



FFTs in Graphics and Vision

More Math Review

Announcements



We now have a Piazza page:

<http://piazza.com/jhu/spring2019/601760>



Outline

Inner Product Spaces

- Real Inner Products
- Hermitian Inner Products
- Orthogonal Transforms
- Unitary Transforms
- Function Spaces



Inner Product Spaces

Given a real vector space V , a *real inner product* is a function $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$ that takes a pair of vectors and returns a real value.



Inner Product Spaces

An inner product is a map $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$ that is:

1. Linear in the first term: For all $u, v, w \in V$ and all $\alpha \in \mathbb{R}$:

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$$

2. Symmetric: For all $v, w \in V$:

$$\langle v, w \rangle = \langle w, v \rangle$$

3. Positive Definite: For all $v \in V$:

$$\langle v, v \rangle \geq 0$$

$$\langle v, v \rangle = 0 \Leftrightarrow v = 0$$



Inner Product Spaces

Note:

Combining linearity in the first term with symmetry, gives:

$$\begin{aligned}\langle u, v + w \rangle &= \langle v + w, u \rangle \\ &= \langle v, u \rangle + \langle w, u \rangle \\ &= \langle u, v \rangle + \langle u, w \rangle\end{aligned}$$

$$\begin{aligned}\langle u, \alpha v \rangle &= \langle \alpha v, u \rangle \\ &= \alpha \langle v, u \rangle \\ &= \alpha \langle u, v \rangle\end{aligned}$$

⇒ Linearity in the second term



Inner Product Spaces

An inner product defines a notion of distance on a vector space by setting:

$$\text{Dist}(v, w) = \sqrt{\langle v - w, v - w \rangle} = \|v - w\|$$



Inner Product Spaces

Examples:

1. On n -dimensional, \mathbb{R} -valued, arrays, the standard inner product is:

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b} \rangle &= \mathbf{a}_1 \mathbf{b}_1 + \cdots + \mathbf{a}_n \mathbf{b}_n \\ &= \mathbf{a}^t \mathbf{b}\end{aligned}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$

2. On continuous, \mathbb{R} -valued functions, defined on a circle, the standard inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot g(\theta) d\theta$$



Inner Product Spaces

Examples:

3. We will think of a \mathbb{R} -valued, n -dimensional array, $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{R}^n$, as a sampling of a continuous \mathbb{R} -valued function on the circle, $f: [0, 2\pi) \rightarrow \mathbb{R}$:

$$\mathbf{a}_j = f\left(\frac{2\pi j}{n}\right)$$

\Downarrow

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b} \rangle_{[0, 2\pi)} &= \frac{2\pi}{n} (\mathbf{a}_1 \mathbf{b}_1 + \dots + \mathbf{a}_n \mathbf{b}_n) \\ &= \frac{2\pi}{n} \mathbf{a}^t \mathbf{b}\end{aligned}$$



Inner Product Spaces

Examples:

4. On n -dimensional, \mathbb{R} -valued arrays, suppose we have a matrix:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \cdots & \mathbf{M}_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{n1} & \cdots & \mathbf{M}_{nn} \end{pmatrix}$$

Does the map:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^t \mathbf{M} \mathbf{b}$$

define an inner product?



Inner Product Spaces

Examples:

4. Does the map:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^t \mathbf{M} \mathbf{b}$$

define an inner product?

- Is it linear?
- Is it symmetric?
- Is it positive definite?



Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^t \mathbf{M} \mathbf{b}$$

Is it linear?

$$\begin{aligned} \langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}} &= (\mathbf{a} + \mathbf{b})^t \mathbf{M} \mathbf{c} \\ &= (\mathbf{a}^t + \mathbf{b}^t) \mathbf{M} \mathbf{c} \\ &= \mathbf{a}^t \mathbf{M} \mathbf{c} + \mathbf{b}^t \mathbf{M} \mathbf{c} \\ &= \langle \mathbf{a}, \mathbf{c} \rangle_{\mathbf{M}} + \langle \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}} \end{aligned}$$



Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^t \mathbf{M} \mathbf{b}$$

Is it linear?

$$\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}} = \langle \mathbf{a}, \mathbf{c} \rangle_{\mathbf{M}} + \langle \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}}$$

$$\begin{aligned} \langle \alpha \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} &= (\alpha \mathbf{a})^t \mathbf{M} \mathbf{b} \\ &= \alpha \mathbf{a}^t \mathbf{M} \mathbf{b} \\ &= \alpha \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \end{aligned}$$



Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^t \mathbf{M} \mathbf{b}$$

Is it linear? Yes

$$\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}} = \langle \mathbf{a}, \mathbf{c} \rangle_{\mathbf{M}} + \langle \mathbf{b}, \mathbf{c} \rangle_{\mathbf{M}}$$

$$\langle \alpha \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} = \alpha \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}}$$



Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^t \mathbf{M} \mathbf{b}$$

Is it symmetric?

$$\begin{aligned} \langle \mathbf{b}, \mathbf{a} \rangle_{\mathbf{M}} &= \mathbf{b}^t \mathbf{M} \mathbf{a} \\ &= (\mathbf{b}^t \mathbf{M} \mathbf{a})^t \\ &= \mathbf{a}^t \mathbf{M}^t \mathbf{b} \\ &= \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}^t} \end{aligned}$$



Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^t \mathbf{M} \mathbf{b}$$

Is it symmetric? Only if \mathbf{M} is ($\mathbf{M} = \mathbf{M}^t$)

$$\langle \mathbf{b}, \mathbf{a} \rangle_{\mathbf{M}} = \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}^t}$$



Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^t \mathbf{M} \mathbf{b}$$

Assuming \mathbf{M} is symmetric, is the map positive definite?

If \mathbf{M} is symmetric, there exists an orthogonal basis $\{v^1, \dots, v^n\} \subset \mathbb{R}^n$ with respect to which \mathbf{M} is diagonal:

$$\mathbf{M} = \mathbf{B}^t \mathbf{\Lambda} \mathbf{B} = \mathbf{B}^t \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix} \mathbf{B}$$



Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^t \mathbf{M} \mathbf{b}$$

Assuming \mathbf{M} is symmetric, is the map $\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}}$ positive definite?

Setting $\mathbf{M} = \mathbf{B}^t \mathbf{\Lambda} \mathbf{B}$, we have:

$$\begin{aligned} \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} &= \mathbf{a}^t \mathbf{M} \mathbf{a} \\ &= \mathbf{a}^t \mathbf{B}^t \mathbf{\Lambda} \mathbf{B} \mathbf{a} \\ &= \langle \mathbf{B} \mathbf{a}, \mathbf{B} \mathbf{a} \rangle_{\mathbf{\Lambda}} \end{aligned}$$

Writing $\mathbf{b} = \mathbf{B} \mathbf{a}$, we get:

$$\begin{aligned} \langle \mathbf{a}, \mathbf{a} \rangle_{\mathbf{M}} &= \langle \mathbf{B} \mathbf{a}, \mathbf{B} \mathbf{a} \rangle_{\mathbf{\Lambda}} = \langle \mathbf{b}, \mathbf{b} \rangle_{\mathbf{\Lambda}} \\ &= \lambda_1 \mathbf{b}_1^2 + \cdots + \lambda_n \mathbf{b}_n^2 \end{aligned}$$



Inner Product Spaces

Examples:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} \equiv \mathbf{a}^t \mathbf{M} \mathbf{b}$$

Assuming \mathbf{M} is symmetric, is the map $\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}}$ positive definite? Only if $\lambda_i > 0$ for all i .

Writing $\mathbf{b} = \mathbf{B} \mathbf{a}$, we get:

$$\langle \mathbf{a}, \mathbf{a} \rangle_{\mathbf{M}} = \lambda_1 \mathbf{b}_1^2 + \cdots + \lambda_n \mathbf{b}_n^2$$



Inner Product Spaces

Examples:

5. On continuous, \mathbb{R} -valued functions, defined on a circle, given $\omega: [0, 2\pi) \rightarrow \mathbb{R}$, does the map:

$$\langle f, g \rangle_{\omega} = \int_0^{2\pi} f(\theta) \cdot g(\theta) \cdot \omega(\theta) d\theta$$

define an inner product?



Inner Product Spaces

Examples:

5. On continuous, \mathbb{R} -valued functions, defined on a circle, given $\omega: [0, 2\pi) \rightarrow \mathbb{R}$, does the map:

$$\langle f, g \rangle_{\omega} = \int_0^{2\pi} f(\theta) \cdot g(\theta) \cdot \omega(\theta) d\theta$$

define an inner product? No



Inner Product Spaces

Examples:

5. On continuous, \mathbb{R} -valued functions, defined on a circle, given $\omega: [0, 2\pi) \rightarrow \mathbb{R}$, does the map:

$$\langle f, g \rangle_{\omega} = \int_0^{2\pi} f(\theta) \cdot g(\theta) \cdot \omega(\theta) d\theta$$

define an inner product? No

What if $\omega(\theta) > 0$?



Inner Product Spaces

Examples:

5. On continuous, \mathbb{R} -valued functions, defined on a circle, given $\omega: [0, 2\pi) \rightarrow \mathbb{R}$, does the map:

$$\langle f, g \rangle_{\omega} = \int_0^{2\pi} f(\theta) \cdot g(\theta) \cdot \omega(\theta) d\theta$$

define an inner product? No

What if $\omega(\theta) > 0$? Yes



Hermitian Inner Product Spaces

Given a complex vector space V , a *Hermitian inner product* is a function $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$ that takes a pair of vectors and returns a complex value.



Hermitian Inner Product Spaces

A Hermitian inner product is a map $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$ that is:

1. Linear in the first term: For all $u, v, w \in V$ and any $\alpha \in \mathbb{C}$:

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$$

2. Conjugate Symmetric: For all $u, v \in V$:

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

3. Positive Definite: For all $v \in V$:

$$\langle v, v \rangle \geq 0$$

$$\langle v, v \rangle = 0 \Leftrightarrow v = 0$$



Hermitian Inner Product Spaces

Note:

Combining linearity in the first term with conjugate symmetry, gives:

$$\begin{aligned}\langle u, v + w \rangle &= \overline{\langle v + w, u \rangle} \\ &= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} \\ &= \langle u, v \rangle + \langle u, w \rangle\end{aligned}$$

$$\begin{aligned}\langle u, \alpha v \rangle &= \overline{\langle \alpha v, u \rangle} \\ &= \overline{\alpha \langle v, u \rangle} \\ &= \bar{\alpha} \langle u, v \rangle\end{aligned}$$

⇒ Conjugate linearity in the second term



Hermitian Inner Product Spaces

As in the real case, a Hermitian inner product defines a notion of distance on a complex vector space by setting:

$$\text{Dist}(v, w) = \sqrt{\langle v - w, v - w \rangle} = \|v - w\|$$



Hermitian Inner Product Spaces

Examples:

1. On n -dimensional, \mathbb{C} -valued arrays, the standard Hermitian inner product is:

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b} \rangle &= \mathbf{a}_1 \bar{\mathbf{b}}_1 + \cdots + \mathbf{a}_n \bar{\mathbf{b}}_n \\ &= \mathbf{a}^t \bar{\mathbf{b}}\end{aligned}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$

2. On continuous, \mathbb{C} -valued functions, defined on a circle, the standard Hermitian inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot \overline{g(\theta)} d\theta$$



Hermitian Inner Product Spaces

Examples:

3. We will think of a \mathbb{C} -valued n -dimensional arrays $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{C}^n$ as a sampling of a continuous, \mathbb{C} -valued functions on the circle, $f: [0, 2\pi) \rightarrow \mathbb{C}$:

$$\mathbf{a}_j = f\left(\frac{2\pi j}{n}\right)$$

⇓

$$\begin{aligned} \langle \mathbf{a}, \bar{\mathbf{b}} \rangle_{[0, 2\pi)} &= \frac{2\pi}{n} (\mathbf{a}_1 \bar{\mathbf{b}}_1 + \dots + \mathbf{a}_n \bar{\mathbf{b}}_n) \\ &= \frac{2\pi}{n} \mathbf{a}^t \bar{\mathbf{b}} \end{aligned}$$



Structure Preservation

Recall:

If we have an n -dimensional vector space V then a linear map L is a function $L: V \rightarrow V$ that preserves the linear structure:

$$L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)$$

for all $v, w \in V$ and all scalars α and β .

If L is invertible, then we can think of L as a function that “renames” all the elements in V while preserving the underlying vector space structure.



Structure Preservation

Orthogonal Transformations:

For a real vector space V that has an inner product, we would also like to consider those functions that “rename” the elements of V while preserving the underlying structure.

If R is such a function, then:

- R must be an invertible linear operator, in order to preserve the underlying vector space structure.
- R must also preserve the underlying inner product.



Structure Preservation

Orthogonal Transformations:

For a real inner-product space V , a linear operator R is called orthogonal if it preserves the inner product:

$$\langle R(v), R(w) \rangle = \langle v, w \rangle$$

for all $v, w \in V$.



Structure Preservation

Example:

On n -dimensional, \mathbb{R} -valued, arrays the matrix $\mathbf{R} \in \mathbb{R}^{n \times n}$ is orthogonal if for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$:

$$\langle \mathbf{R}\mathbf{a}, \mathbf{R}\mathbf{b} \rangle_{[0,2\pi)} = \langle \mathbf{a}, \mathbf{b} \rangle_{[0,2\pi)}$$

$$\Leftrightarrow$$

$$\frac{2\pi}{n} (\mathbf{R}\mathbf{a})^t (\mathbf{R}\mathbf{b}) = \frac{2\pi}{n} \mathbf{a}^t \mathbf{b}$$

$$\Leftrightarrow$$

$$\mathbf{a}^t \mathbf{R}^t \mathbf{R} \mathbf{b} = \mathbf{a}^t \mathbf{b}$$

$$\Leftrightarrow$$

$$\mathbf{R}^t = \mathbf{R}^{-1}$$



Structure Preservation

Example:

On the space of \mathbb{R} -valued, n -dimensional arrays, a matrix is orthogonal if and only if:

$$\mathbf{R}^t = \mathbf{R}^{-1}$$

Note:

The determinant of an orthogonal matrix always has absolute value 1:

$$\begin{aligned} |\det(\mathbf{R})|^2 &= \det(\mathbf{R}) \cdot \det(\mathbf{R}^t) \\ &= \det(\mathbf{R}) \cdot \det(\mathbf{R}^{-1}) \\ &= \det(\mathbf{R}\mathbf{R}^{-1}) \\ &= 1 \end{aligned}$$



Structure Preservation

Example:

On the space of real-valued, n -dimensional arrays, a matrix is orthogonal if and only if:

$$\mathbf{R}^t = \mathbf{R}^{-1}$$

Note: The determinant of an orthogonal matrix always has absolute value 1.

If the determinant of an orthogonal matrix is equal to 1, the matrix is called a rotation.



Orthogonal Matrices and Eigenvalues

If $R: V \rightarrow V$ is an orthogonal transformation and R has an eigenvalue λ , then $|\lambda| = 1$.

To see this, let v be an eigenvector of R corresponding to the eigenvalue λ .

Since R is orthogonal, we have:

$$\begin{aligned}\langle v, v \rangle &= \langle Rv, Rv \rangle \\ &= \langle \lambda v, \lambda v \rangle \\ &= \lambda^2 \langle v, v \rangle\end{aligned}$$

$$\Rightarrow |\lambda| = 1.$$



Structure Preservation

Unitary Transformations:

For a complex vector space V , an invertible linear operator R is called unitary if it preserves the hermitian inner product:

$$\langle Rv, Rw \rangle = \langle v, w \rangle$$

for all $v, w \in V$.



Structure Preservation

Example:

On n -dimensional, \mathbb{C} -valued, arrays the matrix $\mathbf{R} \in \mathbb{C}^{n \times n}$ is unitary if for all $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$:

$$\langle \mathbf{R}\mathbf{a}, \mathbf{R}\mathbf{b} \rangle_{[0,2\pi)} = \langle \mathbf{a}, \mathbf{b} \rangle_{[0,2\pi)}$$

$$\Leftrightarrow$$

$$\frac{2\pi}{n} (\mathbf{R}\mathbf{a})^t (\overline{\mathbf{R}\mathbf{b}}) = \frac{2\pi}{n} \mathbf{a}^t \bar{\mathbf{b}}$$

$$\Leftrightarrow$$

$$\mathbf{a}^t \mathbf{R}^t \overline{\mathbf{R}\mathbf{b}} = \mathbf{a}^t \bar{\mathbf{b}}$$

$$\Leftrightarrow$$

$$\bar{\mathbf{R}}^t = \mathbf{R}^{-1}$$



Structure Preservation

Example:

On n -dimensional, \mathbb{C} -valued, arrays the matrix $\mathbf{R} \in \mathbb{C}^{n \times n}$ is unitary if:

$$\overline{\mathbf{R}}^t = \mathbf{R}^{-1}$$

Note:

The determinant of a unitary matrix has norm 1:

$$\begin{aligned} |\det(\mathbf{R})|^2 &= \det(\mathbf{R}) \cdot \overline{\det(\mathbf{R})} \\ &= \det(\mathbf{R}) \cdot \det(\overline{\mathbf{R}}^t) \\ &= \det(\mathbf{R}) \cdot \det(\mathbf{R}^{-1}) \\ &= \det(\mathbf{R}\mathbf{R}^{-1}) \\ &= 1 \end{aligned}$$



Unitary Matrices and Eigenvalues

If $R: V \rightarrow V$ is a unitary transformation and R has an eigenvalue λ , then $|\lambda| = 1$.

To see this, let v be an eigenvector of R corresponding to the eigenvalue λ .

Since R is unitary, we have:

$$\begin{aligned}\langle v, v \rangle &= \langle Rv, Rv \rangle \\ &= \langle \lambda v, \lambda v \rangle \\ &= \lambda \bar{\lambda} \langle v, v \rangle \\ &= |\lambda|^2 \langle v, v \rangle\end{aligned}$$

$$\Rightarrow |\lambda| = 1.$$



Function Spaces

In this course, the vector spaces we will be looking at most often are the vector spaces of continuous (complex-valued) functions defined over some domain:

- S^1 : The unit circle
- D^2 : The unit disk
- T^2 : The (flat) torus
- S^2 : The unit sphere
- B^3 : The unit ball



Function Spaces: Unit Circle

$$\underline{S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}}:$$

If we have functions $f, g: S^1 \rightarrow \mathbb{C}$ then:

$$\langle f, g \rangle = \int_{p \in S^1} f(p) \cdot \overline{g(p)} dp$$

Or, we can represent points on the circle in terms of angle $\theta \in [0, 2\pi)$:

$$\theta \rightarrow (\cos \theta, \sin \theta)$$

For $f, g: [0, 2\pi) \rightarrow \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot \overline{g(\theta)} d\theta$$



Function Spaces: Unit Circle

$$\underline{S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}}:$$

The space of functions on $[0, 2\pi)$ is equivalent to the space of periodic functions on the line:

$$\{f: \mathbb{R} \rightarrow \mathbb{C} \mid f(x) = f(x + 2\pi)\}$$

Or, we can represent points on the circle in terms of angle $\theta \in [0, 2\pi)$:

$$\theta \rightarrow (\cos \theta, \sin \theta)$$

For $f, g: [0, 2\pi) \rightarrow \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot \overline{g(\theta)} d\theta$$



Function Spaces: Unit Disk

$$\underline{D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}}:$$

If we have functions $f, g: D^2 \rightarrow \mathbb{C}$ then:

$$\langle f, g \rangle = \int_{p \in D^2} f(p) \cdot \overline{g(p)} dp$$

Or, we can represent points on the disk in terms of radius $r \in [0, 1]$ and angle $\theta \in [0, 2\pi)$:

$$(r, \theta) \rightarrow (r \cdot \cos \theta, r \cdot \sin \theta)$$

For $f, g: [0, 1] \times [0, 2\pi) \rightarrow \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^1 f(r, \theta) \cdot \overline{g(r, \theta)} \cdot r dr d\theta$$



Function Spaces: (Flat) Torus

$$\underline{T^2 = S^1 \times S^1:}$$

If we have functions $f, g: T^2 \rightarrow \mathbb{C}$ then:

$$\langle f, g \rangle = \int_{p \in T^2} f(p) \cdot \overline{g(p)} dp$$

Or, we can represent points on the circle in terms of angle $(\theta, \phi) \in [0, 2\pi) \times [0, 2\pi)$:

$$(\theta, \phi) \rightarrow (\cos \theta, \sin \theta, \cos \phi, \sin \phi)$$

For $f, g: [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot \overline{g(\theta)} d\theta$$



Function Spaces: (Flat) Torus

$$\underline{T^2 = S^1 \times S^1:}$$

The space of functions on $[0, 2\pi) \times [0, 2\pi)$ is equivalent to the space of periodic functions in the plane:

$$\{f: \mathbb{R}^2 \rightarrow \mathbb{C} \mid f(x, y) = f(x + 2\pi, y) = f(x, y + 2\pi)\}$$

Or, we can represent points on the circle in terms of angle $(\theta, \phi) \in [0, 2\pi) \times [0, 2\pi)$:

$$(\theta, \phi) \rightarrow (\cos \theta, \sin \theta, \cos \phi, \sin \phi)$$

For $f, g: [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot \overline{g(\theta)} d\theta$$



Function Spaces: Unit Sphere

$$\underline{S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}}$$

If we have functions $f, g: S^2 \rightarrow \mathbb{C}$ then:

$$\langle f, g \rangle = \int_{p \in S^2} f(p) \cdot \overline{g(p)} \, dp$$

Or, we can represent points on the sphere in terms of spherical angle $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$:

$$(\theta, \phi) \rightarrow (\sin \theta \cdot \cos \phi, \cos \theta, \sin \theta \cdot \sin \phi)$$

For $f, g: [0, \pi] \times [0, 2\pi) \rightarrow \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^\pi f(\theta, \phi) \cdot \overline{g(\theta, \phi)} \cdot \sin(\theta) \, d\theta \, d\phi$$



Function Spaces: Unit Ball

$$\underline{B^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}}$$

If we have functions $f, g: B^3 \rightarrow \mathbb{C}$ then:

$$\langle f, g \rangle = \int_{p \in B^3} f(p) \cdot \overline{g(p)} dp$$

Or, representing points in the ball in terms of radius $r \in [0, 1]$ and spherical angle $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$:
 $(r, \theta, \phi) \rightarrow (r \cdot \sin \theta \cdot \cos \phi, r \cdot \cos \theta, r \cdot \sin \theta \cdot \sin \phi)$

For $f, g: [0, 1] \times [0, \pi] \times [0, 2\pi) \rightarrow \mathbb{C}$ the inner product is:

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^\pi \int_0^1 f(r, \theta, \phi) \cdot \overline{g(r, \theta, \phi)} \cdot r^2 \cdot \sin(\theta) dr d\theta d\phi$$



Function Spaces

Examples

If we consider the space of continuous, \mathbb{C} -valued functions on the unit circle:

- Is the map:

$$f(p) \rightarrow f(p) + 1$$

a linear transformation?



Function Spaces

Examples

If we consider the space of continuous, \mathbb{C} -valued functions on the unit circle:

- Is the map:

$$f(p) \rightarrow f(p) + 1$$

a linear transformation? **No**



Function Spaces

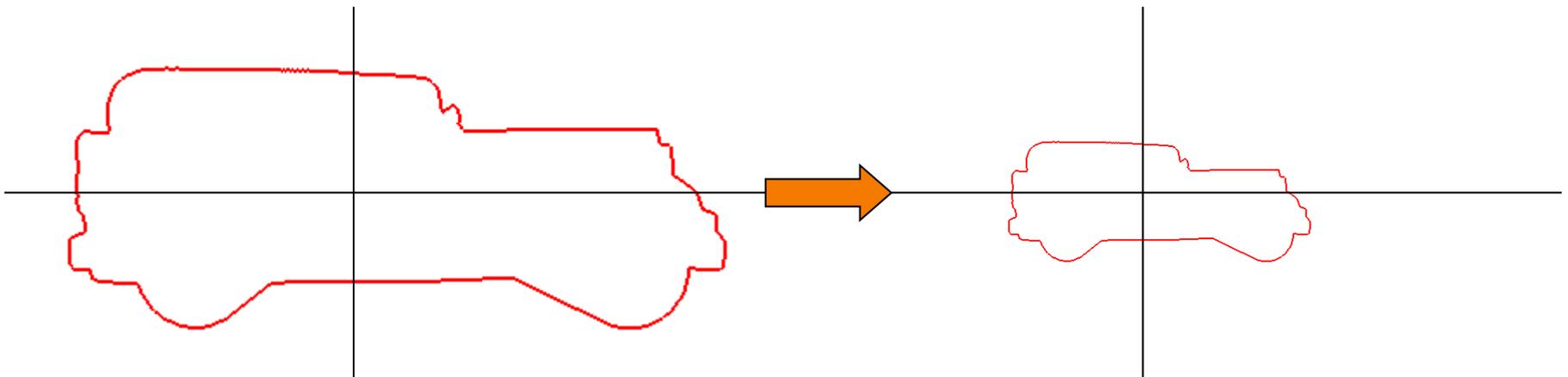
Examples

If we consider the space of continuous, \mathbb{C} -valued functions on the unit circle:

- For any scalar value λ , is:

$$f(p) \rightarrow \lambda \cdot f(p)$$

a linear transformation?





Function Spaces

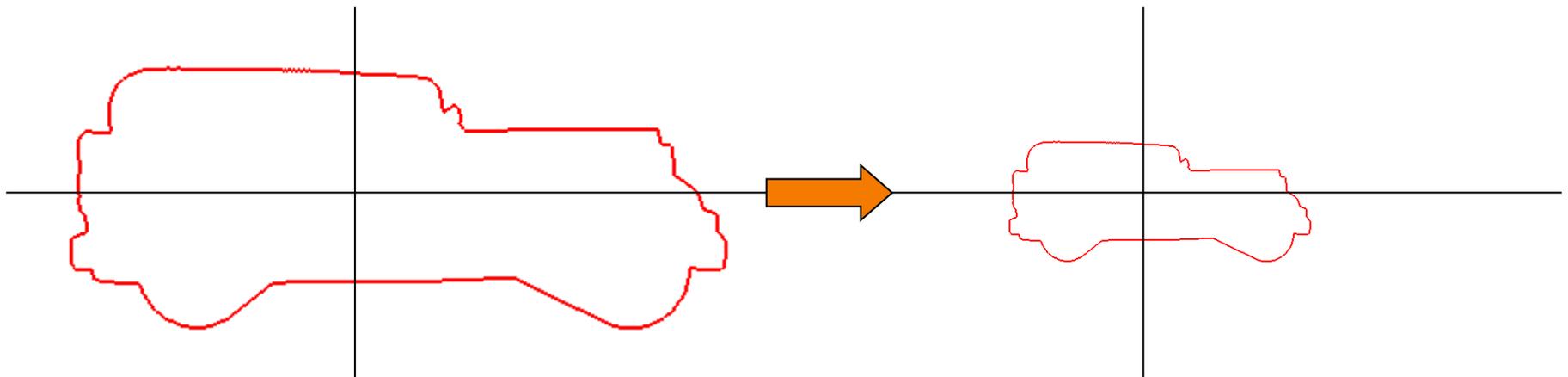
Examples

If we consider the space of continuous, \mathbb{C} -valued functions on the unit circle:

- For any scalar value λ , is:

$$f(p) \rightarrow \lambda \cdot f(p)$$

a linear transformation? **Yes**





Function Spaces

Examples

If we consider the space of continuous, \mathbb{C} -valued functions on the unit circle:

- For any scalar value λ , is:

$$f(p) \rightarrow \lambda \cdot f(p)$$

a linear transformation? **Yes**

- Is it unitary?



Function Spaces

Examples

If we consider the space of continuous, \mathbb{C} -valued functions on the unit circle:

- For any scalar value λ , is:

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a linear transformation? **Yes**

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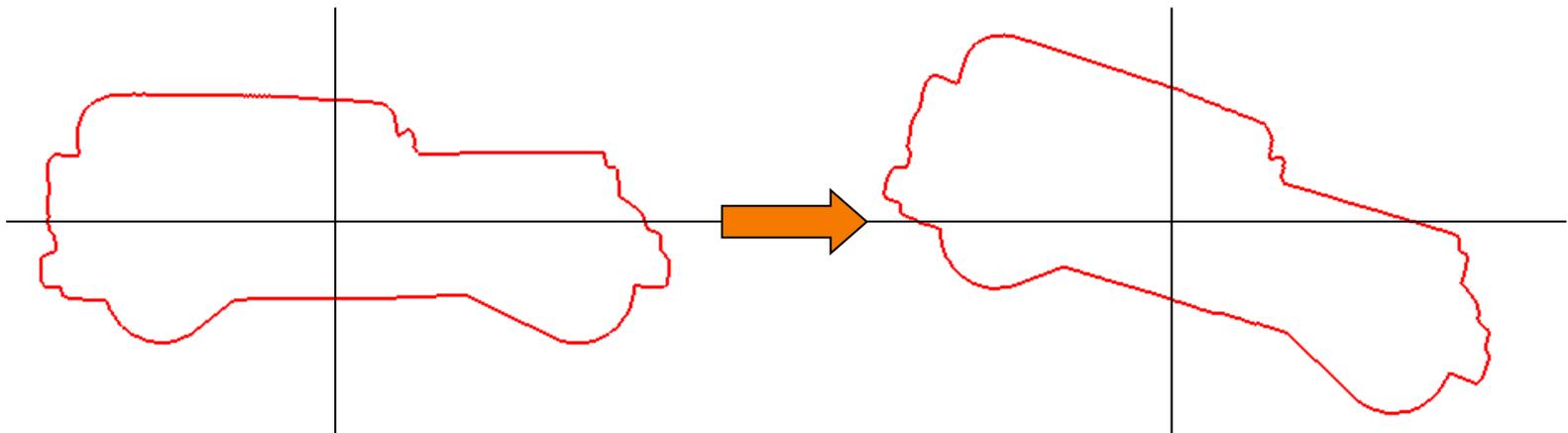
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If we consider the space of continuous, \mathbb{C} -valued functions on the unit circle:

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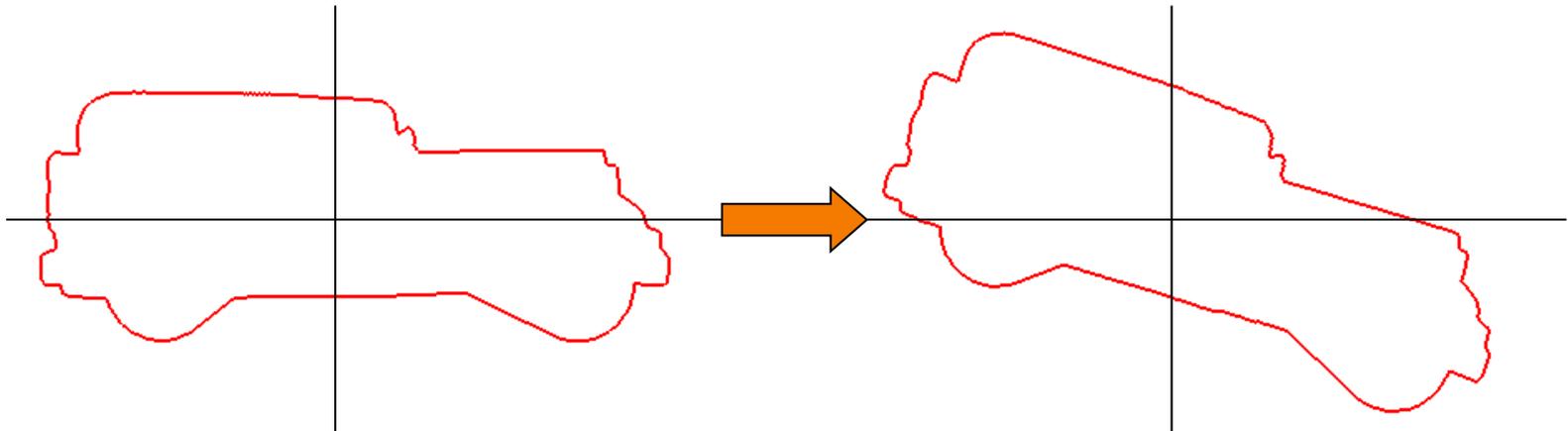
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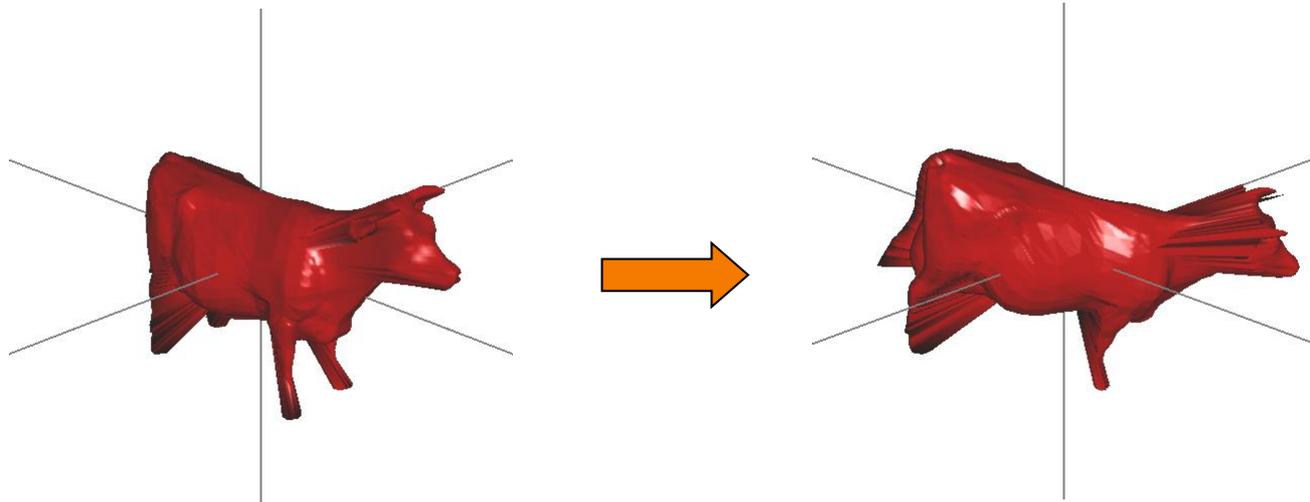
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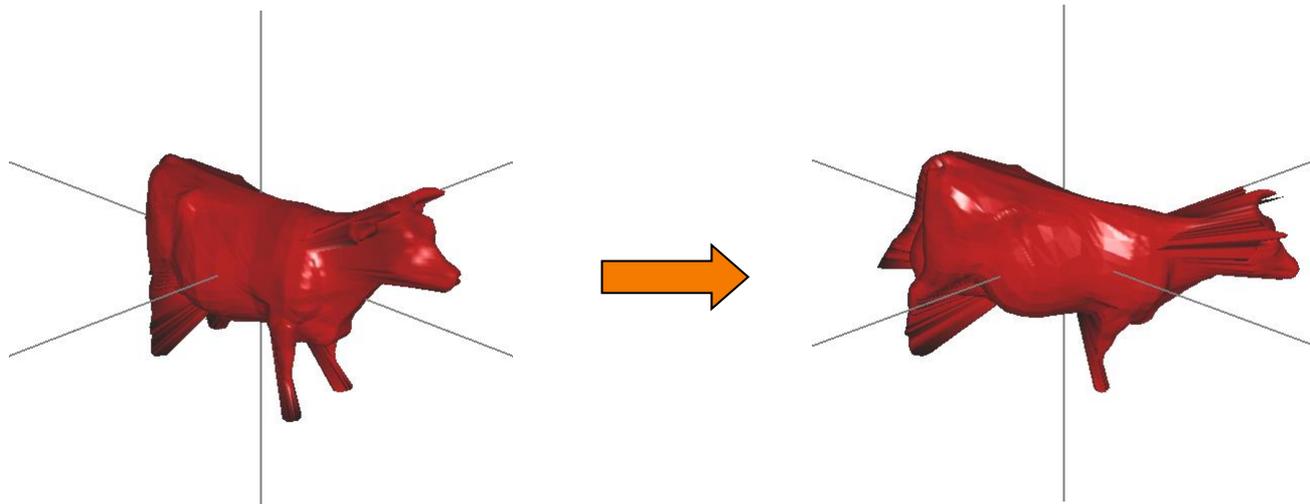
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