# FFTs in Graphics and Vision

Fast Alignment of Spherical Functions

# Outline

- Math Review
- Fast Rotational Alignment

# Review

Recall 1:

We can represent any rotation $R$ in terms of the triplet of Euler angles $(\theta, \phi, \psi)$, with the correspondence defined by:

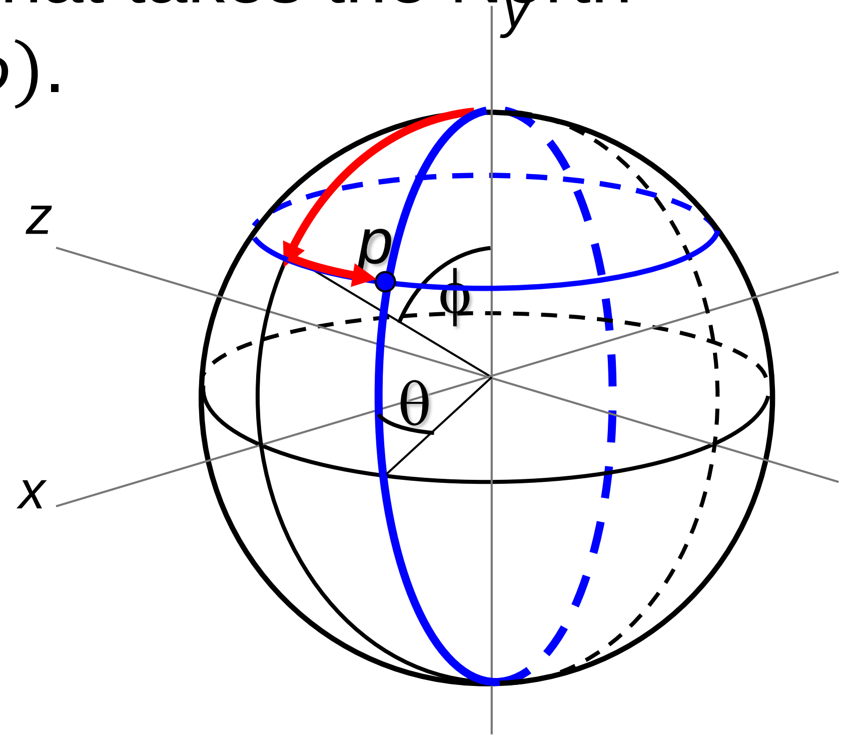
$$R(\theta, \phi, \psi) = R_y(\theta) \cdot R_z(\phi) \cdot R_y(\psi)$$

where $R_y(\alpha)$ is the rotation about the $y$-axis by an angle of $\alpha$, and $R_z(\beta)$ is the rotation about the $z$-axis by an angle of $\beta$.

# Review

Recall 2:

If we express a rotation in terms of its Euler angles $(\theta, \phi, \psi)$, then the angles $(\theta, \phi)$ correspond to the rotation that takes the North pole to the point $p = \Phi(\theta, \phi)$.



# Review

<u>Recall 3</u>:

If we represent a rotation $R$ in terms of the Euler angles $(\theta,\phi,\psi)$, then the inverse of $R$ can be represented by the Euler angles $(-\psi,-\phi,-\theta)$:

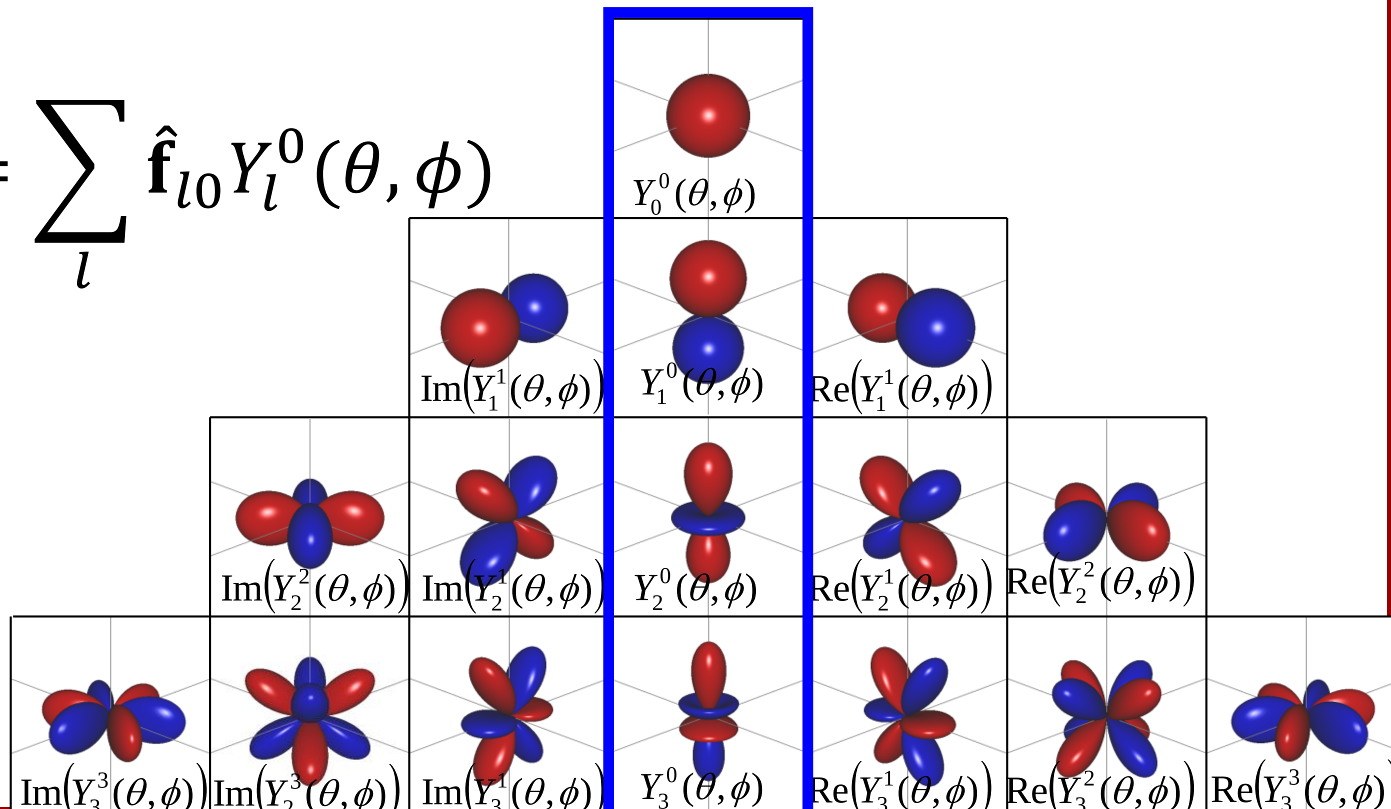
$$\begin{aligned}
R^{-1}(\theta,\phi,\psi) &= \left(R_y(\theta)\cdot R_z(\phi)\cdot R_y(\psi)\right)^{-1}\\
&= R_y^{-1}(\psi)\cdot R_z^{-1}(\phi)\cdot R_y^{-1}(\theta)\\
&= R_y(-\psi)\cdot R_z(-\phi)\cdot R_y(-\theta)
\end{aligned}$$

# Review

<u>Recall 4</u>:

A function $f$ is axially symmetric about the $y$-axis if and only if it is composed entirely of the zonal harmonics:

$$f(\theta,\phi)=\sum_l \hat{\mathbf{f}}_{l0} Y_l^0(\theta,\phi)$$



# Review

Recall 5:

Rotating the spherical harmonic $Y_l^m$ about the $y$-axis by an angle of $\alpha$ is equivalent to multiplying it by $e^{-im\alpha}$.

Expressing the spherical harmonic in terms of the associated Legendre polynomials, we get:

$$Y_l^m(\theta, \phi) = P_l^m(\cos\phi) \cdot e^{im\theta}$$

# Review

Recall 5:

Rotating the spherical harmonic $Y_l^m$ about the $y$-axis by an angle of $\alpha$ is equivalent to multiplying it by $e^{-im\alpha}$.

So rotating by $\alpha$ about the $y$-axis gives:

$$\begin{aligned}\rho_{R_y(\alpha)}(Y_l^m)(\theta,\phi) &= Y_l^m(\theta-\alpha,\phi)\\ &= P_l^m(\cos\phi)\cdot e^{im(\theta-\alpha)}\\ &= e^{-im\alpha}\cdot P_l^m(\cos\phi)\cdot e^{im\theta}\\ &= e^{-im\alpha}\cdot Y_l^m(\theta,\phi)\end{aligned}$$

# Review

<u>Recall 6</u>:

If $f$ is axially symmetric about the $y$-axis, then a rotation of $f$ by a rotation with Euler angles $(\theta, \phi, \psi)$ is independent of the value of $\psi$:

$$\begin{aligned}\rho_{R(\theta,\phi,\psi)}(f) &= \rho_{R_y(\theta)\cdot R_z(\phi)\cdot R_y(\psi)}(f) \\ &= \rho_{R_{y(\theta)}}\left(\rho_{R_{z(\phi)}}\left(\rho_{R_{y(\psi)}}(f)\right)\right) \\ &= \rho_{R_{y(\theta)}}\left(\rho_{R_{z(\phi)}}(f)\right)\end{aligned}$$

# Review

<u>Recall 7</u>:

Given a spherical function $f$ of frequency $l$:

$$f=\sum_{m=-l}^{l}\hat{\mathbf{f}}_{lm}Y_l^m$$

correlating the $l$-th zonal harmonic with $f$ is the same as scaling the conjugate of $f$:

$$\left\langle\rho_{R(\theta,\phi,\psi)}\left(Y_l^0\right),\bar{f}\right\rangle=\sqrt{\frac{4\pi}{2l+1}}f(\theta,\phi)$$

# Review

Recall 8:

Given spherical functions $f$ and $g$, if $f$ is axially symmetric about the $y$-axis, we can compute the correlation of $f$ with $g$ in $O(N^2 \log^2 N)$.

In terms of the spherical harmonic decomposition, this equation becomes:

$$D_{f,g}(R) = \langle \rho_R(f), g \rangle$$

$$\Downarrow$$

$$D_{f,g}(\theta,\phi,\psi) = \left\langle \rho_{R(\theta,\phi,\psi)}\left(\sum_l \hat{\mathbf{f}}_{l0} Y_l^0\right), \sum_l \sum_{m=-l}^{l} \hat{\mathbf{g}}_{lm} Y_l^m \right\rangle$$

# Review

<u>Recall 8</u>:

Given spherical functions $f$ and $g$, if $f$ is axially symmetric about the $y$-axis, we can compute the correlation of $f$ with $g$ in $O(N^2 \log^2 N)$.

By the conjugate linearity and the fact that harmonics of degree $l$ form a sub-representation:

$$D_{f,g}(\theta,\phi,\psi) = \left\langle \rho_{R(\theta,\phi,\psi)}\left(\sum_l \hat{\mathbf{f}}_{l0} Y_l^0\right), \sum_l \sum_{m=-l}^{l} \hat{\mathbf{g}}_{lm} Y_l^m \right\rangle$$

$$\Downarrow$$

$$D_{f,g}(\theta,\phi,\psi) = \sum_l \sum_{m=-l}^{l} \hat{\mathbf{f}}_{l0} \overline{\hat{\mathbf{g}}_{lm}} \left\langle \rho_{R(\theta,\phi,\psi)}\left(Y_l^0\right), Y_l^m \right\rangle$$

# Review

Recall 8:

Given spherical functions $f$ and $g$, if $f$ is axially symmetric about the $y$-axis, we can compute the correlation of $f$ with $g$ in $O(N^2 \log^2 N)$.

Which simplifies to:

$$D_{f,g}(\theta,\phi,\psi) = \sum_{l} \sum_{m=-l}^{l} \hat{\mathbf{f}}_{l0} \overline{\hat{\mathbf{g}}_{lm}} \left\langle \rho_{R(\theta,\phi,\psi)}\left(Y_l^0\right), Y_l^m \right\rangle$$

$$\Downarrow$$

$$D_{f,g}(\theta,\phi,\psi) = \sum_{l} \sum_{m=-l}^{l} \hat{\mathbf{f}}_{l0} \overline{\hat{\mathbf{g}}_{lm}} \sqrt{\frac{4\pi}{2l+1}} \cdot \overline{Y_l^m(\theta,\phi)}$$

# Review

Recall 8:

$$D_{f,g}(\theta,\phi,\psi) = \sum_{l} \sum_{m=-l}^{l} \hat{\mathbf{f}}_{l0} \overline{\hat{\mathbf{g}}_{lm}} \sqrt{\frac{4\pi}{2l+1}} \cdot \overline{Y_l^m(\theta,\phi)}$$

So we can compute the correlation by:

- Computing the spherical harmonic transforms.
  $O(N^2 \log^2 N)$
- Scaling the $(l,m)$-th harmonic coefficient of $g$ by the $(l,0)$-th coefficient of $f$ times $\sqrt{4\pi/2l+1}$.
  $O(N^2)$
- Computing the conjugate of the inverse transform.
  $O(N^2 \log^2 N)$

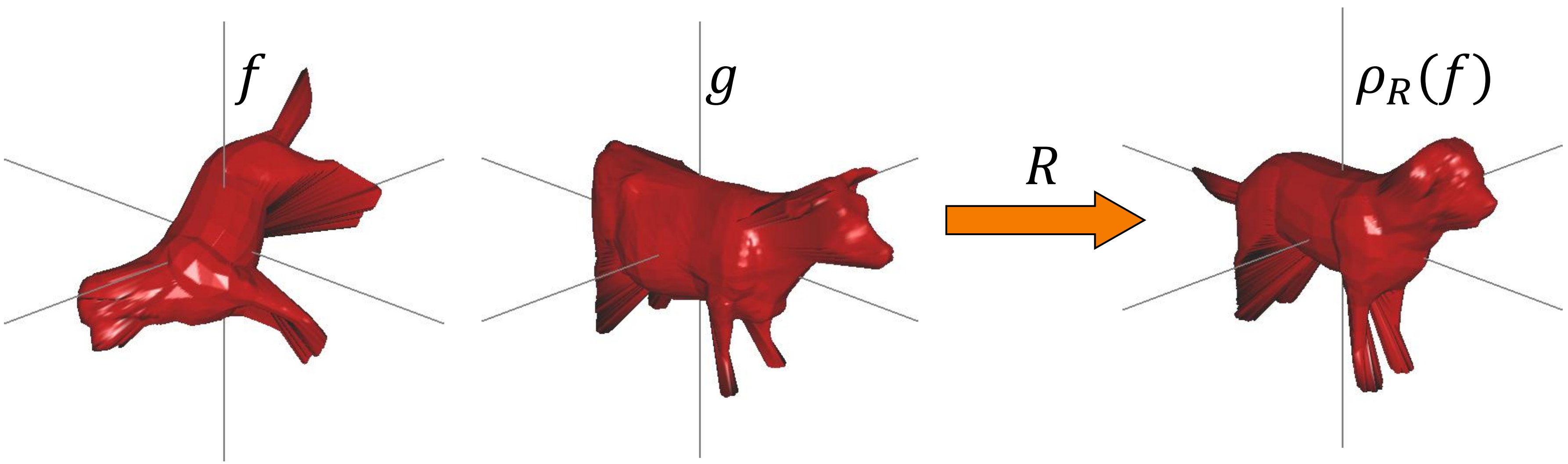
# Outline

- Math Review
- Fast Rotational Alignment

# Goal

Given two spherical functions $f$ and $g$, we would like to find the rotation $R$ that aligns $f$ to $g$:

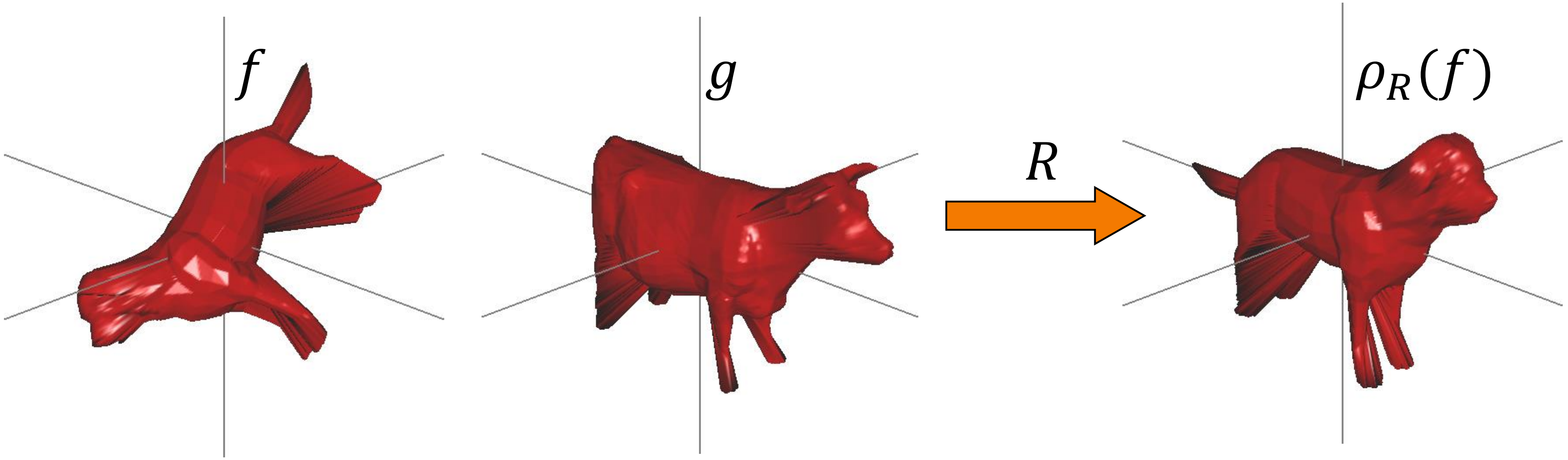
$$R = \underset{R \in SO(3)}{\arg\min} \|\rho_R(f) - g\|^2$$



# Approach

We had shown that finding the rotation minimizing the difference is equivalent to finding the rotation maximizing the correlation:

$$R = \underset{R \in SO(3)}{\arg\max} \langle \rho_R(f), g \rangle$$



# Approach

Solving for the aligning rotation can be done by computing the function on the space of rotations:

$$D_{f,g}(R) = \langle \rho_R(f), g \rangle$$

and finding the rotation maximizing this function.

# Approach

Brute Force:

If the resolution of the spherical grid is $N$, then we can find the optimal rotation in $O(N^5)$ time by:

- For each of $O(N^3)$ rotations
  - Compute the appropriate $O(N^2)$ dot-product

# Approach

<u>Fast Spherical Correlation</u>:

Using the Wigner $D$-Transform, we can implement this in $O(N^3 \log^2 N)$ time by:

- Get the spherical harmonic coefficients of $f$ and $g$.
  $O(N^2 \log^2 N)$
- Cross multiply the coefficients within each frequency to get the Wigner $D$-coefficients.
  $O(N^3)$
- Perform the inverse Wigner $D$-Transform to get the value of the correlation at every rotation.
  $O(N^3 \log^2 N)$

# Efficiency

Although the Wigner $D$-Transform provides an algorithm that is faster than brute force, for many applications, a cubic algorithm is still be too slow.

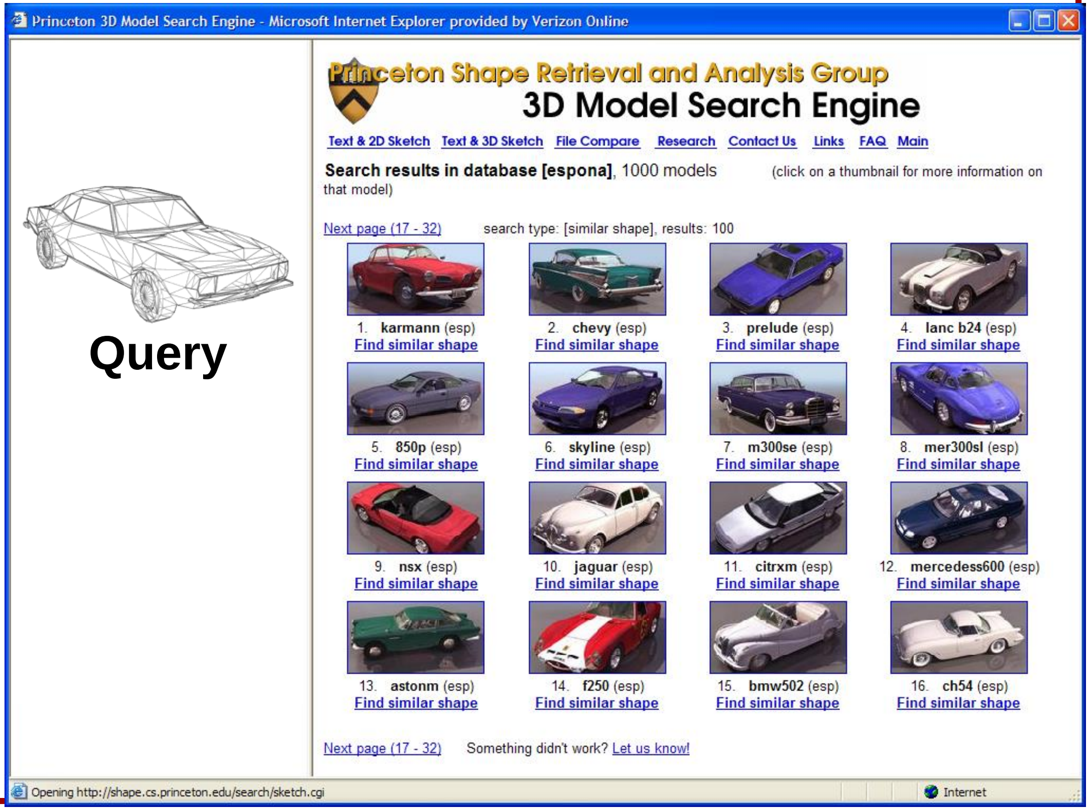
What we would like is an algorithm for aligning two functions that is on the order of the size of the spherical functions (i.e. quadratic in $N$).

# Efficiency

Example:

For database retrieval, we would like to minimize the amount of work that needs to be done online.

We can afford to do a lot of work on a per-model basis in pre-processing, but we can't spend too much time aligning pairs of models for matching.



# Efficiency

Observation:

In using the Wigner $D$-Transform, we obtain the alignment error at every rotation.

All we want is the single, optimal rotation.

# Parameter Optimization

<u>Parameter Splitting</u>:

Given a function $F(x, y)$, we would like to find the parameters $(x^*, y^*)$ at which $F$ is maximal:

$$(x^*, y^*) = \underset{(x,y)\in\mathbb{R}^2}{\arg\max}\big(F(x, y)\big)$$

We can find the parameters $(x^*, y^*)$ by searching over the entirety of the parameter domain to find the parameters at which $F$ is maximal.

This would require a search over a large space of parameters.

# Parameter Optimization

Parameter Splitting:

Given a function $F(x, y)$, we would like to find the parameters $(x^*, y^*)$ at which $F$ is maximal:

$$(x^*, y^*) = \underset{(x,y)\in\mathbb{R}^2}{\arg\max}\big(F(x, y)\big)$$

Instead, we can try to decompose the problem of optimization into two parts:

- First, find the optimal value for $x^*$, and then
- Holding $x^*$ fixed, find the optimal value for $y^*$.

This way, we trade one search over a large space, for two searches over smaller spaces.

# Parameter Optimization

Parameter Splitting:

To do this, we need to define a 1D function $G(x)$ with the property that if $(x^*, y^*)$ maximizes $F(x, y)$ then $x^*$ maximizes $G(x)$.

$$(x^*, y^*) = \underset{(x,y)\in\mathbb{R}^2}{\arg\max}\big(F(x, y)\big)$$

$$\Downarrow$$

$$x^* = \underset{x\in\mathbb{R}}{\arg\max}(G(x))$$

$$y^* = \underset{y\in\mathbb{R}}{\arg\max}\big(F(x^*, y)\big)$$

# Parameter Optimization

Application to Rotational Alignment:

To find the optimal alignment, we would like to find the Euler angles $(\theta^*, \phi^*, \psi^*)$ that maximize the correlation:

$$(\theta^*, \phi^*, \psi^*) = \underset{(\theta,\phi,\psi)}{\arg\max} \left\langle \rho_{R_y(\theta)\cdot R_z(\phi)\cdot R_y(\psi)}(f), g \right\rangle$$

# Parameter Optimization

Application to Rotational Alignment:

Instead of trying to optimize over all three parameters simultaneously, we can optimize over two of the parameters.

$$(\theta^*, \phi^*) = \underset{(\theta,\phi)}{\arg\max}\Big(G(\theta, \phi)\Big)$$

Then fixing the two optimal parameters, optimize over the third:

$$\psi^* = \underset{\psi}{\arg\max}\left\langle \rho_{R_y(\theta^*)\cdot R_z(\phi^*)\cdot R_y(\psi)}(f), g \right\rangle$$

# Parameter Optimization

<u>Application to Rotational Alignment</u>:

To define the function $G(\theta,\phi)$ we choose a function that represents correlation information related to rotations defined by $\theta$ and $\phi$.

Specifically, if we let $h$ be the component of $f$ that is axially symmetric about the $y$-axis:

$$h(\theta,\phi)=\sum_l \hat{\mathbf{f}}_{l0} Y_l^0(\theta,\phi)$$

we can define:

$$G(\theta,\phi)=\left\langle \rho_{R(\theta,\phi,0)}(h),g\right\rangle$$

# Parameter Optimization

Application to Rotational Alignment:

$$G(\theta, \phi) = \langle \rho_{R(\theta,\phi,0)}(h), g \rangle$$

The function $G$ has two properties:

- If $f$ is already axially symmetric about the $y$-axis (i.e. $h = f$), the optimizing angles $(\theta^*, \phi^*)$ are guaranteed to define the optimal transformation.
- Since $h$ is axially symmetric about, we can find the optimizing angles $(\theta^*, \phi^*)$ in $O(N^2 \log^2 N)$ using the fast spherical harmonic transform.

# Parameter Optimization

Application to Rotational Alignment:

Having solved for the optimal angles $(\theta^*, \phi^*)$, we can solve for the optimal $\psi_0$ by solving:

$$\psi^* = \arg\max_{\psi} \left\langle \rho_{R_y(\theta^*)\cdot R_z(\phi^*)\cdot R_y(\psi)}(f), g \right\rangle$$

Since the representation is unitary, this becomes:

$$\psi^* = \arg\max_{\psi} \left\langle \rho_{R_y(\psi)}(f), \rho_{R_z(-\phi^*)\cdot R_y(-\theta^*)}(g) \right\rangle$$

In terms of the spherical harmonics coefficients:

$$\psi^* = \arg\max_{\psi} \left\langle \sum_{l} \sum_{m=-l}^{l} \hat{\mathbf{f}}_{lm} \rho_{R_y(\psi)}(Y_l^m), \rho_{R_z(-\phi^*)\cdot R_y(-\theta^*)}(g) \right\rangle$$

# Parameter Optimization

Application to Rotational Alignment:

Using the fact that a rotation of the spherical harmonic $Y_l^m$ about the $y$-axis by an angle of $\alpha$ corresponds to multiplication by $e^{-im\alpha}$:

$$\psi^* = \arg\max_{\psi} \left\langle \sum_l \sum_{m=-l}^{l} \hat{\mathbf{f}}_{lm} \rho_{R_y(\psi)}(Y_l^m), \rho_{R_z(-\phi^*)\cdot R_y(-\theta^*)}(g) \right\rangle$$

$$\Downarrow$$

$$\psi^* = \arg\max_{\psi} \left\langle \sum_l \sum_{m=-l}^{l} \hat{\mathbf{f}}_{lm} e^{-im\psi} \cdot Y_l^m, \rho_{R_z(-\phi^*)\cdot R_y(-\theta^*)}(g) \right\rangle$$

# Parameter Optimization

<u>Application to Rotational Alignment</u>:

Thus, to find $\psi^*$, we need to maximize:

$$\sum_{l}\sum_{m=-l}^{l}\left\langle \hat{\mathbf{f}}_{lm}Y_l^m, \rho_{R_z(-\phi^*)\cdot R_y(-\theta^*)}(g)\right\rangle e^{-im\psi}$$

But this is just an expression for a function of $\psi$ as a sum of complex exponentials.

So we can get the values at every angle $\psi$ by computing the inverse Fourier transform.

# Parameter Optimization

Application to Rotational Alignment:

Thus, we can align to spherical function $f$ and $g$ in $O(N^2 \log^2 N)$ time by:

- Correlating $f$ with the component of $g$ that is axially symmetric about the $y$-axis
  $O(N^2 \log^2 N)$
- Getting the Fourier coefficients of the function in $\psi$
  $O(N^2)$
- Computing the inverse Fourier transform
  $O(N \log N)$
- Finding the $\psi$ maximizing the function
  $O(N)$

# Parameter Optimization

How well does this work in practice?

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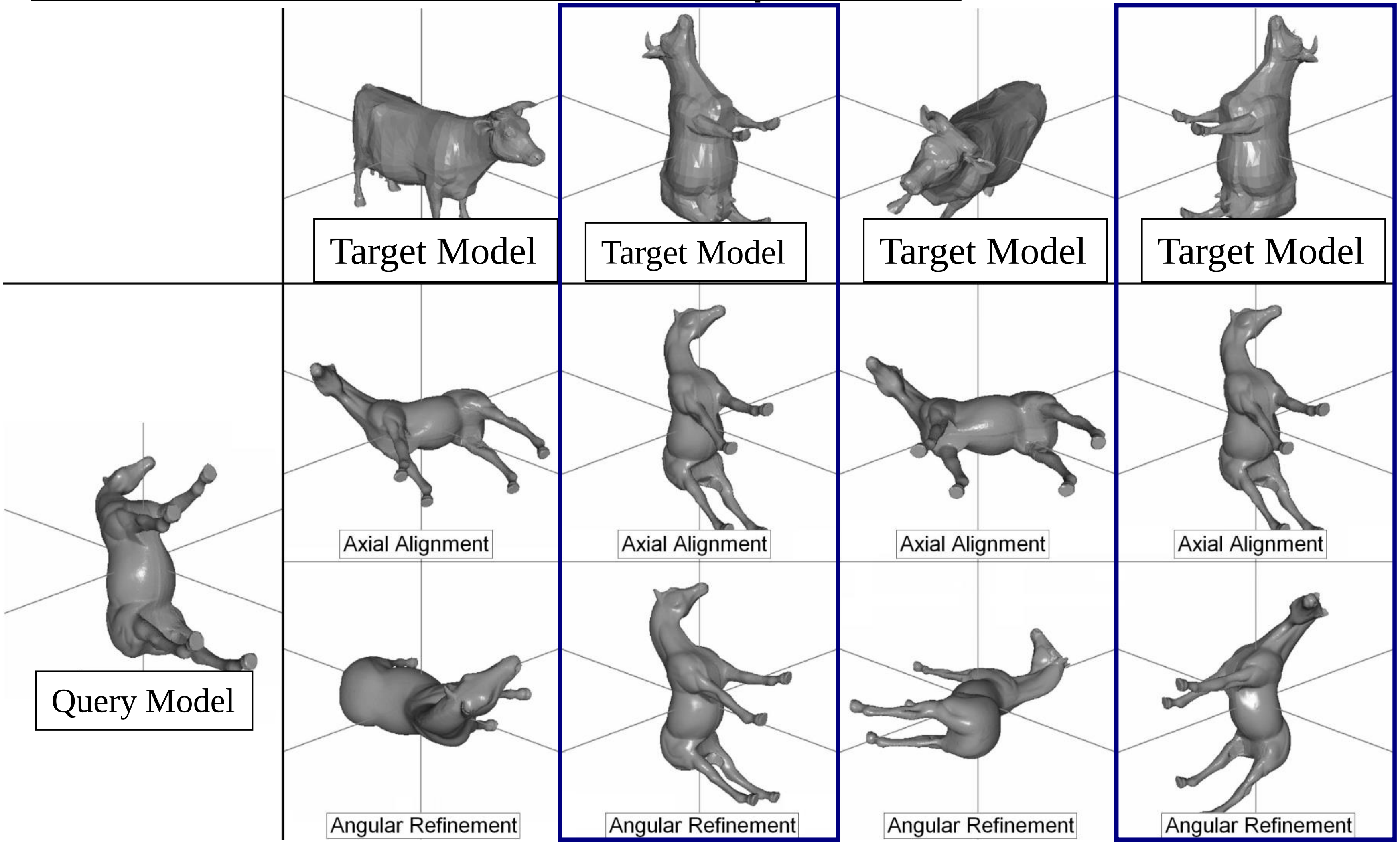


# Parameter Optimization

How well does this work in practice?

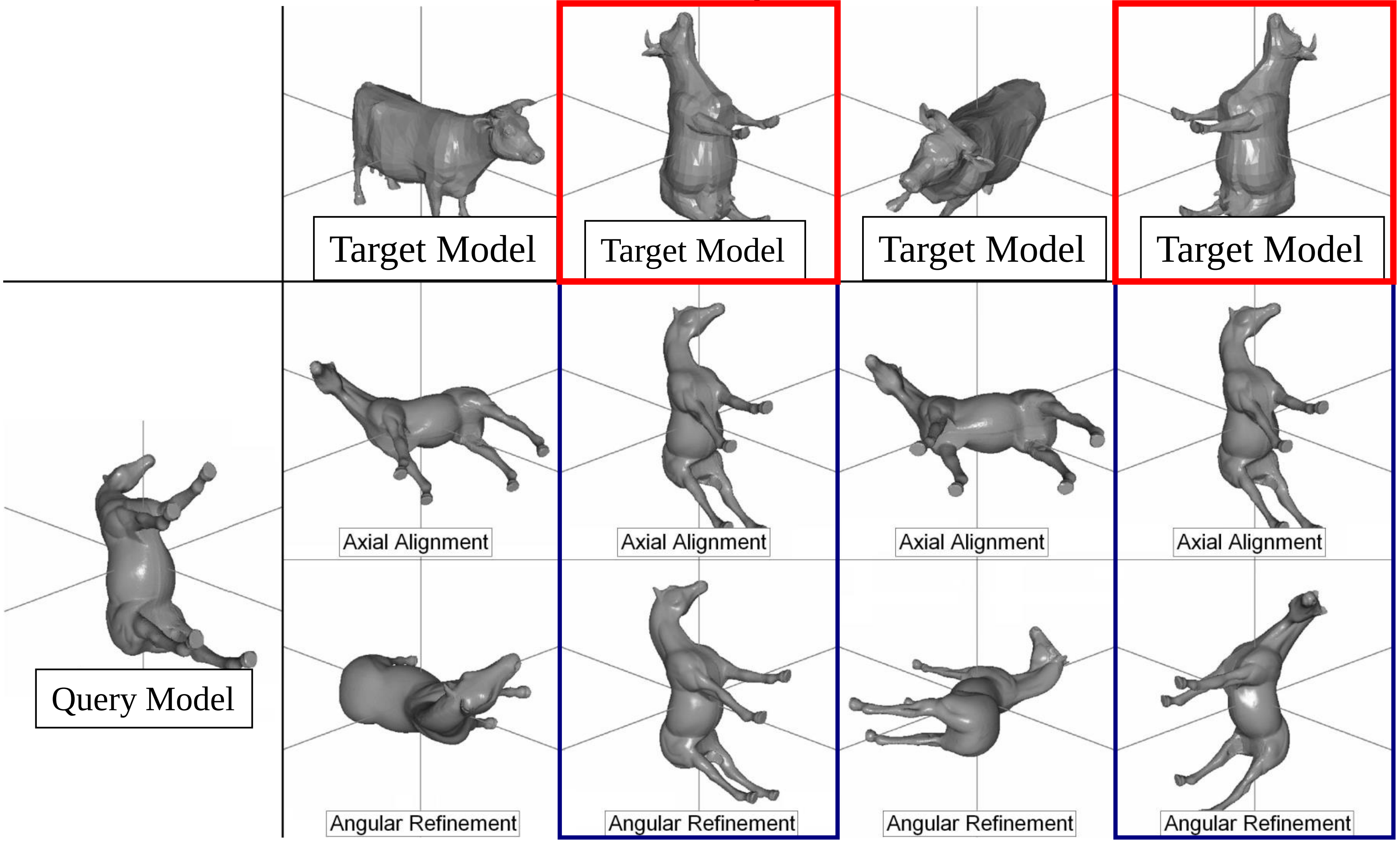
The quality of this method depends on how well $G(\theta,\phi)$ captures the behavior of the $(\theta,\phi)$ components of the rotational alignment.

When we optimize $G(\theta,\phi)$, we are looking for the rotation that best aligns the $y$-axially symmetric component of $f$ to the function $g$.

So if the function $f$ is (nearly) axially symmetric about the $y$-axis, the method will perform well.

# Parameter Optimization

How well does this work in practice?



# Parameter Optimization

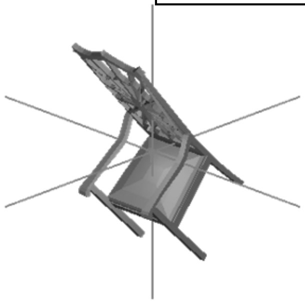
How well does this work in practice?

⇒ In a pre-processing step we can align the function $f$ so that the axis with maximal axial symmetry gets mapped to the $y$-axis.

# Parameter Optimization

How well does this work in practice?

| Pre-Processing | Run-Time |
| --- | --- |

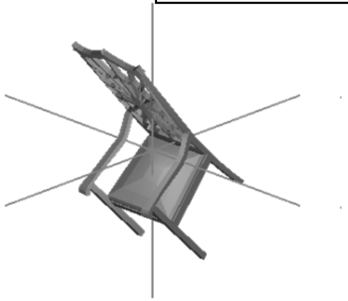


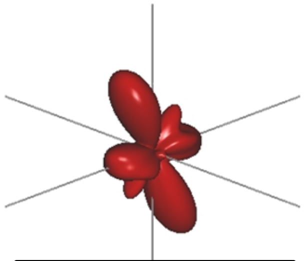
# Parameter Optimization

How well does this work in practice?

Pre-Processing

Run-Time





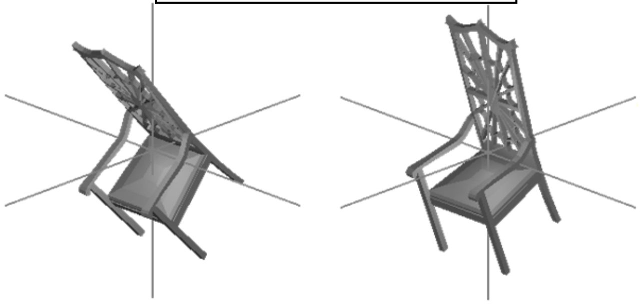
axial symmetry descriptor

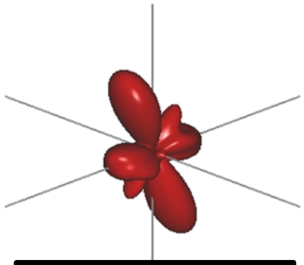
# Parameter Optimization

## How well does this work in practice?

Pre-Processing

Run-Time





axial symmetry descriptor

# Parameter Optimization

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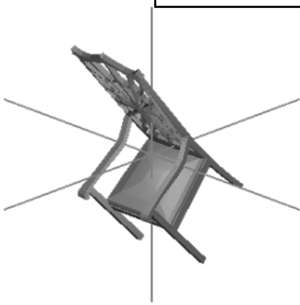
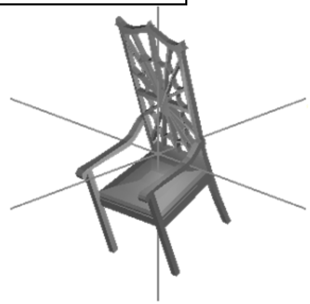
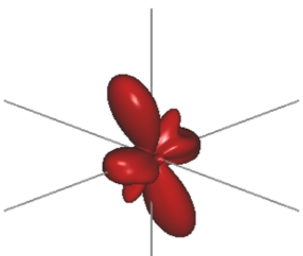
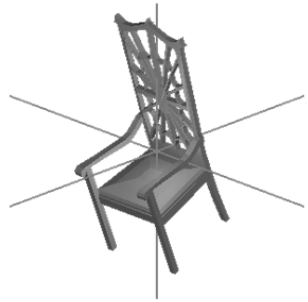


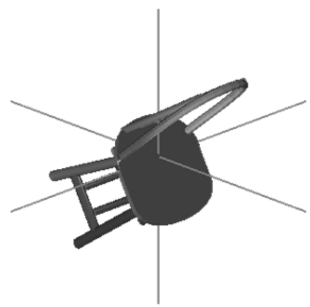




axial symmetry descriptor

Run-Time





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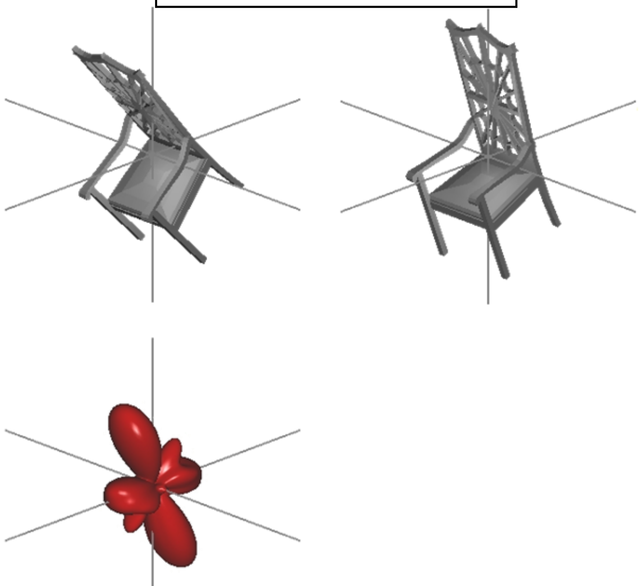
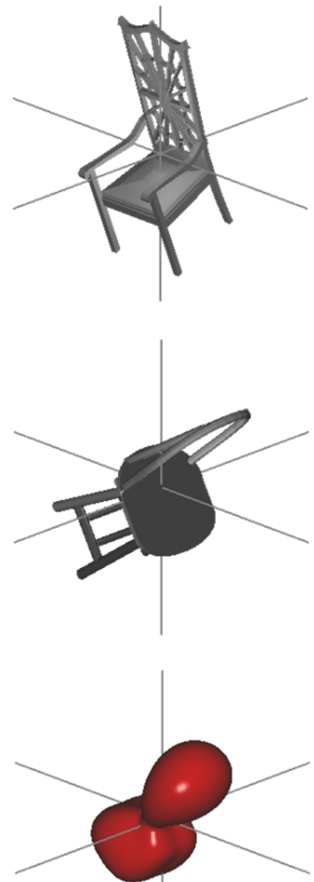


axial symmetry descriptor

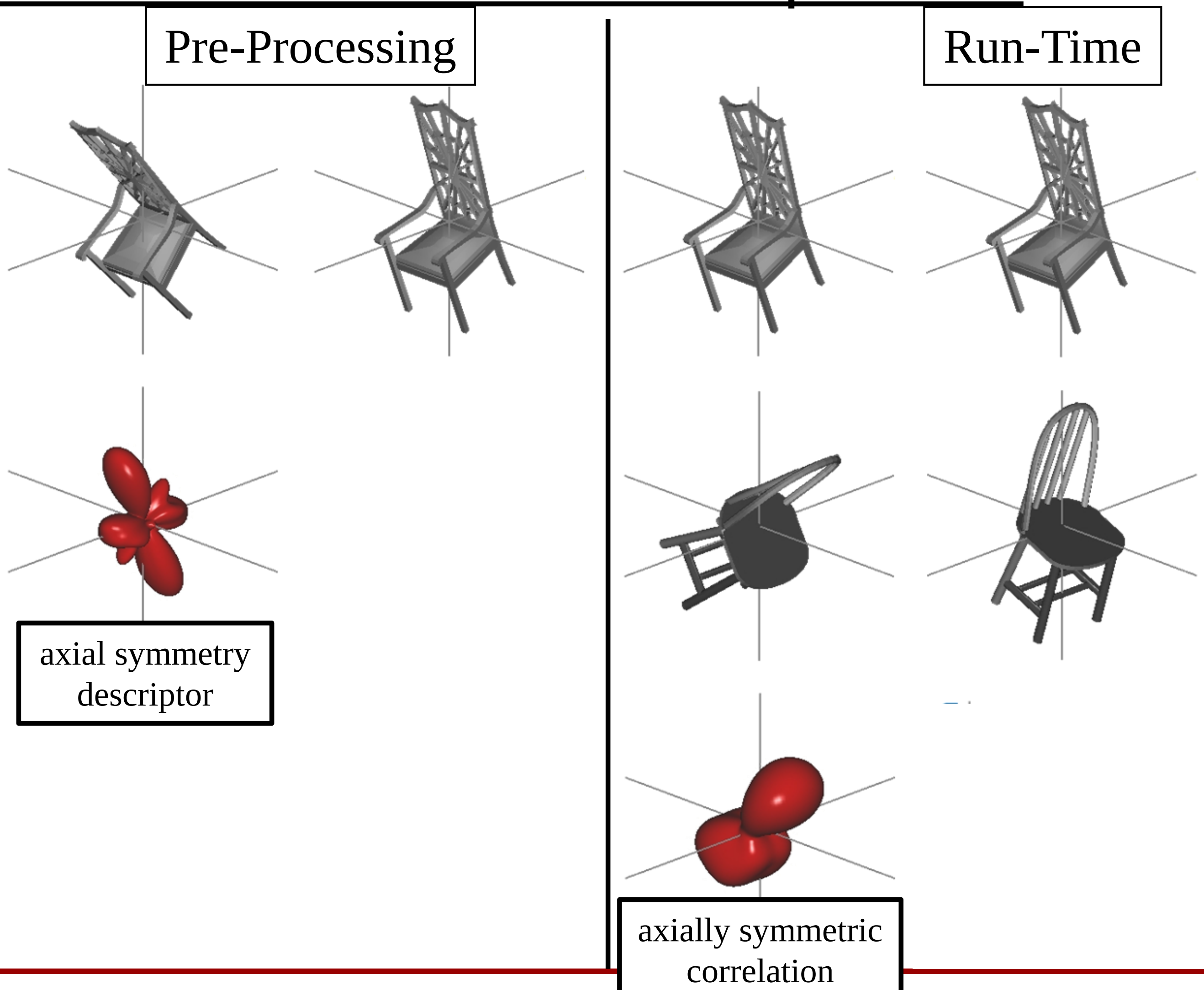
Run-Time



axially symmetric correlation

# Parameter Optimization

How well does this work in practice?



# Parameter Optimization

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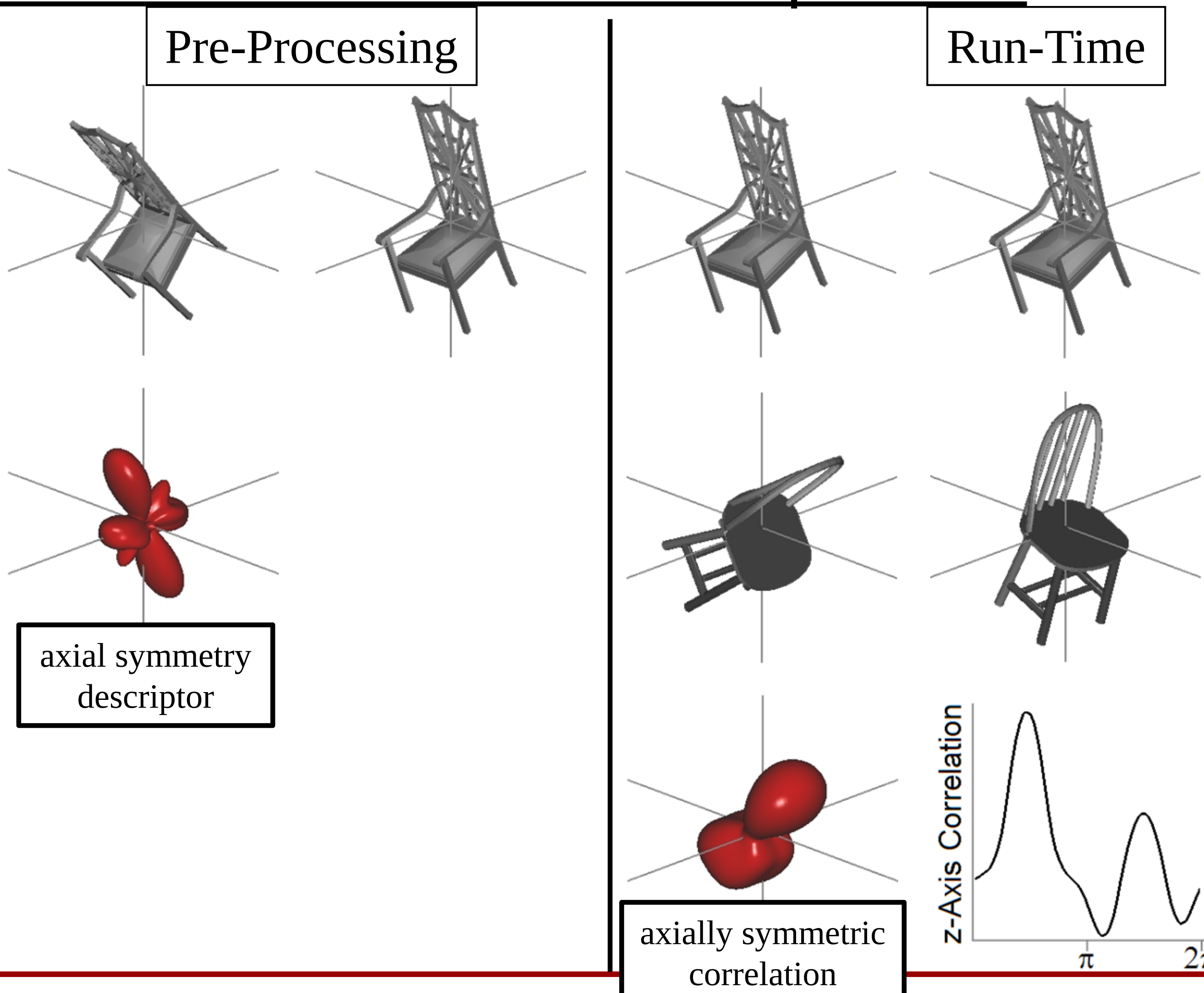
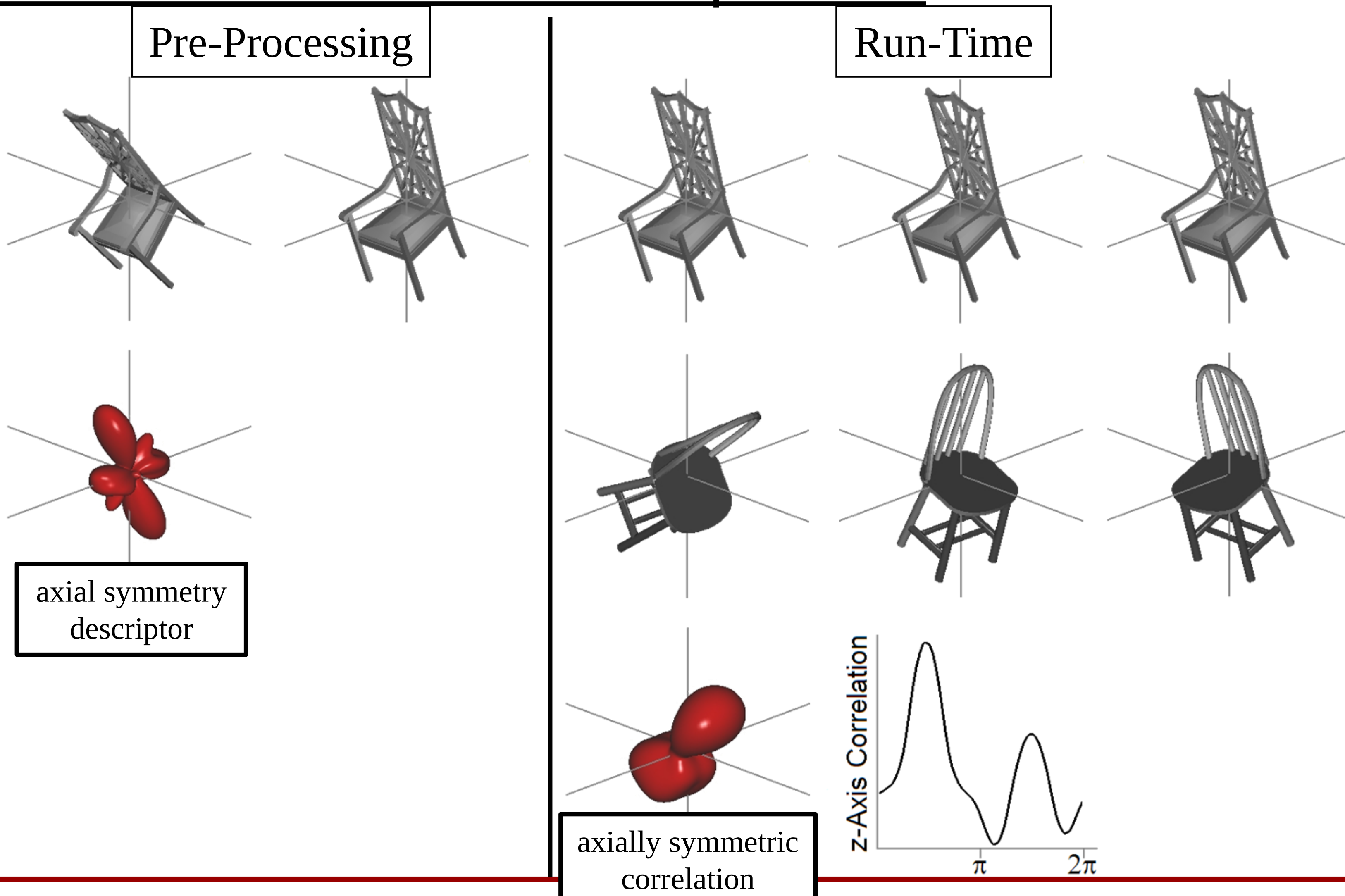


# Parameter Optimization

## How well does this work in practice?



# Parameter Optimization

<u>Performance</u>:

In order to perform the alignment we need to pre-align the models so that there major axis of axial symmetry is aligned with the $y$-axis.

- ☒ This requires computing the axial symmetry descriptor which takes $O(N^3 \log^2 N)$
- ☑ This needs to be done on a per-model basis so this can be done offline.

The online running time of the alignment algorithm remains $O(N^2 \log^2 N)$