FFTs in Graphics and Vision

Correlation

Outline

Review

Correlation:

- One-Dimensional (Continuous)
- One-Dimensional (Discrete)
- Higher-Dimensional
- Computational Complexity

Convolution

Applications in 1D

Representations

A <u>representation</u> of a group G <u>on a vector space V</u>, denoted (ρ, V), is a map ρ that sends every element in G to an invertible linear transformation on V, satisfying:

$$\rho(g \cdot h) = \rho(g) \cdot \rho(h) \quad \forall g, h \in G.$$

Sub-Representation

Given a representation (ρ, V) of a group G, if there exists a subspace $W \subset V$ such that the representation fixes W:

$$\rho_g(w) \in W \qquad \forall g \in G;\ w \in W$$

then we say that W is a sub-representation of V.

Irreducible Representations

Given a representation (ρ, V) of a group G, the representation is said to be irreducible if the only subspaces of V that are sub-representations are:

$$W = V \quad \text{and} \quad W = \{0\}$$

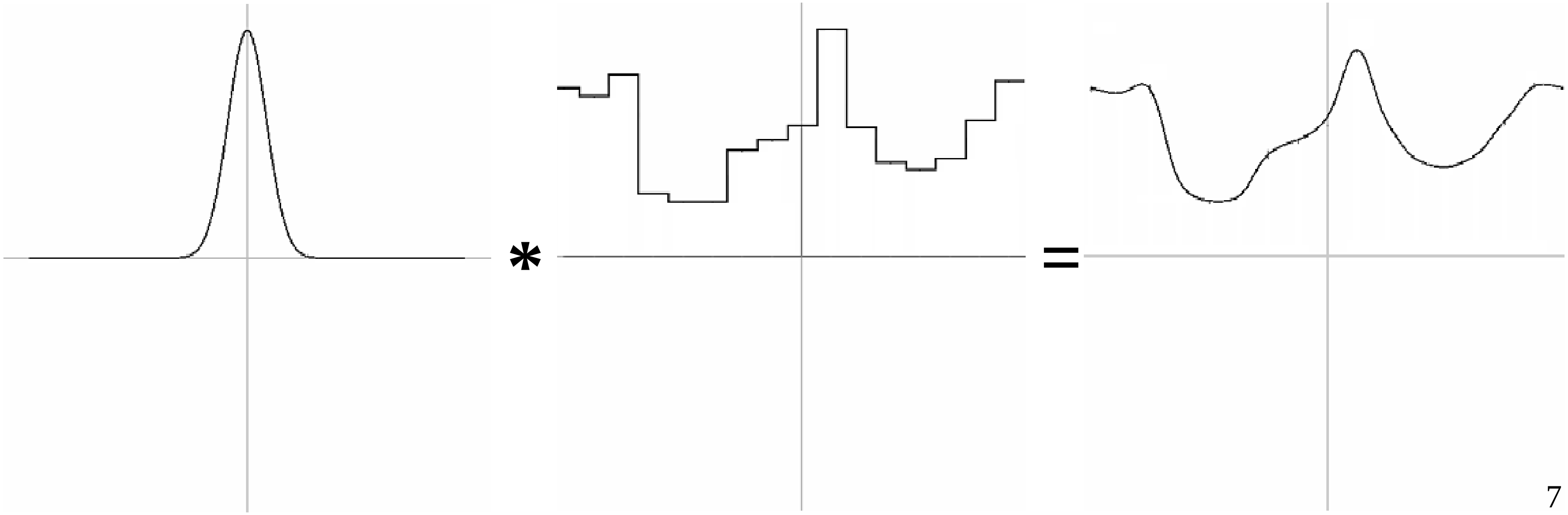
Schur's Lemma (Corollary)

If (ρ, V) is an irreducible, (unitary), representation of a commutative group G, then V must be one-dimensional.

Why do we care?

In signal/image/voxel processing, we are often interested in applying a filter to some initial data.

E.g. Smoothing:



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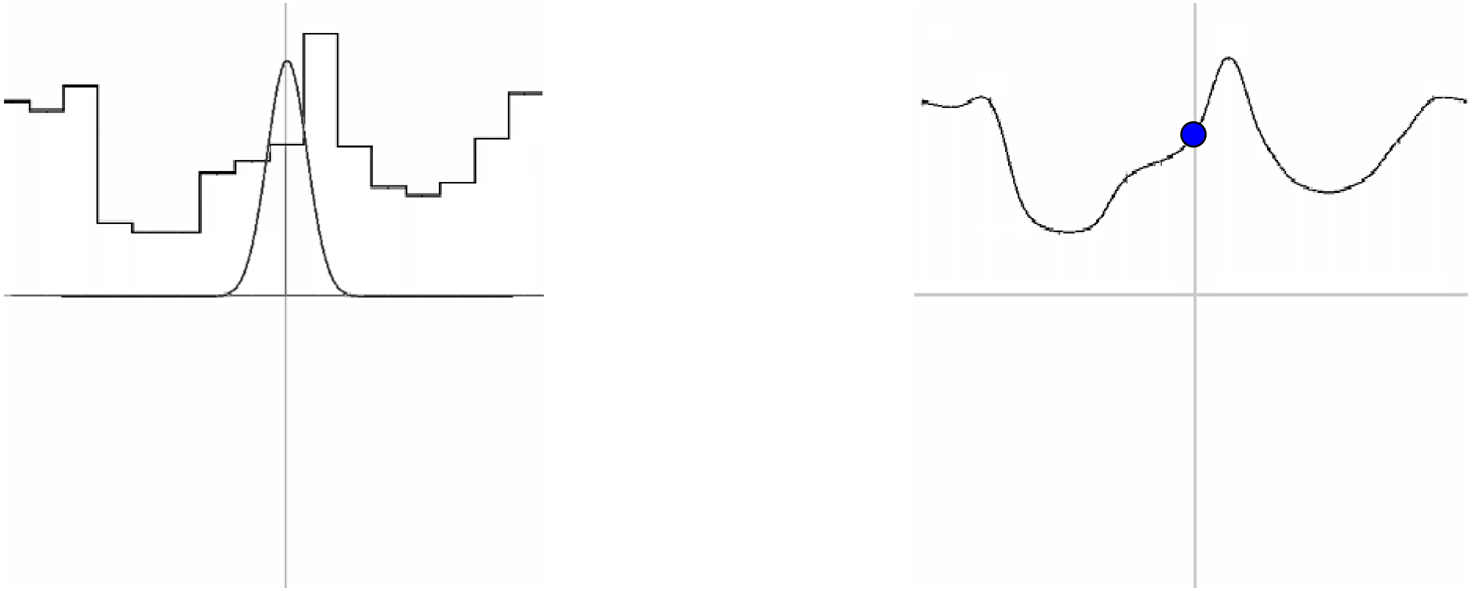

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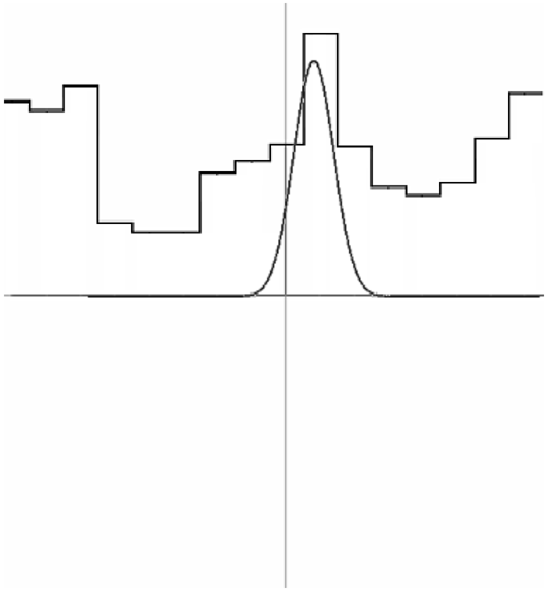

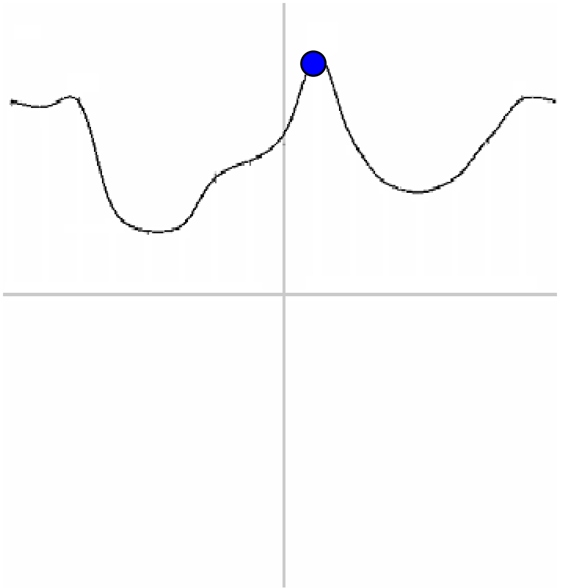

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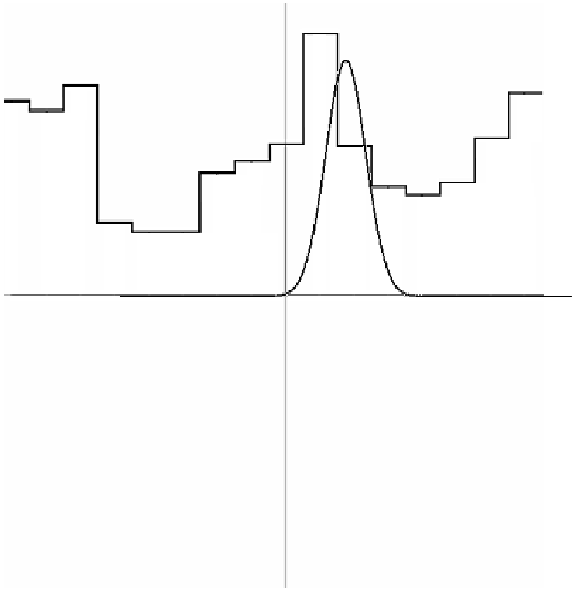

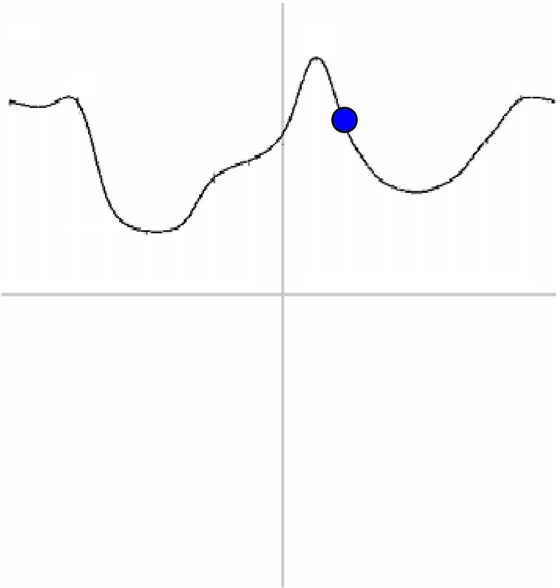

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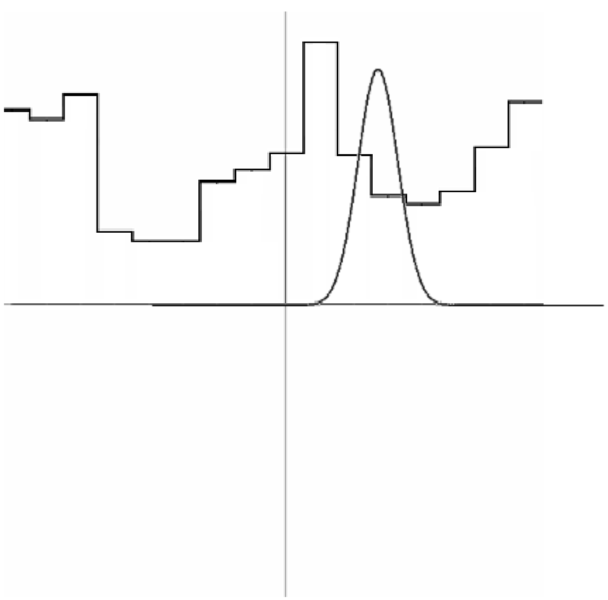

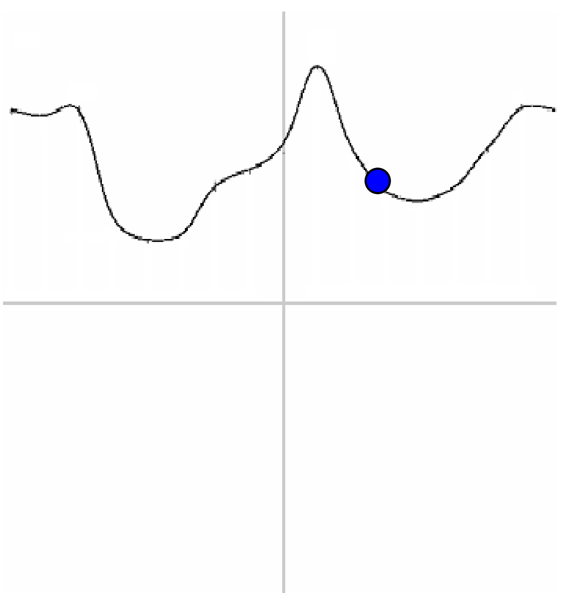

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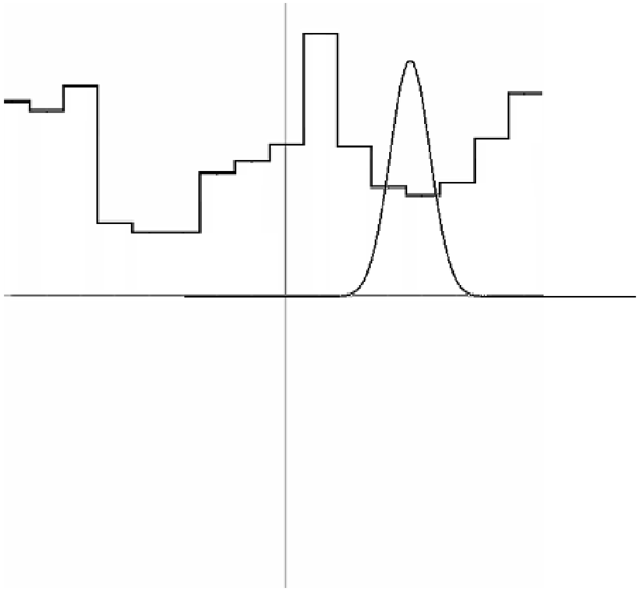

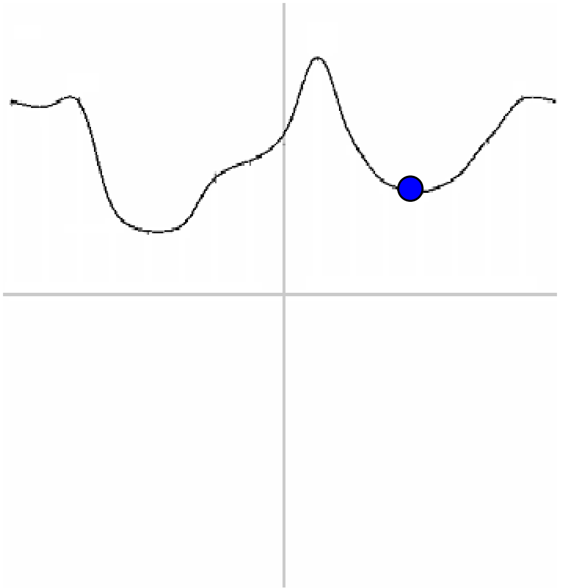

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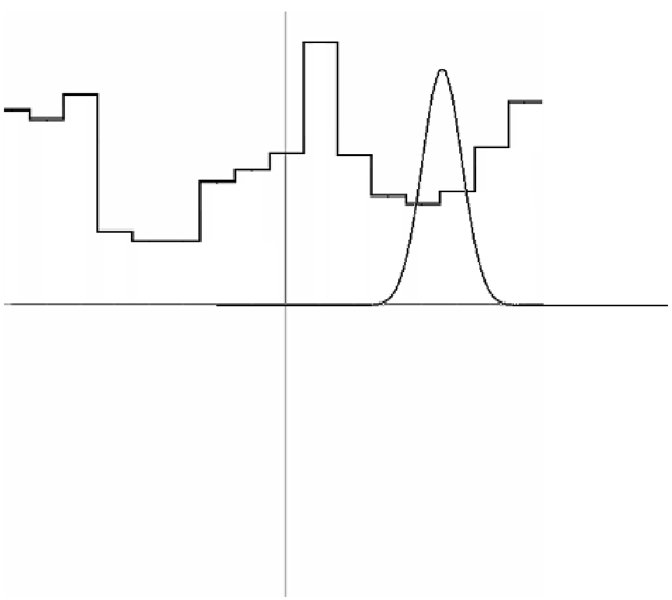

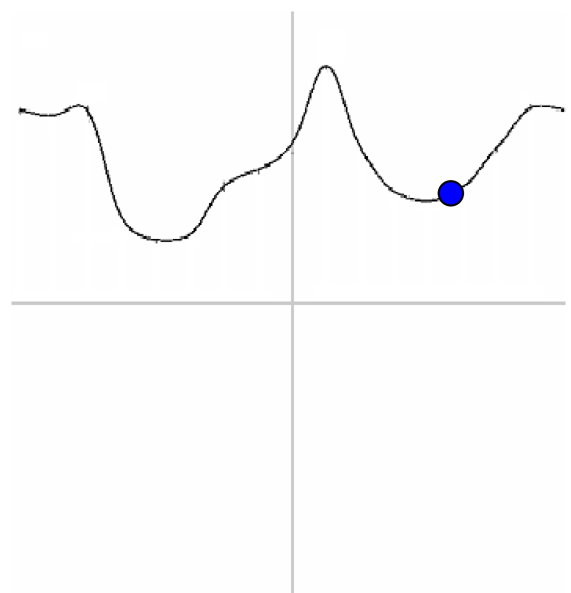

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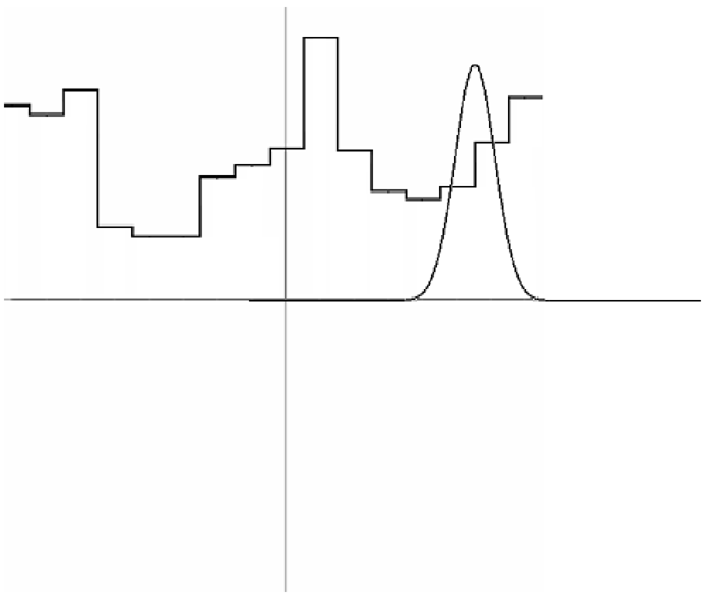
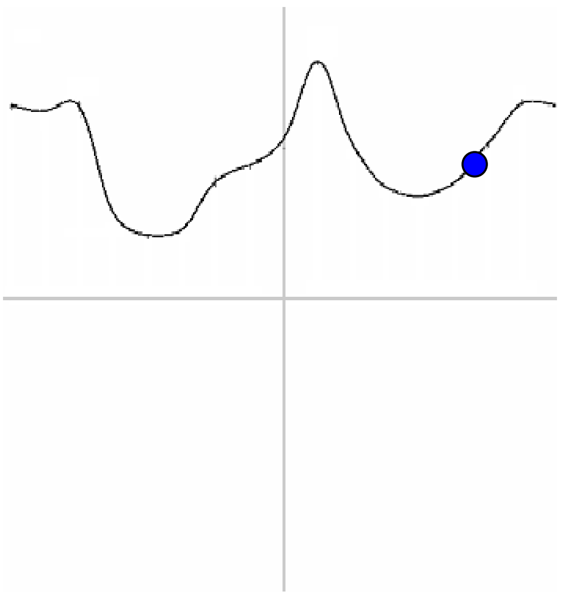
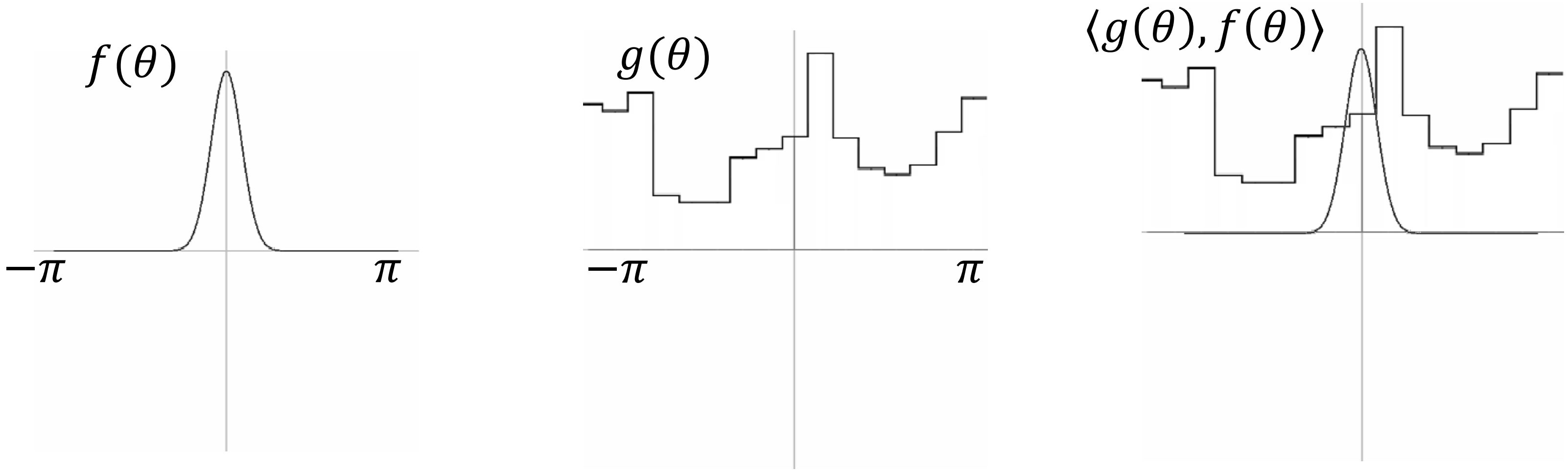




Correlation

What we are really doing is computing a moving inner product:



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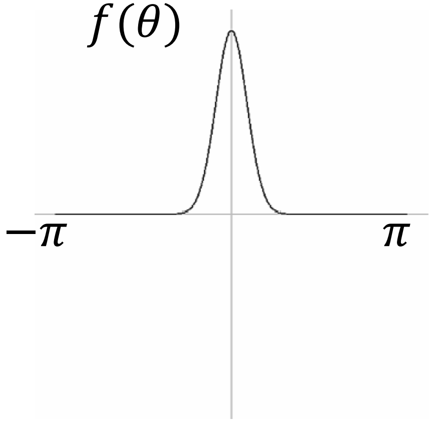

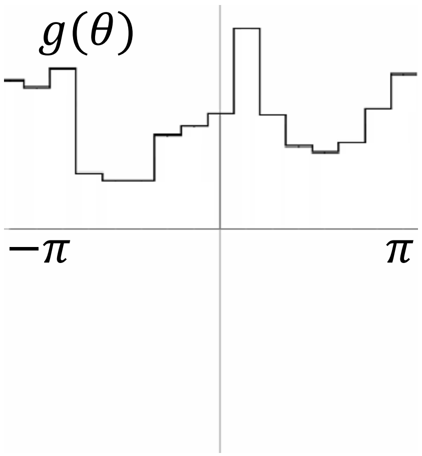

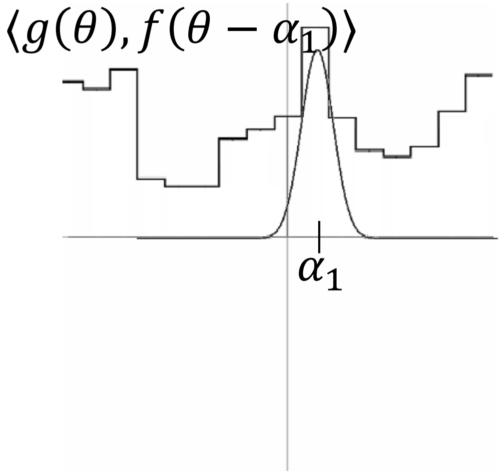

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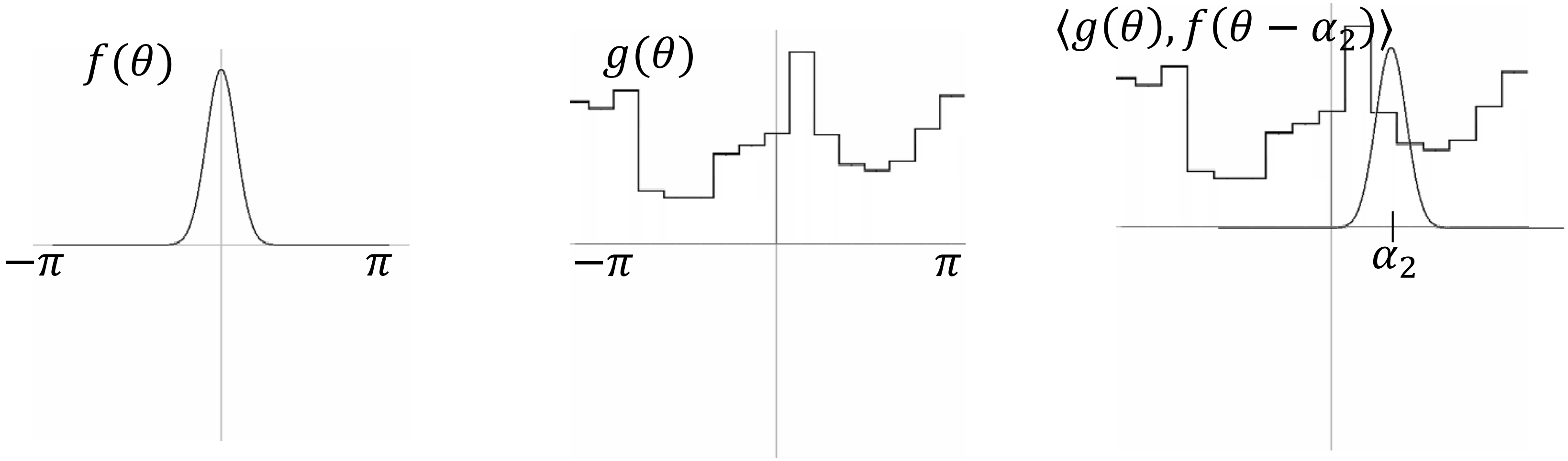

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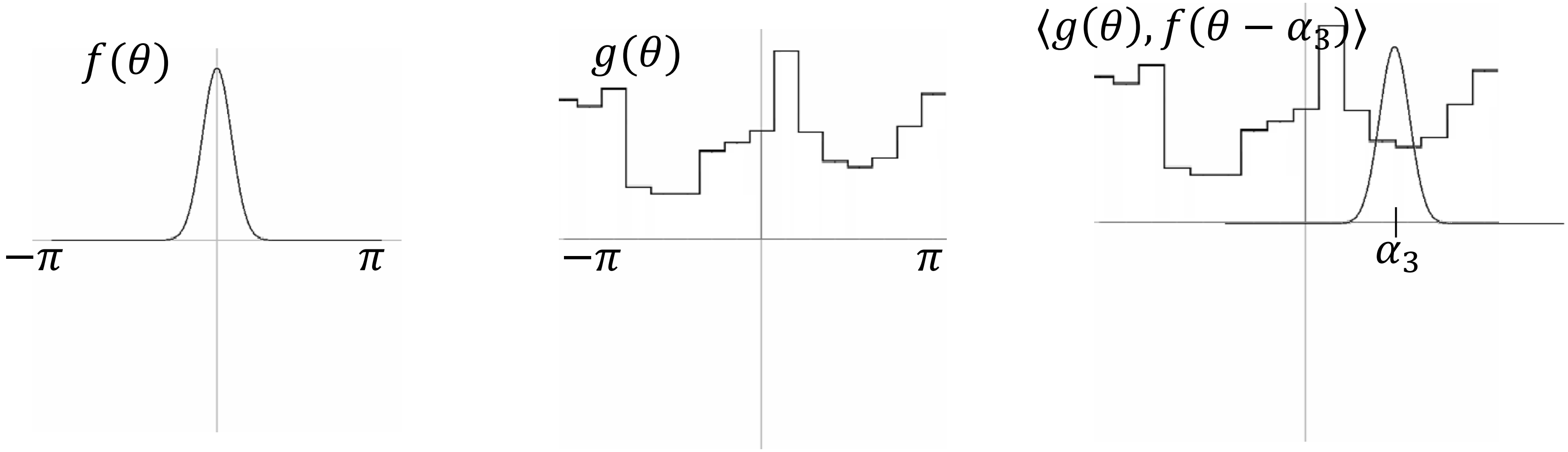
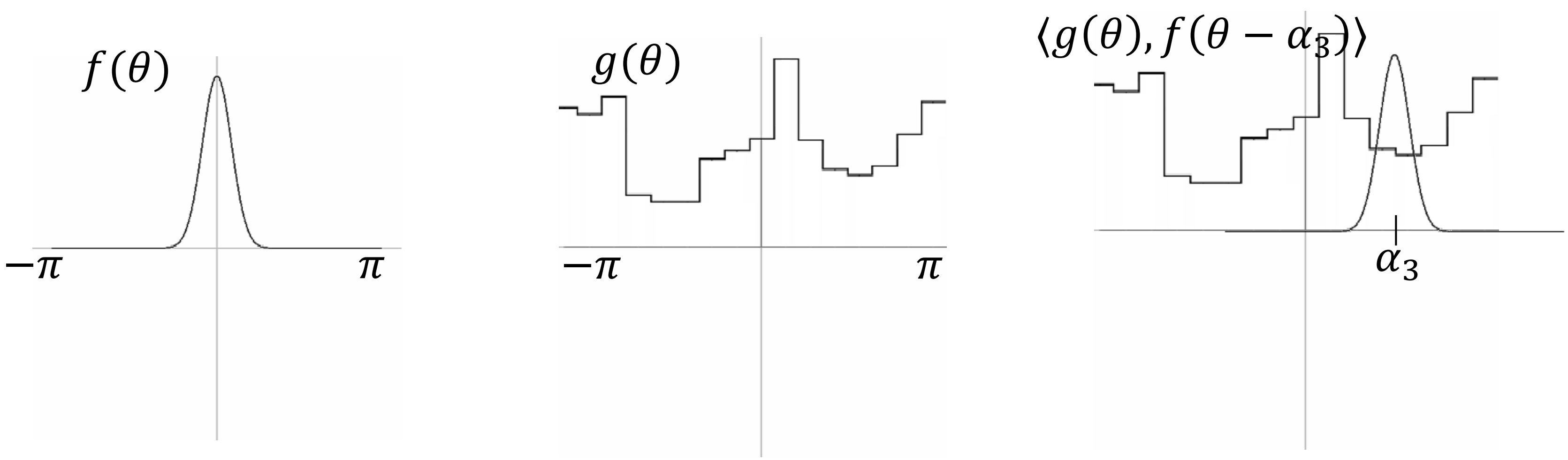


Correlation

We can write out the operation of smoothing a signal g by a filter f as:

$$(g \star f)(\alpha) = \langle g, \rho_\alpha(f) \rangle$$

where ρ_α is the linear transformation that translates a periodic function by α.



Correlation

We can think of this as a representation:

- $V = L^2(\mathbb{R}^2/(2\pi\mathbb{Z}^2))$ is the space of periodic functions on the line
- $G = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$ is the group $\mathbb{R}^2$ modulo addition by integer multiples of 2π in either coordinate
- ρ_α is the representation translating a function by α.

This is a representation of a commutative group…

Warning:

The domain of functions in V and the space G are both parametrized by points in the range $[0,2\pi)$.

- Though the parameters domains are the same, we should think of them as distinct. (The former is the circle S^1, the latter is the rotation group $SO(2)$.)

Correlation

⇒ There exist orthogonal one-dimensional (complex) subspaces $V_1, \cdots, V_n \subset V$ that are the irreducible representations of V.*

Setting $\zeta^j \in V_j$ to be a unit-vector, we know that the group acts on ζ^j by scalar multiplication.

That is, there exist $\chi^j: G \to \mathbb{C}$ s.t.:

$$\rho_\alpha(\zeta^j) = \chi^j(\alpha) \cdot \zeta^j$$

Since the ζ^j are unit vectors:

$$\chi^j(\alpha) = \langle \rho_\alpha(\zeta^j), \zeta^j \rangle$$

*In reality, there are infinitely many such subspaces.

Correlation

⇒ There exist orthogonal one-dimensional (complex) subspaces $V_1, \cdots, V_n \subset V$ that are the irreducible representations of V.*

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Since

> <u>Note</u>:
> Since the V_i are orthogonal, the function basis $\{\zeta^1, \cdots, \zeta^n\}$ is orthonormal.

$$\chi^j(\alpha) = \langle \rho_\alpha(\zeta^j), \zeta^j \rangle$$

*In reality, there are infinitely many such subspaces.

Correlation

Setting $\zeta^j \in V_j$ to be a unit-vector, we know that the group acts on ζ^j by scalar multiplication:

$$\rho_\alpha(\zeta^j) = \chi^j(\alpha) \cdot \zeta^j$$

We can write out vectors $f, g \in V$ in the basis $\{\zeta^1, \ldots, \zeta^n\}$ as:

$$f = \hat{\mathbf{f}}_1 \zeta^1 + \cdots + \hat{\mathbf{f}}_n \zeta^n$$
$$g = \hat{\mathbf{g}}_1 \zeta^1 + \cdots + \hat{\mathbf{g}}_n \zeta^n$$

with $\hat{\mathbf{f}}, \hat{\mathbf{g}} \in \mathbb{C}^n$.

Correlation

Then the correlation can be written as:

$$(g \star f)(\alpha) = \langle g, \rho_\alpha(f) \rangle$$

Expanding in the function basis $\{\zeta^1, \ldots, \zeta^n\}$:

$$(g \star f)(\alpha) = \left\langle \sum_j \hat{\mathbf{g}}_j \zeta^j, \rho_\alpha\left(\sum_k \hat{\mathbf{f}}_k \zeta^k\right) \right\rangle$$

<u>Key Idea:</u>

Since the subspaces V_i are orthogonal sub-representations, we shouldn't have to consider the inner-product between vectors from different subspaces.

Correlation

$$(g \star f)(\alpha) = \left\langle \sum_j \hat{\mathbf{g}}_j \zeta^j , \rho_\alpha \left(\sum_k \hat{\mathbf{f}}_k \zeta^k \right) \right\rangle$$

Using the linearity of ρ_α and the (conjugate)-symmetry of the inner-product:

$$= \left\langle \sum_j \hat{\mathbf{g}}_j \zeta^j , \sum_k \hat{\mathbf{f}}_k \rho_\alpha\left(\zeta^k\right) \right\rangle$$

$$= \sum_j \hat{\mathbf{g}}_j \left\langle \zeta^j , \sum_k \hat{\mathbf{f}}_k \rho_\alpha\left(\zeta^k\right) \right\rangle$$

$$= \sum_{j,k} \hat{\mathbf{g}}_j \cdot \bar{\hat{\mathbf{f}}}_k \left\langle \zeta^j , \rho_\alpha\left(\zeta^k\right) \right\rangle$$

Correlation

$$(g \star f)(\alpha) = \sum_{j,k} \hat{\mathbf{g}}_j \cdot \bar{\hat{\mathbf{f}}}_k \langle \zeta^j, \rho_\alpha(\zeta^k) \rangle$$

Because ρ_α is scalar multiplication in V_i:

$$(g \star f)(\alpha) = \sum_{j,k} \hat{\mathbf{g}}_j \cdot \bar{\hat{\mathbf{f}}}_k \langle \zeta^j, \chi^k(\alpha) \zeta^k \rangle$$

$$= \sum_{j,k} \hat{\mathbf{g}}_j \cdot \bar{\hat{\mathbf{f}}}_k \cdot \bar{\chi}^k(\alpha) \langle \zeta^j, \zeta^k \rangle$$

And finally, by the orthonormality of $\{\zeta^1, \ldots, \zeta^n\}$:

$$= \sum_{j} \hat{\mathbf{g}}_j \cdot \bar{\hat{\mathbf{f}}}_j \cdot \bar{\chi}^j(\alpha)$$

Correlation

$$(g \star f)(\alpha) = \sum_j \hat{\mathbf{g}}_j \cdot \bar{\hat{\mathbf{f}}}_j \cdot \bar{\chi}^j(\alpha)$$

This implies that we can compute the correlation by multiplying the coefficients of f and g.

Correlation in the spatial domain is multiplication in the frequency domain!

Correlation

What is $\chi^j(\alpha)$?

Since the representation is unitary, $\left|\chi^j(\alpha)\right| = 1$.

$$\Downarrow$$

$$\exists \tilde{\chi}^j : G \to \mathbb{R} \quad \text{s.t.} \quad \chi^j(\alpha) = e^{-i\tilde{\chi}^j(\alpha)}$$

Correlation

What is $\chi^j(\alpha)$?

$$\chi^j(\alpha) = e^{-i\tilde{\chi}^j(\alpha)} \text{ for some } \tilde{\chi}^j : G \to \mathbb{R}.$$

Since it's a representation:

$$\Downarrow$$

$$\chi^j(\alpha+\beta) = \chi^j(\alpha) \cdot \chi^j(\beta) \quad \forall \alpha, \beta \in G$$

$$\Downarrow$$

$$\tilde{\chi}^j(\alpha+\beta) = \tilde{\chi}^j(\alpha) + \tilde{\chi}^j(\beta)$$

$$\Downarrow$$

$$\exists \kappa_j \in \mathbb{R} \quad \text{s.t.} \quad \tilde{\chi}^j(\alpha) = \kappa_j \cdot \alpha$$

Correlation

What is $\chi^j(\alpha)$?

$$\chi^j(\alpha) = e^{-i\kappa_j\alpha} \text{ for some } \kappa_j \in \mathbb{R}.$$

Since it's a representation:

$$\Downarrow$$

$$1 = \chi^j(2\pi) = e^{-i\kappa_j 2\pi}$$

$$\Downarrow$$

$$\kappa_j \in \mathbb{Z}$$

Correlation

What is $\chi^j(\alpha)$?

$$(g \star f)(\alpha) = \sum_j \hat{\mathbf{g}}_j \cdot \bar{\hat{\mathbf{f}}}_j \cdot \bar{\chi}^j(\alpha)$$

Thus, the correlation of the signals $f, g: S^1 \to \mathbb{C}$ can be expressed as:

$$(g \star f)(\alpha) = \sum_j \hat{\mathbf{g}}_j \cdot \bar{\hat{\mathbf{f}}}_j \cdot e^{i\kappa_j\alpha}$$

where $\kappa_j \in \mathbb{Z}$.

Correlation

What is ζ^j?

By definition of χ^j, we have:

$$\rho_\alpha\left(\zeta^j\right) = \chi^j(\alpha) \cdot \zeta^j = e^{-ik_j\alpha} \cdot \zeta^j$$

for some $k_j \in \mathbb{Z}$.

On the other hand, we have:

$$\left[\rho_\alpha\left(\zeta^j\right)\right](\theta) = \zeta^j(\theta - \alpha)$$
$$= \zeta^j(\theta) \cdot e^{-ik_j\alpha}$$
$$\Downarrow$$
$$\zeta^j(\theta) = c_j \cdot e^{ik_j\theta}$$

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- One-Dimensional (Discrete)
- Higher-Dimensional
- Computational Complexity

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Applications in 1D

Correlation (Periodic Functions)

Let's consider the case of periodic functions in more detail:

Correlation (Periodic Functions)

Let's revisit periodic functions in more detail:

- $V = L^2(\mathbb{R}/2\pi\mathbb{Z})$ is the space of periodic functions on the line
  - $G = \mathbb{R}/2\pi\mathbb{Z}$ is the group $\mathbb{R}$ modulo addition by integer multiples of 2π
- ρ_α is the representation translating a function by α:

$$\big(\rho_\alpha(f)\big)(\theta) = f(\theta - \alpha)$$

The irreducible representations are:

$$V_j = \mathrm{Span}\big(\tilde{\zeta}^j(\theta) = e^{ik_j\theta}\big).$$

The corresponding scaling functions are:

$$\chi^j(\alpha) = e^{-ik_j\alpha}.$$

Correlation (Periodic Functions)

$$V_j = \text{Span}\left(\tilde{\zeta}^j(\theta) = e^{ik_j\theta}\right)$$
$$\chi^j(\alpha) = e^{-ik_j\alpha}$$

The one-dimensional sub-space:

$$\text{Span}\left(\tilde{\zeta}^j(\theta) = e^{ik\theta}\right)$$

is a sub-representation for <u>every</u> integer $k \in \mathbb{Z}$.

$$\Downarrow$$

$$V_j = \text{Span}\left(\tilde{\zeta}^j(\theta) = e^{ij\theta}\right)$$
$$\chi^j(\alpha) = e^{-ij\alpha}$$

Correlation (Periodic Functions)

Note:

The periodic functions:

$$\tilde{\zeta}^j(\theta) = e^{ij\theta}$$

do not have unit norm!

$$\begin{aligned}\left\|\tilde{\zeta}^j\right\|^2 &= \int_0^{2\pi} e^{ij\theta} \cdot \overline{e^{ij\theta}}\, d\theta \\ &= \int_0^{2\pi} 1\, d\theta \\ &= 2\pi\end{aligned}$$

Correlation (Periodic Functions)

Note:

The periodic functions:

$$\tilde{\zeta}^j(\theta) = e^{ij\theta}$$

do not have unit norm!

⇒ Normalize to make the functions unit-norm:

$$\zeta^j(\theta) = \frac{e^{ij\theta}}{\sqrt{2\pi}}$$

$$\chi^j(\alpha) = e^{-ij\alpha} = \sqrt{2\pi} \cdot \zeta^j(\alpha)$$

Correlation (Periodic Functions)

Remark:

The functions $\left\{\zeta^j\right\}_{j=-\infty}^{\infty}$ form a (orthonormal) basis called the Fourier basis.

Correlation (Periodic Functions)

Given functions, $f, g \in L^2\left(\left(\mathbb{R}^2/(2\pi\mathbb{Z}^2)\right)\right)$, we can expand them in the basis $\left\{\zeta^k\right\}_{k=-\infty}^{\infty}$:

$$f = \sum_{k=-\infty}^{\infty} \hat{\mathbf{f}}_k \cdot \zeta^k \quad \text{and} \quad g = \sum_{k=-\infty}^{\infty} \hat{\mathbf{g}}_k \cdot \zeta^k$$

This gives:

$$(g \star f)(\alpha) = \sum_{k=-\infty}^{\infty} \hat{\mathbf{g}}_k \cdot \bar{\hat{\mathbf{f}}}_k \cdot \bar{\chi}^k(\alpha)$$

$$= \sum_{k=-\infty}^{\infty} \sqrt{2\pi} \cdot \hat{\mathbf{g}}_k \cdot \bar{\hat{\mathbf{f}}}_k \cdot \zeta^k(\alpha)$$

Correlation (Periodic Functions)

$$\rho_\alpha\left(\zeta^j\right) = e^{-ij\alpha} \cdot \zeta^j$$

What's really going on here?

If we express a complex number in terms of radius and angle (r, θ), then rotation by α degrees corresponds to the map:

$$(r, \theta) \to (r, \theta + \alpha)$$

$$\Updownarrow$$

$$re^{i\theta} \to re^{i(\theta+\alpha)} = e^{i\alpha} \cdot re^{i\theta}$$

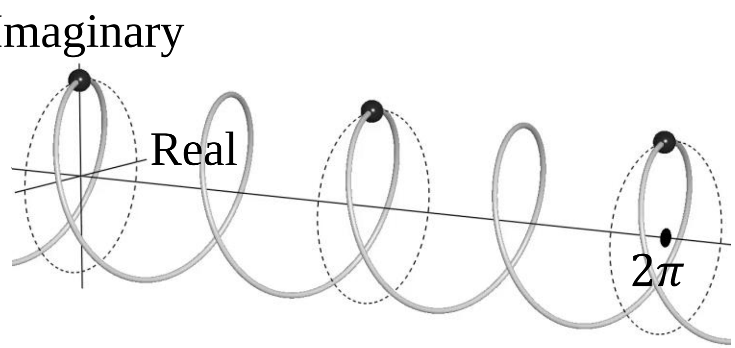
⇒ Multiplication by a complex, unit-norm, number is the same as rotation

Correlation (Periodic Functions)

$$\rho_\alpha(\zeta^j) = e^{-ij\alpha} \cdot \zeta^j$$

What's really going on here?

- Visualize complex-valued (periodic) functions the line by drawing the values in the perpendicular plane.
- A complex exponential becomes a helix.
- We can translate the helix along the line.
- This is the same as rotating in the (complex) plane that is perpendicular to the line.



$f(\theta) = e^{i4\theta}$

Correlation (Periodic Functions)

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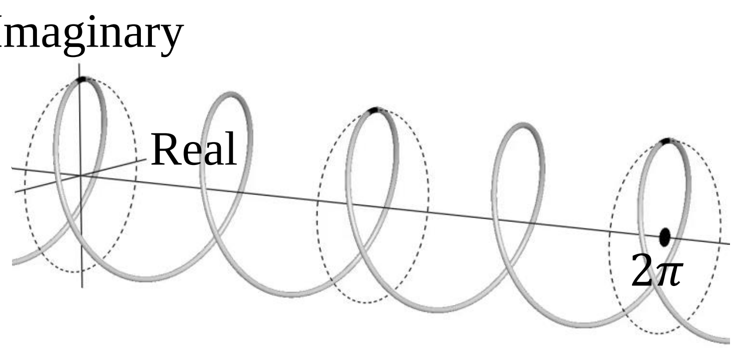

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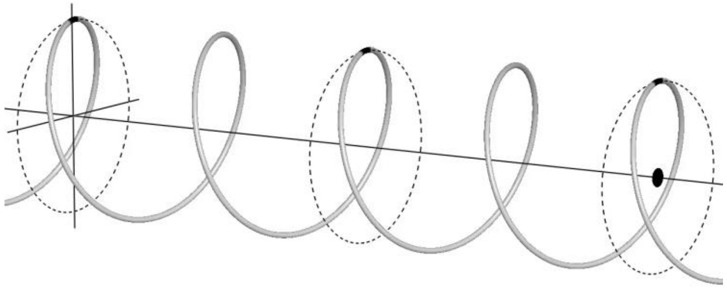

Translating the domain of a complex exponential

⇕

Multiplying its value by a (unit-normal) complex number

Outline

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Applications in 1D

Correlation (Periodic Arrays)

In practice, we don't have infinite precision, and we discretize the function space and the group:

- $V = \mathbb{R}^n$ is the space of periodic n-dimensional arrays
- $G = \mathbb{Z}/n\mathbb{Z}$ is the group of integers modulo n
- ρ_j is the representation shifting the entries in the array by j positions

What are the irreducible representations V_j?

What are the corresponding scaling functions $\mathbf{x}^j \in \mathbb{C}^n$?

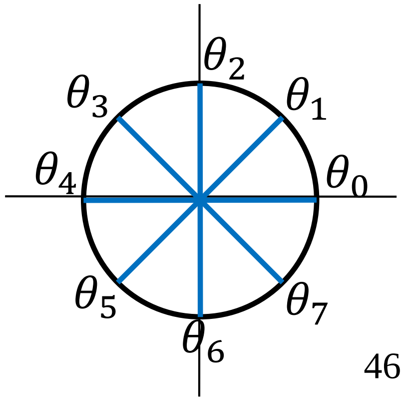
Correlation (Periodic Arrays)

We set V_k to be the (1D) spaces spanned by the <u>discretizations</u> of the complex exponentials:

$$V_k = \text{Span}\left(\tilde{\mathbf{z}}^k\right)$$

where $\tilde{\mathbf{z}}^k$ is defined by regularly sampling the k-th complex exponential:

$$\tilde{\mathbf{z}}^k = \left(e^{ik\theta_0}, \cdots, e^{ik\theta_{n-1}}\right) \quad \text{with } \theta_j = \frac{2\pi j}{n}$$



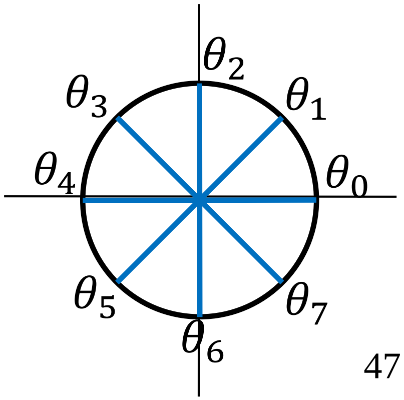
Correlation (Periodic Arrays)

Applying ρ_j to $\tilde{\mathbf{z}}^k$, we get:

$$\rho_j\left(\tilde{\mathbf{z}}^k\right) = \left(e^{ik\theta_{0-j}}, \cdots, e^{ik\theta_{n-1-j}}\right)$$

We can write out:

$$\begin{aligned}\theta_{m-j} &= \frac{2\pi(m-j)}{n} \\ &= \frac{2\pi m}{n} + \frac{-2\pi j}{n} \\ &= \theta_m + \theta_{-j}\end{aligned}$$



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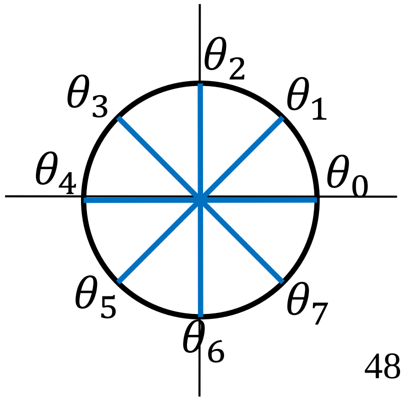
We can write out:

$$\theta_{m-j} = \theta_m + \theta_{-j}$$

Thus:

$$\begin{aligned}\rho_j\left(\tilde{\mathbf{z}}^k\right) &= \left(e^{ik\theta_0} \cdot e^{ik\theta_{-j}}, \cdots, e^{ik\theta_{n-1}} \cdot e^{ik\theta_{-j}}\right)\\ &= e^{ik\theta_{-j}} \cdot \tilde{\mathbf{z}}^k\end{aligned}$$

$$\mathbf{x}_j^k = e^{-ik\theta_j}$$



Correlation (Periodic Arrays)

Note 1:

The periodic arrays:

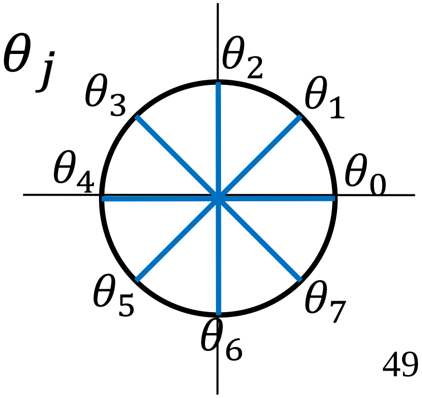
$$\tilde{\mathbf{z}}^k = \left(e^{ik\theta_0}, \cdots, e^{ik\theta_{n-1}}\right)$$

do not have unit norm!

$$\left\langle \tilde{\mathbf{z}}^k, \tilde{\mathbf{z}}^k \right\rangle_{[0,2\pi)} = \frac{2\pi}{n} \cdot \sum_{j=0}^{n-1} \tilde{\mathbf{z}}_j^k \cdot \bar{\tilde{\mathbf{z}}}_j^k$$

$$= \frac{2\pi}{n} \cdot \sum_{j=0}^{n-1} e^{ik\theta_j} \cdot e^{-ik\theta_j}$$

$$= 2\pi$$



Correlation (Periodic Arrays)

Note 1:

The periodic arrays:

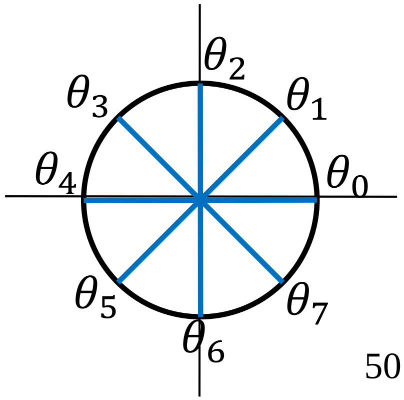
$$\tilde{\mathbf{z}}^k = \left(e^{ik\theta_0}, \cdots, e^{ik\theta_{n-1}}\right)$$

do not have unit norm!

$$\mathbf{x}^k = \sqrt{2\pi} \cdot \mathbf{z}^{-k}$$

We need to normalize these vectors to make them unit-norm:

$$\mathbf{z}^k = \left(\frac{e^{ik\theta_0}}{\sqrt{2\pi}}, \cdots, \frac{e^{ik\theta_{n-1}}}{\sqrt{2\pi}}\right)$$



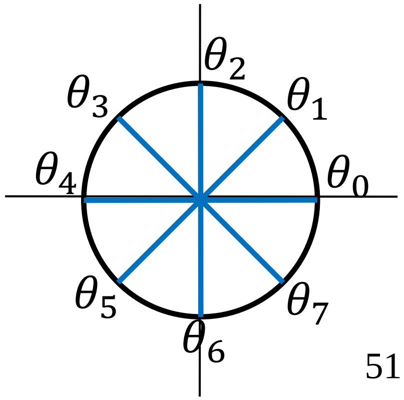
Correlation (Periodic Arrays)

Note 2:

The arrays $\mathbf{z}^k$ and $\mathbf{z}^{k+n}$ are the same array:

$$\mathbf{z}^{k+n} = \left(\frac{e^{i(k+n)\theta_0}}{\sqrt{2\pi}}, \cdots, \frac{e^{i(k+n)\theta_{n-1}}}{\sqrt{2\pi}}\right)$$

$$= \left(\frac{e^{ik\theta_0}}{\sqrt{2\pi}} \cdot e^{in\theta_0}, \cdots, \frac{e^{ik\theta_{n-1}}}{\sqrt{2\pi}} \cdot e^{in\theta_{n-1}}\right)$$



Correlation (Periodic Arrays)

Note 2:

The arrays $\mathbf{z}^k$ and $\mathbf{z}^{k+n}$ are the same array:

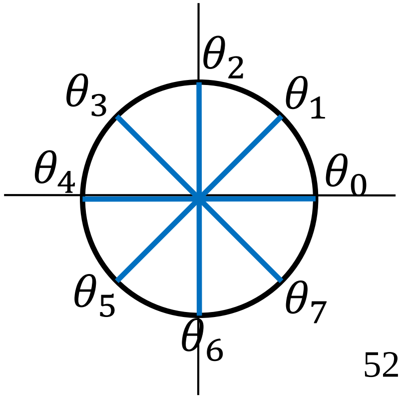
$$\mathbf{z}^{k+n} = \left(\frac{e^{ik\theta_0}}{\sqrt{2\pi}} \cdot e^{in\theta_0}, \ldots, \frac{e^{ik\theta_{n-1}}}{\sqrt{2\pi}} \cdot e^{in\theta_{n-1}} \right)$$

But $n\theta_j$ is a multiple of 2π:

$$n\theta_j = \frac{n2\pi j}{n} = 2\pi j$$

$$\Updownarrow$$

$$e^{in\theta_j} = 1$$



Correlation (Periodic Arrays)

Note 2:

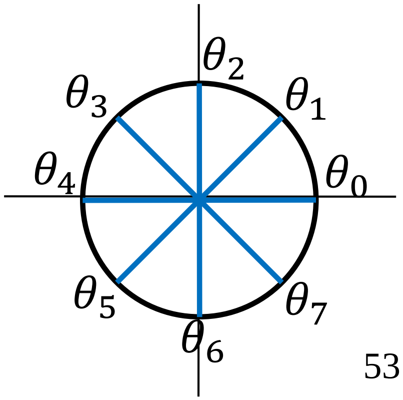
The arrays $\mathbf{z}^k$ and $\mathbf{z}^{k+n}$ are the same array:

$$e^{in\theta_j} = 1$$

$$\mathbf{z}^{k+n} = \left(\frac{e^{ik\theta_0}}{\sqrt{2\pi}} \cdot e^{in\theta_0}, \cdots, \frac{e^{ik\theta_{n-1}}}{\sqrt{2\pi}} \cdot e^{in\theta_{n-1}} \right)$$

$$= \left(\frac{e^{ik\theta_0}}{\sqrt{2\pi}}, \cdots, \frac{e^{ik\theta_{n-1}}}{\sqrt{2\pi}} \right)$$

$$= \mathbf{z}^k$$



Correlation (Periodic Arrays)

Note 3:

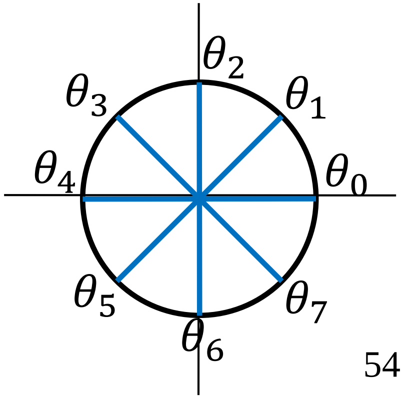
The arrays $\mathbf{z}^k$ and $\mathbf{z}^{-k}$ are conjugate.

$$\mathbf{z}^{-k} = \left(\frac{e^{i(-k)\theta_0}}{\sqrt{2\pi}}, \cdots, \frac{e^{i(-k)\theta_{n-1}}}{\sqrt{2\pi}}\right)$$

$$= \left(\frac{e^{-ik\theta_0}}{\sqrt{2\pi}} \cdot e^{in\theta_0}, \cdots, \frac{e^{-ik\theta_{n-1}}}{\sqrt{2\pi}}\right)$$

$$= \left(\frac{\overline{e^{ik\theta_0}}}{\sqrt{2\pi}} \cdot e^{in\theta_0}, \cdots, \frac{\overline{e^{ik\theta_{n-1}}}}{\sqrt{2\pi}}\right)$$

$$= \bar{\mathbf{z}}^k$$

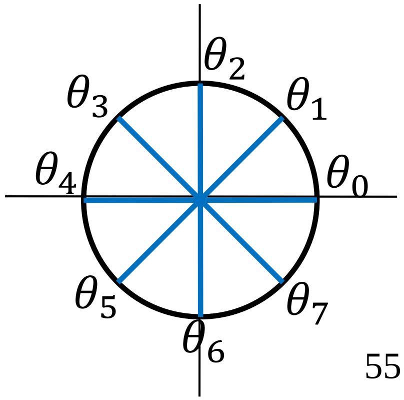


Correlation (Periodic Arrays)

Note 4:

Reflecting the coefficients of array $\mathbf{z}^k$ through the origin gives:

$$\mathbf{z}^k_{-j} = \frac{e^{ik\theta_{-j}}}{\sqrt{2\pi}} = \frac{e^{-ik\theta_j}}{\sqrt{2\pi}} = \overline{\frac{e^{ik\theta_j}}{\sqrt{2\pi}}} = \bar{\mathbf{z}}^k_j$$



Correlation (Periodic Arrays)

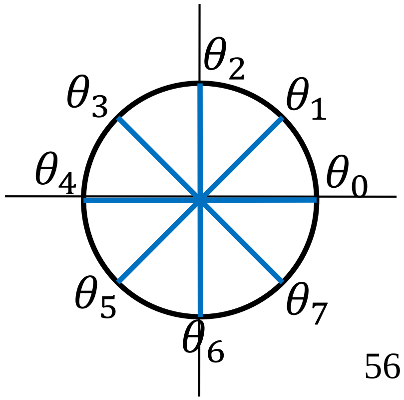
Note 5:

The arrays $\{\mathbf{z}^0, \cdots, \mathbf{z}^{n-1}\}$ are linearly independent.

(More specifically, the arrays are orthonormal.)

⇒ Since the space V is n-dimensional, the arrays form a basis.

This is the discrete Fourier basis.



Correlation (Periodic Arrays)

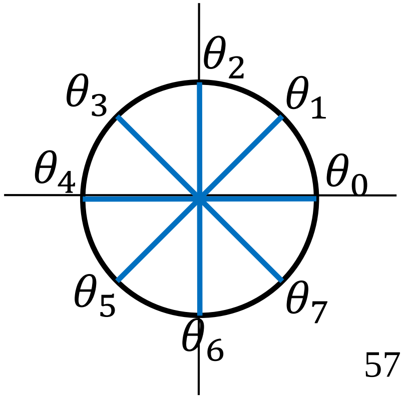
Thus, given two n-dimensional arrays, $\mathbf{f}$ and $\mathbf{g}$, we can expand:

$$\mathbf{f} = \sum_{k=0}^{n-1} \hat{\mathbf{f}}_k \cdot \mathbf{z}^k \quad \text{and} \quad \mathbf{g} = \sum_{k=0}^{n-1} \hat{\mathbf{g}}_k \cdot \mathbf{z}^k$$

This gives:

$$(\mathbf{g} \star \mathbf{f})_j = \sum_{k=0}^{n-1} \hat{\mathbf{g}}_k \cdot \bar{\hat{\mathbf{f}}}_k \cdot \bar{\mathbf{x}}_j^k$$

$$= \sqrt{2\pi} \cdot \sum_{k=0}^{n-1} \hat{\mathbf{g}}_k \cdot \bar{\hat{\mathbf{f}}}_k \cdot \mathbf{z}_j^k$$



Outline

Review

Correlation:

- One-Dimensional (Continuous)
- One-Dimensional (Discrete)
- Higher-Dimensional
- Computational Complexity

Convolution

Applications in 1D

Correlation (Higher Dimensions)

The same kind of method can be used for higher dimensions:

- Periodic functions in 2D

$$\zeta^{l,m}(\theta,\phi)=\sqrt{\frac{1}{(2\pi)^2}}\cdot e^{il\theta}\cdot e^{im\phi}$$

$$\chi^{l,m}(\alpha,\beta)=\sqrt{(2\pi)^2}\zeta^{-l,-m}(\alpha,\beta)$$

- Periodic functions in 3D

$$\zeta^{l,m,n}(\theta,\phi,\psi)=\sqrt{\frac{1}{(2\pi)^3}}\cdot e^{il\theta}\cdot e^{im\phi}\cdot e^{in\psi}$$

$$\chi^{l,m,n}(\alpha,\beta,\gamma)=\sqrt{(2\pi)^3}\cdot\zeta^{-l,-m,-n}(\alpha,\beta,\gamma)$$

Outline

Review

Correlation:

- One-Dimensional (Continuous)
- One-Dimensional (Discrete)
- Higher-Dimensional
- Computational Complexity

Convolution

Applications in 1D

Computational Complexity

To compute the correlation of two periodic, n-dimensional arrays $\mathbf{f}, \mathbf{g} \in \mathbb{C}^n$:

1. Express $\mathbf{f}$ and $\mathbf{g}$ in the basis $\{\mathbf{z}^0, \dots, \mathbf{z}^{n-1}\} \subset \mathbb{C}^n$:
$$\mathbf{f} = \sum_{k=0}^{n-1} \hat{\mathbf{f}}_k \cdot \mathbf{z}^k \quad \text{and} \quad \mathbf{g} = \sum_{k=0}^{n-1} \hat{\mathbf{g}}_k \cdot \mathbf{z}^k$$
2. Multiply (and scale) the coefficients:
$$(\mathbf{g} \star \mathbf{f}) = \sqrt{2\pi} \cdot \sum_{k=0}^{n-1} \hat{\mathbf{g}}_k \cdot \bar{\hat{\mathbf{f}}}_k \cdot \mathbf{z}^k$$
3. Evaluate at every index j:
$$(\mathbf{g} \star \mathbf{f})_j = \sqrt{2\pi} \cdot \sum_{k=0}^{n-1} \hat{\mathbf{g}}_k \cdot \bar{\hat{\mathbf{f}}}_k \cdot \mathbf{z}_j^k$$

Computational Complexity

To compute the correlation of two periodic, n-dimensional arrays $\mathbf{f}, \mathbf{g} \in \mathbb{C}^n$:

- The first and third steps are a change of bases.
  - These can be implemented as matrix multiplication and (naively) are quadratic in n.

Computational Complexity

To compute the correlation of two periodic, n-dimensional arrays $\mathbf{f},\mathbf{g}\in\mathbb{C}^n$:

1. Express $\mathbf{f}$ and $\mathbf{g}$ in the basis $\{\mathbf{z}^0,\ldots,\mathbf{z}^{n-1}\}\subset\mathbb{C}^n$:

$$\mathbf{f}=\sum_{k=0}^{n-1}\hat{\mathbf{f}}_k\cdot\mathbf{z}^k \quad \text{and} \quad \mathbf{g}=\sum_{k=0}^{n-1}\hat{\mathbf{g}}_k\cdot\mathbf{z}^k$$

$O(N^2)$

2. Multiply (and scale) the coefficients:

$$(\mathbf{g}\star\mathbf{f})=\sqrt{2\pi}\cdot\sum_{k=0}^{n-1}\hat{\mathbf{g}}_k\cdot\bar{\hat{\mathbf{f}}}_k\cdot\mathbf{z}^k$$

$O(N)$

3. Evaluate at every index j:

$$(\mathbf{g}\star\mathbf{f})_j=\sqrt{2\pi}\cdot\sum_{k=0}^{n-1}\hat{\mathbf{g}}_k\cdot\bar{\hat{\mathbf{f}}}_k\cdot\mathbf{z}_j^k$$

$O(N^2)$

Computational Complexity

To compute the correlation of two periodic, n-dimensional arrays $\mathbf{f}, \mathbf{g} \in \mathbb{C}^n$:

The Fast Fourier Transform (FFT) is an algorithm for expressing an array represented by samples at $\{\theta_0, \cdots, \theta_{n-1}\}$ as a linear sum of $\{\mathbf{z}^0, \ldots, \mathbf{z}^{n-1}\}$.

The Fast Inverse Fourier Transform (IFFT) is an algorithm for expressing an array represented as a linear sum of $\{\mathbf{z}^0, \ldots, \mathbf{z}^{n-1}\}$ by samples at $\{\theta_0, \cdots, \theta_{n-1}\}$.

Both take $O(N \log N)$ time.

Computational Complexity

To compute the correlation of two periodic, n-dimensional arrays $\mathbf{f}, \mathbf{g} \in \mathbb{C}^n$:

1. Express $\mathbf{f}$ and $\mathbf{g}$ in the basis $\{\mathbf{z}^0, \dots, \mathbf{z}^{n-1}\} \subset \mathbb{C}^n$:

$$\mathbf{f} = \sum_{k=0}^{n-1} \hat{\mathbf{f}}_k \cdot \mathbf{z}^k \quad \text{and} \quad \mathbf{g} = \sum_{k=0}^{n-1} \hat{\mathbf{g}}_k \cdot \mathbf{z}^k$$

$O(N \log N)$

2. Multiply (and scale) the coefficients:

$$(\mathbf{g} \star \mathbf{f}) = \sqrt{2\pi} \cdot \sum_{k=0}^{n-1} \hat{\mathbf{g}}_k \cdot \bar{\hat{\mathbf{f}}}_k \cdot \mathbf{z}^k$$

$O(N)$

3. Evaluate at every index j:

$$(\mathbf{g} \star \mathbf{f})_j = \sqrt{2\pi} \cdot \sum_{k=0}^{n-1} \hat{\mathbf{g}}_k \cdot \bar{\hat{\mathbf{f}}}_k \cdot \mathbf{z}_j^k$$

$O(N \log N)$

Outline

Review

Correlation:

- One-Dimensional (Continuous)
- One-Dimensional (Discrete)
- Higher-Dimensional
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Convolution

Applications in 1D

Convolution

<u>Recall</u>:

The Fourier basis is defined by the functions:

$$\zeta^j(\theta) = \frac{e^{ij\theta}}{\sqrt{2\pi}} \qquad \forall j \in \mathbb{Z}$$

These functions have the properties that their reflections through the origin are their conjugates:

$$\begin{aligned} \zeta^j(-\theta) &= \frac{e^{-ij\theta}}{\sqrt{2\pi}} \\ &= \overline{\frac{e^{ij\theta}}{\sqrt{2\pi}}} \\ &= \bar{\zeta}^j(\theta) \end{aligned}$$

Convolution

Given a function f, the conjugate of its reflection through the origin can be expressed as:

$$\begin{aligned}
\bar{f}(-\theta) &= \overline{\sum_j \mathbf{f}_j \zeta^j(-\theta)} \\
&= \sum_j \bar{\mathbf{f}}_j \bar{\zeta}^j(-\theta) \\
&= \sum_j \bar{\mathbf{f}}_j \bar{\bar{\zeta}}^j(\theta) \\
&= \sum_j \bar{\mathbf{f}}_j \zeta^j(\theta)
\end{aligned}$$

Convolution

Given complex valued functions f and g on the circle, we define the convolution of g with f as the result obtained by first reflecting and conjugating f and then correlating g with the transformed f:

$$g * f = g \star \tilde{f} \quad \text{with } \tilde{f}(\theta) = \bar{f}(-\theta)$$

$$\Updownarrow$$

$$g * f = g \star \tilde{f} \quad \text{with } \hat{\tilde{\mathbf{f}}}_k = \bar{\hat{\mathbf{f}}}_k$$

Convolution

$$g * f = g \star \tilde{f} \quad \text{with } \hat{\tilde{\mathbf{f}}}_k = \overline{\hat{\mathbf{f}}}_k$$

Plugging this into equation for correlation:

$$(g * f) = \left(g \star \tilde{f}\right) = \sqrt{2\pi} \sum_k \hat{\mathbf{g}}_k \cdot \overline{\hat{\tilde{\mathbf{f}}}}_k \cdot \zeta^k$$

$$= \sqrt{2\pi} \sum_k \hat{\mathbf{g}}_k \cdot \overline{\overline{\hat{\mathbf{f}}}}_k \cdot \zeta^k$$

$$= \sqrt{2\pi} \sum_k \hat{\mathbf{g}}_k \cdot \hat{\mathbf{f}}_k \cdot \zeta^k$$

Convolution

$$g * f = g \star \tilde{f} \quad \text{with } \tilde{\hat{\mathbf{f}}}_k = \overline{\hat{\mathbf{f}}}_k$$

<u>Note</u>:

1. Unlike correlation, convolution is symmetric:

$$g * f = f * g$$

2. If f is real and symmetric with respect to reflection, then convolution and correlation are the same thing:

$$g * f = g \star f$$

$$= \sqrt{2\pi} \sum_k \hat{\mathbf{g}}_k \cdot \hat{\mathbf{f}}_k \cdot \zeta^k$$

Outline

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Correlation:

- One-Dimensional (Continuous)
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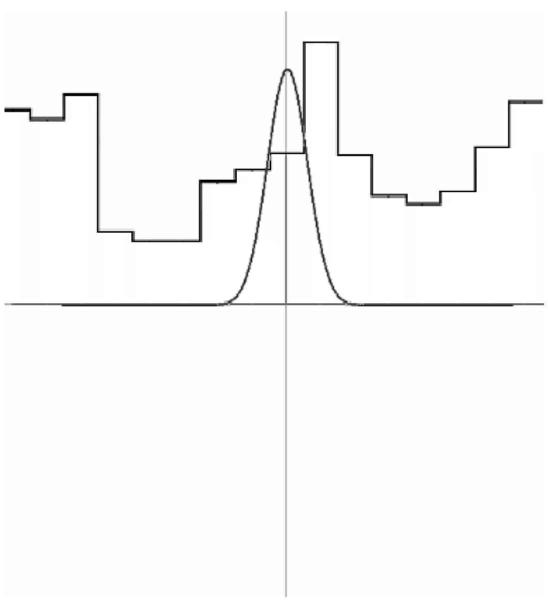
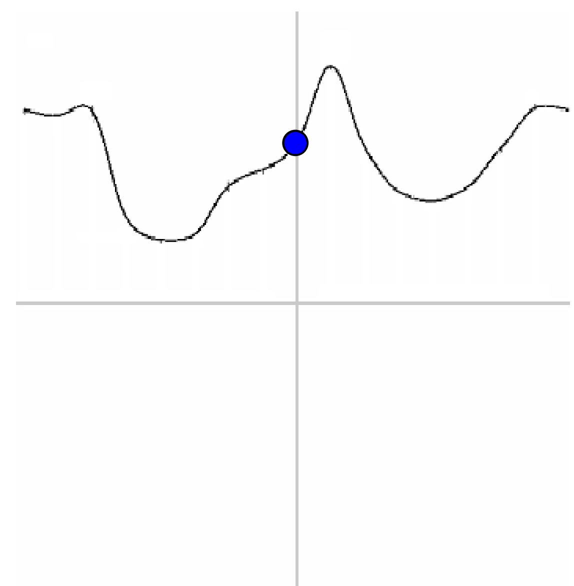
Applications in 1D

Applications of the FFT

- Correlation

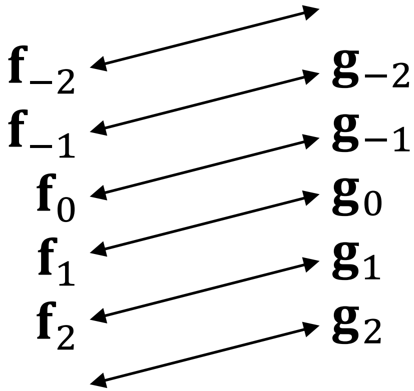
$$
\begin{aligned}
\mathbf{f}_{-2} &\longleftrightarrow \mathbf{g}_{-2}\\
\mathbf{f}_{-1} &\longleftrightarrow \mathbf{g}_{-1}\\
\mathbf{f}_{0} &\longleftrightarrow \mathbf{g}_{0}\\
\mathbf{f}_{1} &\longleftrightarrow \mathbf{g}_{1}\\
\mathbf{f}_{2} &\longleftrightarrow \mathbf{g}_{2}
\end{aligned}
$$

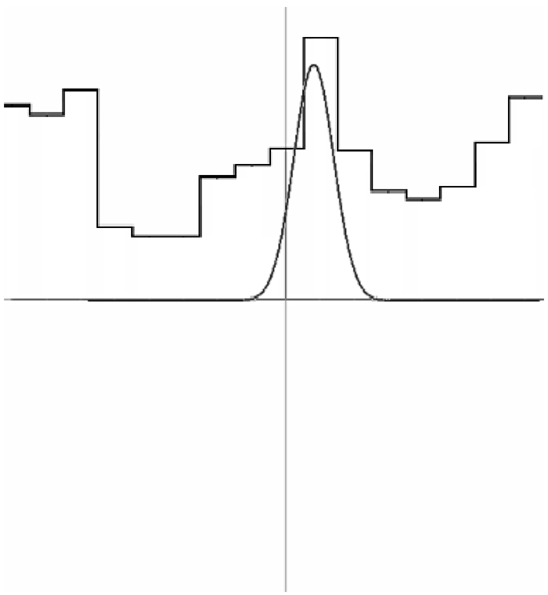


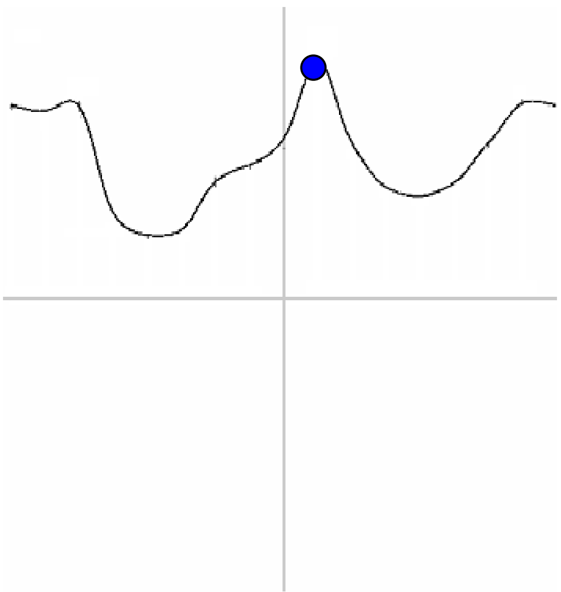


Applications of the FFT

- Correlation

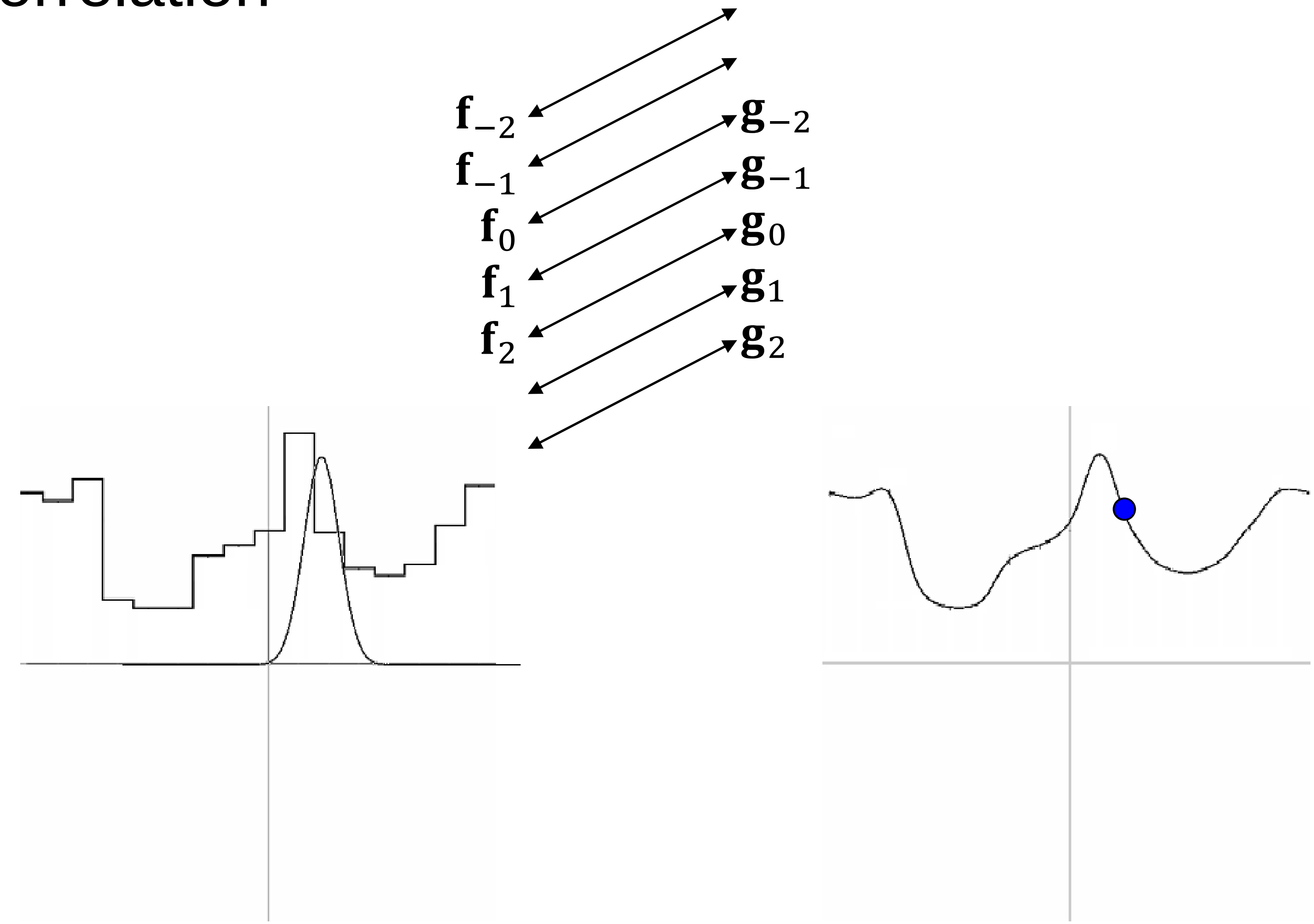






Applications of the FFT

- Correlation

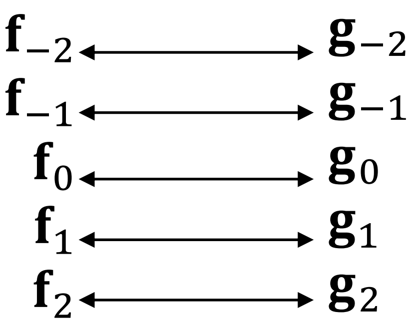


Applications of the FFT

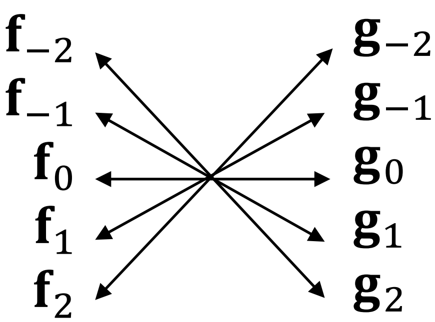
- Correlation
- Convolution

  This is like correlation, except that we flip the entries of the array **f** before correlating.

Correlation



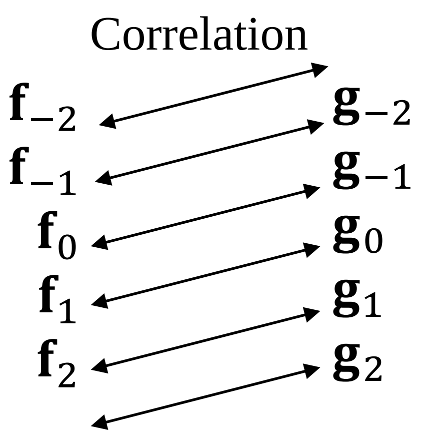
Convolution

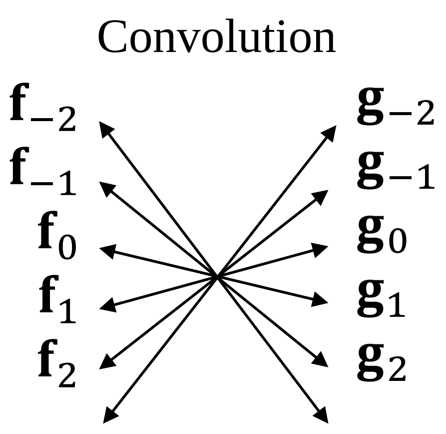


Applications of the FFT

- Correlation
- Convolution

  This is like correlation, except that we flip the entries of the array $\mathbf{f}$ before correlating.





Applications of the FFT

- Correlation
- Convolution

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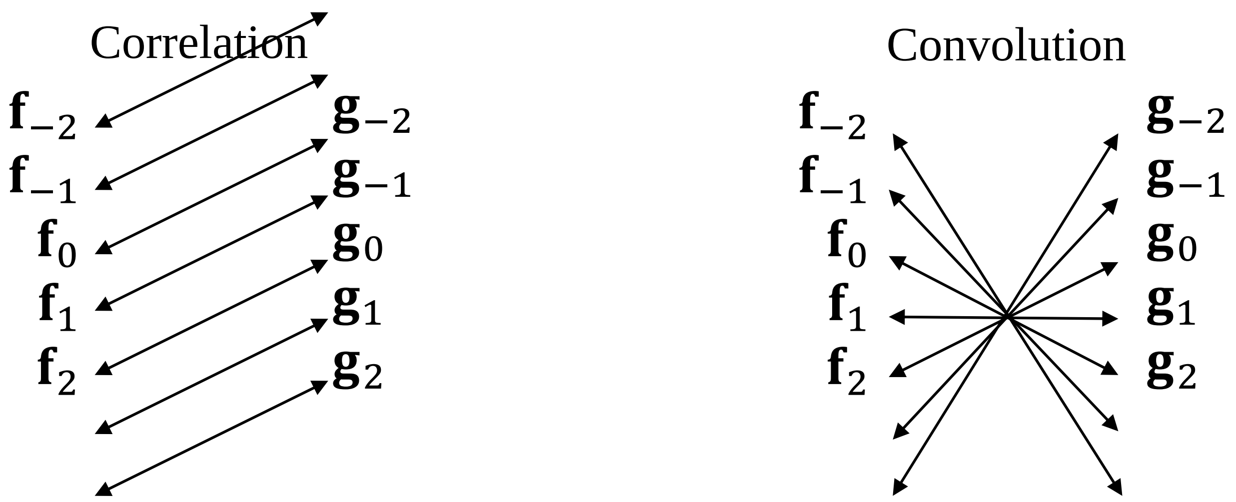


Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication

Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication

Given two polynomials:

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$
$$q(x) = b_0 + b_1 x + \cdots + b_n x^n$$

we can represent the polynomials $p(x)$ and $q(x)$ by $(2n+1)$-dimensional arrays:

$$p(x) \rightarrow (a_{-n}, \cdots, a_{-1}, a_0, a_1, \cdots, a_n)$$
$$q(x) \rightarrow (b_{-n}, \cdots, b_{-1}, b_0, b_1, \cdots, b_n)$$

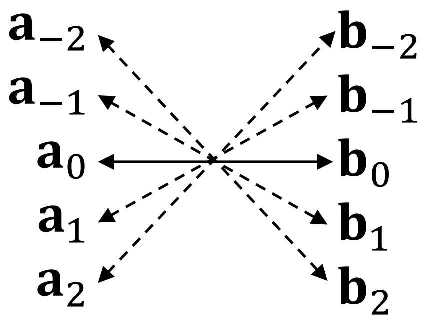
with:

$$a_{-n} = \cdots = a_{-1} = b_{-n} = \cdots = b_{-1} = 0$$

Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication

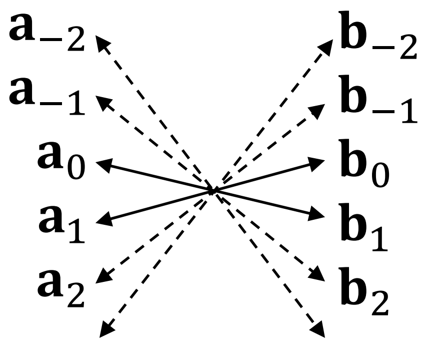
  The 0^{th} order coefficient of the product is:



Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication

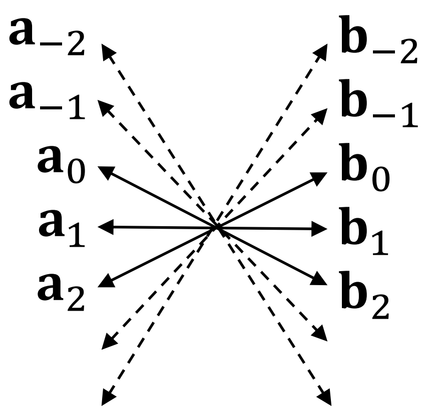
The 1st order coefficient of the product is:



Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication

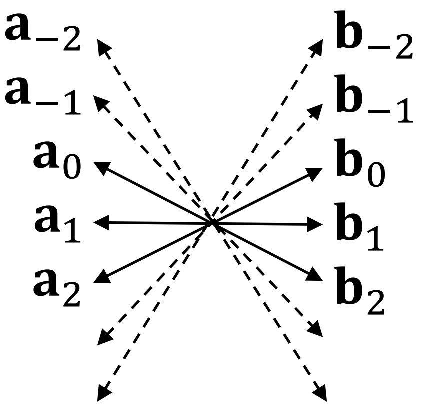
  The 2nd order coefficient of the product is:



Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication

  The coefficients of the product can be computed by convolving the arrays corresponding to the coefficients of the original polynomials.



Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication
- Big Integer Multiplication

Given an integer, we can treat it as a polynomial.

Example:

$$47601345 = 5\cdot 10^0 + 4\cdot 10^1 + 3\cdot 10^2 + 1\cdot 10^3 + \cdots$$

Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication
- Big Integer Multiplication

  Given an integer, we can treat it as a polynomial.
  To multiply two integers, we need to figure out what the new value in the 1s place,

  Example:

$$47601345 = 5 \cdot 10^0 + 4 \cdot 10^1 + 3 \cdot 10^2 + 1 \cdot 10^3 + \cdots$$

$$\updownarrow$$

$$46018729 = 9 \cdot 10^0 + 2 \cdot 10^1 + 7 \cdot 10^2 + 8 \cdot 10^3 + \cdots$$

Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication
- Big Integer Multiplication

Given an integer, we can treat it as a polynomial.
To multiply two integers, we need to figure out what the new value in the 1s place, the 10s place,

Example:

$$47601345 = 5 \cdot 10^0 + 4 \cdot 10^1 + 3 \cdot 10^2 + 1 \cdot 10^3 + \cdots$$

$$46018729 = 9 \cdot 10^0 + 2 \cdot 10^1 + 7 \cdot 10^2 + 8 \cdot 10^3 + \cdots$$

Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication
- Big Integer Multiplication

Given an integer, we can treat it as a polynomial. To multiply two integers, we need to figure out what the new value in the 1s place, the 10s place, the 100s place, etc. will be.

Example:

$$47601345 = 5 \cdot 10^0 + 4 \cdot 10^1 + 3 \cdot 10^2 + 1 \cdot 10^3 + \cdots$$

$$46018729 = 9 \cdot 10^0 + 2 \cdot 10^1 + 7 \cdot 10^2 + 8 \cdot 10^3 + \cdots$$

Applications of the FFT

- Correlation
- Convolution
- Polynomial Multiplication
- Big Integer Multiplication

  Given an integer, we can treat it as a polynomial.
  To multiply two integers, we need to figure out what the new value in the 1s place, the 10s place, the 100s place, etc. will be.

  So big integer multiplication can be implemented as a convolution.