FFTs in Graphics and Vision

Characters of Representations

Outline

- Math Review
- Characters

Math Review

Notation:

Given vector spaces V and W, we define $V \oplus W$ to be the direct sum of the vector spaces:

$$V \oplus W = \{(v, w) | v \in V \text{ and } w \in W\}$$

Given linear maps $\mathcal{L}: V \to V$ and $\mathcal{M}: W \to W$, we define $\mathcal{L} \oplus \mathcal{M}$ to be the map:

$$\begin{aligned} \mathcal{L} \oplus \mathcal{M}: V \oplus W &\to V \oplus W \\ (v, w) &\mapsto \big(\mathcal{L}(v), \mathcal{M}(w)\big) \end{aligned}$$

Math Review

Definition:

Given a matrix $\mathbf{M} \in \mathbb{C}^{n\times n}$, the *trace* of $\mathbf{M}$ is the sum of the diagonal entries:

$$\mathrm{Tr}(M) = \sum_{i=1}^{n} \mathbf{M}_{ii}$$

Math Review

Properties:

1. $\text{Tr}(a \cdot \mathbf{M}) = a \cdot \text{Tr}(\mathbf{M})$

2. $\text{Tr}(\mathbf{M}) = \text{Tr}(\mathbf{M}^t)$

3. $\text{Tr}(\overline{\mathbf{M}}) = \overline{\text{Tr}(\mathbf{M})}$

4. If $\mathbf{M}$ is a unitary matrix, then:
$$\text{Tr}(\mathbf{M}^{-1}) = \text{Tr}\left(\overline{\mathbf{M}^t}\right) = \overline{\text{Tr}(\mathbf{M})}$$

5. $\text{Tr}(\mathbf{M}) = \text{Tr}(\mathbf{N}^{-1} \cdot \mathbf{M} \cdot \mathbf{N}) \quad \forall \mathbf{N} \in GL(n)$

Math Review

$$\mathrm{Tr}(\mathbf{M}) = \mathrm{Tr}(\mathbf{N}^{-1} \cdot \mathbf{M} \cdot \mathbf{N}) \quad \forall \mathbf{N} \in GL(n)$$

Properties:

If $\mathcal{L}: V \to V$ is a linear transformation, $\{v_1, \cdots, v_n\}$ is a basis for V, and $\mathbf{M} \in \mathbb{C}^{n\times n}$ is the representation of $\mathcal{L}$ in the basis, then the trace of $\mathbf{M}$ is independent of the choice of basis.

$\Rightarrow$ The "trace of a linear operator" is well-defined without a matrix representation.

Math Review

Properties:

Given vector spaces V and W and linear maps $\mathcal{L}: V \rightarrow V$ and $\mathcal{M}: W \rightarrow W$ we have:

$$\text{Tr}(\mathcal{L} \oplus \mathcal{M}) = \text{Tr}(\mathcal{L}) + \text{Tr}(\mathcal{M})$$

Math Review

Notation:

Given the space of n-dimensional vectors, we denote by $\mathbf{e}^i \in \mathbb{C}^n$ the vector with a "1" in the i-th entry and "0" everywhere else:

$$\mathbf{e}^i_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Given the space of $n \times n$ matrices, we denote by $\mathbf{E}^{ij} \in \mathbb{C}^{n \times n}$ the matrix with "1" in the i-th column and j-th row and "0" everywhere else:

$$\mathbf{E}^{ij}_{ab} = \delta_{ia} \cdot \delta_{jb}$$

Math Review

Notation:

Given a matrix $\mathbf{N} \in \mathbb{C}^{n \times n}$, we have:

$$\begin{aligned}\left(\mathbf{E}^{ij} \cdot \mathbf{N}\right)_{ab} &= \sum_{c=0}^{n} \mathbf{E}_{cb}^{ij} \cdot \mathbf{N}_{ac} \\ &= \sum_{c=0}^{n} \delta_{ic} \cdot \delta_{jb} \cdot \mathbf{N}_{ac} \\ &= \delta_{jb} \cdot \mathbf{N}_{ai}\end{aligned}$$

This is the matrix whose j-th row contains the i-th row of $\mathbf{N}$.

Math Review

Notation:

Given matrices $\mathbf{M}, \mathbf{N} \in \mathbb{C}^{n\times n}$, we have:

$$\begin{aligned}\left(\mathbf{M}\cdot\mathbf{E}^{ij}\cdot\mathbf{N}\right)_{ab} &= \sum_{c=0}^{n}\mathbf{M}_{cb}\cdot\left(\mathbf{E}^{ij}\cdot\mathbf{N}\right)_{ac}\\ &= \sum_{c=0}^{n}\mathbf{M}_{cb}\cdot\delta_{jc}\cdot\mathbf{N}_{ai}\\ &= \mathbf{M}_{jb}\cdot\mathbf{N}_{ai}\end{aligned}$$

This is the matrix whose (a,b)-th entry is the product of the (j,b)-th entry of $\mathbf{M}$ and the (a,i)-th entry of $\mathbf{N}$.

Functions on Groups

Note:

Given a representation (ρ, V) and given a basis $\{v_1, \cdots, v_n\}$ we can represent each ρ_g as a matrix $\mathbf{M}^\rho(g)$, with coefficients that are functions:

$$\mathbf{M}^\rho_{ij}: G \to \mathbb{C}$$

Since $\mathbf{M}^\rho(g)$ is unitary, we have:

$$\mathbf{M}^\rho(g^{-1}) = \left(\mathbf{M}^\rho(g)\right)^{-1} = \overline{\left(\mathbf{M}^\rho(g)\right)^t}$$

Functions on Groups

Notation:

Given the space of complex-valued functions on G, we can define a scalar product on this space by setting:

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \cdot \overline{\psi(g)}$$

for any functions $\phi, \psi: G \to \mathbb{C}$.

Math Review

Definition:

Given representations (ρ_1, V) and (ρ_2, W) of a group G, a linear map $\mathcal{L}: V \to W$ is G-*linear* if:

$$\rho_2(g) \circ \mathcal{L} = \mathcal{L} \circ \rho_1(g) \quad \forall g \in G$$

$$\Updownarrow$$

$$\mathcal{L} = \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g) \quad \forall g \in G$$

Math Review

Schur's Lemma:

Given irreducible representations (ρ_1, V) and (ρ_2, W) of a group G, if $\mathcal{L}: V \to W$ is G-linear then:

1. If ρ_1 and ρ_2 are not isomorphic, then $\mathcal{L} = 0$.
2. If $V = W$ and $\rho_1 = \rho_2$, then $\mathcal{L} = \lambda \cdot \mathrm{Id}$.

Math Review

Maschke's Theorem:

If W is a sub-representation of V, then the space $W^\perp$ will also be a sub-representation of V.

Corollary:

Given a representation (ρ, V) we can decompose V into a direct-sum of irreducible representations:

$$V = \bigoplus_i V_i$$

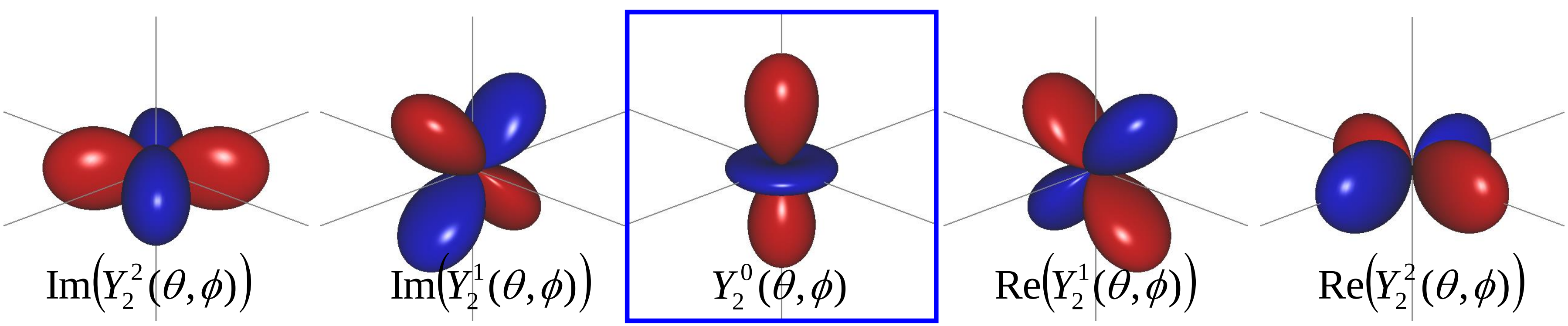
Note that an irreducible representation may occur with multiplicity (i.e. V_i may be isomorphic to V_j).

Math Review

Recall:

Convolving with the l-th zonal harmonic is the same as scaling the l-th spherical frequency:

$$\langle \rho_{R(\theta,\phi)}\left(Y_l^0\right), \overline{Y_{l'}^m}, \rangle = \delta_{l,l'} \cdot \lambda_l \cdot Y_l^m$$



Outline

- Math Review
- Characters

G-Linear Maps by Averaging

Given representations (ρ_1, V) and (ρ_2, W) of a group G, and given a linear map $\mathcal{L}: V \to W$, we can construct a G-linear map $\mathcal{L}^0$ by averaging:

$$\mathcal{L}^0 = \frac{1}{|G|} \sum_{g \in G} \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g)$$

G-Linear Maps by Averaging

Proof:

For any $h \in G$ we have:

$$\begin{aligned}
\rho_2(h^{-1}) \circ \mathcal{L}^0 \circ \rho_1(h) &= \rho_2(h^{-1}) \circ \left(\frac{1}{|G|} \sum_{g \in G} \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g) \right) \circ \rho_1(h) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_2(h^{-1} \cdot g^{-1}) \circ \mathcal{L} \circ \rho_1(g \cdot h) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_2\big((g \cdot h)^{-1}\big) \circ \mathcal{L} \circ \rho_1(g \cdot h) \\
&= \frac{1}{|G|} \sum_{g \in Gh} \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_2(g^{-1}) \circ \mathcal{L} \circ \rho_1(g) \\
&= \mathcal{L}^0
\end{aligned}$$

G-Linear Maps by Averaging

$$\mathcal{L}^0 = \frac{1}{|G|}\sum_{g\in G}\rho_2(g^{-1})\circ\mathcal{L}\circ\rho_1(g)$$

By Schur's Lemma:

1. If ρ_1 and ρ_2 are not isomorphic, then $\mathcal{L}^0 = 0$.
2. If $V = W$ and $\rho_1 = \rho_2$, then $\mathcal{L}^0 = \lambda\cdot\mathrm{Id}$.

   Taking the trace:

$$\mathrm{Tr}(\mathcal{L}^0) = \mathrm{Tr}\left(\frac{1}{|G|}\sum_{g\in G}\rho_1(g^{-1})\circ\mathcal{L}\circ\rho_1(g)\right)$$

$$\mathrm{Tr}(\lambda\cdot\mathrm{Id.}) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}\left(\rho_1^{-1}(g)\circ\mathcal{L}\circ\rho_1(g)\right)$$

$$\lambda\cdot\mathrm{Tr}(\mathrm{Id.}) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(\mathcal{L})$$

$$\lambda\cdot\dim(V) = \mathrm{Tr}(\mathcal{L})$$

G-Linear Maps by Averaging

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1. If ρ_1 and ρ_2 are not isomorphic, then $\mathcal{L}^0 = 0$.
2. If $V = W$ and $\rho_1 = \rho_2$, then $\mathcal{L}^0 = \frac{\mathrm{Tr}(\mathcal{L})}{\dim(V)}\cdot \mathrm{Id}$.

Coefficient Orthogonality

$$\mathcal{L}^0 = \frac{1}{|G|}\sum_{g\in G}\rho_2(g^{-1})\circ\mathcal{L}\circ\rho_1(g)$$

Choosing a basis and taking $\mathcal{L} = \mathbf{E}^{ij}$ we get:

1. If ρ_1 and ρ_2 are not isomorphic, then $\mathcal{L}^0 = 0$:

$$0 = \frac{1}{|G|}\sum_{g\in G}\rho_2(g^{-1})\circ\mathbf{E}^{ij}\circ\rho_1(g)$$

$$\Downarrow$$

$$\begin{aligned}(0)_{ab} &= \frac{1}{|G|}\sum_{g\in G}\mathbf{M}^{\rho_1}_{ai}(g)\cdot\mathbf{M}^{\rho_2}_{jb}(g^{-1})\\ &= \frac{1}{|G|}\sum_{g\in G}\mathbf{M}^{\rho_1}_{ai}(g)\cdot\overline{\mathbf{M}^{\rho_2}_{bj}(g)}\end{aligned}$$

$$\Updownarrow$$

$$0 = \langle\mathbf{M}^{\rho_1}_{ai},\mathbf{M}^{\rho_2}_{bj}\rangle$$

Coefficient Orthogonality

$$\mathcal{L}^0 = \frac{1}{|G|}\sum_{g\in G}\rho_2(g^{-1})\circ\mathcal{L}\circ\rho_1(g)$$

Choosing a basis and taking $\mathcal{L} = \mathbf{E}^{ij}$ we get:

2. If $V = W$ and $\rho_1 = \rho_2$, then $\mathcal{L}^0 = \frac{\mathrm{Tr}(\mathbf{E}^{ij})}{\dim(V)}\cdot \mathrm{Id.}$:

$$\frac{\mathrm{Tr}(\mathbf{E}^{ij})}{\dim(V)}\cdot \mathrm{Id.} = \frac{1}{|G|}\sum_{g\in G}\rho(g^{-1})\circ\mathbf{E}^{ij}\circ\rho(g)$$

$$\Downarrow$$

$$\left(\frac{\delta_{ij}}{\dim(V)}\cdot \mathrm{Id.}\right)_{ab} = \frac{1}{|G|}\sum_{g\in G}\mathbf{M}^{\rho}_{ai}(g)\cdot\overline{\mathbf{M}^{\rho}_{bj}(g)}$$

$$\Updownarrow$$

$$\frac{\delta_{ij}\cdot\delta_{ab}}{\dim(V)} = \langle \mathbf{M}^{\rho}_{ai}, \mathbf{M}^{\rho}_{bj}\rangle$$

Characters

Definition:

Given a representation (ρ, V) of a group G, the *character of the representation* is a map:

$$\begin{aligned}\chi_\rho : G &\to \mathbb{C}\\ g &\mapsto \mathrm{Tr}(\rho_g)\end{aligned}$$

Characters

Claim:

1. If $\chi_1, \chi_2: G \to \mathbb{C}$ are the characters of two non-isomorphic irreducible representations (ρ_1, V_1) and (ρ_2, V_2) then:
$$\langle \chi_1, \chi_2 \rangle = 0$$
2. If $\chi: G \to \mathbb{C}$ is the character of an irreducible representation (ρ, V) then:
$$\langle \chi, \chi \rangle = 1$$

Characters

$$0 = \langle \mathbf{M}_{ai}^{\rho_1}, \mathbf{M}_{bj}^{\rho_2} \rangle$$

1. If $\chi_1, \chi_2 : G \to \mathbb{C}$ are the characters of two non-isomorphic irreducible representations (ρ_1, V_1) and (ρ_2, V_2) then:

$$\langle \chi_1, \chi_2 \rangle = 0$$

Proof:

$$\begin{aligned}\langle \chi_1, \chi_2 \rangle &= \left\langle \sum_{i=1}^{n_1} \mathbf{M}_{ii}^{\rho_1}, \sum_{j=1}^{n_2} \mathbf{M}_{jj}^{\rho_2} \right\rangle \\ &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \langle \mathbf{M}_{ii}^{\rho_1}, \mathbf{M}_{jj}^{\rho_2} \rangle \\ &= 0\end{aligned}$$

Characters

$$\frac{\delta_{ij}\cdot\delta_{ab}}{\dim(V)} = \langle \mathbf{M}^{\rho}_{ai}, \mathbf{M}^{\rho}_{bj}\rangle$$

2. If $\chi: G \to \mathbb{C}$ is the character of an irreducible representation (ρ, V) then:

$$\langle \chi, \chi\rangle = 1$$

Proof:

$$\begin{aligned}\langle \chi, \chi\rangle &= \sum_{i,j=1}^{n} \left\langle \mathbf{M}^{\rho}_{ii}, \mathbf{M}^{\rho}_{jj}\right\rangle \\ &= \sum_{i,j=1}^{n} \frac{\delta_{ij}\cdot\delta_{ij}}{\dim(V)} \\ &= \sum_{i=1}^{n} \frac{1}{\dim(V)} \\ &= 1\end{aligned}$$

Characters

Implications:

Given a representation (ρ, V) of a group G and given some irreducible representation (ρ', V') we would like to know "how many times" the irreducible representation ρ' occurs in ρ.

Characters

Implications:

How many times does ρ' occurs in ρ?

$$V = \bigoplus_i V_i$$

$$\Downarrow$$

$$\chi_\rho = \sum_i \chi_{\rho_i}$$

$$\Downarrow$$

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \sum_i \langle \chi_{\rho_i}, \chi_{\rho'} \rangle$$

$$= \sum_i \begin{cases} 1 & \text{if } (\rho_i, V_i) \approx (\rho', V') \\ 0 & \text{otherwise} \end{cases}$$

Characters

Example:

We know that if G is the group of 2D rotations, $G = SO(2)$, and V is the space of functions on a circle, we have:

$$V = \bigoplus_{k=-\infty}^{\infty} V_k$$

where V_k is the 1D space of functions spanned by the complex exponentials:

$$V_k = \text{Span}\{e^{ik\theta}\}$$

Characters

Example:

Using the fact that $\{e^{ik\theta}\}$ is a basis for V_k, we can express $\rho_k(g)$ as a matrix with respect to this basis. (In this case, a 1×1 matrix.)

Denoting by g_ϕ the rotation by ϕ degrees, we get:

$$\rho_k(g_\phi) = (e^{-ik\phi})$$

So the character of this representation is:

$$\chi_{\rho_k}(g_\phi) = \mathrm{Tr}(e^{-ik\phi}) = e^{-ik\phi}$$

Characters

Example:

We know that if G is the group of 3D rotations, $G = SO(3)$, and V is the space of functions on a circle, we have:

$$V = \bigoplus_{l=0}^{\infty} V_l$$

where V_l is the $(2l + 1)$-dimensional space of functions spanned by the spherical harmonics:

$$V_l = \text{Span}\{Y_l^{-l}, \cdots, Y_l^l\}$$

Characters

<u>Example</u>:

Using the spherical harmonic basis we get:

$$\rho_l(R)\left[\sum_{m=-l}^{m} \mathbf{a}_{lm} Y_l^m\right] = \sum_{m=-l}^{m} \mathbf{a}_{lm} \sum_{m\prime=-l}^{l} \langle R(Y_l^m), Y_l^{m'}\rangle \cdot Y_l^{m'}$$

$$= \sum_{m=-l}^{m} \mathbf{a}_{lm} \sum_{m\prime=-l}^{l} D_l^{m,m'}(R) \cdot Y_l^{m'}$$

So the character of this representation is:

$$\chi_{\rho_l}(R) = \mathrm{Tr}\begin{pmatrix} D_l^{-l,-l}(R) & \cdots & D_l^{-l,l}(R) \\ \vdots & \ddots & \vdots \\ D_l^{l,-l}(R) & \cdots & D_l^{l,l}(R) \end{pmatrix} = \sum_{m=-l}^{l} D_l^{m,m}(R)$$

Characters

Application:

$$\langle \rho_{R(\theta,\phi)}\left(Y_l^0\right), \overline{Y_l^m} \rangle = \lambda_l \cdot Y_l^m(\theta,\phi)$$

What is λ_l?

Characters

$$\langle \rho_{R(\theta,\phi)}\left(Y_l^0\right), \overline{Y_l^m} \rangle = \lambda_l \cdot Y_l^m(\theta,\phi)$$

Since Y_l^0 is real-valued, we can re-write this as:

$$\langle Y_l^m, \rho_{R(\theta,\phi)}\left(Y_l^0\right) \rangle = \lambda_l \cdot Y_l^m(\theta,\phi)$$

Characters

$$\langle Y_l^m, \rho_{R(\theta,\phi)}\left(Y_l^0\right)\rangle = \lambda_l \cdot Y_l^m(\theta,\phi)$$

Recall that:

$$\frac{\delta_{ij}\cdot\delta_{ab}}{\dim(V_l)} = \langle \mathbf{M}_{ai}^{\rho_l}, \boldsymbol{M}_{bj}^{\rho_l}\rangle$$

$$\mathbf{M}_{mm'}^{\rho_l}(R) = \langle R(Y_l^m), Y_l^{m'}\rangle$$

Setting $i, j = 0$, $a, b = m$, and writing the dot-product as an integral:

$$\begin{aligned}\frac{1}{2l+1} &= \langle \mathbf{M}_{m0}^{\rho_l}, \mathbf{M}_{m0}^{\rho_l}\rangle \\ &= \frac{1}{|SO(3)|}\int_{R\in SO(3)} \overline{\langle R\left(Y_l^m\right), Y_l^0\rangle}\cdot\langle R(Y_l^m), Y_l^0\rangle dR\end{aligned}$$

Characters

$$\langle Y_l^m, \rho_{R(\theta,\phi)}\left(Y_l^0\right)\rangle = \lambda_l \cdot Y_l^m(\theta,\phi)$$

Since rotations are orthogonal:

$$\begin{aligned}\frac{1}{2l+1} &= \frac{1}{|SO(3)|}\int_{R\in SO(3)} \overline{\langle R\left(Y_l^m\right), Y_l^0\rangle} \cdot \langle R(Y_l^m), Y_l^0\rangle\, dR \\ &= \frac{1}{|SO(3)|}\int_{R\in SO(3)} \overline{\langle Y_l^m, R^{-1}(Y_l^0)\rangle} \cdot \langle Y_l^m, R^{-1}(Y_l^0)\rangle dR \\ &= \frac{1}{|SO(3)|}\int_{R\in SO(3)} \overline{\langle Y_l^m, R(Y_l^0)\rangle} \cdot \langle Y_l^m, R(Y_l^0)\rangle dR\end{aligned}$$

Characters

$$\langle Y_l^m, \rho_{R(\theta,\phi)}\left(Y_l^0\right)\rangle = \lambda_l \cdot Y_l^m(\theta,\phi)$$

Factoring the integral we get:

$$\frac{1}{2l+1} = \frac{1}{|SO(3)|}\int_{R\in SO(3)} \overline{\langle Y_l^m, R(Y_l^0)\rangle} \cdot \langle Y_l^m, R(Y_l^0)\rangle dR$$

$$\Downarrow$$

$$\frac{1}{2l+1} = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \overline{\langle Y_l^m, R_{\theta,\phi,\psi}\left(Y_l^0\right)\rangle} \cdot \langle Y_l^m, R_{\theta,\phi,\psi}\left(Y_l^0\right)\rangle \, d\psi \sin(\phi) \, d\phi \, d\theta$$

Characters

$$\langle Y_l^m, \rho_{R(\theta,\phi)}\left(Y_l^0\right)\rangle = \lambda_l \cdot Y_l^m(\theta,\phi)$$

Using the invariance of the zonal harmonics to rotations about the y-axis:

$$\frac{1}{2l+1} = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\overline{\langle Y_l^m, R_{\theta,\phi,\psi}\left(Y_l^0\right)\rangle} \cdot \langle Y_l^m, R_{\theta,\phi,\psi}\left(Y_l^0\right)\rangle \, d\psi \sin(\phi)\, d\phi\, d\theta$$

$$\Downarrow$$

$$\frac{1}{2l+1} = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\overline{\langle Y_l^m, R_{\theta,\phi}\left(Y_l^0\right)\rangle} \cdot \langle Y_l^m, R_{\theta,\phi}\left(Y_l^0\right)\rangle \, d\psi \sin(\phi)\, d\phi\, d\theta$$

Characters

$$\langle Y_l^m, \rho_{R(\theta,\phi)}\left(Y_l^0\right)\rangle = \lambda_l \cdot Y_l^m(\theta,\phi)$$

Since the integrand is independent of ψ:

$$\frac{1}{2l+1} = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\overline{\langle Y_l^m, R_{\theta,\phi}\left(Y_l^0\right)\rangle}\cdot\langle Y_l^m, R_{\theta,\phi}\left(Y_l^0\right)\rangle\, d\psi \sin(\phi)\ d\phi\, d\theta$$

$$\Downarrow$$

$$\frac{1}{2l+1} = \frac{2\pi}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\overline{\langle Y_l^m, R_{\theta,\phi}\left(Y_l^0\right)\rangle}\cdot\langle Y_l^m, R_{\theta,\phi}\left(Y_l^0\right)\rangle \sin(\phi)\ d\phi\, d\theta$$

Characters

$$\langle Y_l^m, \rho_{R(\theta,\phi)}\left(Y_l^0\right)\rangle = \lambda_l \cdot Y_l^m(\theta,\phi)$$

Plugging in the equation for zonal convolution:

$$\frac{1}{2l+1} = \frac{2\pi}{8\pi^2}\int_0^{2\pi}\int_0^{\pi} \overline{\langle Y_l^m, R_{\theta,\phi}\left(Y_l^0\right)\rangle} \cdot \langle Y_l^m, R_{\theta,\phi}\left(Y_l^0\right)\rangle \sin(\phi)\; d\phi\, d\theta$$

$$\Downarrow$$

$$\frac{1}{2l+1} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \overline{\lambda_l \cdot Y_l^m(\theta,\phi)} \cdot \lambda_l \cdot Y_l^m(\theta,\phi) \sin(\phi)\; d\phi\, d\theta$$

$$\Downarrow$$

$$\frac{4\pi}{2l+1} = \|\lambda_l\|^2$$

$$\Downarrow$$

$$\lambda_l = \zeta \cdot \sqrt{\frac{4\pi}{2l+1}} \quad \text{with } \zeta \in \mathbb{C} \text{ and } \|\zeta\| = 1$$

Characters

$$\lambda_l = \zeta \cdot \sqrt{\frac{4\pi}{2l+1}} \quad \text{with } \zeta \in \mathbb{C} \text{ and } \|\zeta\| = 1$$

Taking $m = 0$ and $\theta, \phi = 0$, we get:

$$\langle Y_l^m, \rho_{R(\theta,\phi)}\left(Y_l^0\right)\rangle = \lambda_l \cdot Y_l^m(\theta,\phi)$$

$$\Downarrow$$

$$1 = \lambda_l \cdot Y_l^0(0,0)$$

Since the convention is for the zonal harmonics to be real and positive at the north pole:

$$\lambda_l = \sqrt{\frac{4\pi}{2l+1}}$$