



FFTs in Graphics and Vision

Rotational and Reflective
Symmetry Detection



Outline

Representation Theory

Symmetry Detection

- Rotational Symmetry
- Reflective Symmetry



Representation Theory

Recall:

A group is a set of elements G with a binary operation (often denoted “ \cdot ”) such that for all $f, g, h \in G$, the following are satisfied:

- Closure:

$$g \cdot h \in G$$

- Associativity:

$$f \cdot (g \cdot h) = (f \cdot g) \cdot h$$

- Identity: There exists an identity element $1 \in G$ s.t.:

$$1 \cdot g = g \cdot 1 = g$$

- Inverse: Every element g has an inverse g^{-1} s.t.:

$$g \cdot g^{-1} = g^{-1} \cdot g = 1$$



Representation Theory

Observation 1:

Given a group $G = \{g_1, \dots, g_n\}$, for any $g \in G$, the (set-theoretic) map that multiplies the elements of G on the left by g is invertible.

(The inverse is the map multiplying the elements of G on the left by g^{-1} .)



Representation Theory

Observation 1:

In particular, the set $\{g \cdot g_1, \dots, g \cdot g_n\}$ is just a re-ordering of the set $\{g_1, \dots, g_n\}$.

Or more simply, $g \cdot G = G$.

Similarly, the set $\{g_1^{-1}, \dots, g_n^{-1}\}$ is just a re-ordering of the set $\{g_1, \dots, g_n\}$.

Or more simply, $G^{-1} = G$.



Representation Theory

Recall:

A Hermitian inner product is a map from $V \times V$ into the complex numbers that is:

1. Linear: For all $u, v, w \in V$ and any scalar $\lambda \in \mathbb{C}$

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$$

2. Conjugate Symmetric: For all $v, w \in V$

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

3. Positive Definite: For all $v \in V$

$$\langle v, v \rangle \geq 0$$

$$\langle v, v \rangle = 0 \Leftrightarrow v = 0$$



Representation Theory

Observation 2:

Given a Hermitian inner-product space V , and vectors $\{v_1, \dots, v_n\} \subset V$, the vector minimizing the sum of squared distances is the average:

$$\frac{1}{n} \sum_{k=1}^n v_k = \arg \min_{v \in V} \left(\sum_{k=1}^n \|v - v_k\|^2 \right)$$



Representation Theory

Recall:

A unitary representation of a group G on a Hermitian inner-product space V is a map ρ that sends every element in G to an orthogonal transformation on V , satisfying:

$$\rho_{g \cdot h} = \rho_g \cdot \rho_h$$

for all $g, h \in G$.



Representation Theory

Definition:

A vector $v \in V$ is invariant under the action of G if:

$$\rho_g(v) = v$$

for all $g \in G$.

We denote by V_G the set of vectors in V that are invariant under the action of G :

$$V_G = \{v \in V \mid \rho_g(v) = v, \forall g \in G\}$$



Representation Theory

Observation 3:

The set V_G is a vector sub-space of V .

If $v, w \in V_G$, then for any $g \in G$, we have:

$$\rho_g(v) = v \quad \text{and} \quad \rho_g(w) = w$$

And for all scalars α and β we have:

$$\begin{aligned} \rho_g(\alpha \cdot v + \beta \cdot w) &= \alpha \cdot \rho_g(v) + \beta \cdot \rho_g(w) \\ &= \alpha \cdot v + \beta \cdot w \end{aligned}$$

So $\alpha \cdot v + \beta \cdot w \in V_G$ as well.



Representation Theory

Observation 4:

Given a finite group G and given a vector $v \in V$, the average of v over G :

$$\text{Average}(v, G) = \frac{1}{|G|} \sum_{g \in G} \rho_g(v)$$

is invariant under the action of G .



Representation Theory

Observation 4:

Let h be any element in G .

We would like to show that h maps the average back to itself:

$$\text{Average}(v, G) = \rho_h(\text{Average}(v, G))$$



Representation Theory

Observation 4:

$$\begin{aligned}\text{Average}(v, G) &= \rho_h(\text{Average}(v, G)) \\ &= \rho_h\left(\frac{1}{|G|} \sum_{g \in G} \rho_g(v)\right) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_h \cdot \rho_g(v) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_{h \cdot g}(v) \\ &= \frac{1}{|G|} \sum_{g \in h \cdot G} \rho_g(v) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_g(v) \\ &= \text{Average}(v, G)\end{aligned}$$



Representation Theory

Observation 5:

Given a finite group G and given a vector $v \in V$, the average of v over G is the closest G -invariant vector to v :

$$\text{Average}(v, G) = \arg \min_{v_0 \in V_G} (\|v_0 - v\|^2)$$



Representation Theory

Observation 5:

$$\begin{aligned}\|v_0 - v\|^2 &= \frac{1}{|G|} \sum_{g \in G} \|\rho_g(v_0) - v\|^2 \\ &= \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_g^{-1}(v)\|^2 \\ &= \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_{g^{-1}}(v)\|^2 \\ &= \frac{1}{|G|} \sum_{g \in G^{-1}} \|v_0 - \rho_g(v)\|^2 \\ &= \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_g(v)\|^2\end{aligned}$$



Representation Theory

Observation 5:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_g(v)\|^2$$

Thus, v_0 is the G -invariant vector minimizing the squared distance to v if and only if it minimizes the sum of squared distances to the vectors:

$$\{\rho_{g_1}(v), \dots, \rho_{g_n}(v)\}$$

So v_0 must be the average of these vectors:

$$v_0 = \frac{1}{|G|} \sum_{g \in G} \rho_g(v) = \text{Average}(v, G)$$



Representation Theory

Note:

Since the average map:

$$\text{Average}(v, G) = \frac{1}{|G|} = \sum_{g \in G} \rho_g(v)$$

is a linear map returning the closest G -invariant vector to v , it is the projection map from V to V_G .



Outline

Representation Theory

Symmetry Detection

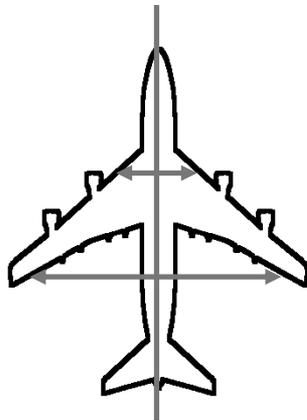
- Rotational Symmetry
- Reflective Symmetry



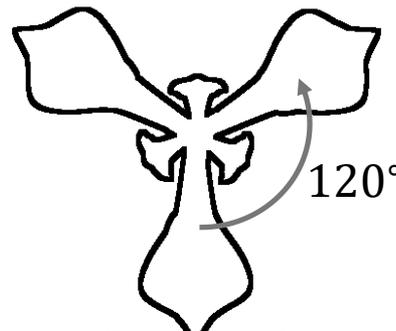
Symmetry Detection

For functions on a circle, we defined measures of:

- Reflective Symmetry: for every axis of reflective symmetry.
- Rotational Symmetry: for every order of rotational symmetry.

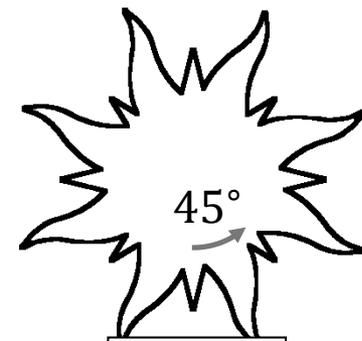


Reflective



3-Fold

Rotational



8-Fold

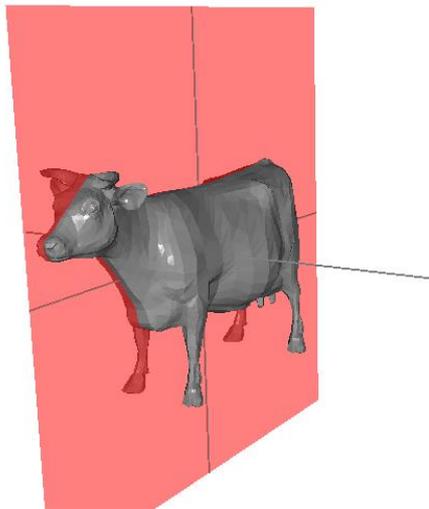
Rotational



Symmetry Detection

For functions on a sphere, we would like to define a measure of:

- Reflective Symmetry: for every plane of reflective symmetry.

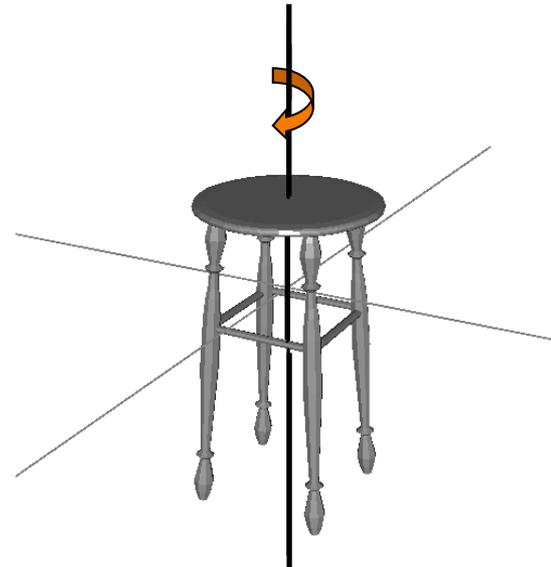
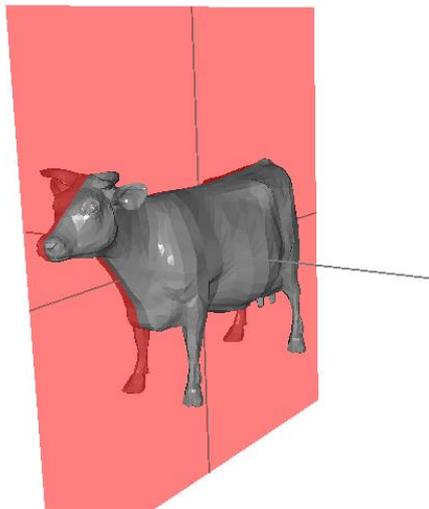




Symmetry Detection

For functions on a sphere, we would like to define a measure of:

- Reflective Symmetry: for *every* plane of reflective symmetry.
- Rotational Symmetry: for *every* axis through the origin and *every* order of rotational symmetry.





Symmetry Detection

Goal:

Reflective Symmetry:

- Compute the spherical function giving the measure of reflective symmetry of every plane passing through the origin.

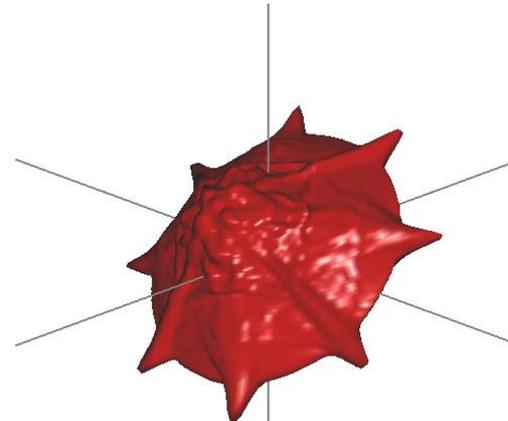
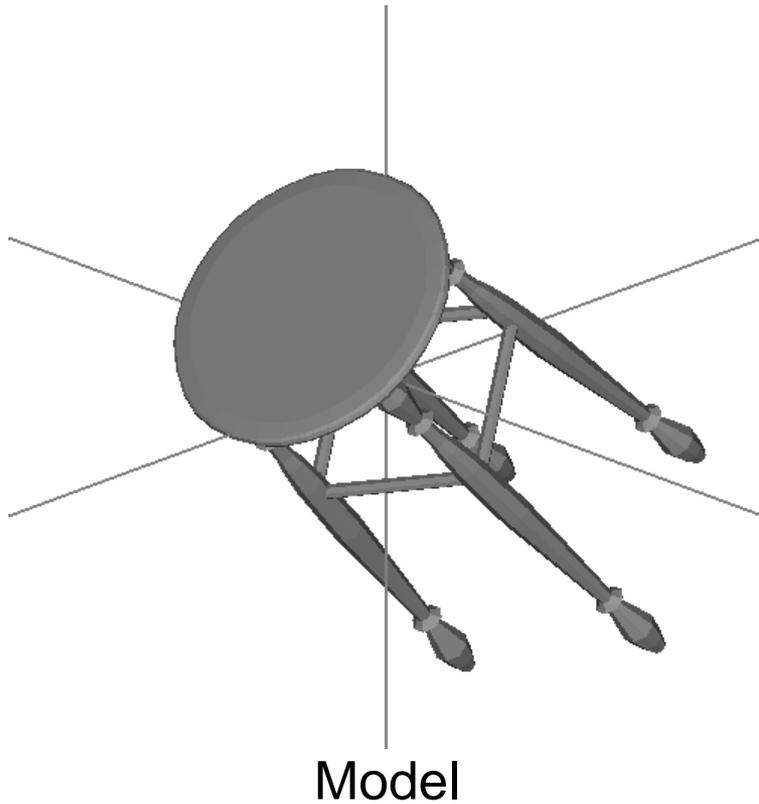
Rotational Symmetry:

- For every order of rotational symmetry k :
 - » Compute the spherical function giving the measure of k -fold symmetry about every axis through the origin.

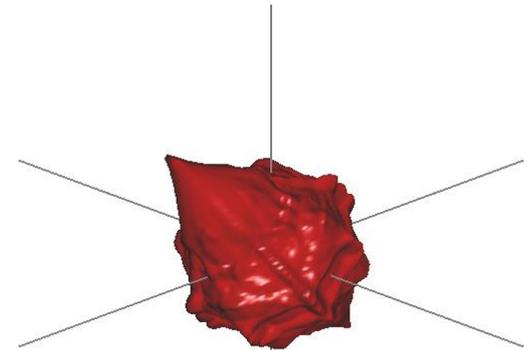


Symmetry Detection

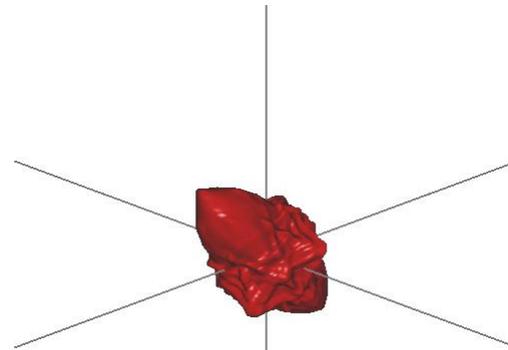
Goal:



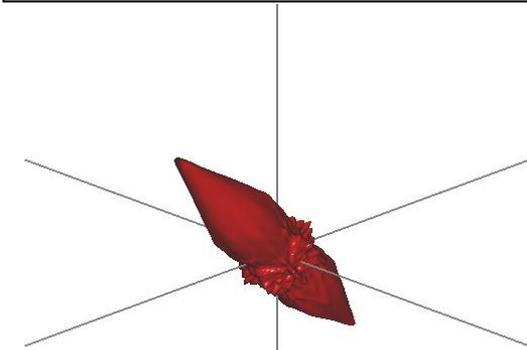
Reflective Symmetries



2-Fold
Rotational Symmetries



3-Fold
Rotational Symmetries



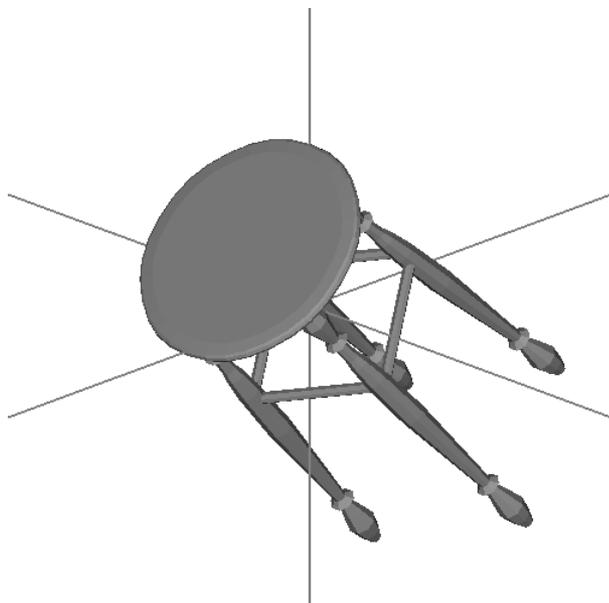
4-Fold
Rotational Symmetries



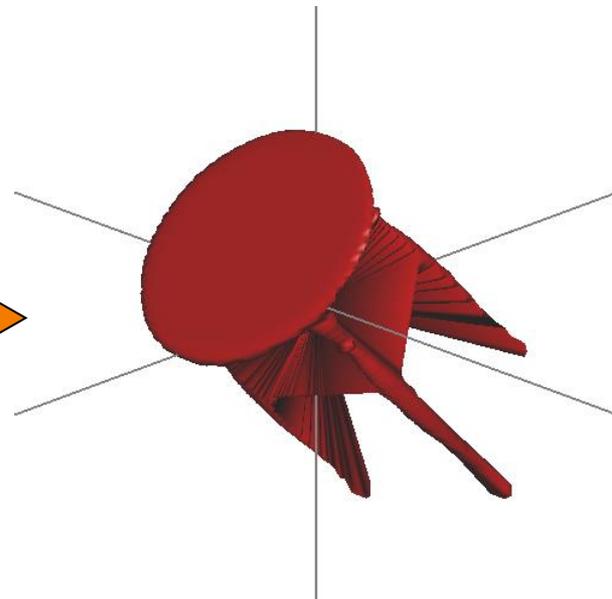
Symmetry Detection

Approach:

As in the 1D case, we will compute the symmetries of a shape by representing the shape by a spherical function.



Model



Spherical Extent Function



Symmetry Detection

Recall:

To measure a function's symmetry we:

- Associated a group G of transformations to each type of symmetry
- Defined the measure of symmetry as the size of the closest G -invariant function:

$$\text{Sym}^2(f, G) = \|\pi_G(f)\|^2$$

Since the nearest symmetric function is the average under the action of the group, we got:

$$\text{Sym}^2(f, G) = \left\| \frac{1}{|G|} \sum_{g \in G} \rho_g(f) \right\|^2$$



Outline

Representation Theory

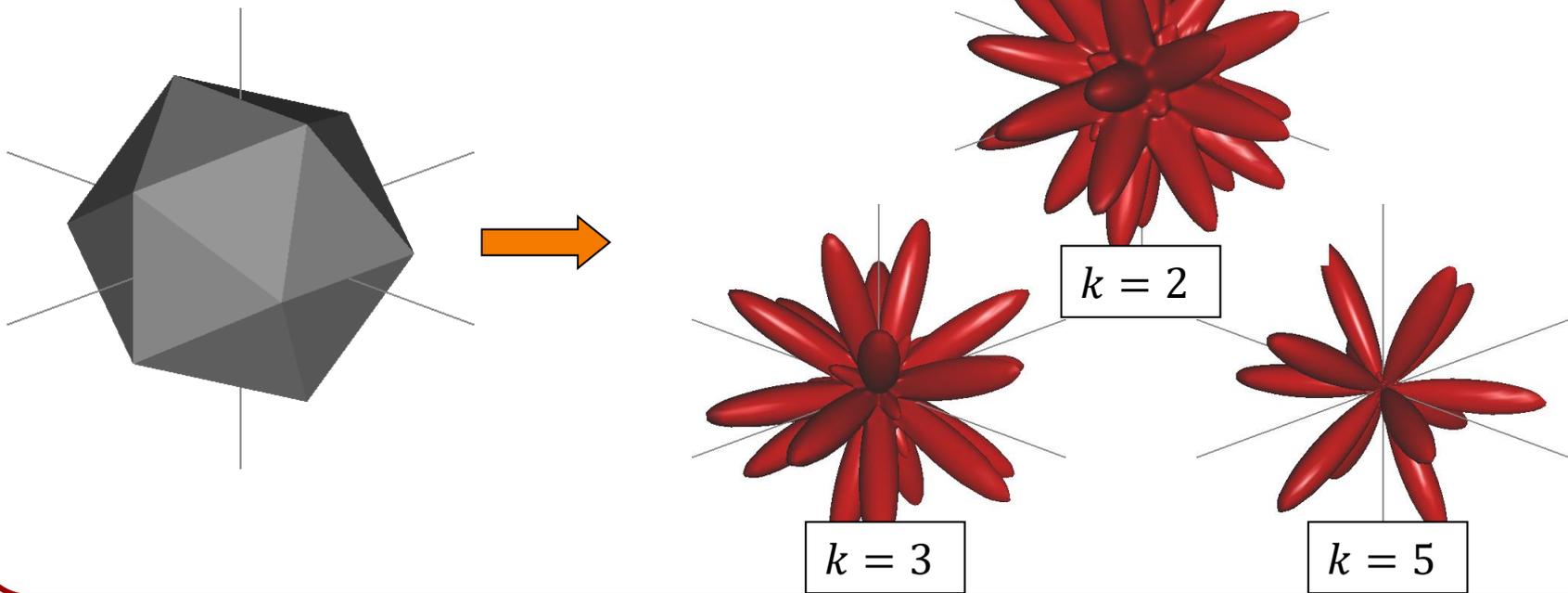
Symmetry Detection

- Rotational Symmetry
- Reflective Symmetry



Rotational Symmetry

Given a function on the sphere, and given a fixed order of rotational symmetry k , define a function whose value at a point is the measure of k -fold rotational symmetry about the associated axis.





Rotational Symmetry

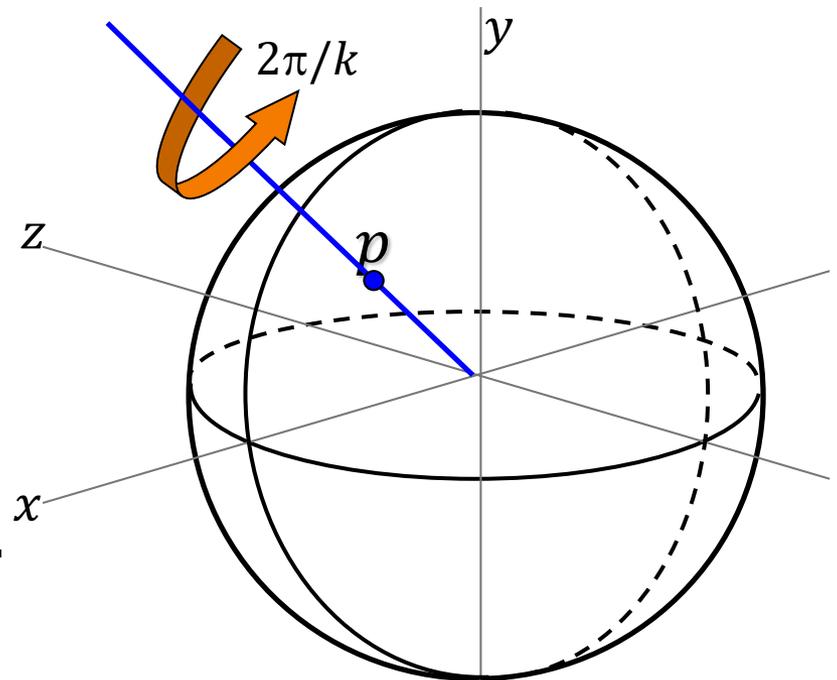
To do this, we need to associate a group to every axis passing through the origin.

We denote by $G_{p,k}$ the group of k -fold rotations about the axis through p .

The elements of the group are the rotations:

$$g_j = R \left(p, \frac{2j\pi}{k} \right)$$

corresponding to rotations about p by the angle $2j\pi/k$.



Rotational Symmetry



$$\begin{aligned}\text{Sym}^2(f, G_{p,k}) &= \left\| \frac{1}{k} \sum_{j=0}^{k-1} \rho_{g_j}(f) \right\|^2 \\ &= \frac{1}{k^2} \left\langle \sum_{i=0}^{k-1} \rho_{g_i}(f), \sum_{j=0}^{k-1} \rho_{g_j}(f) \right\rangle \\ &= \frac{1}{k^2} \sum_{i,j=0}^{k-1} \langle \rho_{g_i}(f), \rho_{g_j}(f) \rangle \\ &= \frac{1}{k^2} \sum_{i,j=0}^{k-1} \langle \rho_{g_{i-j}}(f), f \rangle \\ &= \frac{1}{k} \sum_{j=0}^{k-1} \langle \rho_{g_j}(f), f \rangle\end{aligned}$$



Rotational Symmetry

$$\text{Sym}^2(f, G_{p,k}) = \frac{1}{k} \sum_{j=0}^{k-1} \left\langle \rho_{g_j}(f), f \right\rangle$$

The measure of k -fold rotational symmetry about the axis p can be computed by taking the average of the dot-products of the function f with its k rotations about the axis p .



Rotational Symmetry

$$\text{Sym}^2(f, G_{p,k}) = \frac{1}{k} \sum_{j=0}^{k-1} \langle \rho_{g_j}(f), f \rangle$$

So computing the measures of rotational symmetry reduces to the problem of computing the correlation of f with itself:

$$D_{f,f}(R) = \langle f, \rho_R(f) \rangle$$

And this is something that we can do using the Wigner D -transform from last lecture.



Rotational Symmetry

$$\text{Sym}^2(f, G_{p,k}) = \frac{1}{k} \sum_{j=0}^{k-1} \left\langle \rho_{g_j}(f), f \right\rangle$$

Algorithm:

Given a function f :

- Compute the auto-correlation of f .
- For each order of symmetry k :
 - » Compute the spherical function whose value at p is the average of the correlation values at rotations $R\left(p, \frac{2\pi j}{k}\right)$, with $0 \leq j < k$.



Rotational Symmetry

$$\text{Sym}^2(f, G_{p,k}) = \frac{1}{k} \sum_{j=0}^{k-1} \left\langle \rho_{g_j}(f), f \right\rangle$$

Complexity:

- Compute the auto-correlation: $O(n^3 \log^2 n)$
- For each order of symmetry k :
 - Compute the spherical function: $O(n^2 k)$

Giving a complexity of $O(n^2 K^2 + n^3 \log^2 n)$ to compute rotational symmetries through order K .

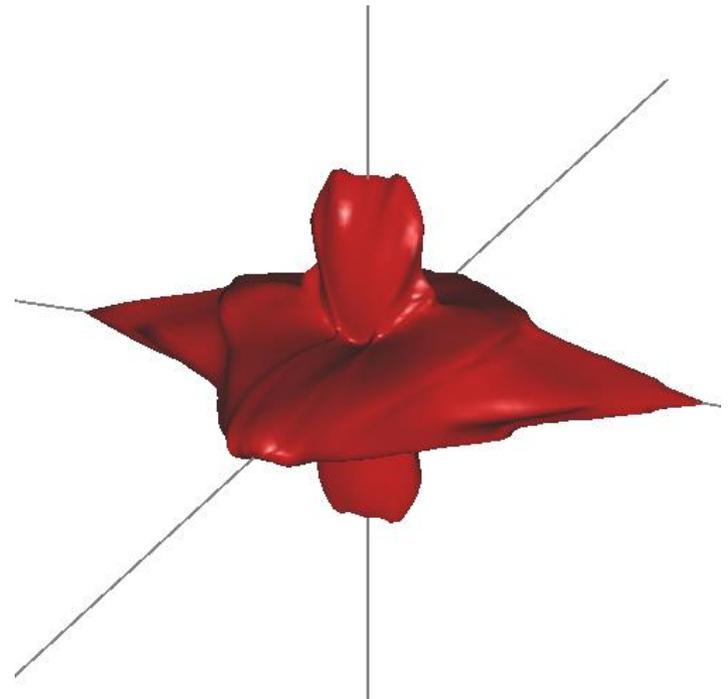
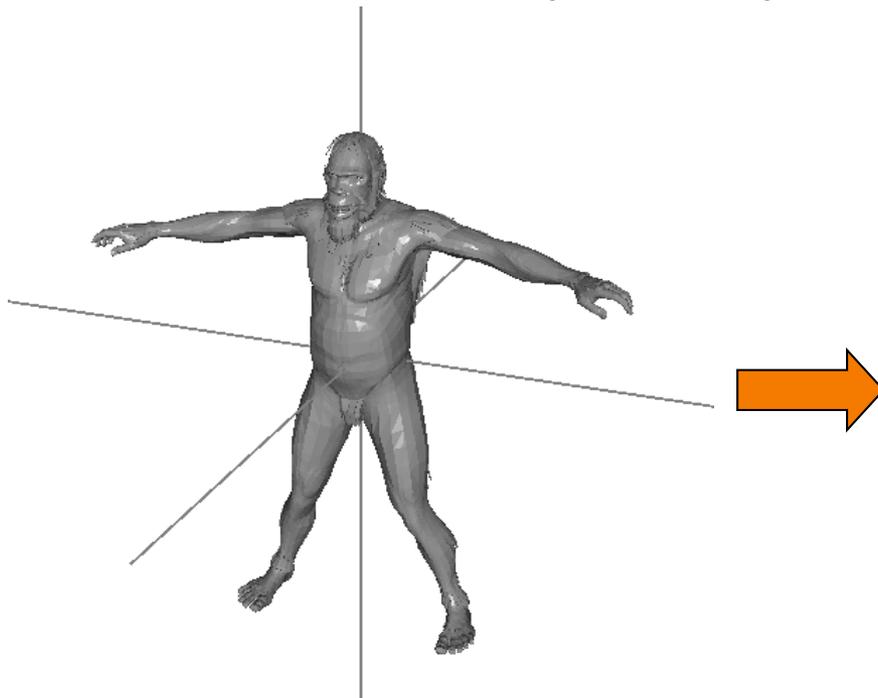


Outline

Representation Theory

Symmetry Detection

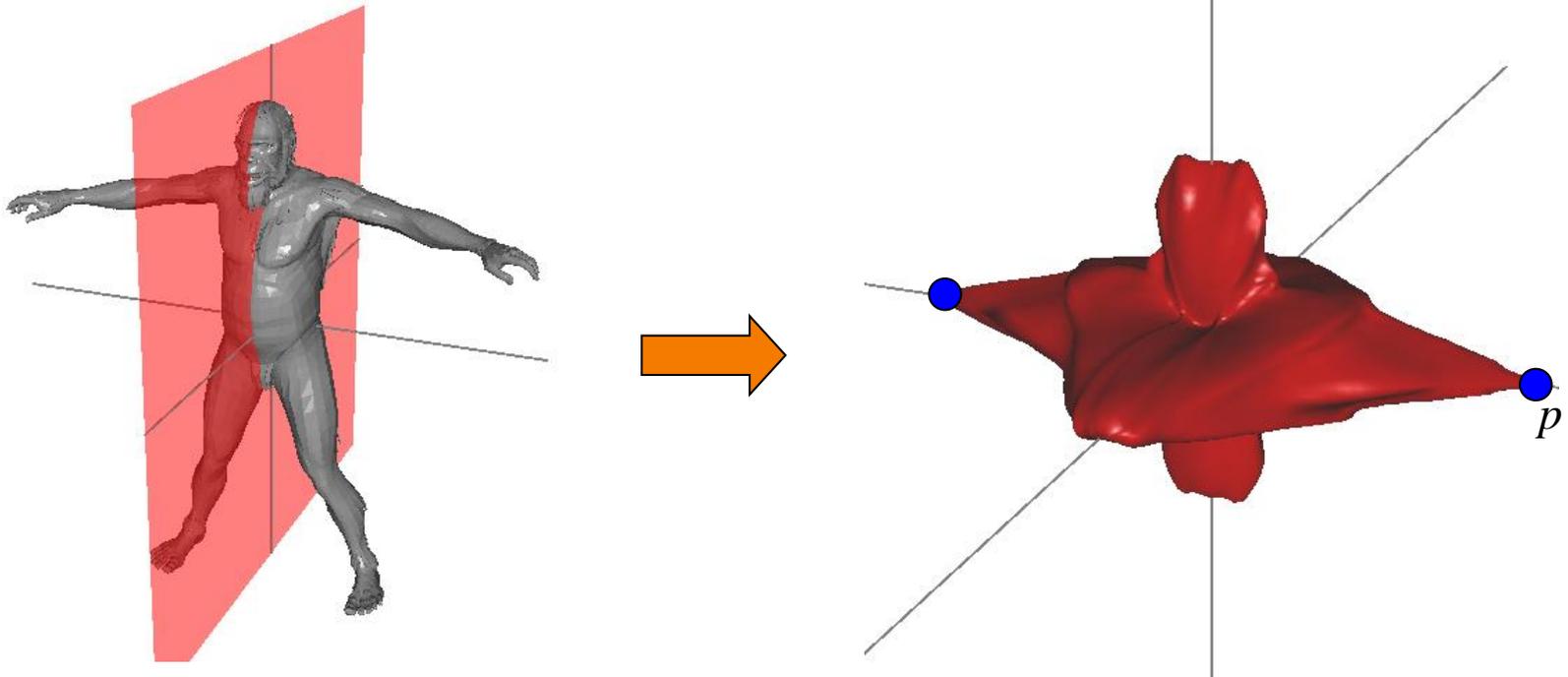
- Rotational Symmetry
- **Reflective Symmetry**





Reflective Symmetry

Given a spherical function f , we would like to compute a function whose value at a point p is the measure of reflective symmetry with respect to the plane perpendicular to p .

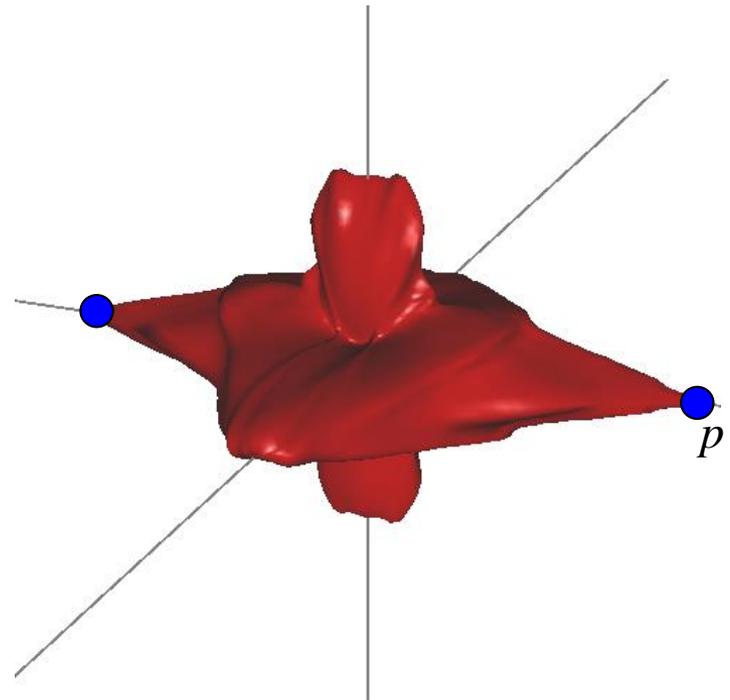
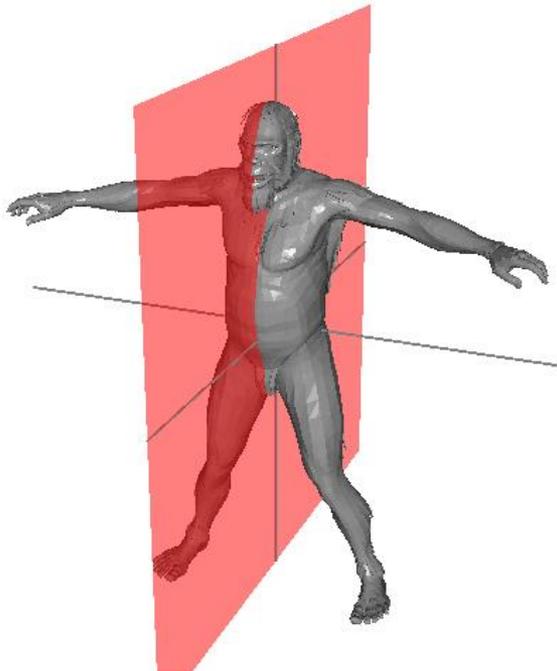




Reflective Symmetry

Reflections through the plane perpendicular to p correspond to a group with two elements:

$$G_p = \{\text{Id}, \text{Ref}_p\}$$





Reflective Symmetry

Reflections through the plane perpendicular to p correspond to a group with two elements:

$$G_p = \{\text{Id}, \text{Ref}_p\}$$

So the measure of reflective symmetry becomes:

$$\begin{aligned} \text{Sym}^2(f, G_p) &= \frac{1}{2} \left(\langle f, f \rangle + \langle \rho_{\text{Ref}_p}(f), f \rangle \right) \\ &= \frac{1}{2} \left(\|f\|^2 + \langle \rho_{\text{Ref}_p}(f), f \rangle \right) \end{aligned}$$



Reflective Symmetry

How do we compute the dot-product of the function f with the reflection of f through the plane perpendicular to p ?

Since reflections are not rotations we cannot use the auto-correlation (directly).



Reflective Symmetry

General Approach:

If we have two orthogonal transformations S and T , with determinant -1 , we can set R to be the transformation:

$$R = T \cdot S$$

Since S and T are both orthogonal, the product R must also be orthogonal.

Since both S and T have determinant -1 , R must have determinant 1 .



Reflective Symmetry

General Approach:

If we have two orthogonal transformations S and T , with determinant -1 , we can set R to be the transformation:

$$R = T \cdot S$$

Thus, R must be a rotation and we have:

$$T = R \cdot S^{-1}$$

⇒ Any orthogonal transformation T with det. -1 can be expressed as the product of a rotation with (the inverse of) a fixed orthogonal transformation with det. -1 .



Reflective Symmetry

General Approach:

Compute the correlation of f with an orthogonal transformation with determinant -1 , $\rho_S(f)$:

$$D_{\rho_S(f),f}(R) = \langle \rho_R(\rho_S(f)), f \rangle$$

Then we can get the dot-product of f with its reflection through the plane perpendicular to p :

$$\begin{aligned} \langle \rho_{\text{Ref}_p}(f), f \rangle &= \langle \rho_{\text{Ref}_p \cdot S^{-1}}(\rho_S(f)), f \rangle \\ &= D_{\rho_S(f),f}(\underbrace{\text{Ref}_p \cdot S^{-1}}_{\text{Rotation}}) \end{aligned}$$



Reflective Symmetry

$$\text{Sym}^2(f, G_p) = \frac{1}{2} \left(\|f\|^2 + D_{\rho_S(f), f}(\text{Ref}_p \cdot S^{-1}) \right)$$

Algorithm:

Given a function f :

- Compute the correlation of f with $\rho_S(f)$
- Compute the spherical function whose value at p is the average of the size of f and the dot-product of f with the rotation of $\rho_S(f)$ by $\text{Ref}_p \cdot S^{-1}$.



Reflective Symmetry

$$\text{Sym}^2(f, G_p) = \frac{1}{2} \left(\|f\|^2 + D_{\rho_S(f), f}(\text{Ref}_p \cdot S^{-1}) \right)$$

Complexity:

- Compute the correlation: $O(n^3 \log^2 n)$
- Compute the spherical function: $O(n^2)$

Giving a complexity of $O(n^3 \log^2 n)$ to compute all reflective symmetries.



Reflective Symmetry

There are many different choices for the reflection S we use to compute:

$$D_{\rho_S(f),f}(R)$$

A simple orthogonal transformation with determinant -1 is the antipodal map:

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Note that $S = S^{-1}$.

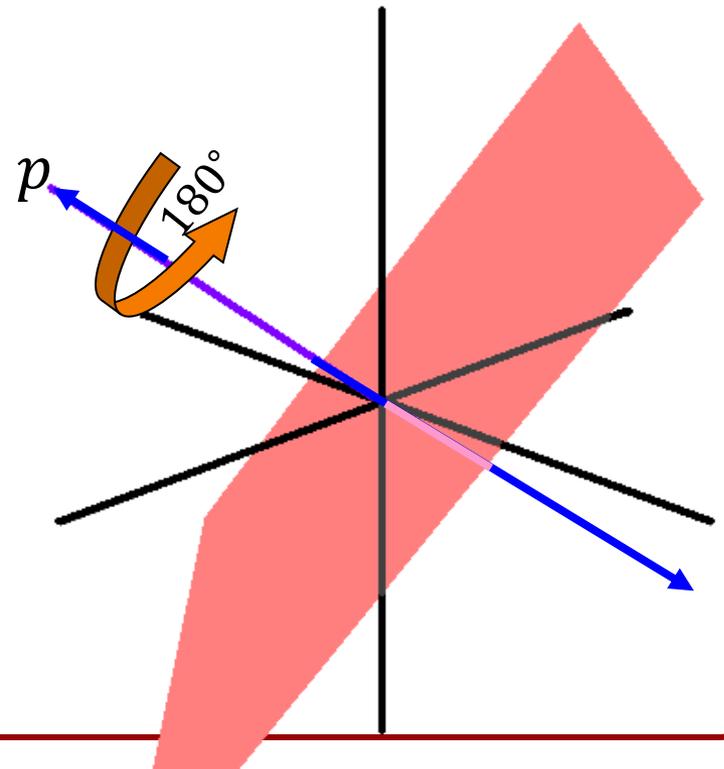


Reflective Symmetry

The advantage of using the antipodal map is that it makes it easy to express $\text{Ref}_p \cdot S$.

Fixing a point p , S is the composition of two maps:

- A reflection through the plane perpendicular to p , and
- A rotation by 180° about the axis through p .

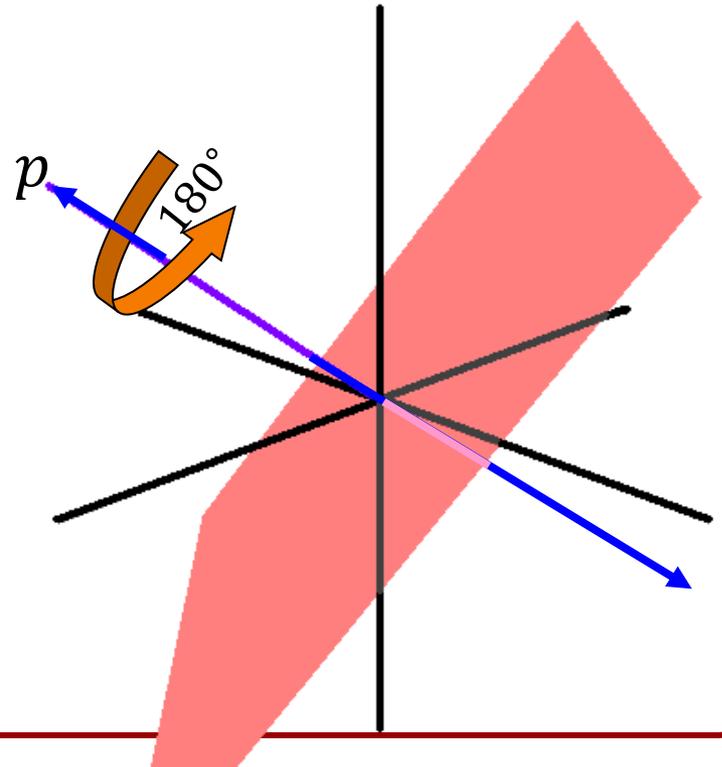




Reflective Symmetry

So a reflection through the plane perpendicular to p is the product of the antipodal map and a rotation by 180° around the axis through p :

$$\text{Ref}_p = R(p, \pi) \cdot S$$





Reflective Symmetry

Setting S to be the antipodal map, we get:

$$\text{Sym}^2(f, G_p) = \frac{1}{2} (\|f\|^2 + \langle \rho_{R(p,\pi)}(\rho_S(f)), f \rangle)$$

Note that evaluating reflective symmetry only requires knowing the correlation values for 180° rotations.

For computing reflective symmetries, the computation of the correlation is overkill as we don't use most of the correlation values.

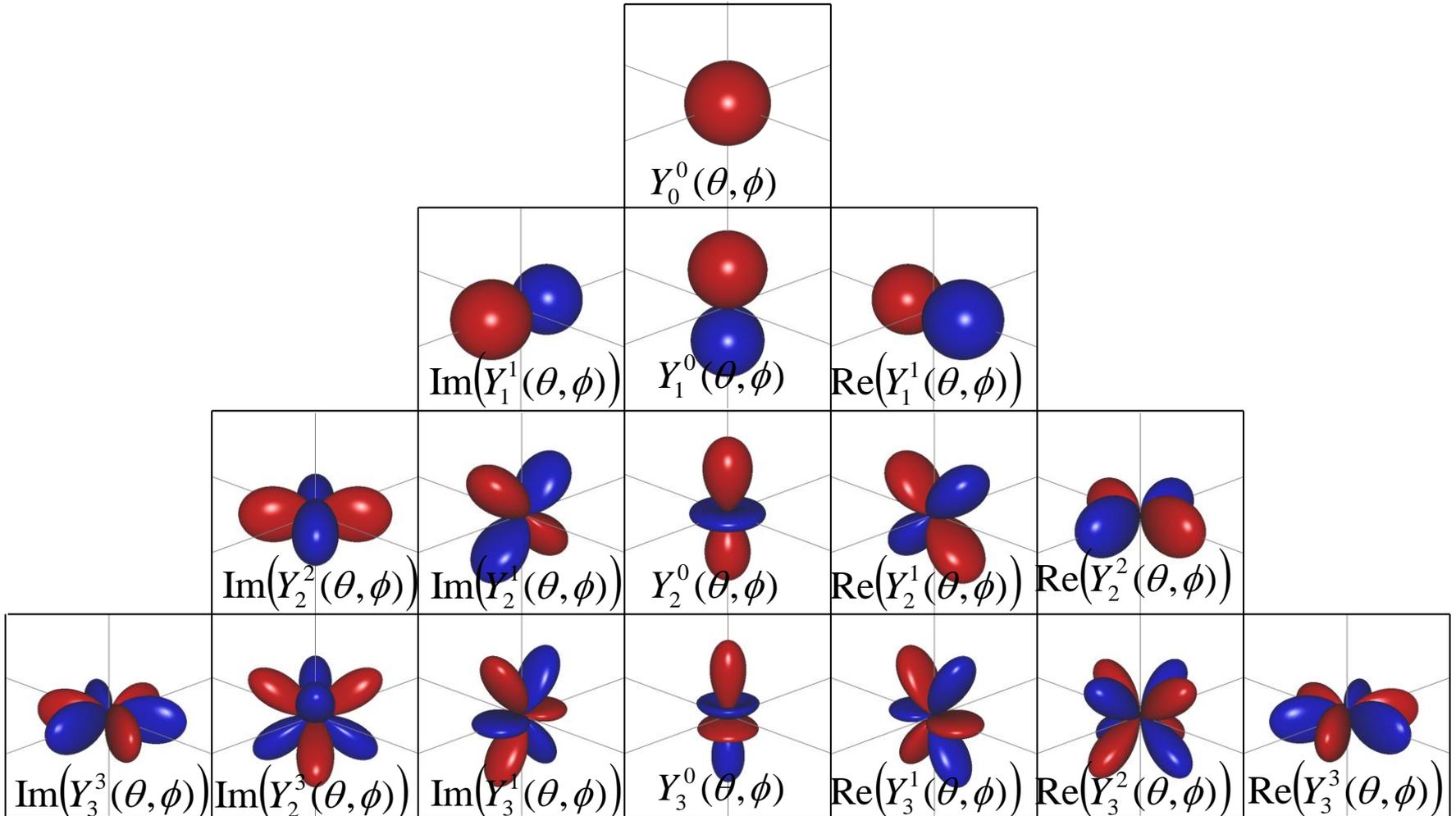


Reflective Symmetry

Since the spherical harmonics of degree l are homogenous polynomials of degree l , we get a simple expression for $\rho_S(f)$:

$$\rho_S(f) = \sum_l (-1)^l \sum_{m=-l}^l \hat{\mathbf{f}}_{lm} Y_l^m$$

Reflective Symmetry





Reflective Symmetry

In particular, if f is antipodally symmetric:

$$\rho_S(f) = f$$

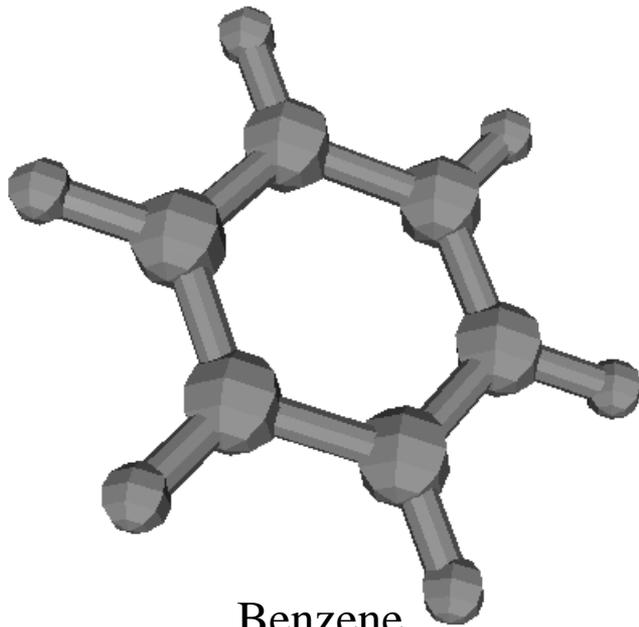
we have:

$$\begin{aligned}\text{Sym}^2(f, G_p) &= \frac{1}{2} (\|f\|^2 + \langle \rho_{R(p,\pi)}(\rho_S(f)), f \rangle) \\ &= \frac{1}{2} (\|f\|^2 + \langle \rho_{R(p,\pi)}(f), f \rangle) \\ &= \text{Sym}^2(f, G_{p,2})\end{aligned}$$

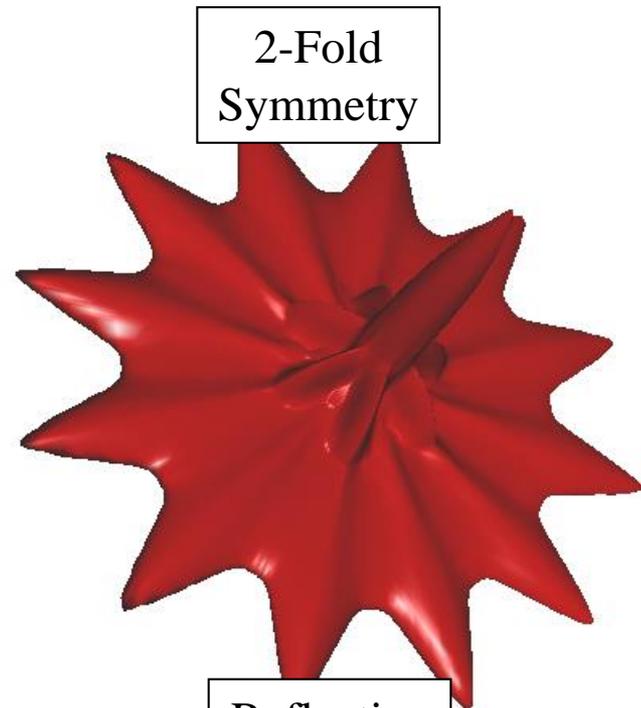


Reflective Symmetry

That is, if f is antipodally symmetric, the 2-fold rotational symmetries of f and the reflective symmetries of f are the same.



Benzene



2-Fold
Symmetry

Reflective
Symmetry