Polygon Triangulation

O’Rourke, Chapter 1

Announcements

- Assignment 1 has been posted

Outline

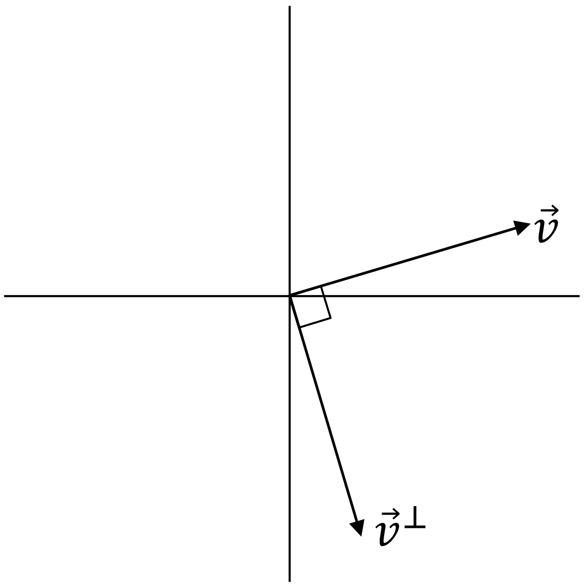
- Polygon Area
- Implementation

Notation

Given a vector $\vec{v} \in \mathbb{R}^2$, we set $\vec{v}^\perp$ to be the clockwise rotation of $\vec{v}$ by $90°$ degrees.

If $\vec{v} = (x, y)$ then we have:

$$\vec{v}^\perp = (y, -x)$$



Triangle Area

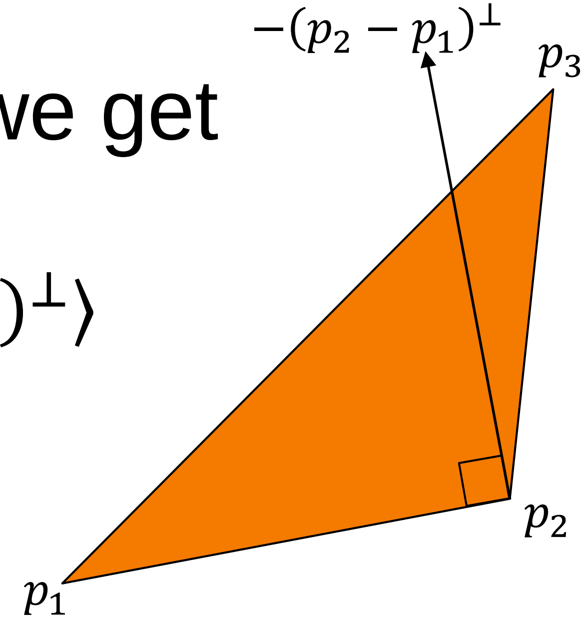
Given a triangle $T = \{p_1, p_2, p_3\}$, the area of the triangle is half the base times the height:

$$2 \cdot |T| = \|p_2 - p_1\| \cdot \left| \left\langle p_3 - p_2, \frac{(p_1 - p_2)^\perp}{\|(p_1 - p_2)^\perp\|} \right\rangle \right|$$
$$= \left| \langle p_3 - p_2, (p_1 - p_2)^\perp \rangle \right|$$

If we drop the absolute value, we get the *signed area*:

$$2 \cdot |T| = \langle p_3 - p_2, (p_1 - p_2)^\perp \rangle$$

This is positive if the vertices are in CCW order.



Triangle Area

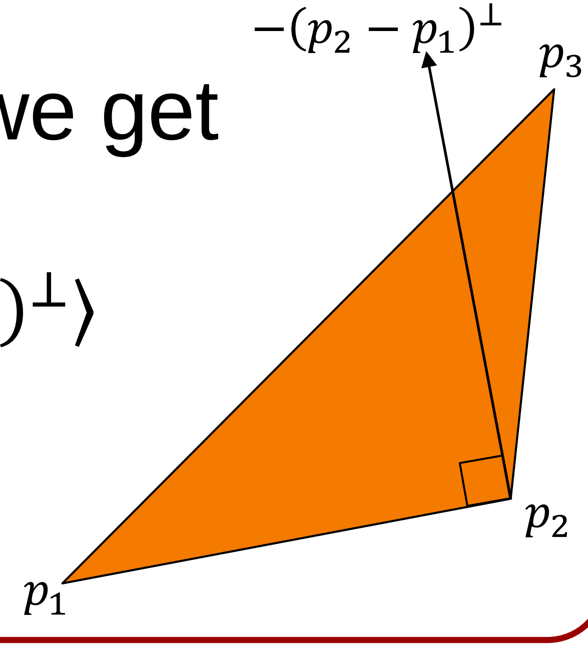
Given a triangle $T = \{p_1, p_2, p_3\}$, the area of the triangle is half the base times the height:

Unless otherwise noted, we will use $|\cdot|$ to denote the <u>signed</u> area.

If we drop the absolute value, we get the *signed area*:

$$2 \cdot |T| = \langle p_3 - p_2, (p_1 - p_2)^{\perp} \rangle$$

This is positive if the vertices are in CCW order.



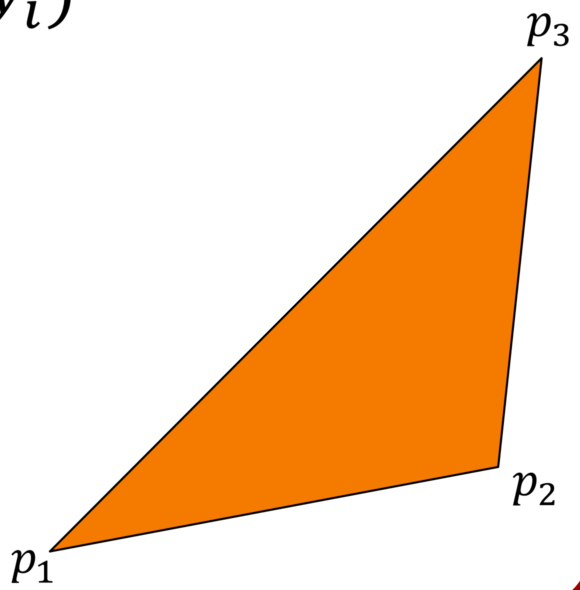
Triangle Area

$$2 \cdot |T| = \langle p_3 - p_2, (p_1 - p_2)^\perp \rangle$$

Setting $p_i = (x_i, y_i)$, this gives:

$$2 \cdot |T| = \langle (x_3 - x_2, y_3 - y_2), (y_1 - y_2, x_2 - x_1) \rangle$$

$$= \sum_{i=1}^{3} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$



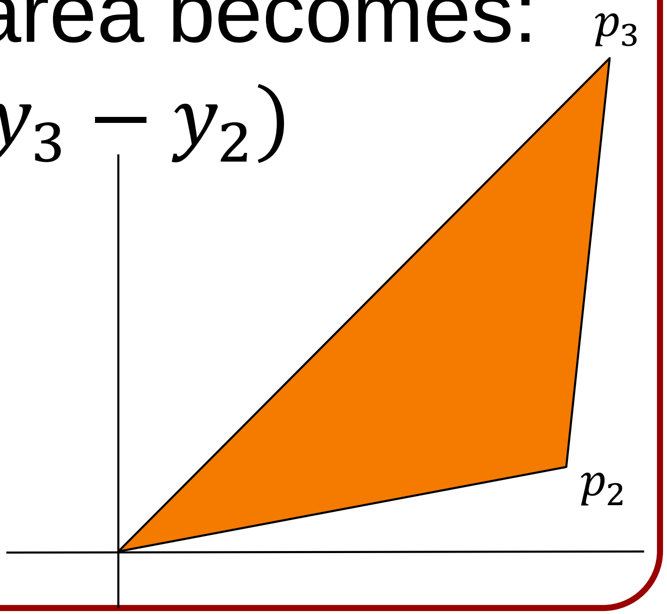
Triangle Area

$$2 \cdot |T| = \sum_{i=1}^{3} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

Note:

If p_1 is at the origin, then the area becomes:

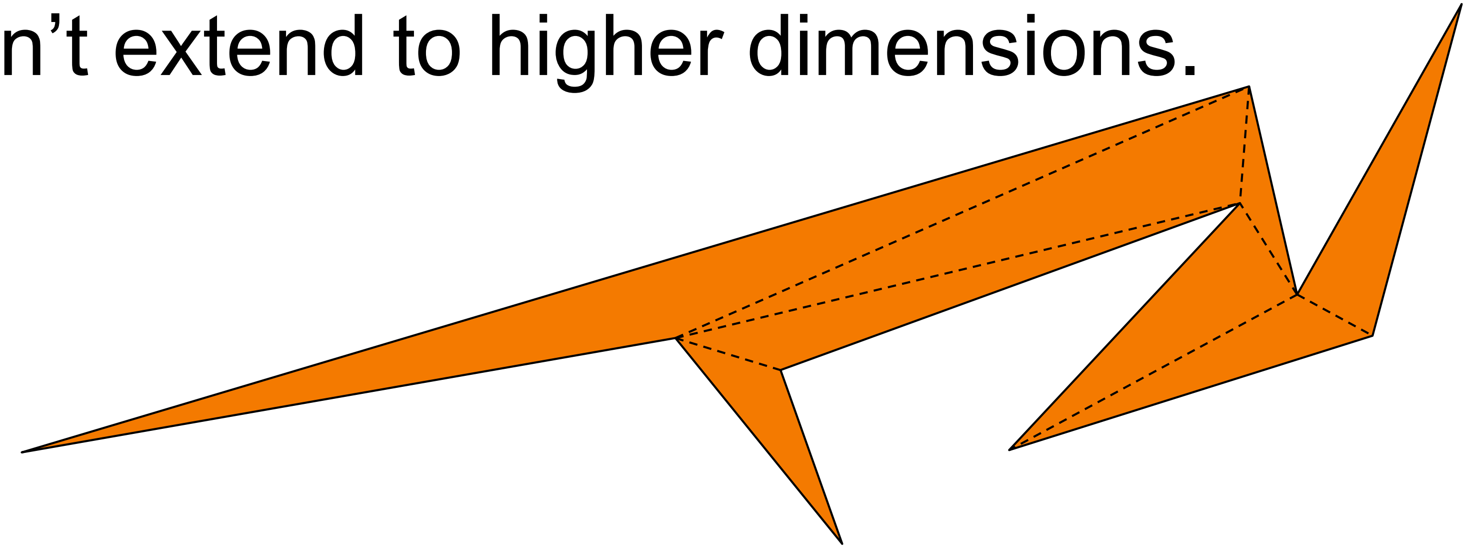
$$2 \cdot |T| = (x_3 + x_2) \cdot (y_3 - y_2)$$



Polygon Area (Take 1)

Triangulate the polygon and compute the sum of the triangle areas.

- ✗ Solving a harder problem than is required.
- ✗ Restricted to “simple” polygons.
- ✗ Doesn’t extend to higher dimensions.



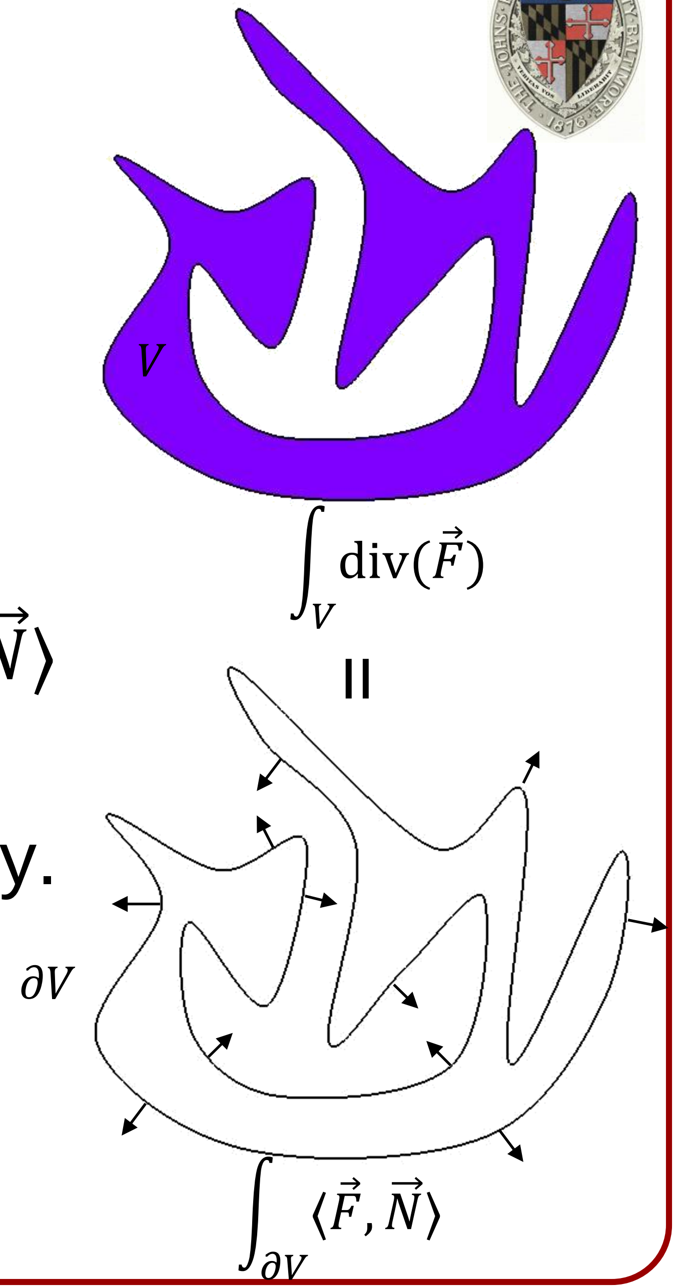
Polygon Area (Take 2)

Divergence Theorem:

Let V be a region in space with boundary ∂V, and let $\vec{F}$ be a vector field on V, then:

$$\int_V \operatorname{div}(\vec{F}) = \int_{\partial V} \langle \vec{F}, \vec{N} \rangle$$

with $\vec{N}$ the normal on the boundary.



Polygon Area (Take 2)

Divergence Theorem:

$$\int_V \mathrm{div}(\vec{F}) = \int_{\partial V} \langle \vec{F}, \vec{N} \rangle$$

Taking $\vec{F}(x, y) = (x, y)$, gives:

$$2 \int_V 1 = \int_{(x,y)\in\partial V} \langle (x, y), \vec{N} \rangle$$

$$2 \cdot |V| = \int_{\partial V} \langle p, \vec{N} \rangle \, dp$$

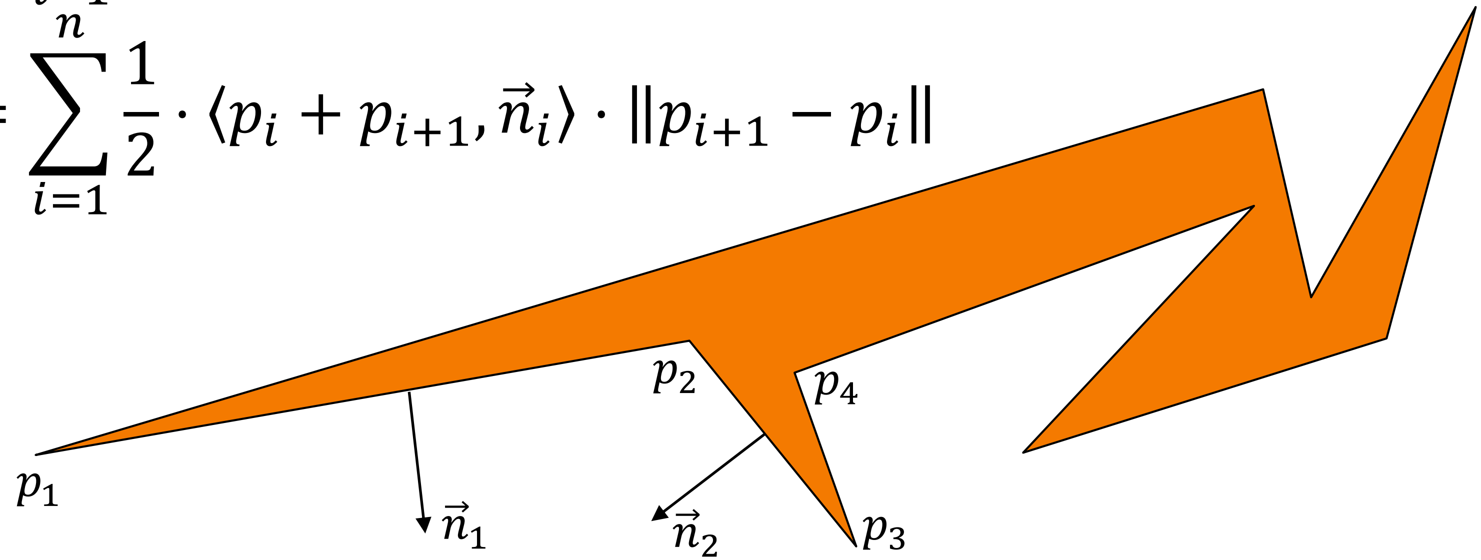
Polygon Area (Take 2)

$$2 \cdot |V| = \int_{\partial V} \langle p, \vec{N} \rangle dp$$

For a polygon $P = \{p_1, \ldots, p_n\}$, we have:

$$2 \cdot |P| = \sum_{i=1}^{n} \int_0^1 \langle (1-t) \cdot p_i + t \cdot p_{i+1}, \vec{n}_i \rangle \cdot \|p_{i+1} - p_i\| \cdot dt$$

$$= \sum_{i=1}^{n} \frac{1}{2} \cdot \langle p_i + p_{i+1}, \vec{n}_i \rangle \cdot \|p_{i+1} - p_i\|$$

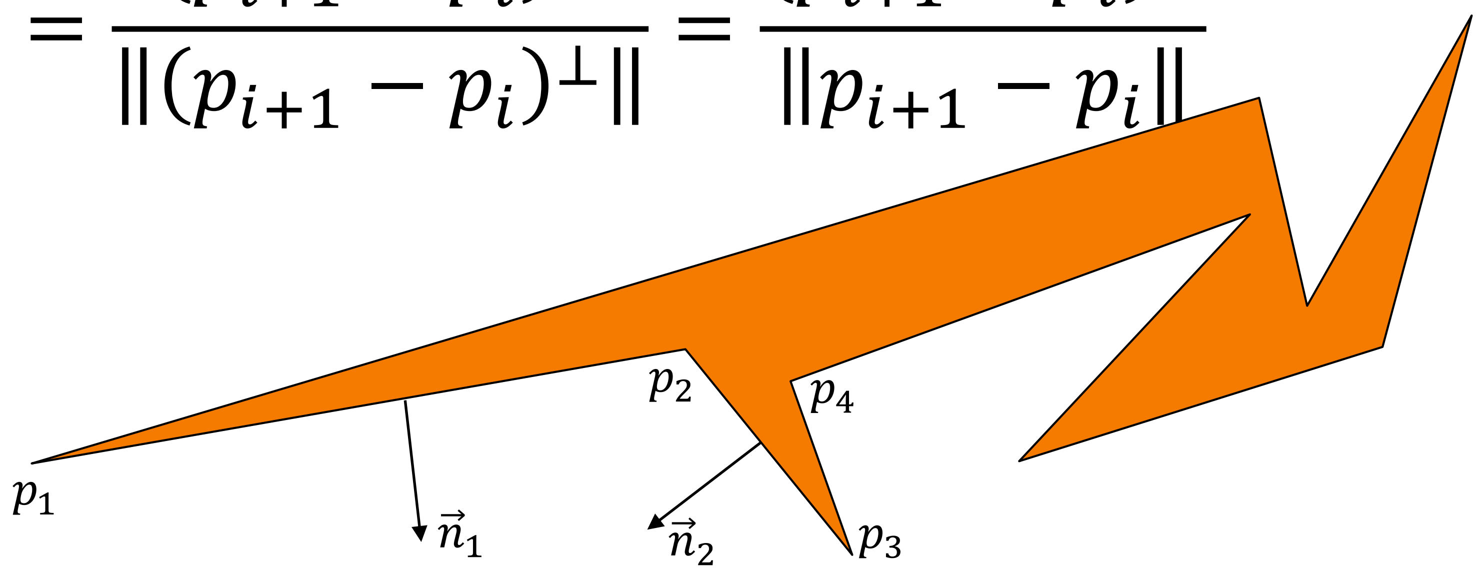


Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} \frac{1}{2} \cdot \langle p_i + p_{i+1}, \vec{n}_i \rangle \cdot \|p_{i+1} - p_i\|$$

Writing the normal as the 90° rotation of the difference (normalized):

$$\vec{n}_i = \frac{(p_{i+1} - p_i)^{\perp}}{\|(p_{i+1} - p_i)^{\perp}\|} = \frac{(p_{i+1} - p_i)^{\perp}}{\|p_{i+1} - p_i\|}$$



Polygon Area (Take 2)

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↓

$$2 \cdot |P| = \sum_{i=1}^{n} \frac{1}{2} \cdot \langle (p_{i+1} + p_i), (p_{i+1} - p_i)^\perp \rangle$$

Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} \frac{1}{2} \cdot \langle (p_{i+1} + p_i), (p_{i+1} - p_i)^{\perp} \rangle$$

Noting that $(x, y)^{\perp} = (y, -x)$ and writing $p_i = (x_i, y_i)$, we get:

$$2 \cdot |P| = \sum_{i=1}^{n} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} \frac{1}{2} \cdot \langle (p_{i+1} + p_i), (p_{i+1} - p_i)^{\perp} \rangle$$

Computing the area of a polygon requires two adds and one multiply per vertex.

$$2 \cdot |P| = \sum_{i=1}^{n} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

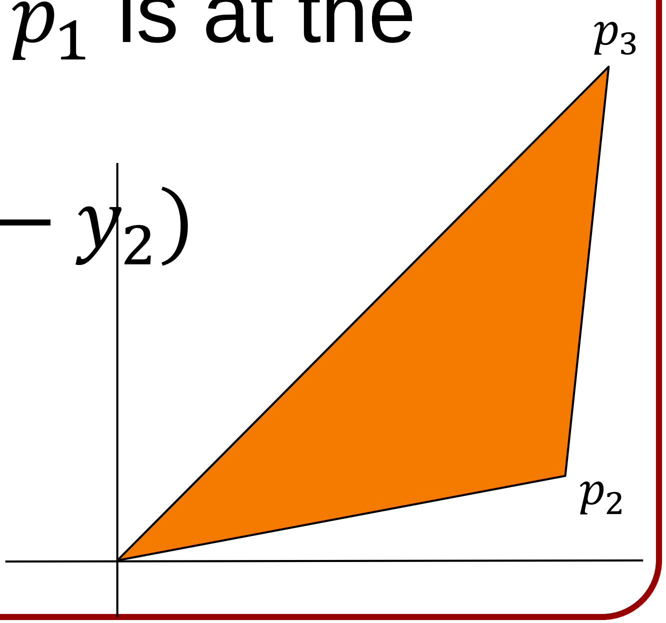
Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

Q: What's really going on?

A: For a triangle $\{p_1, p_2, p_3\}$, if p_1 is at the origin, the area is:

$$2 \cdot |T| = (x_3 + x_2) \cdot (y_3 - y_2)$$

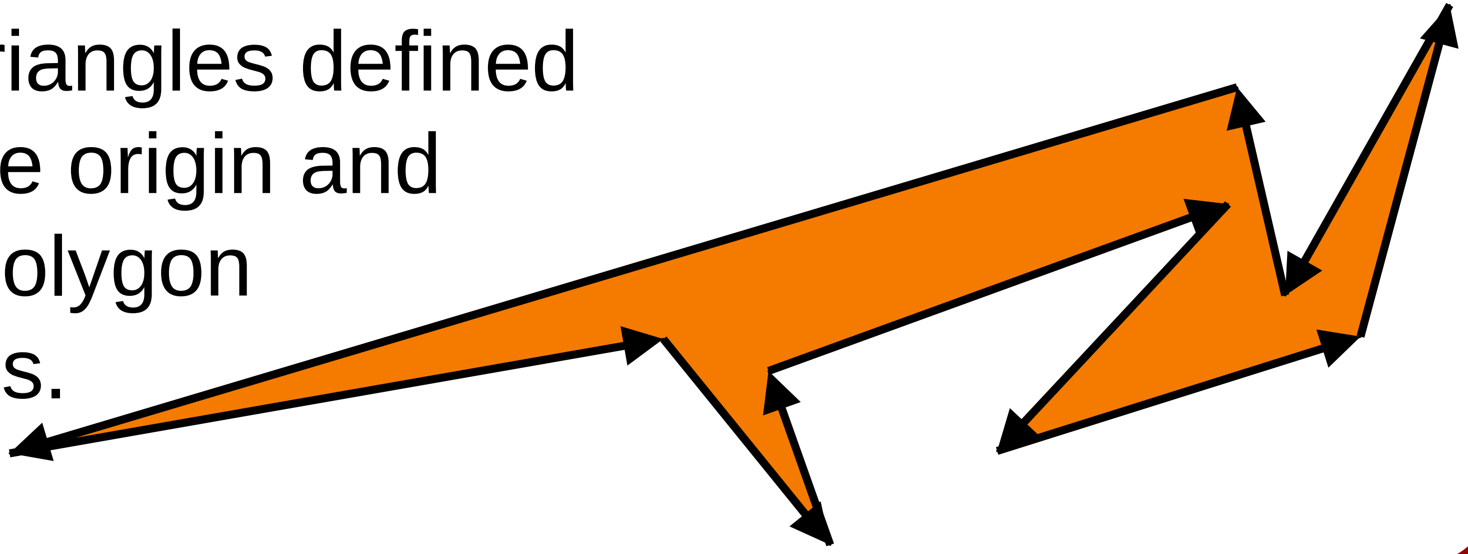


Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

Q: What's really going on?

A: Sum the areas of the triangles defined by the origin and the polygon edges.



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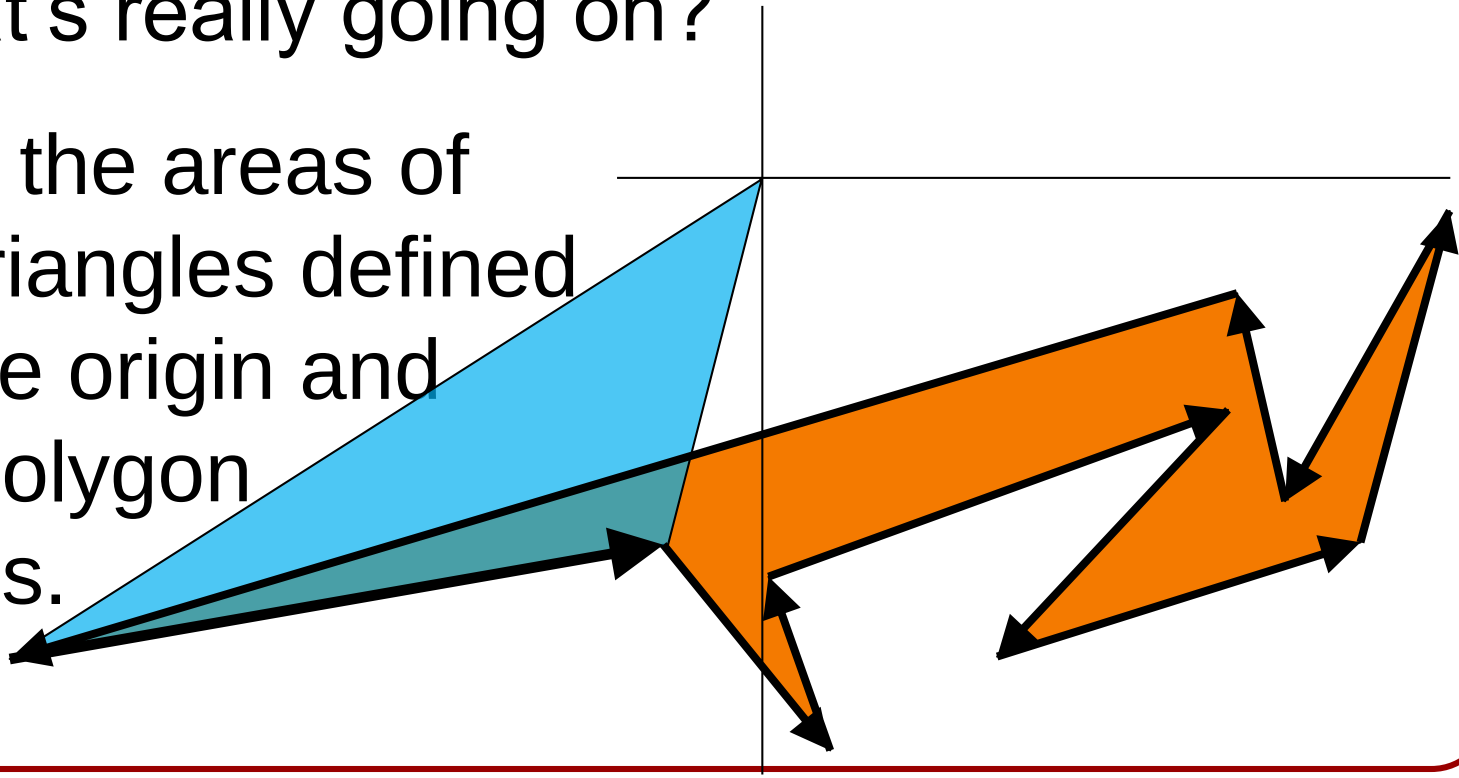

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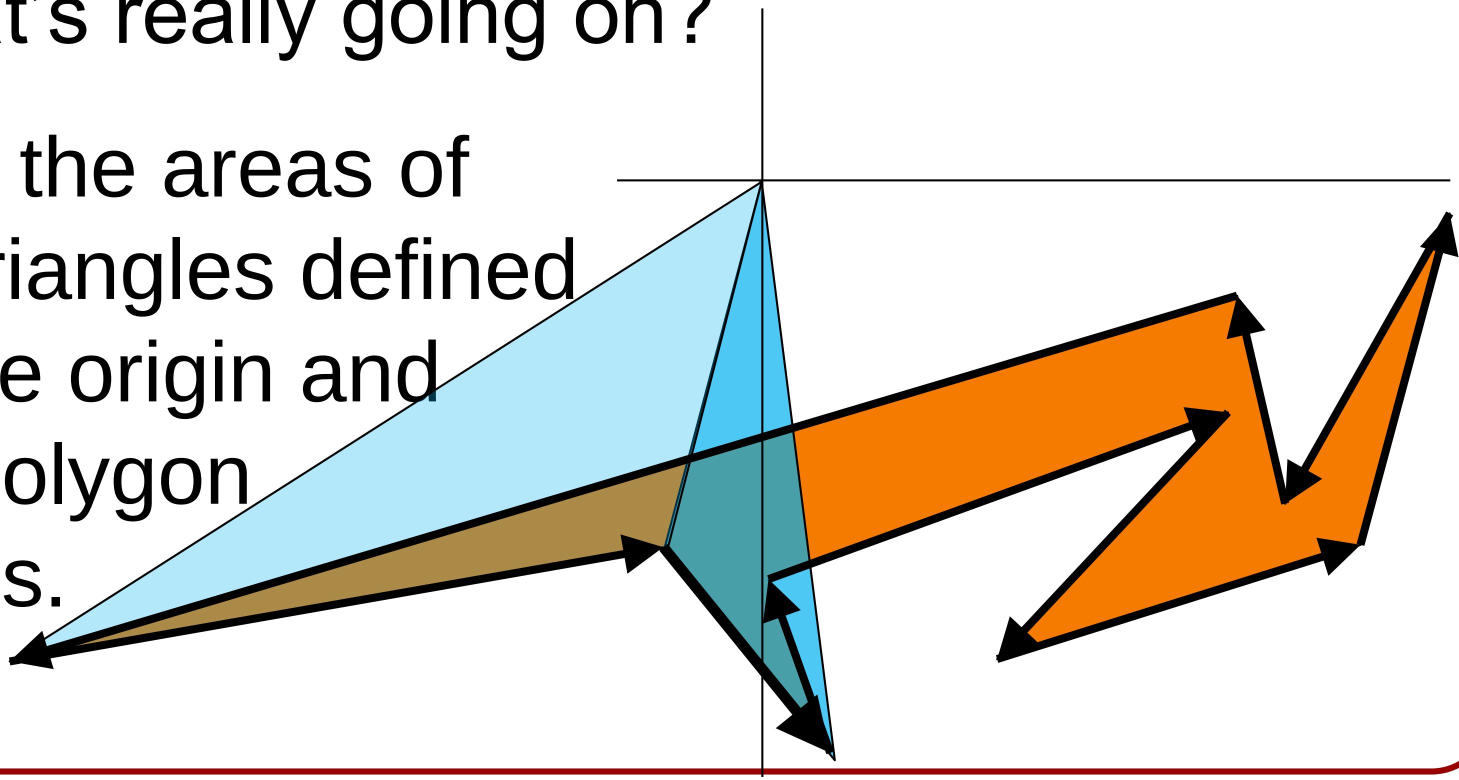

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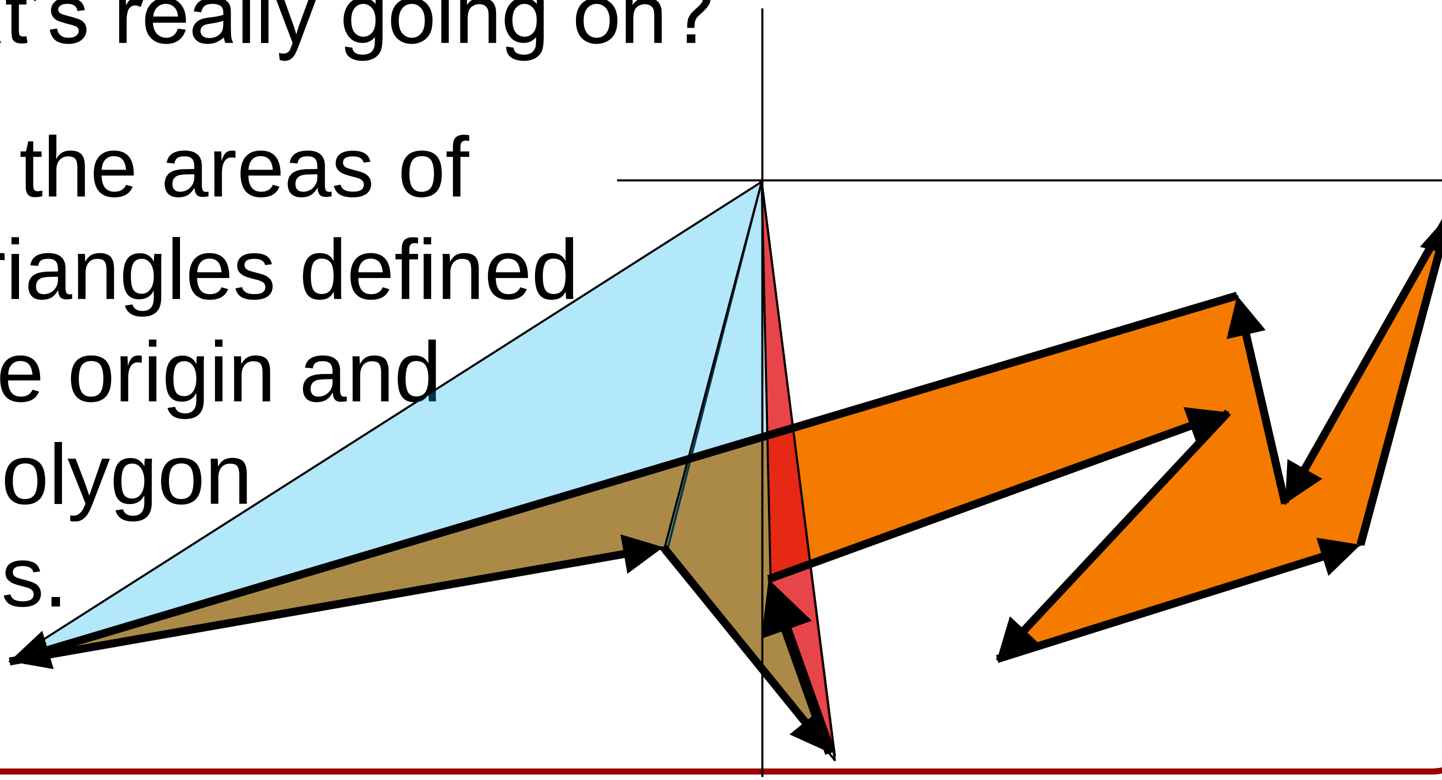

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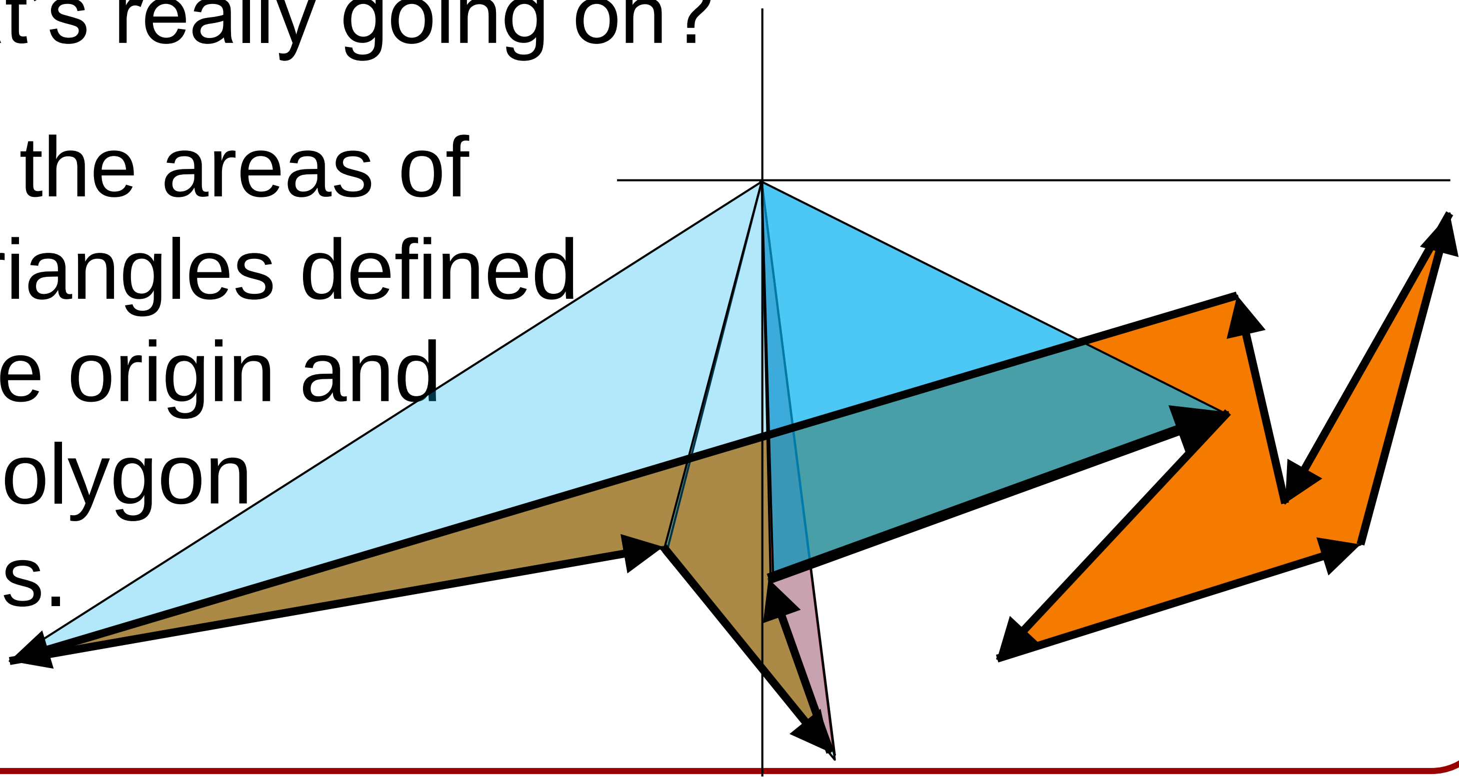

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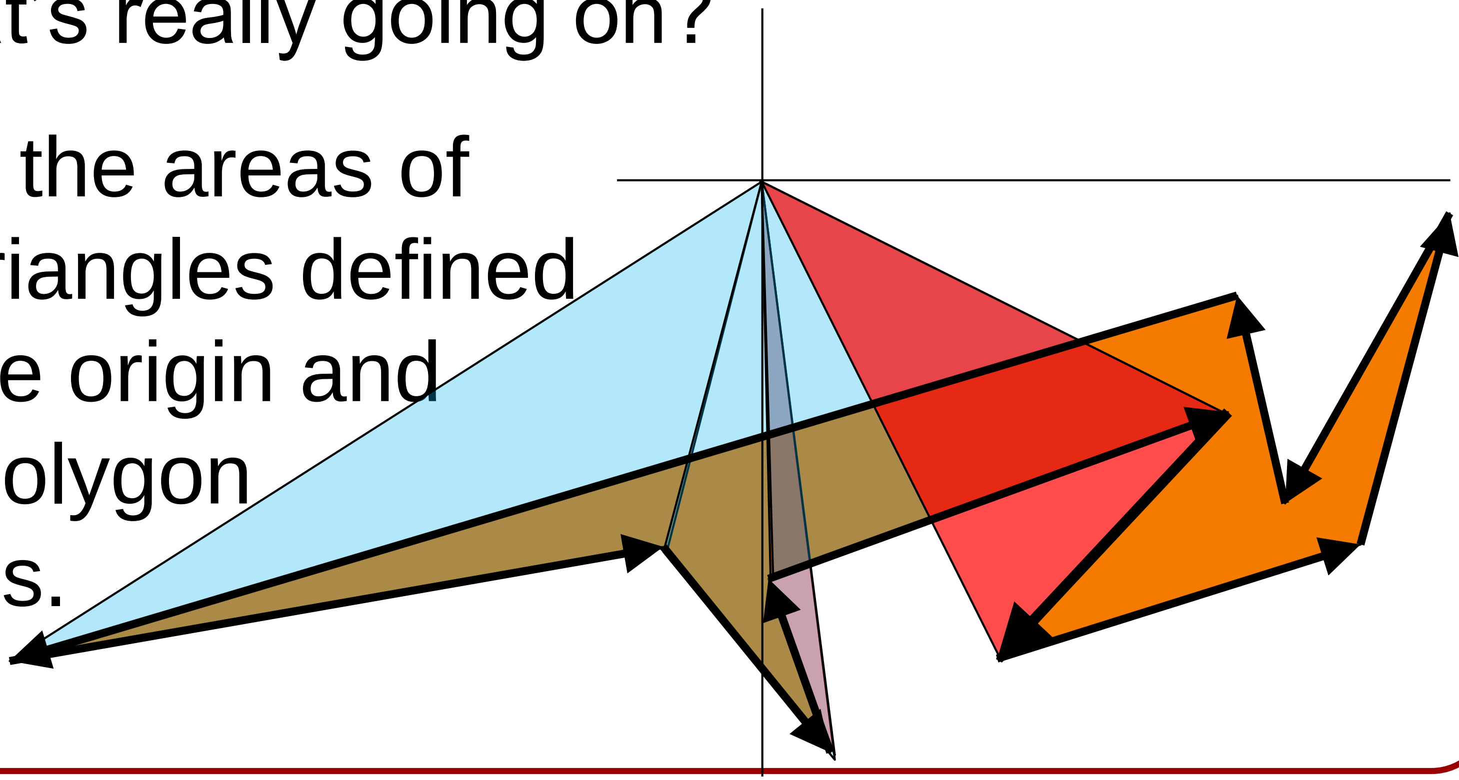

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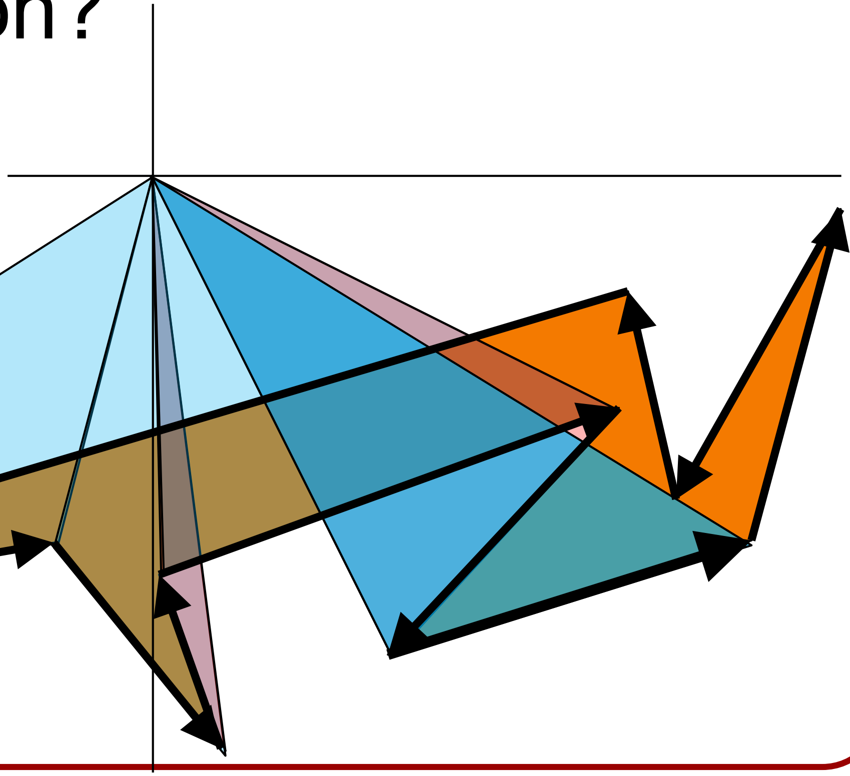

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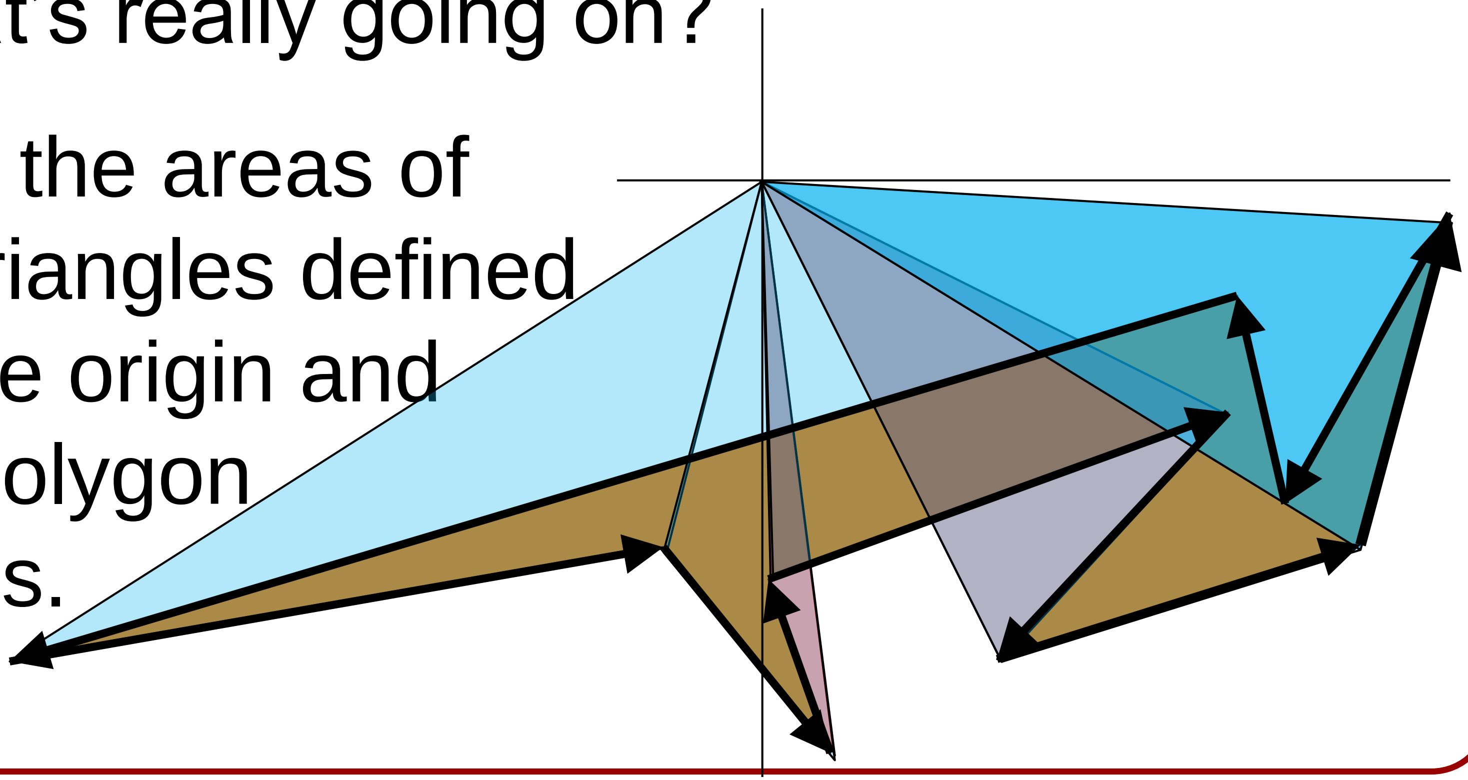

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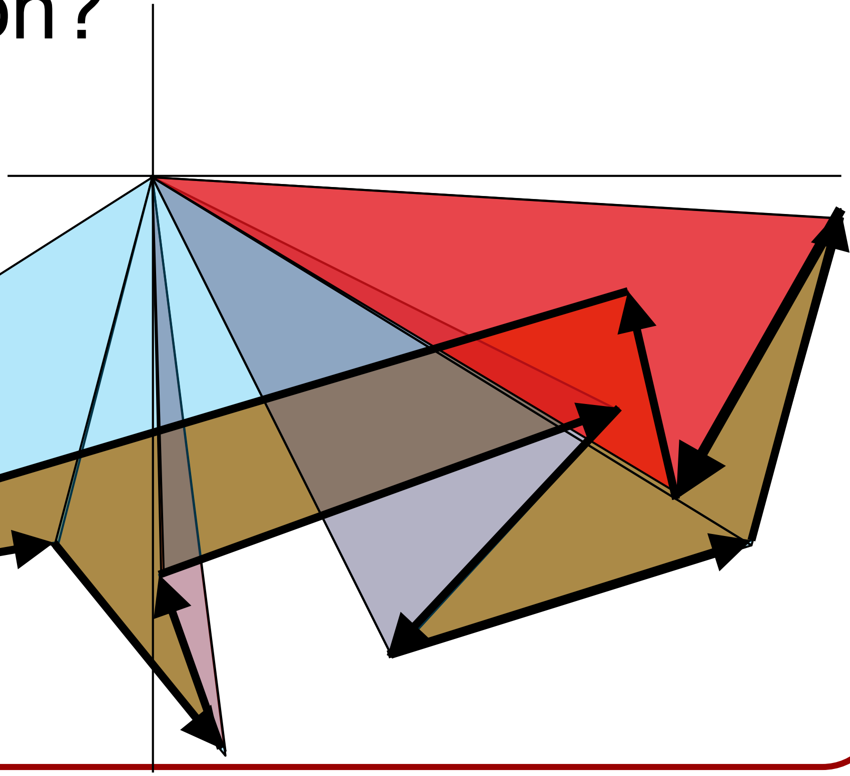

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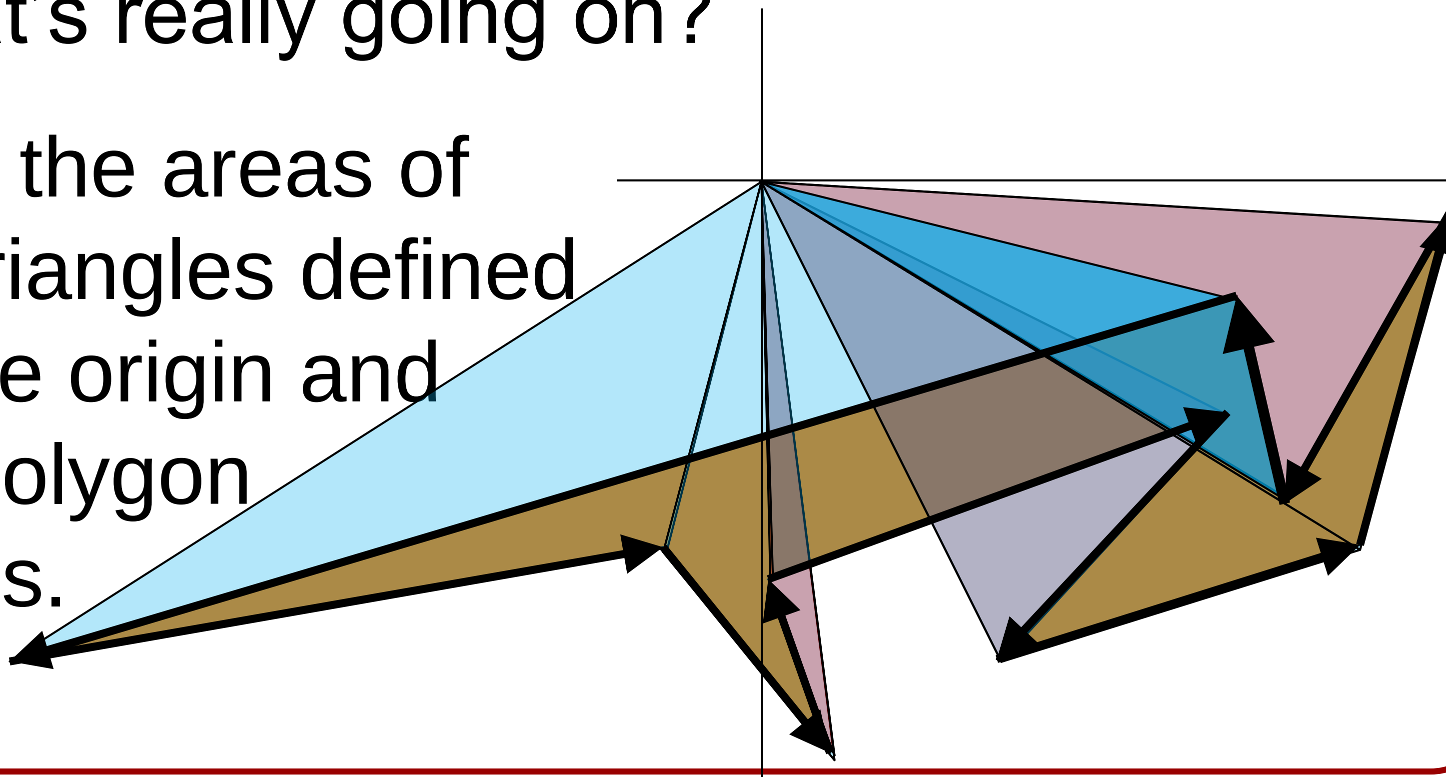

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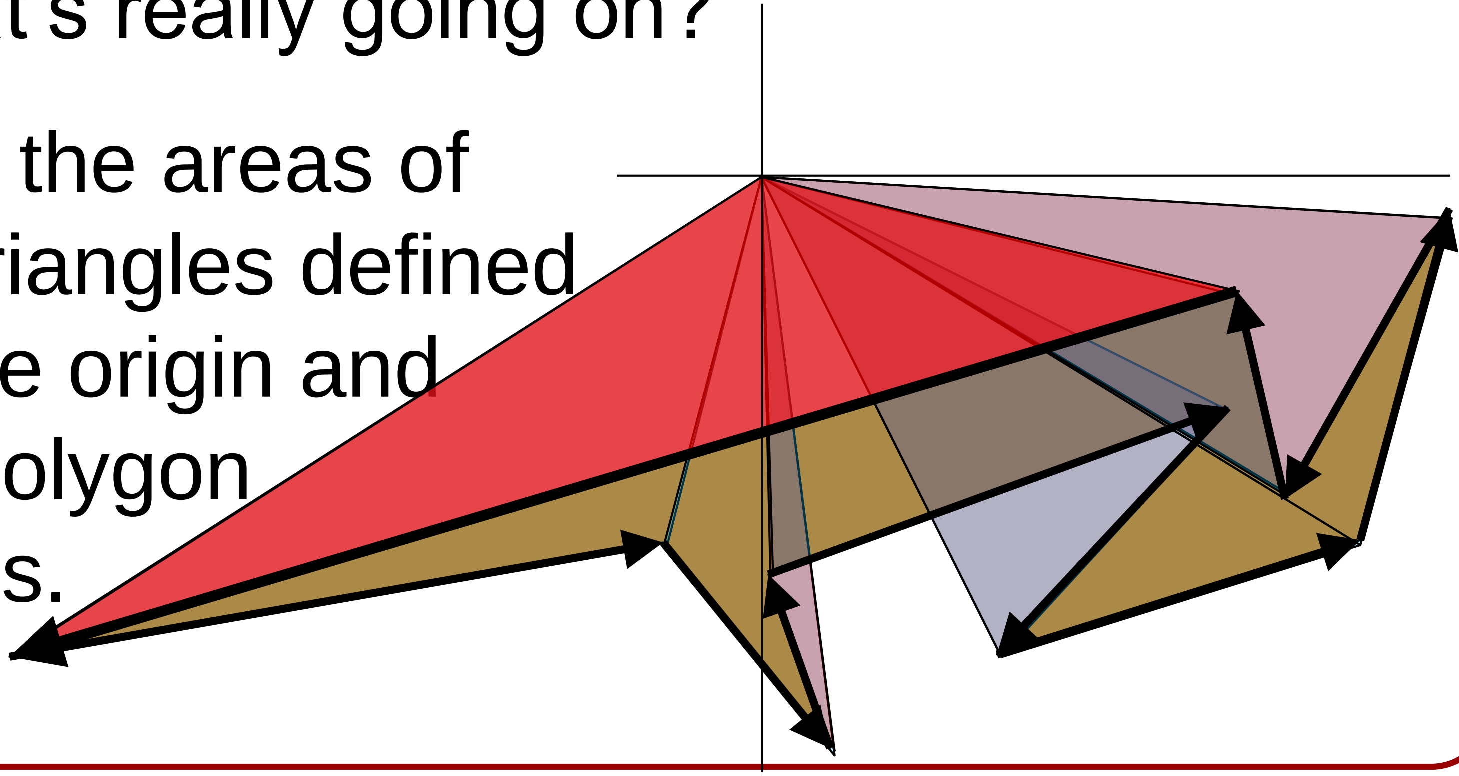
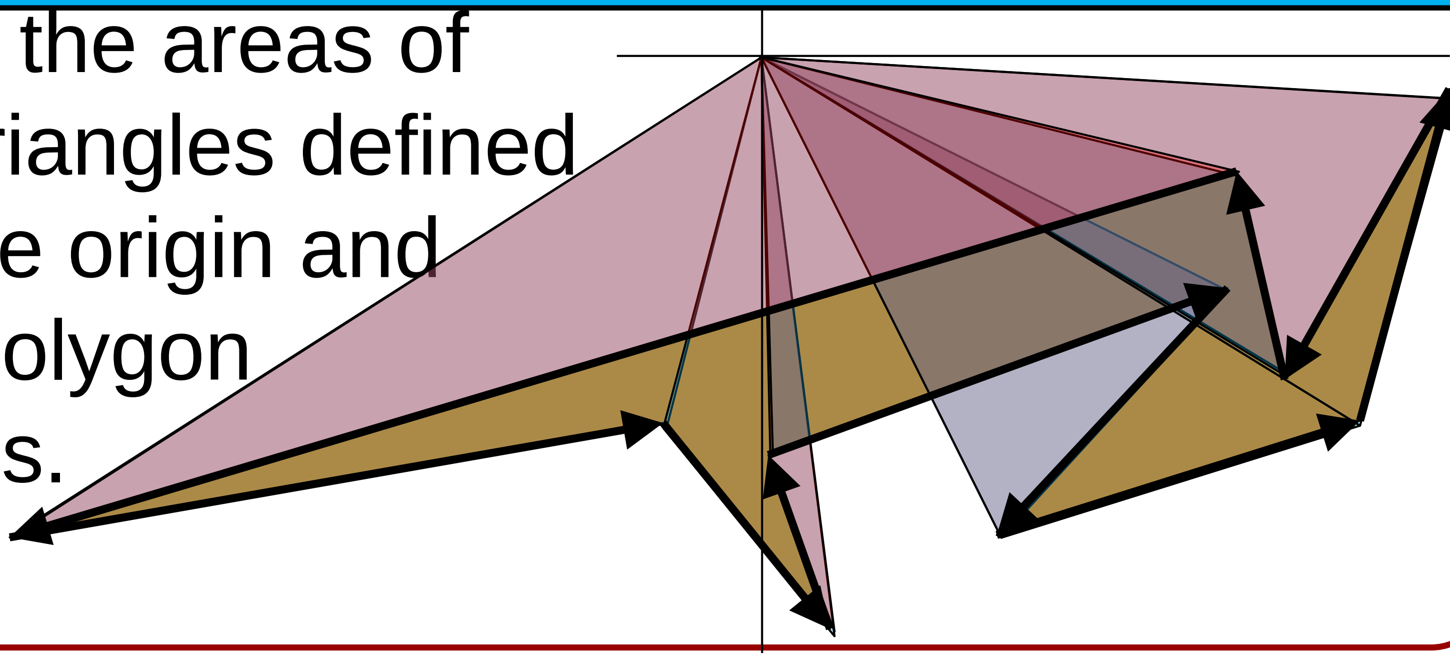


Polygon Area (Take 2)

$$2 \cdot |P| = \sum_{i=1}^{n} (x_{i+1} + x_i) \cdot (y_{i+1} - y_i)$$

In this "triangulation", the use of signed area cancels out the unwanted contribution.

A: Sum the areas of the triangles defined by the origin and the polygon edges.



Polygon Area (Take 2)

Note:

The same approach can be used to compute the volume enclosed by a triangle mesh in 3D:

- Pick a base point.
- Create tetrahedra by joining the base point to the triangles of the mesh.
- Sum the signed volumes of the tetrahedra.

Outline

- Polygon Area
- Implementation

Implementation

```
// A general structure for points with integer coordinates in
// arbitrary dimensions
template< unsigned int D >
struct Point
{
    int c[D];
    Point( void ){ memset( c , 0 , sizeof(int)*D ); }
    int &operator[]( int idx )        { return c[idx]; }
    int   operator[]( int idx ) const { return c[idx]; }
};
```

Implementation

```
long long Area2( Point< 2 > p0 , Point< 2 > p1 , Point< 2 > p2 );
{
    long long a = 0;
    a += ( (long long)( p1[0] + p0[0] ) ) * ( p1[1] - p0[1] );
    a += ( (long long)( p2[0] + p1[0] ) ) * ( p2[1] - p1[1] );
    a += ( (long long)( p0[0] + p2[0] ) ) * ( p0[1] - p2[1] );
    return a;
}
```

Implementation

```
// A circular linked-list structure for representing a vertex
// within a polygon in 2D
struct PVertex
{
    Point< 2 > p;
    PVertex *prev , *next;
    PVertex( Point< 2 > _p );
    PVertex &addBefore( Point< 2 > p );
    unsigned int size( void ) const;
    long long area2( void ) const;
    static PVertex *Remove( PVertex *v );
};
```

Implementation

```
PVertex::PVertex( Point< 2 > _p ){ p=_p , prev = next = this; }
```

Implementation

```
PVertex& PVertex::addBefore( Point< 2 > p )
{
    PVertex *v = new PVertex(p);
    v->prev = prev , v->next = this;
    prev->next = v;
    prev = v;
    return *v;
};
```

Implementation

```
static PVertex *PVertex::Remove( PVertex *v )
{
    PVertex *temp = v->prev;
    v->prev->next = v->next;
    v->next->prev = v->prev;
    delete v;
    return temp==v ? NULL : temp;
}
```

Implementation

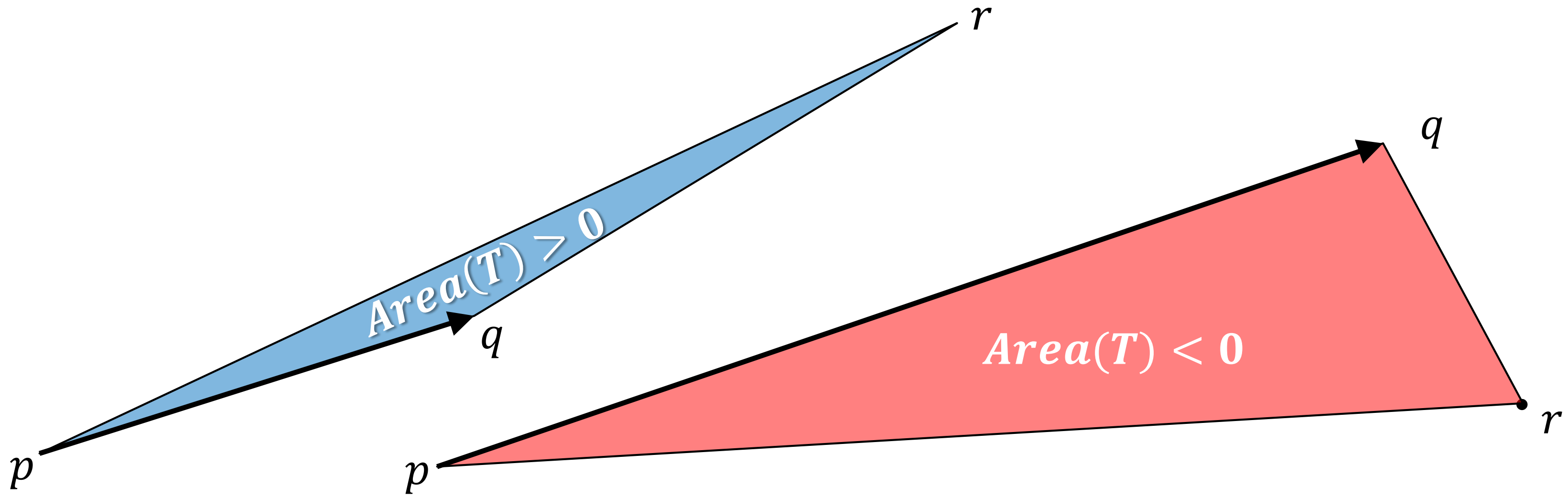
```
unsigned int PVertex::size( void ) const
{
    unsigned int s = 0;
    for( const PVertex *v=this ; ; v=v->next )
    {
        s++;
        if( v->next==this ) break;
    }
    return s;
}
```

Implementation

```
long long PVertex::area2( void ) const
{
    long long a = 0;
    for( const PVertex *v=this ; ; v=v->next )
    {
        a += Area2( Point< 2 >() , v->p , v->next->p );
        if( v->next==this ) break;
    }
    return a;
}
```

Sidedness

Given a line segment, $\overrightarrow{pq}$, and a point r, we can determine if r is to the left of, on, or to the right of $\overrightarrow{pq}$ by testing the sign of the area of triangle Δpqr.



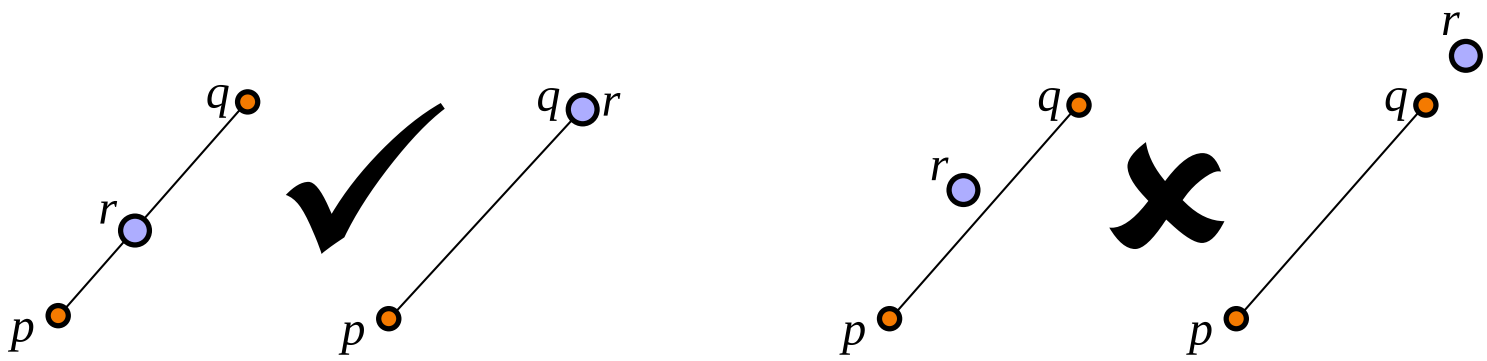
Implementation

```
bool Left( Point< 2 > p , Point< 2 >  q , Point< 2 > r )
{ return Area2( p , q , r ) > 0; }

bool LeftOn( Point< 2 > p , Point< 2 >  q , Point< 2 > r )
{ return Area2( p , q , r ) >= 0; }

bool Collinear( Point< 2 > p , Point< 2 >  q , Point< 2 > r )
{ return Area2( p , q , r ) == 0; }

bool Right( Point< 2 > p , Point< 2 >  q , Point< 2 > r )
{ return Area2( p , q , r ) < 0; }

bool RightOn( Point< 2 > p , Point< 2 >  q , Point< 2 > r )
{ return Area2( p , q , r ) <= 0; }
```

Point on Line Segment

Given a line segment, $\overline{pq}$, a point r is between p and q if:

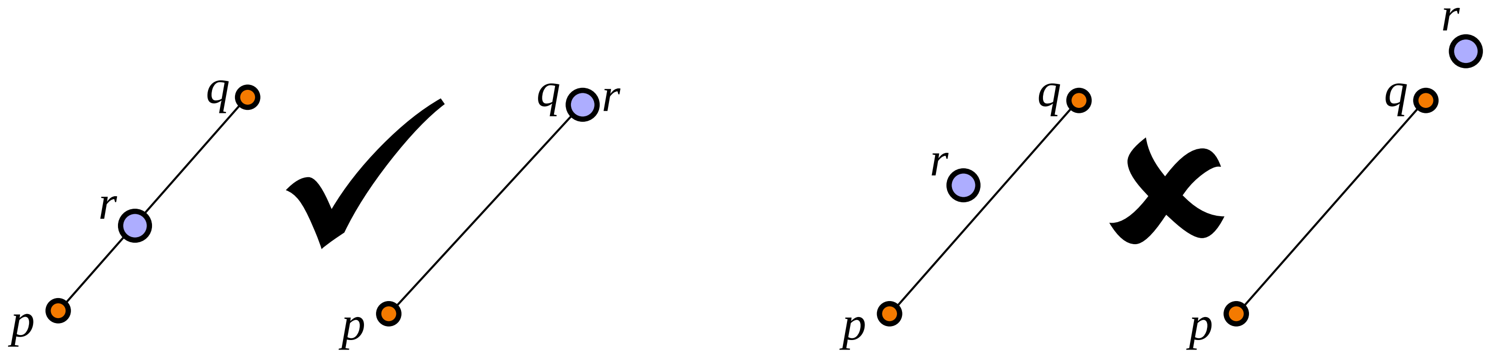
- r is on the line between p and q, and
- the x-coordinate of r is between the x-coordinates of p and q



Point on Line Segment

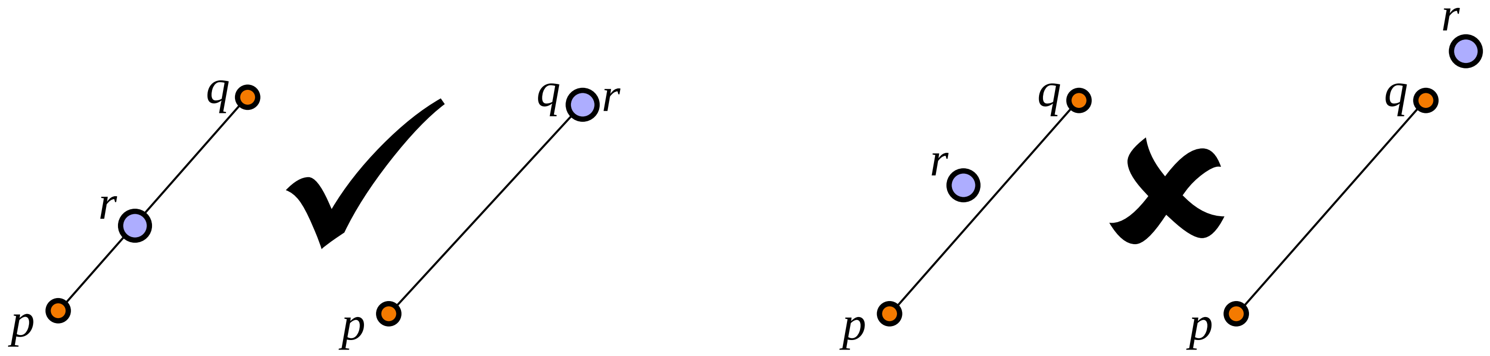
Given a line segment, $\overline{pq}$, a point r is between p and q if:

- r is on the line between p and q, and
- the x-coordinate of r is between the x-coordinates of p and q (if $\overline{pq}$ is not vertical)
- the y-coordinate of r is between the y-coordinates of p and q (if $\overline{pq}$ is vertical)



Implementation

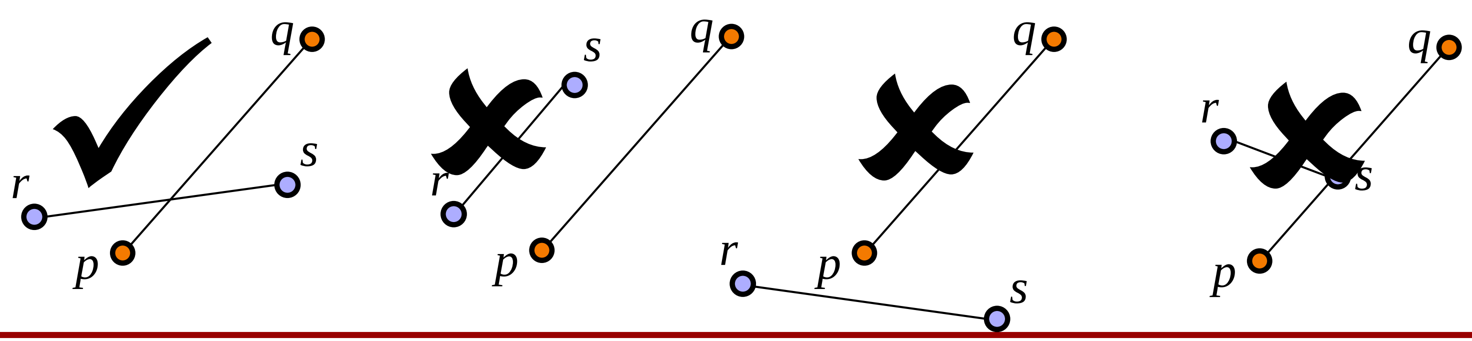
```
bool Between( Point< 2 > p , Point< 2 >  q , Point< 2 > r )
{
    if( !Collinear( p , q , r ) ) return false;
    unsigned int dir = p[0]!=q[0] ? 0 : 1;
    return
        ( p[dir] <= r[dir] && r[dir] <= q[dir] ) ||
        ( q[dir] <= r[dir] && r[dir] <= p[dir] );
}
```



Proper Intersection

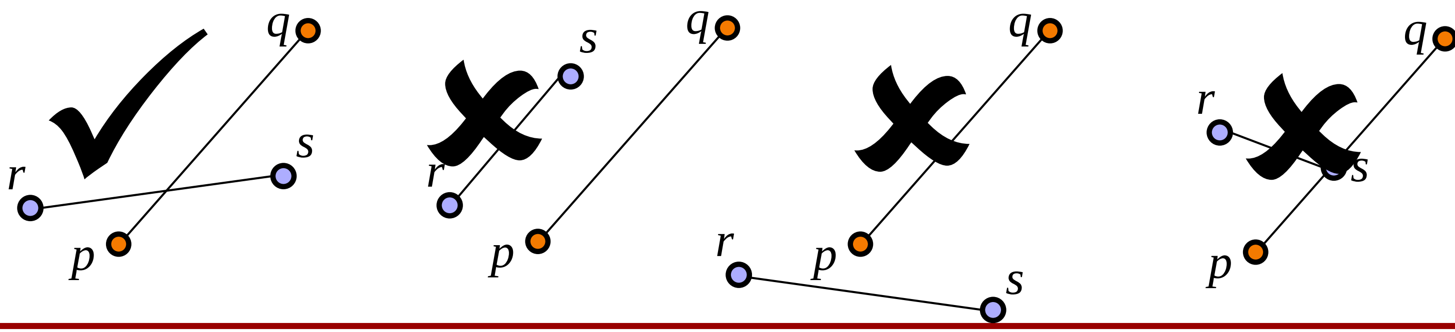
Line segments $\overline{pq}$ and $\overline{rs}$, *intersect properly* if they intersect in their interior:

- Neither r nor s is on the segment $\overline{pq}$.
- Neither p nor q is on the segment $\overline{rs}$.
- p and q are on different sides of $\overline{rs}$, and r and s are on different sides of $\overline{pq}$.



Implementation

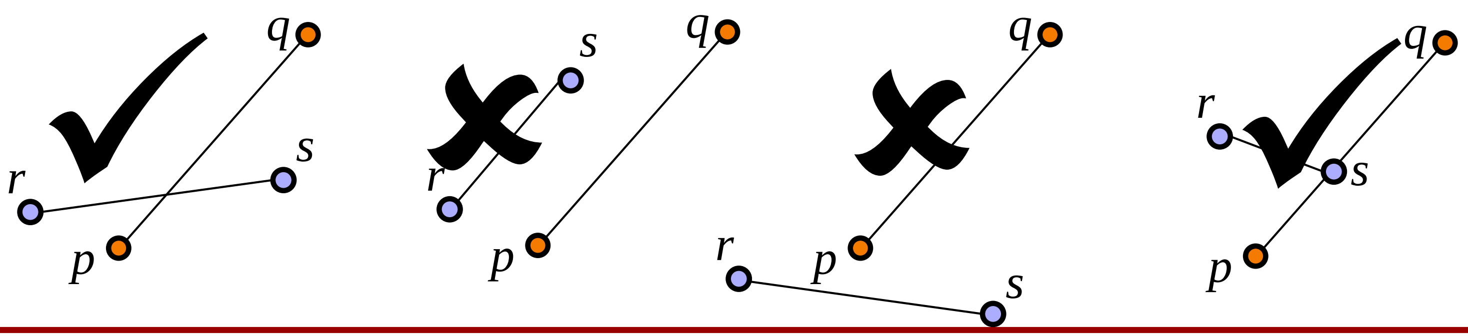
```
bool IsectProper( Point< 2 > p , Point< 2 >  q , Point< 2 > r , Point< 2 > s )
{
    if( Collinear( p , q , r ) || Collinear( p , q , s ) ) return false;
    if( Collinear( r , s , p ) || Collinear( r , s , q ) ) return false;
    if( Left( p , q , r ) == Left( p , q , s ) ) return false;
    if( Left( r , s , p ) == Left( r , s , q ) ) return false;
    return true;
}
```



Intersection

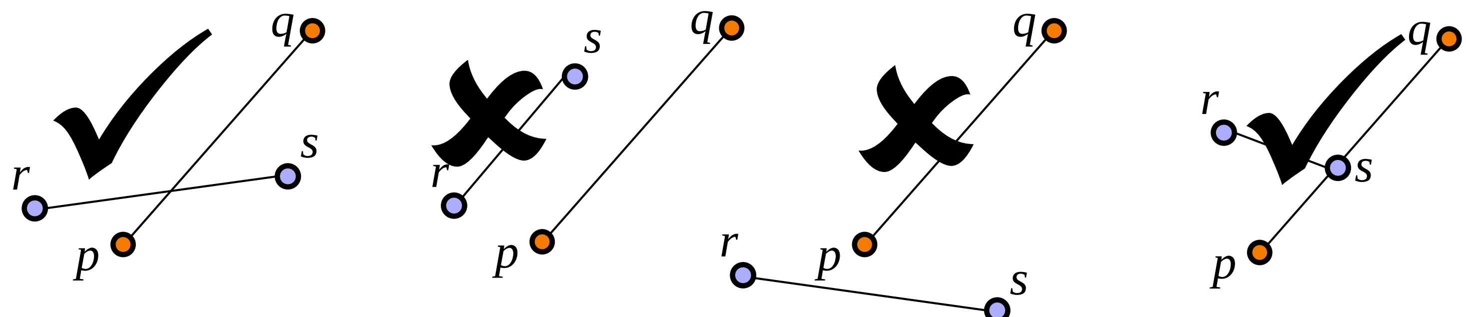
Line segments $\overline{pq}$ and $\overline{rs}$, *intersect* if:

- p is between r and s, or
- q is between r and s, or
- r is between p and q, or
- s is between p and q, or
- they intersect properly.



Implementation

```
bool Isect ( Point< 2 > p , Point< 2 >  q , Point< 2 > r , Point< 2 > s )
{
    return
        IsectProper( p , q , r , s ) ||
        Between( p , q , r ) || Between( p , q , s ) ||
        Between( r , s , p ) || Between( r , s , q );
}
```



Diagonal

Property:

Given a polygon, $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$, an edge $\overline{p_i p_j}$ is a *diagonal* if:

1. $\forall p_k \in P$ w/ $k, k+1 \notin \{i, j\}$: $\overline{p_i p_j} \cap \overline{p_k p_{k+1}} = \emptyset$
2. $\overline{p_i p_j}$ is internal to P around p_i and p_j

Edge Intersection

To test the first property:

1. $\forall p_k \in P$ w/ $k, k+1 \notin \{i, j\}$: $\overline{p_i p_j} \cap \overline{p_k p_{k+1}} = \emptyset$

we check for the intersection of $\overline{p_i p_j}$ with all edges.

Implementation

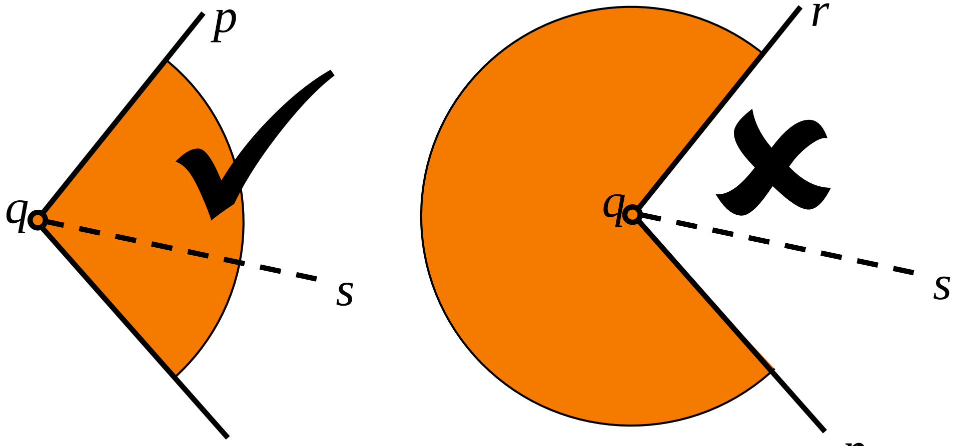
```
bool DiagonalIsect( const PVertex< 2 > *r , const PVertex< 2  > *s )
{
    for( const PVertex< 2 > *v=r ;  ; v=v->next )
    {
        if( v->prev!=r && v->prev!=s && v!=r && v!=s )
            if( Isect( r->p , s->p , v->prev->p , v->p ) ) return true;
        if( v->next==r ) break;
    }
    return false;
}
```

Complexity:
$O(n)$

Cone Interior

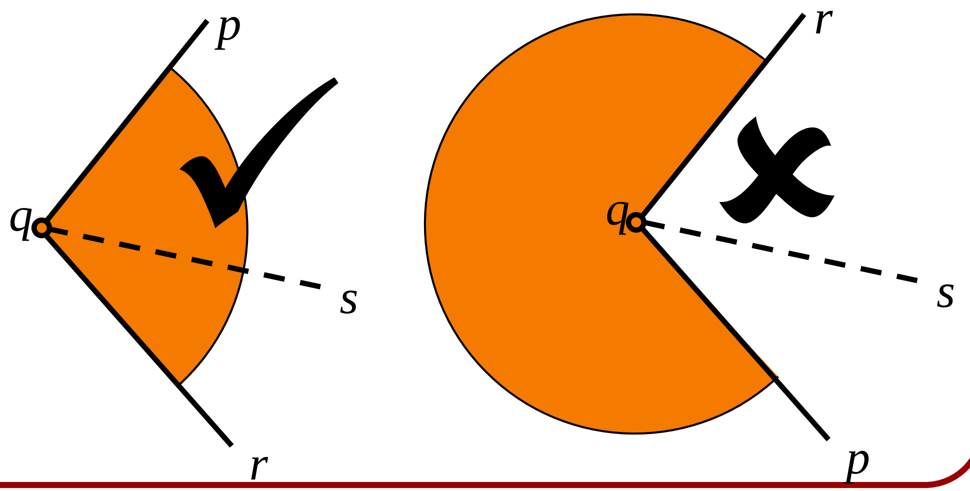
Given points p, q, and r, a line segment $\overline{qs}$ is *in the cone of* pqr if $\overline{qs}$ is strictly interior to the region swept out CW from $\overrightarrow{qp}$ to $\overrightarrow{qr}$.

- If $\angle pqr$ is a left turn (i.e. q is convex):
  s must be to the left of both $\overrightarrow{pq}$ and $\overrightarrow{qr}$.
- Otherwise:
  s cannot be to the right of or on both $\overrightarrow{pq}$ and $\overrightarrow{qr}$.



Implementation

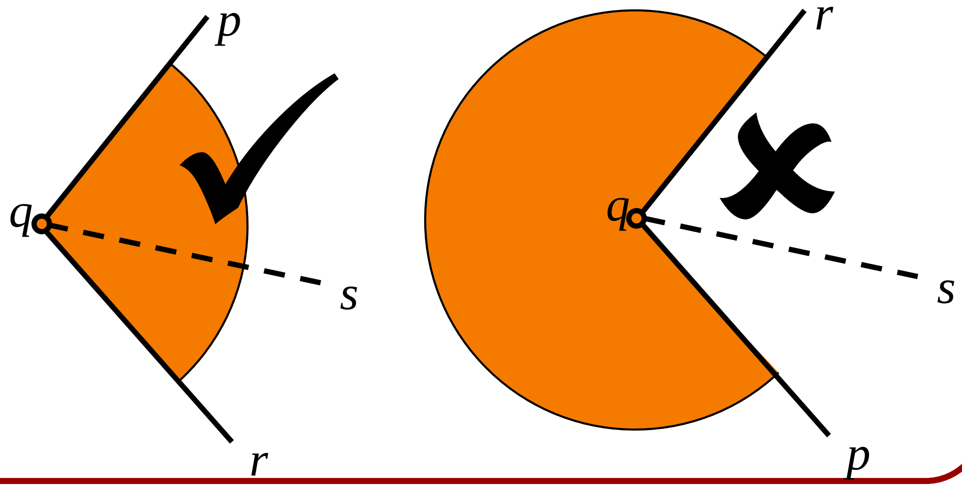
```
bool InCone( Point< 2 > p , Point< 2 >  q , Point< 2 > r , Point< 2 > s )
{
    if( Left( p , q , r ) )
        return  ( Left( p , q , s ) && Left( q , r , s ) );
    else
        return !( RightOn( p , q , s ) && RightOn( q , r , s ) );
}
```



Implementation

```
bool InCones( const PVertex< 2 >* r , const PVertex< 2  >* s )
{
    return
        InCone( r->prev->p , r->p , r->next->p , s->p ) &&
        InCone( s->prev->p , s->p , s->next->p , r->p );
}
```

Complexity:
O(1)



Implementation

```
bool IsDiagonal( const PVertex< 2 >* r , const PVertex< 2 >* s )
{
    return InCones( r , s ) && !DiagonalIsect( r , s );
}
```

Complexity:
O(n)

Trangulation (Naïve)

Recursively:

1. If the polygon is a triangle, output the triangle.
2. Otherwise
   a. Find diagonal.
   b. Split the polygon in two.

Implementation

```
void OutputTriangulation( PVertex< 2 > *poly )
{
    if( poly->size()>3 )
    {
        PVertex< 2 > *r , *s , *poly1 , *poly2;
        GetDiagonal( poly , r , s )
        SplitOnDiagonal( poly , r , s , poly1 , poly2 );
        OutputTriangulation( poly1 );
        OutputTriangulation( poly2 );
    }
    else Output( poly );
}
```

Complexity:
$O(n^4)$

Triangulation (Ear Removal)

While there are more than three vertices:

1. Find an ear p_i.
2. Output the triangle $\{p_{i-1}, p_i, p_{i+1}\}$.
3. Remove p_i from the polygon.

Note:

The ear status can only change for the vertices p_{i-1} and p_{i+1}.

Triangulation (Ear Removal)

Initialize the ear status of all vertices.

While there are more than three vertices:

1. Find an ear p_i.
2. Output the triangle $\{p_{i-1}, p_i, p_{i+1}\}$.
3. Remove p_i from the polygon.
4. Update the ear status of p_{i-1} and p_{i+1}.

Implementation

```
// Assumes member:
//        bool PVertex< 2 >::isEar
void InitEars( PVertex< 2 > *poly )
{
    for( PVertex< 2 > *v=poly ; ; v=v->next )
    {
        v->isEar = IsDiagonal( v->prev , v->next );
        if( v->next==poly ) break;
    }
}
```

Complexity:
$O(n^2)$

Implementation

```
PVertex< 2 > *ProcessEar( PVertex< 2 > *e )
{
	Output( e->prev , e , e->next );
	e->prev->isEar = IsDiagonal( e->prev->prev , e->next );
	e->next->isEar = IsDiagonal( e->prev , e->next->next );
	return PVertex< 2 >::Remove( e );
}
```

Complexity: $O(n)$

Implementation

```
void OutputTriangulation( PVertex< 2 > *poly )
{
    InitEars( poly );
    unsigned int sz = poly->size();
    while( sz>=3 )
        for( PVertex< 2 > *v=poly ; ; v=v->next )
        {
            if( v->isEar ){ poly = ProcessEar( v ) ; sz-- ; break; }
            if( v->next==poly ) break;
        }
    }
}
```

Complexity: O(n^2)