Lecture 9

Introduction to Geometry Processing

Spring 2017

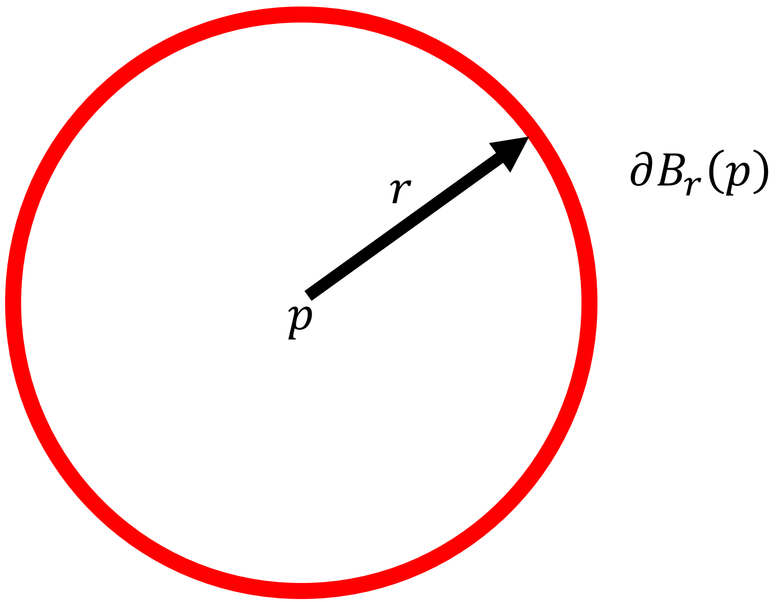
Johns Hopkins University

Stiffness vs Laplacian

Recall our definition:

$$\Delta f = \nabla \cdot \nabla f$$

And our geometric motivation:



$$\bar{f}_{\partial B_r(p)} = f(p) + \Delta f(p) r^2 + O(r^3)$$

Assuming our surface M is closed (i.e., $\partial M = \emptyset$), the divergence theorem tell us:

$$\int_M \nabla \cdot (g \nabla f) = \int_{\partial M} (g \nabla f) d\vec{n} = 0, \qquad \forall f, g$$

Since, $\nabla \cdot (g \nabla f) = \langle \nabla g, \nabla f \rangle + g \Delta f$, we get:

$$\int_M \langle \nabla g, \nabla f \rangle = -\int_M g \Delta f$$

Stiffness vs Laplacian

In the discrete domain: $$\int_M \langle \nabla g, \nabla f \rangle = [g]^\top S[f]$$

$$-\int_M g\Delta f = [g]M[\Delta f]$$

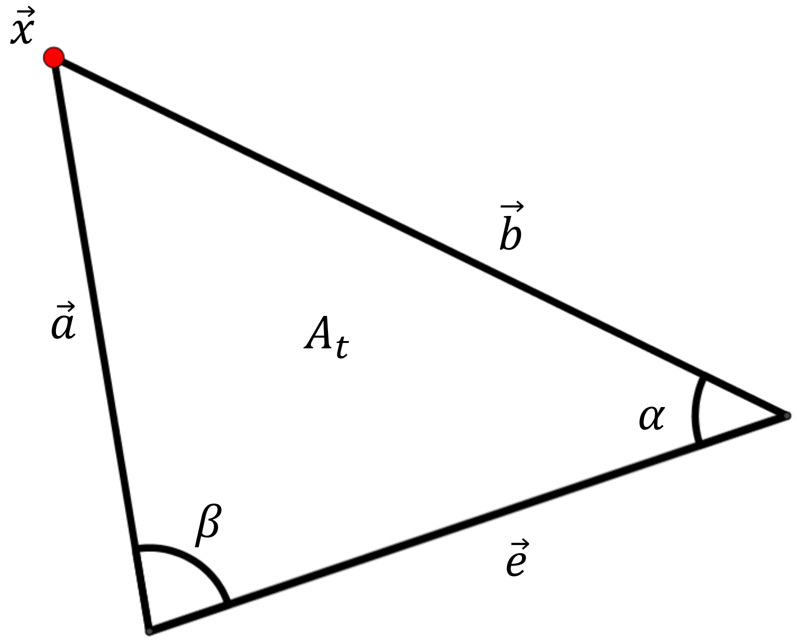
Our discrete Laplacian is given by: $$[\Delta f] = -M^{-1}S[f]$$

REMARK: Laplacian depends on the scale!. Stiffness does not!.

Area gradient

Exercise:

1. Let A_t be the area of a 2D triangle. Compute $\frac{\partial A_t}{\partial \vec{x}}$ in terms of the opposite edge $\vec{e}$.

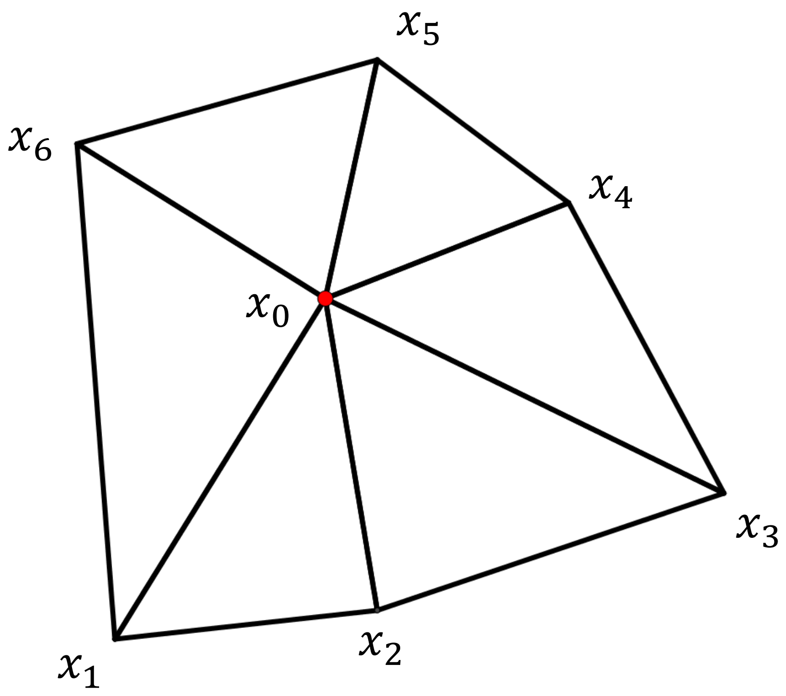


2. Show that $\frac{\partial A_t}{\partial \vec{x}} = \vec{a}\frac{\cot \alpha}{2} + \vec{b}\frac{\cot \beta}{2}$

Area gradient

Exercise:

1. Let A be the area of the one ring neighborhood of a interior vertex in a flat triangulation . Compute $\frac{\partial A}{\partial x_0}$ in terms of x_i and the interior angles.



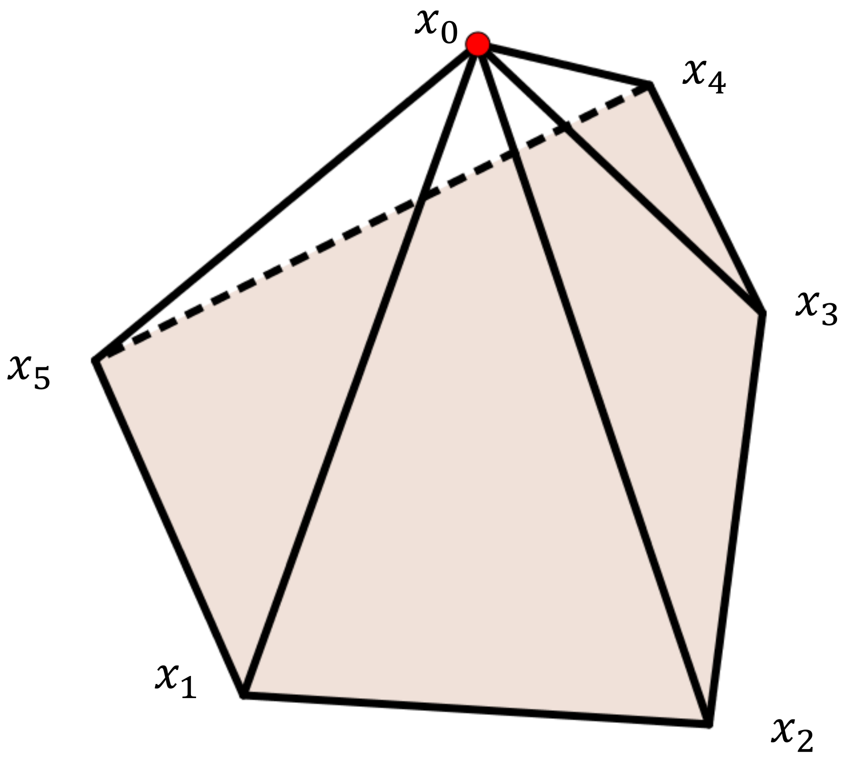
2. Conclude that $\frac{\partial A}{\partial x_k} = (SX)_k = 0$, where S is the stiffness matrix.

3. Let $f: R^2 \to R$ be an affine linear function, i.e., $f(x) = a^\top x + b$, for some $a \in R^2$ and $b \in R$. Show that $Sf = 0$.

Area gradient

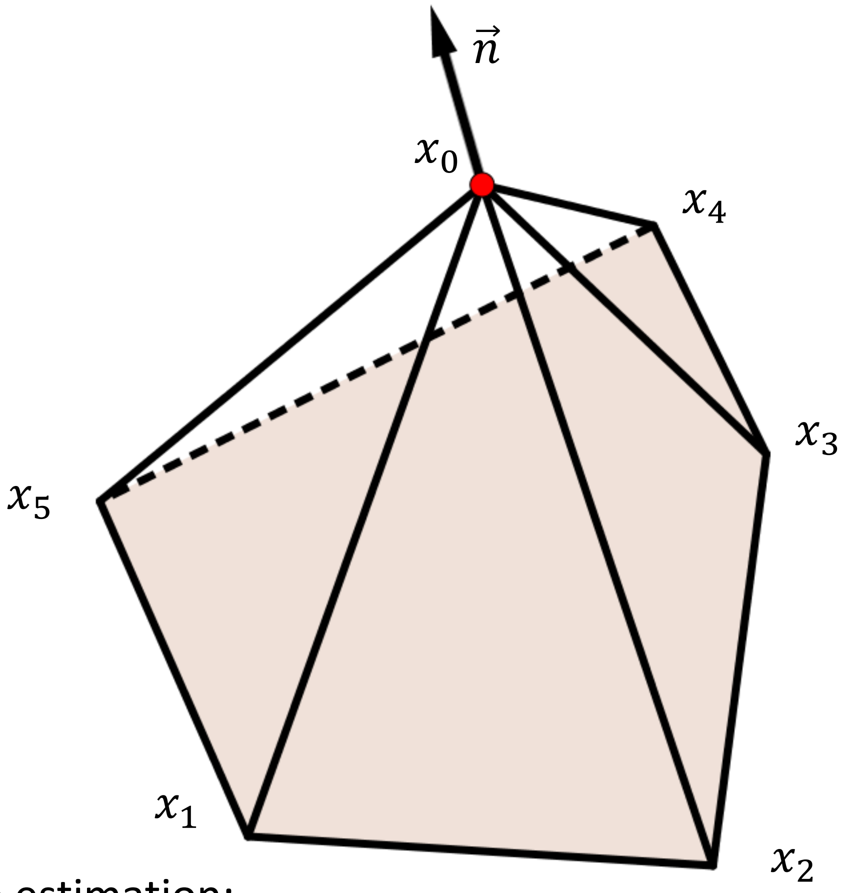
Exercise:

1. What happened if the triangulation is NOT FLAT anymore?. $\frac{\partial A}{\partial x_k} = 0$ holds ? $\frac{\partial A}{\partial x_k} = (SX)_k$ holds?



Mean Curvature Theorem

$$\Delta X(p) = -2\vec{n}(p)H(p)$$

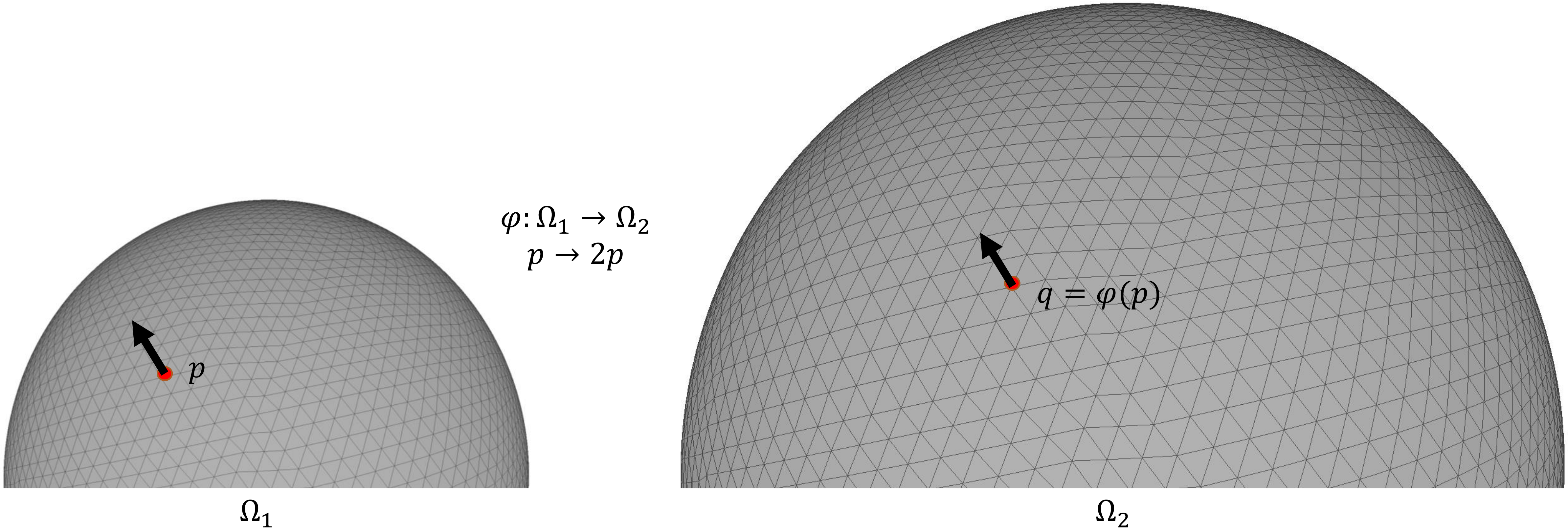


Recall. In the discrete case we use the estimation:

$$\Delta X = -M^{-1}SX$$

Recall. Mean curvature in the continuous case:

$$H(p) = \frac{\kappa_1(p) + \kappa_2(p)}{2}$$



In the continuous case, What is the relation between $H(p)$ and $H(q)$?

Let (M_i, S_i) be the mass and stiffness matrix of mesh Ω_i. What is the relation between M_0 and M_1?. What is the relation between S_0 and S_1?

What is the relation between $M_1^{-1} S_1 X_1$ and $M_2^{-1} S_2 X_2$?

Mean Curvature Flow

$$\frac{\partial X}{\partial t} = -2\vec{n}H = \Delta X$$

Explicit Step:

$$X_t = X_0 + t\Delta X_0$$

$$[X_t] = [X_0] - tM^{-1}S[X_0]$$

What happen if $t \to \infty$?

Implicit Step:

$$X_0 = X_t - t\Delta X_t$$

$$X_t = (I - t\Delta)^{-1}X_0$$

$$X_t = (M + tS)^{-1}MX_0$$

What happen if $t \to \infty$?

Mean Curvature Flow

$$\frac{\partial X}{\partial t} = -2\vec{n}H = \Delta X$$

Implicit Step:

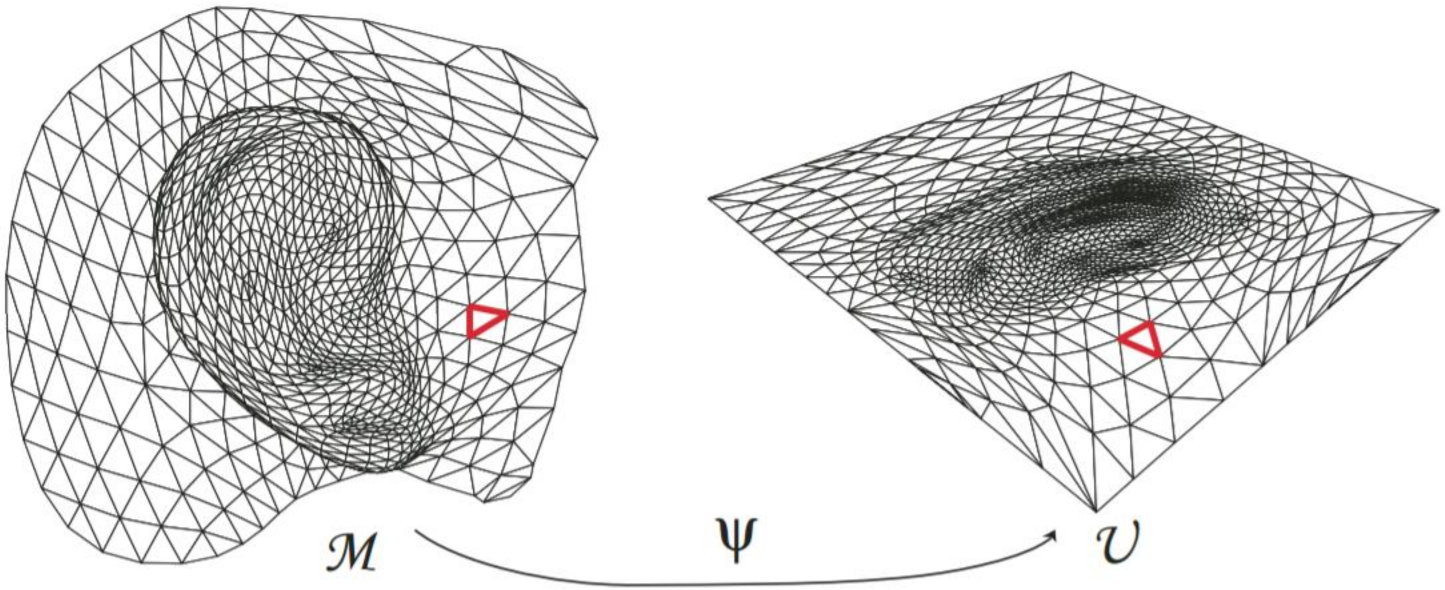
$$X_t = (M + tS)^{-1} M X_0$$

$$X_t = \text{argmin}_X \, |X - X_0|^2 + t|\nabla X\,|^2$$

Mesh Parametrization

Problem: Unwrap a patch of a mesh (topologically a disk) to a flat domain.

$$\Psi: M \;\rightarrow V \subset R^2$$



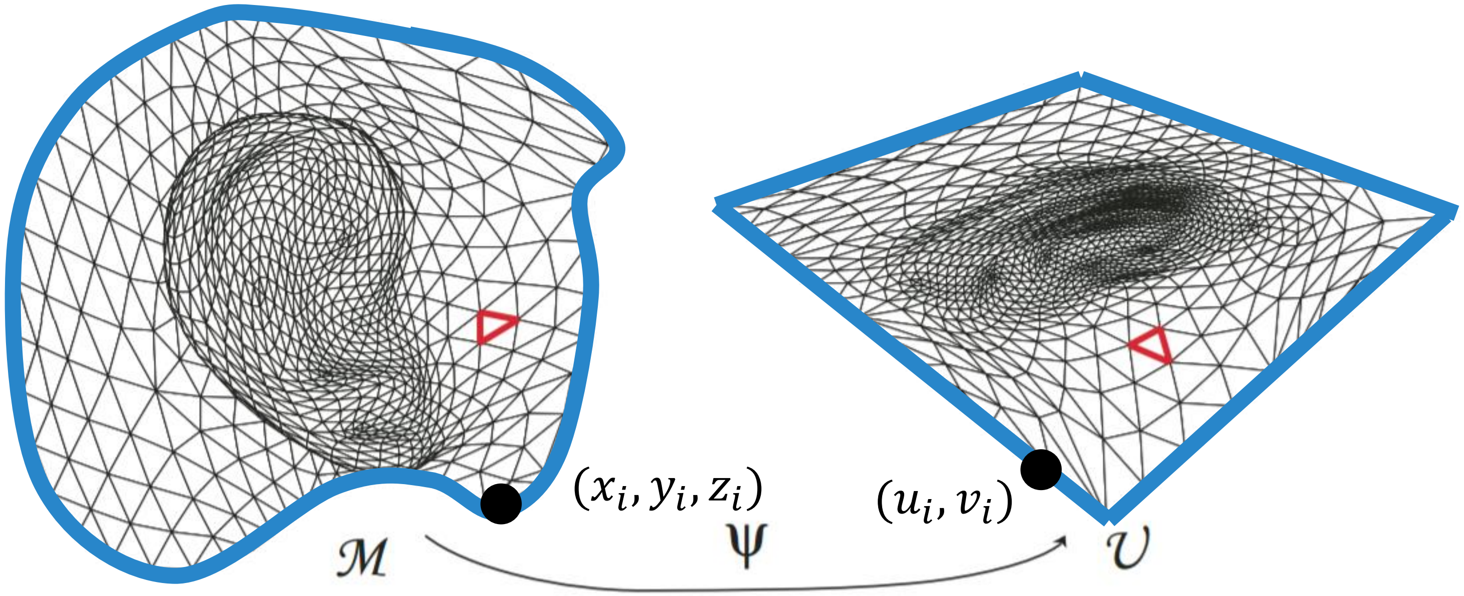
Possible goals:

Preserve angles -> Can this be perfectly achieved?
Preserve areas -> Can this be perfectly achieved?
Minimize distortion -> What is a good metric?

Desbrun et. al, *Intrinsic Parameterizations of Surface Meshes*

Smooth Filling

1. Fix the boundary: $(x_i, y_i, z_i) \in \partial M \to (u_i, v_i) \in \partial V$



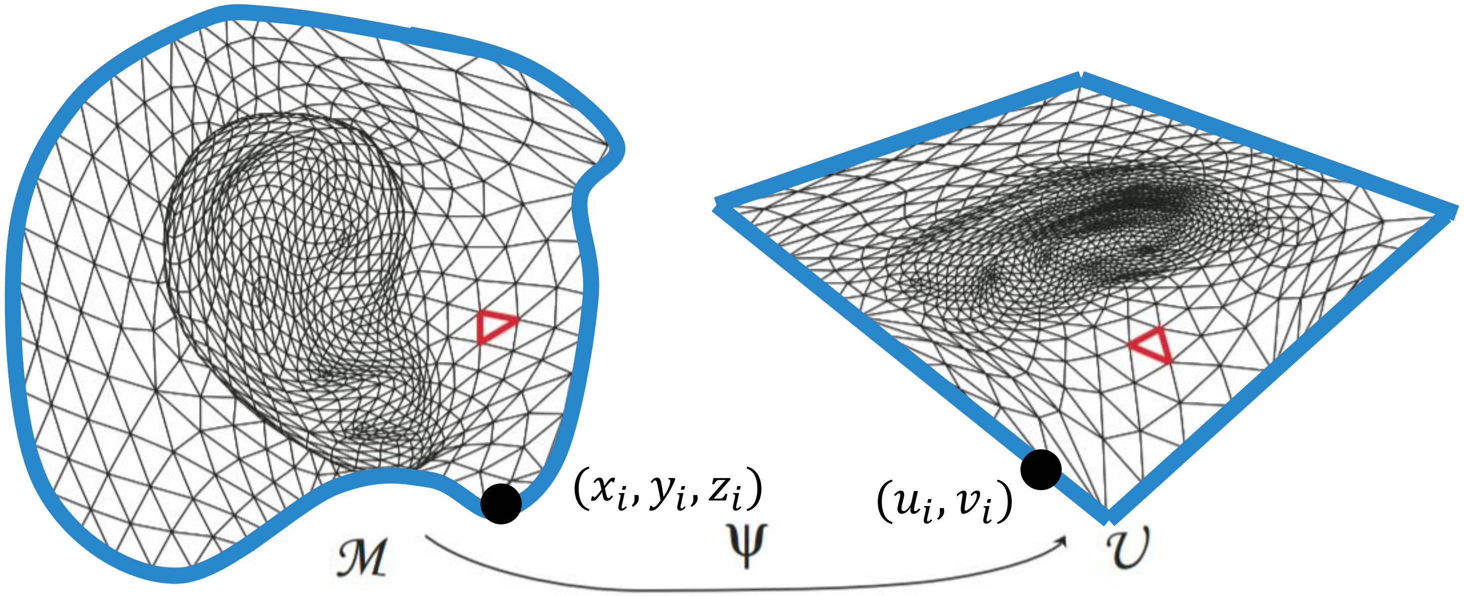
2. Solve for interior:

$$\min_{\Psi=(u,v)} |\nabla u|^2 + |\nabla v|^2$$

$$u(x_i, y_i, z_i) = u_i\,,\; v(x_i, y_i, z_i) = v_i \quad \forall\, (x_i, y_i, z_i) \in \partial M$$

As conformal as possible

1. Fix the boundary: $(x_i, y_i, z_i) \in \partial M \to (u_i, v_i) \in \partial V$



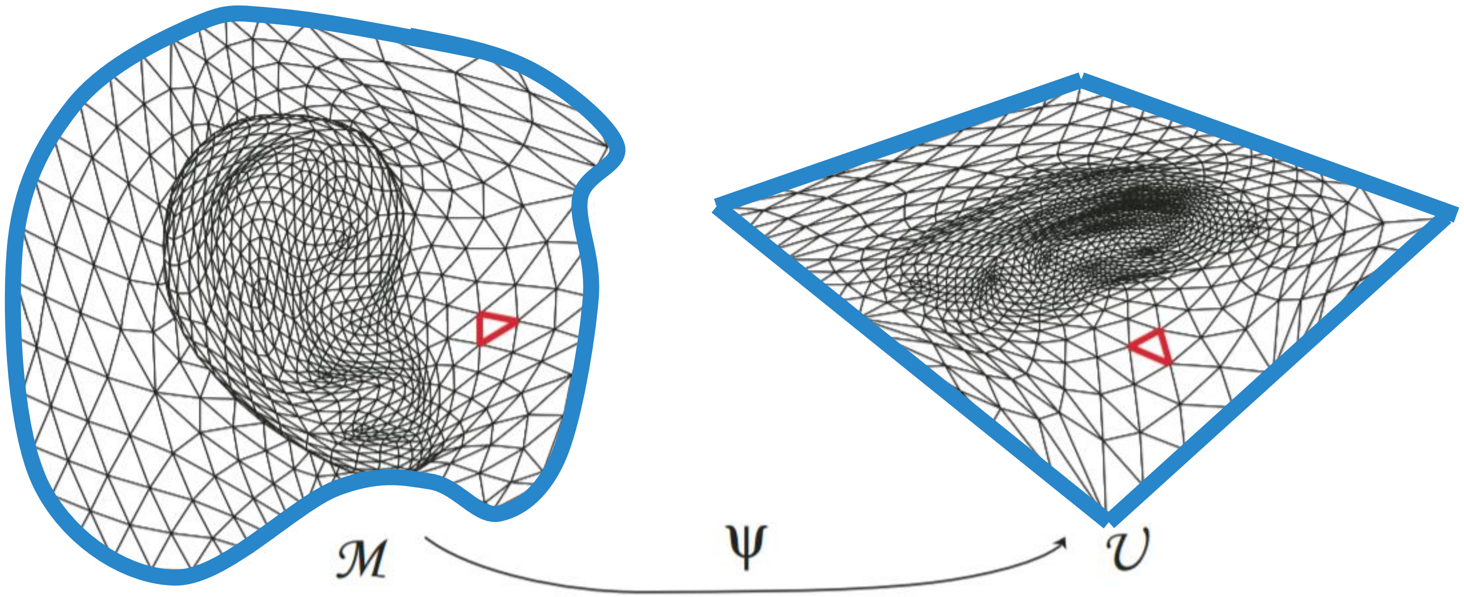
2. Solve for interior:

$$\min_{\Psi=(u,v)} |\nabla u - J \nabla v|^2$$

$$u(x_i, y_i, z_i) = u_i\,,\; v(x_i, y_i, z_i) = v_i \quad \forall\, (x_i, y_i, z_i) \in \partial M$$

Exercise:

Fix the boundary: $(x_i, y_i, z_i) \in \partial M \rightarrow (u_i, v_i) \in \partial V$



Prove: $$|\nabla u - J\nabla v|^2 = |\nabla u|^2 + |\nabla v|^2 - \mathrm{Area}(V)$$

Smooth filling and the as conformal as possible lead to the same parametrization!