



FFTs in Graphics and Vision

Spherical Harmonics
and
Legendre Polynomials



Outline

Math Stuff

Gram-Schmidt Orthogonalization

Completing Homogenous Polynomials

Review

Defining the Harmonics



Recall

Given a monomial x^d , we have:

$$\int_{-1}^1 x^d dx = \frac{1}{d+1} x^{d+1} \Big|_{-1}^1 = \frac{1 - (-1)^{d+1}}{d+1}$$

$\Rightarrow \int_{-1}^1 x^d dx = 0$ if d is odd.

\Rightarrow If $p(x)$ is a polynomial whose even-degree coefficients are all zero:

$$\int_{-1}^1 p(x) dx = 0$$



Gram–Schmidt Orthogonalization

Given an inner product space V , and given a basis $\{v_1, \dots, v_n\}$ we can define an orthonormal basis $\{w_1, \dots, w_n\}$ for V :

$$\langle w_i, w_j \rangle = \delta_{ij}$$

Algorithm:

Start by making v_1 a unit vector:

$$w_1 = \frac{v_1}{\|v_1\|}$$



Gram–Schmidt Orthogonalization

Given an inner product space V , and given a basis $\{v_1, \dots, v_n\}$ we can define an orthonormal basis $\{w_1, \dots, w_n\}$ for V :

$$\langle w_i, w_j \rangle = \delta_{ij}$$

Algorithm:

To get the 2nd basis element, subtract from v_2 the w_1 component and normalize:

$$w_2 = \frac{v_2 - \langle v_2, w_1 \rangle w_1}{\|v_2 - \langle v_2, w_1 \rangle w_1\|}$$



Gram–Schmidt Orthogonalization

Given an inner product space V , and given a basis $\{v_1, \dots, v_n\}$ we can define an orthonormal basis $\{w_1, \dots, w_n\}$ for V :

$$\langle w_i, w_j \rangle = \delta_{ij}$$

Algorithm:

To get the i -th basis element, subtract off all the earlier components from v_i and normalize:

$$w_i = \frac{v_i - \langle v_i, w_{i-1} \rangle w_{i-1} - \dots - \langle v_i, w_1 \rangle w_1}{\|v_i - \langle v_i, w_{i-1} \rangle w_{i-1} - \dots - \langle v_i, w_1 \rangle w_1\|}$$



Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree N on the interval $[-1,1]$, with the inner-product:

$$\langle f(x), g(x) \rangle = \int_{-1}^1 f(x) \cdot g(x) dx$$

We would like to obtain an orthonormal basis:

$$\{p_0(x), \dots, p_N(x)\}$$

Start with the basis of monomials:

$$\{1, x, x^2, \dots, x^N\}$$

and perform Gram-Schmidt orthogonalization.



Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree N on the interval $[-1,1]$, with the inner-product:

By induction, $p_k(x)$ is a polynomial of degree k since G.S. orthogonalization only subtracts off lower-degree basis functions.

We would like to obtain an orthonormal basis:

$$\{p_0(x), \dots, p_N(x)\}$$

Start with the basis of monomials:

$$\{1, x, x^2, \dots, x^N\}$$

and perform Gram-Schmidt orthogonalization.



Gram–Schmidt Orthogonalization

Example:

Starting with the constant term, we get:

$$\begin{aligned} p_0(x) &= \frac{1}{\|1\|} \\ &= \frac{1}{\sqrt{\int_{-1}^1 dx}} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$



Gram–Schmidt Orthogonalization

Example:

For the linear term, we get:

$$\begin{aligned} p_1(x) &= \frac{x - \langle x, p_0(x) \rangle p_0(x)}{\|x - \langle x, p_0(x) \rangle p_0(x)\|} \\ &= \frac{x - \left(\int_{-1}^1 x \cdot \frac{1}{\sqrt{2}} dx \right) \frac{1}{\sqrt{2}}}{\left\| x - \left(\int_{-1}^1 x \cdot \frac{1}{\sqrt{2}} dx \right) \frac{1}{\sqrt{2}} \right\|} \\ &= \frac{x}{\sqrt{\int_{-1}^1 x^2 dx}} \\ &= \sqrt{\frac{3}{2}} x \end{aligned}$$

Gram–Schmidt Orthogonalization



Example:

And for the quadratic term:

$$p_2(x) = \frac{x^2 - \langle x^2, p_1(x) \rangle p_1(x) - \langle x^2, p_0(x) \rangle p_0(x)}{\|x^2 - \langle x^2, p_1(x) \rangle p_1(x) - \langle x^2, p_0(x) \rangle p_0(x)\|}$$

These are the *Legendre Polynomials*.



Legendre Polynomials

Claim:

The monomials comprising the k -th Legendre polynomials have degree with the parity of k :

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \dots$$

Proof by Induction:



Legendre Polynomials

Claim:

The monomials comprising the k -th Legendre polynomials have degree with the parity of k :

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \dots$$

Proof by Induction ($k = 0$):

$$p_0(x) = \frac{1}{\sqrt{2}}$$



Legendre Polynomials

Claim:

The monomials comprising the k -th Legendre polynomials have degree with the parity of k :

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \dots$$

Proof by Induction (assume true for $k = n$):

$$p_{n+1}(x) = \frac{x^{n+1} - \langle x^{n+1}, p_n(x) \rangle p_n(x) - \dots - \langle x^{n+1}, p_0(x) \rangle p_0(x)}{\|x^{n+1} - \langle x^{n+1}, p_n(x) \rangle p_n(x) - \dots - \langle x^{n+1}, p_0(x) \rangle p_0(x)\|}$$

Recall that:

$$\langle x^{n+1}, p_m(x) \rangle = \int_{-1}^1 x^{n+1} \cdot p_m(x) dx$$



Legendre Polynomials

Claim:

The monomials comprising the k -th Legendre polynomials have degree with the parity of k :

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \dots$$

Proof by Induction (assume true for $k = n$):

$$\langle x^{n+1}, p_m(x) \rangle = \int_{-1}^1 x^{n+1} \cdot p_m(x) dx$$

Since $m \leq n$ we can assume that the monomials comprising $p_m(x)$ have the parity of m .



Legendre Polynomials

Claim:

The monomials comprising the k -th Legendre polynomials have degree with the parity of k :

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \dots$$

Proof by Induction (assume true for $k = n$):

So if n and m are both even/odd, then the polynomial $x^{n+1} \cdot p_m(x)$ is comprised of strictly odd-powered monomials:

$$\langle x^{n+1}, p_m(x) \rangle = \int_{-1}^1 x^{n+1} \cdot p_m(x) dx = 0$$



Legendre Polynomials

Claim:

The monomials comprising the k -th Legendre polynomials have degree with the parity of k :

$$p_k(x) = a_k x^k + a_{k-2} x^{k-2} + \dots$$

Proof by Induction (assume true for $k = n$):

$$p_{n+1}(x) = \frac{x^{n+1} - \langle x^{n+1}, p_n(x) \rangle p_n(x) - \langle x^{n+1}, p_{n-1}(x) \rangle p_{n-1}(x) - \dots}{\|x^{n+1} - \langle x^{n+1}, p_n(x) \rangle p_n(x) - \langle x^{n+1}, p_{n-1}(x) \rangle p_{n-1}(x) - \dots\|}$$

$\Rightarrow p_{n+1}(x)$ is obtained by starting with the monomial x^{n+1} and subtracting off monomials with the same parity.



Gram–Schmidt Orthogonalization

Example:

Consider the space of polynomials of degree N on the interval $[-1,1]$, with an inner-product giving more weight to points near the origin:

$$\langle f(x), g(x) \rangle_m = \int_{-1}^1 (1 - x^2)^m \cdot f(x) \cdot g(x) dx$$

We would like to obtain an orthonormal basis:

$$\{p_0^m(x), \dots, p_N^m(x)\}$$



Gram–Schmidt Orthogonalization

Example:

$$\langle f(x), g(x) \rangle_m = \int_{-1}^1 (1 - x^2)^m \cdot f(x) \cdot g(x) dx$$

We proceed as before with the new inner-product.

Since the weighting function is even, if f is an even function and g is an odd function (or vice-versa), the inner product must be zero:

$$\langle f(x), g(x) \rangle_m = 0$$

As before, the degree of the monomials comprising $p_l^m(x)$ must all have the parity of l .



Completing Homogenous Polynomials

Consider a polynomial $p(x, y, z)$ of degree d , consisting of monomials of degree $d, d - 2, \dots$:

$$p(x, y, z) = \sum_{k=0}^{\lfloor d/2 \rfloor} \left(\sum_{l+m+n=d-2k} a_{lmn} \cdot x^l \cdot y^m \cdot z^n \right)$$

This is *not* a homogenous polynomial.

However, if we restrict it to the sphere, we can think of it as homogenous:

$$p(x, y, z) = \sum_{k=0}^{\lfloor d/2 \rfloor} (x^2 + y^2 + z^2)^k \left(\sum_{l+m+n=d-2k} a_{lmn} \cdot x^l \cdot y^m \cdot z^n \right)$$



Completing Homogenous Polynomials

Example:

$$p(x, y, z) = x^2y + y + z$$

is not a homogenous polynomial.

But on the unit-sphere, the polynomial:

$$q(x, y, z) = x^2y + (y + z)(x^2 + y^2 + z^2)$$

has identical values and is homogenous of degree 3.



Outline

Math Stuff

Review

Spherical Harmonics

Defining the Harmonics



Spherical Harmonics

For each non-negative integer l , there are $2l + 1$ spherical harmonics of degree l .

1. Each spherical harmonic of degree l can be expressed as the restriction of a homogenous polynomial of degree l to the unit-sphere.
2. The different spherical harmonics are orthogonal to each other.



Spherical Harmonics

We saw that by considering just rotations about the y -axis, we could factor the spherical harmonics as:

$$\begin{aligned} Y_l^m(\theta, \phi) &= e^{im\theta} \cdot g_l^m(\phi) \\ &= (\cos \theta + i \sin \theta)^m \cdot P_l^m(\cos \phi) \end{aligned}$$

where $|m| \leq l$.

The functions $P_l^m(x)$ are the associated Legendre polynomials:

$$P_l^k(x) = \frac{(-1)^k}{2^l l!} (1 - x^2)^{k/2} \frac{d^{l+k}}{dx^{l+k}} (x^2 - 1)^l$$

Outline

Math Stuff

Review

Defining the Harmonics





Defining the Harmonics ($m \geq 0$)

To define the spherical harmonics, we would like to express the function:

$$Y_l^m(\theta, \phi) = (\cos \theta + i \sin \theta)^m \cdot P_l^m(\cos \phi)$$

as the restriction of a homogenous polynomial of degree l to the unit sphere.



Defining the Harmonics ($m \geq 0$)

Using the parameterization of the unit-sphere

$$\Phi(\theta, \phi) = (\cos \theta \cdot \sin \phi, \cos \phi, \sin \theta \cdot \sin \phi)$$

we get:

$$\begin{aligned} Y_l^m(\theta, \phi) &= (\cos \theta + i \sin \theta)^m \cdot P_l^m(\cos \phi) \\ &= \left(\frac{x}{\sin \phi} + i \frac{z}{\sin \phi} \right)^m \cdot P_l^m(y) \\ &= (x + iz)^m \cdot \frac{P_l^m(y)}{\sin^m \phi} \\ &= (x + iz)^m \cdot \frac{P_l^m(y)}{\left(\sqrt{1 - y^2} \right)^m} \end{aligned}$$



Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = (x + iz)^m \cdot \frac{P_l^m(y)}{(1 - y^2)^{m/2}}$$

This • is a homogenous polynomial of degree l .

This • is a homogenous polynomial of degree m .

So we want:

1. This • to complete to a homogenous polynomial of degree $l - m$.
2. The different Y_l^m to be orthogonal



Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = (x + iz)^m \cdot \frac{P_l^m(y)}{(1 - y^2)^{m/2}}$$

Homogeneous Completion:

To satisfy the homogeneity constraint, we need:

$$\frac{P_l^m(y)}{(1 - y^2)^{m/2}} = a_{l-m}y^{l-m} + a_{l-m-2}y^{l-m-2} + \dots$$

\Downarrow

$$P_l^m(y) = q_l^m(y) \cdot (1 - y^2)^{m/2}$$

for some polynomial:

$$q_l^m(y) = a_{l-m}y^{l-m} + a_{l-m-2}y^{l-m-2} + \dots$$



Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos \phi)$$

Orthogonality:

To satisfy the orthogonality constraint, we need:

$$\langle Y_l^m(\theta, \phi), Y_{l'}^{m'}(\theta, \phi) \rangle = 0 \quad \forall l \neq l' \text{ or } m \neq m'$$



Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos \phi)$$

Orthogonality ($m \neq m'$):

Since we have:

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos \phi)$$

we know that:

$$\begin{aligned} \langle Y_l^m(\theta, \phi), Y_{l'}^{m'}(\theta, \phi) \rangle &= \int_0^\pi \int_0^{2\pi} e^{im\theta} \cdot P_l^m(\cos \phi) \cdot \overline{e^{im'\theta} \cdot P_{l'}^{m'}(\cos \phi)} d\theta \sin \phi d\phi \\ &= \left(\int_0^\pi P_l^m(\cos \phi) \cdot \overline{P_{l'}^{m'}(\cos \phi)} \sin \phi d\phi \right) \cdot \left(\int_0^{2\pi} e^{i(m-m')\theta} d\theta \right) \end{aligned}$$



Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos \phi)$$

Orthogonality ($m \neq m'$):

$$\langle Y_l^m(\theta, \phi), Y_l^{m'}(\theta, \phi) \rangle = \left(\int_0^\pi P_l^m(\cos \phi) \cdot \overline{P_l^{m'}(\cos \phi)} \sin \phi d\phi \right) \boxed{\left(\int_0^{2\pi} e^{i(m-m')\theta} d\theta \right)}$$

But this • is zero whenever $m \neq m'$:

$$\int_0^{2\pi} e^{i(m-m')\theta} d\theta = \frac{1}{i(m-m')} \cdot e^{i(m-m')\theta} \bigg|_0^{2\pi} = 0$$



Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos \phi)$$

Orthogonality ($m = m'$ and $l \neq l'$):

We have to choose the function:

$$P_l^m(y) = q_l^m(y) \cdot (1 - y^2)^{m/2}$$

so that:

$$0 = \int_0^\pi \int_0^{2\pi} e^{im\theta} \cdot P_l^m(\cos \phi) \cdot \overline{e^{im\theta} \cdot P_{l'}^m(\cos \phi)} d\theta \sin \phi d\phi$$

\Downarrow

$$0 = 2\pi \int_0^\pi P_l^m(\cos \phi) \cdot \overline{P_{l'}^m(\cos \phi)} \cdot \sin \phi d\phi$$



Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos \phi)$$

Orthogonality ($m = m'$ and $l \neq l'$):

We have to choose the function:

$$P_l^m(y) = q_l^m(y) \cdot (1 - y^2)^{m/2}$$

Changing variables:

$$\begin{aligned} 0 &= \int_0^\pi P_l^m(\cos \phi) \cdot \overline{P_{l'}^m(\cos \phi)} \cdot \sin \phi \, d\phi \\ &= \int_{-1}^1 P_l^m(y) \cdot \overline{P_{l'}^m(y)} \, dy \\ &= \int_{-1}^1 q_l^m(y) \cdot \overline{q_{l'}^m(y)} \cdot (1 - y^2)^m \, dy \end{aligned}$$



Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos \phi)$$

$$P_l^m(y) = q_l^m(y) \cdot (1 - y^2)^{m/2}$$

Thus, the polynomials $q_l^m(y)$ should:

1. Complete to homogeneous polynomials of degree $l - m$:

$$q_l^m(y) = a_{l-m}y^{l-m} + a_{l-m-2}y^{l-m-2} + \dots$$

2. Satisfy the orthogonality condition:

$$0 = \int_{-1}^1 q_l^m(y) \cdot \overline{q_{l'}^m(y)} \cdot (1 - y^2)^m dy$$



Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos \phi)$$
$$P_l^m(y) = q_l^m(y) \cdot (1 - y^2)^{m/2}$$

This is what we get with G.S. orthogonalization $\{1, y, y^2, \dots\} \rightarrow \{p_0^m(y), p_1^m(y), p_2^m(y), \dots\}$ relative to the inner-product:

$$\langle f(y), g(y) \rangle_m = \int_{-1}^1 f(y) \cdot g(y) \cdot (1 - y^2)^m dy$$

and set:

$$q_l^m(y) = p_{l-m}^m(y)$$



Defining the Harmonics ($m \geq 0$)

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot P_l^m(\cos \phi)$$
$$P_l^m(y) = q_l^m(y) \cdot (1 - y^2)^{m/2}$$

In sum, we get an expression for the spherical harmonics as:

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot p_{l-m}^m(\cos \phi) \cdot \left(\sqrt{1 - \cos^2 \phi}\right)^m$$
$$= e^{im\theta} \cdot p_{l-m}^m(\cos \phi) \cdot \sin^m \phi$$

where $p_{l-m}^m(y)$ is a polynomial of degree $l - m$ whose monomials have degree with the same parity as $l - m$.



Defining the Harmonics

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot p_{l-m}^m(\cos \phi) \cdot \sin^m \phi$$

Note:

For $m < 0$ we can define:

$$Y_l^m(\theta, \phi) = \pm \overline{Y_l^m}(\theta, \phi)$$

- These are orthogonal to each other (same argument as above).
- They are orthogonal to the spherical harmonics with $m \geq 0$, because for $m' \neq m$:

$$\int_0^{2\pi} e^{i(m-m')\theta} d\theta = 0$$



Defining the Harmonics

$$Y_l^m(\theta, \phi) = \sin^m \phi \cdot p_{l-m}^m(\cos \phi) \cdot e^{im\theta}$$

Examples ($l = 0$):

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}} = \sin^0 \phi \cdot \left(\frac{1}{\sqrt{4\pi}} \right) \cdot e^{i(0)\theta}$$



Defining the Harmonics

$$Y_l^m(\theta, \phi) = \sin^m \phi \cdot p_{l-m}^m(\cos \phi) \cdot e^{im\theta}$$

Examples ($l = 1$):

$$Y_1^{-1}(\theta, \phi) = \sin^1 \phi \cdot \left(\sqrt{\frac{3}{8\pi}} \right) \cdot e^{i(-1)\theta}$$

$$Y_1^0(\theta, \phi) = \sin^0 \phi \cdot \left(\sqrt{\frac{3}{4\pi}} \cos \phi \right) \cdot e^{i(0)\theta}$$

$$Y_1^1(\theta, \phi) = \sin^1 \phi \cdot \left(\sqrt{\frac{3}{8\pi}} \right) \cdot e^{i(1)\theta}$$



Defining the Harmonics

$$Y_l^m(\theta, \phi) = e^{im\theta} \cdot p_{l-m}^m(\cos \phi) \cdot \sin^m \phi$$

Examples ($l = 2$):

$$Y_2^{-2}(\theta, \phi) = \sin^2(\phi) \cdot \left(\sqrt{\frac{15}{32\pi}} \right) \cdot e^{i(-2)\theta}$$

$$Y_2^{-1}(\theta, \phi) = \sin^1 \phi \cdot \left(\sqrt{\frac{15}{8\pi}} \cos \phi \right) \cdot e^{i(-1)\theta}$$

$$Y_2^0(\theta, \phi) = \sin^0 \phi \cdot \left(\sqrt{\frac{5}{16\pi}} (3 \cos^2 \phi - 1) \right) \cdot e^{i(0)\theta}$$

$$Y_2^1(\theta, \phi) = \sin^1 \phi \cdot \left(\sqrt{\frac{15}{8\pi}} \cos \phi \right) \cdot e^{i(1)\theta}$$

$$Y_2^2(\theta, \phi) = \sin^2 \phi \cdot \left(\sqrt{\frac{15}{32\pi}} \right) \cdot e^{i2\theta}$$