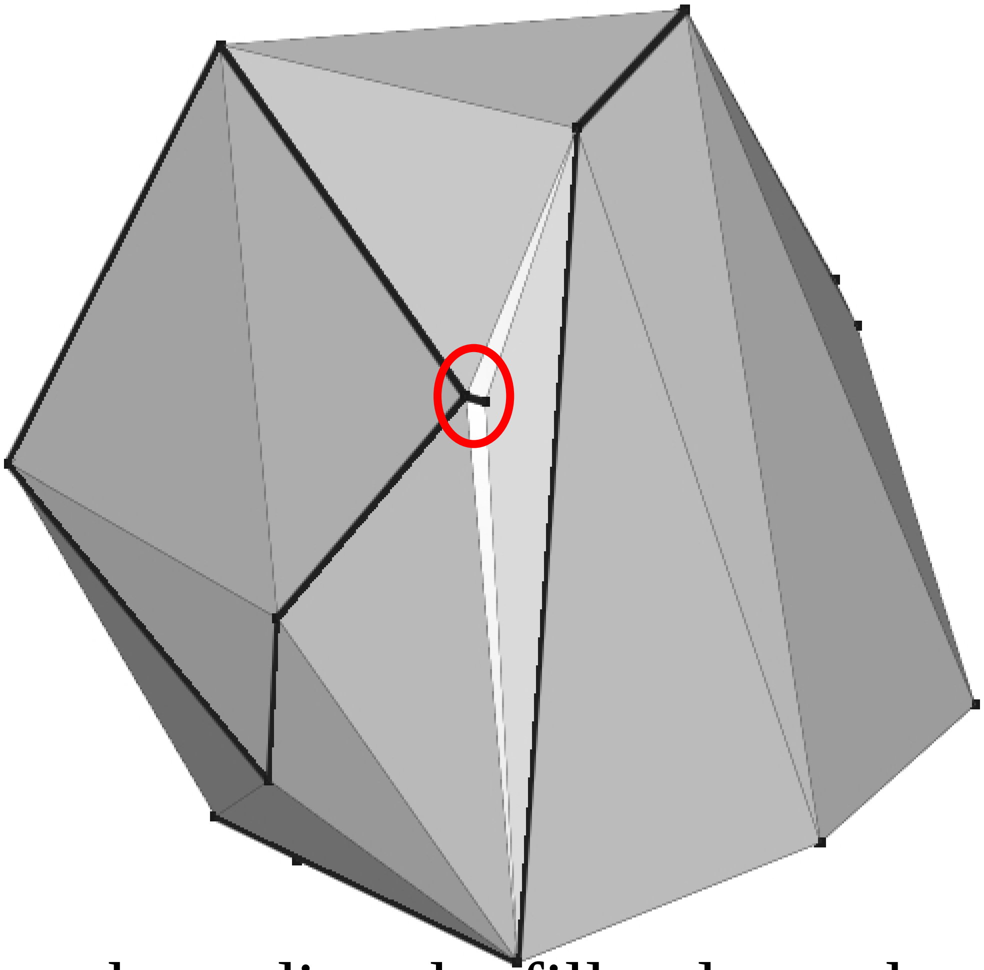

Note: The curves bounding the fillet do not have to be simple